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DOI:

[10.1137/17m1112595](https://doi.org/10.1137/17m1112595)

## Document Version

Accepted author manuscript

[Link to publication record in Manchester Research Explorer](#)

## Citation for published version (APA):

Jasra, A., Kamatani, K., Law, K., & Zhou, Y. (2018). Bayesian Static Parameter Estimation for Partially Observed Diffusions via Multilevel Monte Carlo. *SIAM Journal on Scientific Computing*, 40(2), A887-A902. <https://doi.org/10.1137/17m1112595>

## Published in:

SIAM Journal on Scientific Computing

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## BAYESIAN STATIC PARAMETER ESTIMATION FOR PARTIALLY OBSERVED DIFFUSIONS VIA MULTILEVEL MONTE CARLO\*

AJAY JASRA<sup>†</sup>, KENGO KAMATANI<sup>‡</sup>, KODY LAW<sup>§</sup>, AND YAN ZHOU<sup>†</sup>

**Abstract.** In this article we consider static Bayesian parameter estimation for partially observed diffusions that are discretely observed. We work under the assumption that one must resort to discretizing the underlying diffusion process, for instance, using the Euler–Maruyama method. Given this assumption, we show how one can use Markov chain Monte Carlo (MCMC) and particularly particle MCMC [C. Andrieu, A. Doucet, and R. Holenstein, *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 72 (2010), 269–342] to implement a new approximation of the multilevel (ML) Monte Carlo (MC) collapsing sum identity. Our approach comprises constructing an approximate coupling of the posterior density of the joint distribution over parameter and hidden variables at two different discretization levels and then correcting by an importance sampling method. The variance of the weights are independent of the length of the observed data set. The utility of such a method is that, for a prescribed level of mean square error, the cost of this MLMC method is provably less than i.i.d. sampling from the posterior associated to the most precise discretization. However the method here comprises using only known and efficient simulation methodologies. The theoretical results are illustrated by inference of the parameters of two prototypical processes given noisy partial observations of the process: the first is an Ornstein–Uhlenbeck process and the second is a more general Langevin equation.

**Key words.** multilevel Monte Carlo, Markov chain Monte Carlo, diffusion processes

**AMS subject classifications.** 65C05, 65C20, 65C30, 65C35, 65C40, 68U20, 82C80, 60H35, 60H10

**DOI.** 10.1137/17M112595

**1. Introduction.** The hidden Markov model (HMM) is widely used in many disciplines, including applied mathematic, statistics, economics, and finance; see [2] for an overview. In this article, we are interested in HMMs that are induced by diffusions which are partially observed, discretely in time. In particular, we assume that in order to fit the model to the data, one *must* resort to a discretization of the diffusion, for instance, using Euler–Maruyama. In addition, we assume that associated to the model is a static (non-time-varying) finite dimensional parameter, which one is interested to infer given a fixed length data record of length  $n$  and given a prior on the parameter. In simple terms, the discretization of level  $h$  (independent of all other parameters of the model), where as  $h \rightarrow 0$  one obtains the exact diffusion, induces a posterior  $\Pi_h$  on the static parameter  $\theta$  and hidden states at some observation times  $X_0, X_1, \dots, X_n$  denoted by  $X_{0:n}$ . Without any loss of generality we have assumed the time window in between observations is 1, and we will always assume that  $h < 1$ . We seek to approximate  $\mathbb{E}_{\Pi_h}[\varphi(\theta, X_{0:n})]$  for appropriately defined

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\*Submitted to the journal’s Methods and Algorithms for Scientific Computing section January 20, 2017; accepted for publication (in revised form) November 13, 2017; published electronically DATE.  
<http://www.siam.org/journals/sisc/x-x/M11259.html>

**Funding:** The first and fourth authors’ work was supported by an AcRF tier 2 grant: R-155-000-161-112. The first and third authors’ work was supported by CREST. The third author’s work was supported by ORNL LDRD Strategic Hire grant 32112580.

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real-valued functions. Ultimately, one might seek to remove the dependence upon  $h$  and get the exact expectation with no discretization bias. We remark that the model will be formally introduced in the next section. This framework is relevant to a broad range of applications in science and engineering; see [2, 18]

The task of computing the expectation for any fixed  $h > 0$  is a nontrivial task, which often requires quite advanced Monte Carlo methods. As has been remarked in many articles in the literature, often the joint correlation between  $\theta$  and  $X_{0:n}$  means even standard MCMC methods may produce very inaccurate or inefficient approximations of the expectation of interest, despite their theoretical validity. An important algorithm that has, to an extent, helped to alleviate these difficulties is the particle MCMC (PMCMC) methods of [1] and their subsequent developments (e.g., [4]). Intrinsically, this method uses a sequential Monte Carlo (SMC) (e.g., [7]) method to help move the samples around the state-space, for instance, inside a Metropolis–Hastings acceptance/rejection scheme, although Gibbs versions also exist. PMCMC delivers a Markov chain which provides consistent estimates of expectations of the form  $\mathbb{E}_{\Pi_h}[\varphi(\theta, X_{0:n})]$ , for any fixed  $h$  SMC methods are well-known as being efficient techniques for filtering, when the state-variable at time  $k$ ,  $X_k$ , is of moderate to low dimension and all the static parameters are fixed.

In the context of this article, there is an additional degree of freedom, which can be utilized to further enhance the PMCMC method. This is associated to the discretization level  $h$ . We consider using the multilevel Monte Carlo (MLMC) framework [8, 9, 12]. This allows one to leverage in an optimal way the nested problems arising in this context, hence minimizing the necessary cost to obtain a given level of mean square error. Set  $\Pi_0$  as the posterior on  $\theta, X_{0:n}$  with no discretization bias and  $\Pi_{h_l}$  as the time-discretized posterior on  $\theta, X_{0:n}$  with time discretization  $h_l$ , and one has for an integrable, real-valued function  $\varphi$  and  $+\infty > h_0 > h_1 > \dots > h_L > 0$  (the levels)

$$(1) \quad \mathbb{E}_{\Pi_{h_L}}[\varphi(\theta, X_{0:n})] = \sum_{l=0}^L \{\mathbb{E}_{\Pi_{h_l}}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_{h_{l-1}}}[\varphi(\theta, X_{0:n})]\},$$

where  $\mathbb{E}$  is the expectation operator and  $\mathbb{E}_{\Pi_{h_{-1}}}[\varphi(\theta, X_{0:n})] := 0$ . The idea of MLMC is then to approximate each summand by independently simulating  $N_l$  samples from a dependent coupling of  $(\Pi_{h_l}, \Pi_{h_{l-1}})$ . In such scenarios, one can show that the overall mean square error (MSE) associated to the approximation of  $\mathbb{E}_{\Pi_0}[\varphi(\theta, X_{0:n})]$  is

$$(2) \quad \text{MSE} = \text{Bias}(L, \varphi)^2 + \sum_{l=0}^L \frac{V_l}{N_l},$$

where

$$(3) \quad \text{Bias}(L, \varphi) = |\mathbb{E}_{\Pi_{h_L}}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_0}[\varphi(\theta, X_{0:n})]|,$$

and  $0 < V_l < +\infty$  are a collection of constants. It is remarked that it is the coupled samples which induce  $V_l$  to be a function of  $h_l$  which is often critical as we explain below. Assuming the cost of  $C_l$  per level, per sample, the cost of the algorithm is then  $\sum_{l=0}^L C_l N_l$ . Fixing  $\epsilon > 0$  and given an appropriate parameterization of  $h_l$  (e.g.,  $h_l = 2^{-l}$ ), one then chooses  $L$  to ensure that  $\text{Bias}(L, \varphi)^2 = \mathcal{O}(\epsilon^2)$  and then given  $C_l, V_l$  characterized as a function of  $h_l$  optimizes  $N_0, \dots, N_L$  to minimize the cost so that the term  $\sum_{l=0}^L \frac{V_l}{N_l} = \mathcal{O}(\epsilon^2)$ ; Giles [8] gives the solution to this constrained optimization problem. In many scenarios of practical interest the associated MLMC

algorithm can achieve a MSE of  $\mathcal{O}(\epsilon^2)$  at a cost which is less than i.i.d. sampling from  $\Pi_{h_L}$ ; note that this has not yet been established in the problem under study here. The main issue is that sampling independently from the couples  $(\Pi_{h_l}, \Pi_{h_{l-1}})$  is not possible in our context.

In this paper we show how to implement a new approximation of the multilevel collapsing sum identity. Our approach comprises constructing an approximate coupling of the posterior density of the joint on the parameter and hidden space at two different discretization levels and then correcting by an importance sampling method, whose variance of the weights are independent of the length of the observed data set. The utility of such a method is that it comprises using known and efficient simulation methodologies, instead of coupling algorithms as explored in [14, 15, 16, 20]. In particular, our approach facilitates a mathematical analysis which allows us to establish that our approach can be better than sampling (e.g., by PMCMC) from the posterior associated to the most precise discretization. The algorithm presented here is distinct from either of the previously introduced multilevel MCMC (MLMCMC) algorithms [13, 17] and may be generalized.

This article is structured as follows. In section 2 the model is described. In section 3 we describe our approach and give a mathematical result associated to the MSE of the method. In section 4 we give practical simulations to establish the theory. The appendix contains some of the proofs for the result of section 3 and some details on coupling Euler approximations and verifying the assumptions for our main result.

**2. Model.** We consider the following partially observed diffusion process:

$$(4) \quad dX_t = a_\theta(X_t)dt + b_\theta(X_t)dW_t$$

with  $X_t \in \mathbb{R}^d = \mathbb{S}$ ,  $t \geq 0$ ,  $X_0$  has initial measure  $\mu_\theta(dx)$  and  $\{W_t\}_{t \in [0, T]}$  a Brownian motion of appropriate dimension.  $\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$  is a static parameter of interest. The following assumptions will be made on the diffusion process.

*Assumption 2.1.* Measurable functions  $a_\cdot : \Theta \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $b_\cdot : \Theta \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  satisfy the following:

- (i) *Global Lipschitz property:* There is a  $C > 0$  such that  $|a_\theta(x) - a_\theta(y)| + |b_\theta(x) - b_\theta(y)| \leq C|x - y|$  for all  $x, y \in \mathbb{S}$  and all  $\theta \in \Theta$ .
- (ii) *Bounded moments:*  $\sup_{\theta \in \Theta} \mathbb{E}_\theta |X_0|^p < \infty$  for all  $p \geq 1$ .

Notice that (i) and (ii) together imply that  $\mathbb{E}_\theta |X_t|^p < \infty$  for all  $t \geq 0$ .

It will be assumed that the data are regularly spaced (i.e., in discrete time) observations  $y_1, \dots, y_n$ ,  $y_k \in \mathbb{R}^m = \mathbb{T}$ . These are observed at one unit of time. It is noted that we assume that one does not have access to a nonnegative and unbiased estimate of the transition density of the diffusion and we are forced to work with a discretized process.

The above formulation can then be summarized as follows, on discretizing the diffusion process with discretization level  $h > 0$ ;  $h = 0$  corresponds to the limiting diffusion process. We have a pair of discrete-time stochastic processes,  $\{X_n\}_{n \geq 0}$  and  $\{Y_n\}_{n \geq 1}$ , where  $X_n \in \mathbb{S}$  (with associated  $\sigma$ -algebra  $\mathcal{S}$ ) is an unobserved process and  $y_n \in \mathbb{T}$  (with associated  $\sigma$ -algebra  $\mathcal{T}$ ) is observed. Let  $\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$  be a parameter. The hidden process  $\{X_n\}$  is a Markov chain with initial measure  $\mu_\theta(dx)$  at time 0 and transition kernel  $F_{\theta, h}(x_{p-1}, dx_p)$ , i.e., for each  $\theta \in \Theta$

$$(5) \quad \mathbb{P}_{\theta, h}(X_0 \in A) = \mu_\theta(A) \quad \text{and} \quad \mathbb{P}_{\theta, h}(X_p \in A | X_{p-1} = x_{p-1}) = F_{\theta, h}(x_{p-1}, A), \quad p \geq 1,$$

where  $\mathbb{P}_{\theta, h}$  denotes probability,  $A \in \mathcal{S}$ . Note that under most discretizations (e.g.,

Euler) the transition density is not analytically tractable for many diffusion processes, but can be simulated exactly. In addition, the observations  $\{Y_n\}_{n \geq 1}$  conditioned upon  $\{X_n\}_{n \geq 0}$  are statistically independent and have conditional measure with probability density function  $g_\theta(x, y)$  with respect to a dominating  $\sigma$ -finite measure  $dy$  on  $\mathbb{T}$ , i.e.,

$$(6) \quad \mathbb{P}_{\theta, h}(Y_n \in B | \{X_k\}_{k \geq 0} = \{x_k\}_{k \geq 0}) = \int_B g_\theta(x_n, y) dy$$

with  $B \in \mathcal{T}$ . We assume  $g : \Theta \times \mathbb{S} \times \mathbb{T}$  is jointly measurable. The HMM is given by (5)–(6) and is often referred to in the literature as a state-space model. In our context  $\theta \in \Theta$  is a parameter of interest with prior measure  $\Pi(d\theta)$  with probability density function  $\pi(\theta)$  with respect to the Lebesgue measure  $d\theta$  on  $\mathbb{R}^{d_\theta}$ .

Given the joint measure on  $\mathbb{U} := \Theta \times \mathbb{S}^{n+1}$

$$(7) \quad \Pi_h(d\theta, dx_{0:n}) \propto \Pi(d\theta) \mu_\theta(dx_0) \prod_{p=1}^n g_\theta(x_p, y_p) F_{\theta, h}(x_{p-1}, dx_p)$$

for  $\varphi \in \mathcal{B}_b(\mathbb{U}) \cap \text{Lip}(\mathbb{U})$ , where  $\mathcal{B}_b(\mathbb{U})$  are the bounded and real-valued measurable functions on  $\mathbb{U}$  and  $\text{Lip}(\mathbb{U})$  are the Lipschitz, measurable functions on  $\mathbb{U}$ , and for  $+\infty > h_0 > \dots > h_L > 0$  we would like to compute

$$(8) \quad \mathbb{E}_{\Pi_{h_L}}[\varphi(\theta, X_{0:n})] = \sum_{l=0}^L \left\{ \mathbb{E}_{\Pi_{h_l}}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_{h_{l-1}}}[\varphi(\theta, X_{0:n})] \right\},$$

where  $\mathbb{E}_{\Pi_{h_{-1}}}[\cdot] = 0$ . We will use the MLMC approach.

Consider only a single pair  $\mathbb{E}_{\Pi_h}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_{h'}}[\varphi(\theta, X_{0:n})]$ ,  $h < h'$ . It is well known that if one can sample from a dependent coupling of  $(\Pi_h, \Pi_{h'})$ , such as the maximal coupling, then Monte Carlo estimation of such a difference can be performed at a lower cost than i.i.d sampling from the independent coupling of  $(\Pi_h, \Pi_{h'})$  [8, 9]. The main issue is that such couplings are typically not available, even up-to a nonnegative and unbiased estimator. We consider the scenario where one samples from a sensible, approximate, coupling and corrects via importance sampling.

### 3. Method and analysis.

**3.1. Method.** We are to approximate the identity (8). Our procedure, when considering the summands from  $1, \dots, L$  will be to run  $L$  independent pairs of the idea to be described below. The case  $l = 0$  is simply using (e.g.,) PMCMC to approximate  $\mathbb{E}_{\Pi_{h_0}}[\varphi(\theta, X_{0:n})]$ ; we refer the reader to [1] for details on PMCMC—a simple description is below. We only consider a pair  $\mathbb{E}_{\Pi_h}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_{h'}}[\varphi(\theta, X_{0:n})]$ ,  $h < h'$ . The methodology and analysis in this context of one pair will suffice to justify our approach as we will explain below.

Let  $z = (x, x') \in \mathbb{S} \times \mathbb{S} = \mathbb{Z}$  and  $Q_{\theta, h, h'}(z, d\bar{z})$  be any coupling (other than the independent one) of  $(F_{\theta, h}(x, d\bar{x}), F_{\theta, h'}(x', d\bar{x}'))$ . For instance, in the context of an Euler discretization a description can be found in [16] (see also Appendix B). Let  $G_{p, \theta}(z) = \max\{g_\theta(x, y_p), g_\theta(x', y_p)\}$ . (Note that alternative choices of  $G_{p, \theta}$  are possible, and this point is revisited below.) We propose to sample from the probability measure on  $\mathbb{V} = \Theta \times \mathbb{S}^{2n+2}$  (write the associated  $\sigma$ -algebra as  $\mathcal{V}$ )

$$\Pi_{h, h'}(d\theta, dz_{0:n}) \propto \Pi(d\theta) \nu_\theta(dz_0) \prod_{p=1}^n G_{p, \theta}(z_p) Q_{\theta, h, h'}(z_{p-1}, dz_p),$$

where  $\nu_\theta$  is the initial coupling

$$\nu_\theta(d(x, x')) = \mu_\theta(dx)\delta_x(dx').$$

Here,  $\delta_x$  is the Dirac measure. Then for  $\varphi \in \mathcal{B}_b(\mathbf{U}) \cap \text{Lip}(\mathbf{U})$ ,

$$(9) \quad \begin{aligned} & \mathbb{E}_{\Pi_h}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_{h'}}[\varphi(\theta, X_{0:n})] \\ &= \frac{\mathbb{E}_{\Pi_{h,h'}}[\varphi(\theta, X_{0:n})H_{1,\theta}(\theta, Z_{0:n})]}{\mathbb{E}_{\Pi_{h,h'}}[H_{1,\theta}(\theta, Z_{0:n})]} - \frac{\mathbb{E}_{\Pi_{h,h'}}[\varphi(\theta, X'_{0:n})H_{2,\theta}(\theta, Z_{0:n})]}{\mathbb{E}_{\Pi_{h,h'}}[H_{2,\theta}(\theta, Z_{0:n})]}, \end{aligned}$$

where

$$\begin{aligned} H_{1,\theta}(\theta, z_{0:n}) &= \prod_{p=1}^n \frac{g_\theta(x_p, y_p)}{G_{p,\theta}(z_p)}, \\ H_{2,\theta}(\theta, z_{0:n}) &= \prod_{p=1}^n \frac{g_\theta(x'_p, y_p)}{G_{p,\theta}(z_p)}. \end{aligned}$$

We note that our choice of  $G_{p,\theta}(z)$  ensures that  $H_{1,\theta}$  and  $H_{2,\theta}$  are uniformly upper-bounded by 1 and hence that the variance w.r.t. any probability does not grow without bound as  $n$  increases. Another possible choice would be  $G_{p,\theta}(z) = \min\{g_\theta(x, y_p), g_\theta(x', y_p)\}$ , but this choice does not share the desirable stability property which ours does. The exploration of alternative importance distributions, for instance, minimizing the variance, could be adopted.

**3.1.1. Particle MCMC.** Let  $(W, \mathcal{W})$  be a measurable space such that  $\mathbf{V} \subseteq W$ . Let  $K : W \times W \rightarrow [0, 1]$  be any ergodic Markov kernel of invariant measure  $\eta$  such that one can consistently estimate expectations w.r.t.  $\Pi_{h,h'}$ . For instance, if for every  $A \in \mathcal{V}$

$$\int_{A \times (W \setminus \mathbf{V})} \eta(dw) = \Pi_{h,h'}(A).$$

Our construction allows a particle MCMC approach to be adopted, which is not quite as the displayed equation, but nonetheless allows one to infer  $\Pi_{h,h'}$ . We focus on one particle MCMC method for completeness, but we reiterate that one can use the analysis here for more advanced versions of the algorithm, or indeed, any MCMC of the form above.

We will now describe the particle marginal Metropolis–Hastings (PMMH) algorithm. Let  $M \geq 1$  and  $\theta$  be fixed, and introduce random variables  $a_{0:n-1} \in \{1, \dots, M\}^n$ , which will denote the indices of the selected particles upon resampling at the given steps. One can run a particle filter [5] to approximate

$$\Pi_{h,h'}(dz_{0:n}|\theta) \propto \nu_\theta(dz_0) \prod_{p=1}^n G_{p,\theta}(z_p) Q_{\theta,h,h'}(z_{p-1}, dz_p)$$

by sampling from the following joint, on the space  $\{1, \dots, M\}^n \times Z^{M(n+1)}$

$$(10) \quad \begin{aligned} & P(\{a_{0:n-1}^{1:M}\}, dz_{0:n}^{1:M}|\theta) \\ &= \left( \prod_{i=1}^M \nu_\theta(dz_0^i) \right) \prod_{p=1}^n \prod_{i=1}^M \left( \frac{G_{p-1,\theta}(z_{p-1}^{a_{p-1}^i})}{\sum_{j=1}^M G_{p-1,\theta}(z_{p-1}^j)} Q_{\theta,h,h'}(z_{p-1}^{a_{p-1}^i}, dz_p^i) \right), \end{aligned}$$

where  $G_{0,\theta} := 1$ . Note that this concise equation unambiguously defines the bootstrap particle filter algorithm at the same time as its joint distribution, i.e.,  $z_{p-1}^{a_{p-1}^i}$  is resampled with the probability  $\frac{G_{p-1,\theta}(z_{p-1}^{a_{p-1}^i})}{\sum_{j=1}^M G_{p-1,\theta}(z_{p-1}^j)}$ . Better algorithms can be constructed, but we just present the most simple approach. We remark that

$$(11) \quad p_{h,h'}^M(y_{0:n}|\theta) = \prod_{p=1}^n \left( \frac{1}{M} \sum_{j=1}^M G_{p,\theta}(z_p^j) \right)$$

is an unbiased estimator of  $p_{h,h'}(y_{0:n}|\theta) = \int_{Z^{n+1}} \nu_\theta(dz_0) \prod_{p=1}^n G_{p,\theta}(z_p) Q_{\theta,h,h'}(z_{p-1}, dz_p)$ ; see [5].

The PMMH algorithm works as follows. The superscripts for  $(\theta, k)$  are the iteration (time) counter of the MCMC.

1. Initialize: Sample  $\theta^0$  from the prior and then sample  $(a_{0:n-1}^{1:M}, z_{0:n}^{1:M})$  from  $p(a_{0:n-1}^{1:M}, z_{0:n}^{1:M}|\theta^0)$  as in (10), and store  $p_{h,h'}^M(y_{0:n}|\theta^0)$  as in (11). Select a path  $\hat{z}_{0:n}^j$ , constructed by drawing  $\hat{z}_n^j$  with probability proportional to  $G_{n,\theta^0}(\hat{z}_n^j)$ , then redefining  $\hat{z}_{n-1}^j = z_{n-1}^{a_{n-1}^j}$ ,  $\hat{z}_{n-2}^j = z_{n-2}^{a_{n-2}^j}$ , and so on; set  $k^0$  as the index of the selected path. Set  $i = 1$ .
2. Iterate: Sample  $\theta^i|\theta^{i-1}$  according to a proposal with conditional density  $R(d\theta^i|\theta^{i-1}) = r(\theta^i|\theta^{i-1})d\theta^i$  and then from  $P(da_{0:n-1}^{1:M}, dz_{0:n}^{1:M}|\theta^i)$  as in (10). Select a path  $\hat{z}_{0:n}^j$  with probability proportional to  $G_{n,\theta^i}(\hat{z}_n^j)$  and constructed as described above; set  $k^i$  as the index of the selected path. Set  $\theta^i = \theta'$ ,  $k^i = k'$  with probability

$$\min \left\{ 1, \frac{p_{h,h'}^M(y_{0:n}|\theta')}{p_{h,h'}^M(y_{0:n}|\theta^{i-1})} \frac{\pi(\theta')r(\theta^{i-1}|\theta')}{\pi(\theta^{i-1})r(\theta^i|\theta^{i-1})} \right\};$$

otherwise  $\theta^i = \theta^{i-1}$ ,  $k^i = k^{i-1}$ . Set  $i = i + 1$  and return to the start of 2.

We denote by  $K$  the PMMH kernel and denote by  $(\mathbb{W}, \mathcal{W})$  the measurable space for which it is defined upon. The invariant measure is denoted  $\eta$ . For the analysis, we assume the MCMC algorithm is started in stationarity.

Then one estimates (9) by

$$\frac{\frac{1}{N} \sum_{i=1}^N \varphi(\theta^i, x_{0:n}^{k^i}) H_{1,\theta}(\theta^i, z_{0:n}^{k^i})}{\frac{1}{N} \sum_{i=1}^N H_{1,\theta}(\theta^i, z_{0:n}^{k^i})} - \frac{\frac{1}{N} \sum_{i=1}^N \varphi(\theta^i, x_{0:n}^{k^i}) H_{2,\theta}(\theta^i, z_{0:n}^{k^i})}{\frac{1}{N} \sum_{i=1}^N H_{2,\theta}(\theta^i, z_{0:n}^{k^i})}.$$

This estimate is consistent in the limit as  $N$  grows; see [1]. To simplify the notation we replace  $k^i$  in the superscripts by  $i$  from here on.

**3.2. Multilevel considerations.** As described for MLMC in the introduction, we will approximate the expectation using the telescopic sum identity given in (1). We will establish error estimates for

$$(12) \quad \sum_{l=0}^L \bar{E}_l^{N_l}(\varphi), \quad \bar{E}_l^{N_l}(\varphi) = E_l^{N_l}(\varphi) - E_l(\varphi),$$

where

$$(13) \quad E_l^{N_l}(\varphi) = \frac{\frac{1}{N_l} \sum_{i=1}^{N_l} \varphi(\theta^i, x_{0:n}^i) H_{1,\theta}(\theta^i, z_{0:n}^i)}{\frac{1}{N_l} \sum_{i=1}^{N_l} H_{1,\theta}(\theta^i, z_{0:n}^i)} - \frac{\frac{1}{N_l} \sum_{i=1}^{N_l} \varphi(\theta^i, x_{0:n}^i) H_{2,\theta}(\theta^i, z_{0:n}^i)}{\frac{1}{N_l} \sum_{i=1}^{N_l} H_{2,\theta}(\theta^i, z_{0:n}^i)}$$

is a consistent estimator of  $E_l(\varphi) := \mathbb{E}_{\Pi_{h_l}}[\varphi(\theta, X_{0:n})] - \mathbb{E}_{\Pi_{h_{l-1}}}[\varphi(\theta, X_{0:n})]$ . Therefore (12) is a consistent estimator of  $\mathbb{E}_{\Pi_{h_L}}[\varphi(\theta, X_{0:n})]$  and the MSE (2) can be bounded, up to a constant, by the sum of the squared error of (12) and  $\text{Bias}(L, \varphi)^2$ , as given by (3), which is  $\mathcal{O}(h_L)$ , for example, using Euler–Maruyama.

Using  $\mathbb{E}$  to denote the expectation w.r.t. the law associated to our algorithm, assuming the Markov chain is started in stationarity, our objective is therefore to investigate

$$(14) \quad \mathbb{E} \left[ \left( \sum_{l=0}^L \bar{E}_l^{N_l}(\varphi) \right)^2 \right] = \sum_{l=0}^L \mathbb{E} \left[ \bar{E}_l^{N_l}(\varphi)^2 \right]$$

so as to optimally allocate  $N_0, \dots, N_L$  as described in the introduction. Thus we must investigate terms such as  $\mathbb{E}[\bar{E}_l^{N_l}(\varphi)^2]$  for a given  $l$ .

**3.3. Algorithm analysis.** Below  $\mathcal{P}(\mathbb{W})$  are the collection of probability measures on  $(\mathbb{W}, \mathcal{W})$ . The labelling of the assumptions here is different to that of Assumption 2.1 as they are made about the discretized model (7) and PMCMC kernel, not necessarily directly about the underlying diffusion process.

(A1) For every  $y \in \mathbb{T}$  there exist  $0 < \underline{C} < \bar{C} < +\infty$  such that for every  $x \in \mathbb{S}$ ,  $\theta \in \Theta$ ,

$$\underline{C} \leq g_\theta(x, y) \leq \bar{C}.$$

For every  $y \in \mathbb{T}$ ,  $g_\theta(x, y)$  is globally Lipschitz on  $\mathbb{S} \times \Theta$ .

(A2) For any  $0 \leq k \leq n$ ,  $q \in \{1, 2\}$  there exists a  $\beta > 0$  such that for any  $\varphi \in \mathcal{B}_b(\Theta \times \mathbb{S}^{k+1}) \cap \text{Lip}(\Theta \times \mathbb{S}^{k+1})$  there exists a  $C < +\infty$

$$\left( \int_{\Theta \times \mathbb{S}^{2k+2}} |\varphi(\theta, x_{0:k}) - \varphi(\theta, x'_{0:k})|^q \Pi(d\theta) \nu_\theta(dz_0) \prod_{p=1}^k Q_{\theta, h, h'}(z_{p-1}, dz_p) \right)^{3-q} \leq C(h')^\beta.$$

(A3) Suppose that for any  $n > 0$  there exist a  $\xi \in (0, 1)$  and  $\nu \in \mathcal{P}(\mathbb{W})$  such that for each  $w \in \mathbb{W}$ ,  $\varphi \in \mathcal{B}_b(\mathbb{W}) \cap \text{Lip}(\mathbb{W})$ ,  $h, h'$ :

$$\int_{\mathbb{W}} \varphi(w') K(w, dw') \geq \xi \int_{\mathbb{W}} \varphi(v) \nu(dv).$$

$K$  is  $\eta$ -reversible, that is,  $\int_{w \in B} \eta(dw) K(w, A) = \int_{w \in A} \eta(dw) K(w, B)$  for any  $A, B \in \mathcal{W}$ .

We note that (A1) can be verified for some state-space models (especially if  $\mathbb{T}$  and  $\Theta$  are compact) and (A3) can be verified for a PMCMC kernel if  $\Theta, \mathbb{S}$  are compact—indeed, the constants would all be independent of  $n$  under appropriate settings of the algorithm. It is also noted that under the Assumption (A3), allowing the chain to start in stationarity is not overly strong. However, to deal with the case where the chain is not started in stationarity requires a very intricate control, for instance, of



the solution to the Poisson equation (e.g., [10]) in  $h'$ . This is undoubtedly possible, but would require technical details that are beyond the scope of the article. Finally note that conditions (A2) and (A3) are verified in Appendix C in the case of an Euler discretization and coupling (see Appendix B).

**THEOREM 3.1.** *Assume (A1)–(A3). Then for any  $n > 0$ , there exists a  $\beta > 0$  such that for any  $\varphi \in \mathcal{B}_b(\Theta \times \mathbb{S}^{n+1}) \cap \text{Lip}(\Theta \times \mathbb{S}^{n+1})$  there exists a  $C < +\infty$  such that*

$$\begin{aligned} & \mathbb{E} \left[ \left( \frac{\frac{1}{N} \sum_{i=1}^N \varphi(\theta^i, x_{0:n}^i) H_{1,\theta}(\theta^i, z_{0:n}^i)}{\frac{1}{N} \sum_{i=1}^N H_{1,\theta}(\theta^i, z_{0:n}^i)} - \frac{\frac{1}{N} \sum_{i=1}^N \varphi(\theta^i, x_{0:n}^i) H_{2,\theta}(\theta^i, z_{0:n}^i)}{\frac{1}{N} \sum_{i=1}^N H_{2,\theta}(\theta^i, z_{0:n}^i)} \right. \right. \\ & \quad \left. \left. - \left( \frac{\mathbb{E}_{\Pi_{h,h'}}[\varphi(\theta, X_{0:n}) H_{1,\theta}(\theta, Z_{0:n})]}{\mathbb{E}_{\Pi_{h,h'}}[H_{1,\theta}(\theta, Z_{0:n})]} - \frac{\mathbb{E}_{\Pi_{h,h'}}[\varphi(\theta, X'_{0:n}) H_{2,\theta}(\theta, Z_{0:n})]}{\mathbb{E}_{\Pi_{h,h'}}[H_{2,\theta}(\theta, Z_{0:n})]} \right) \right)^2 \right] \\ & \leq \frac{C(h')^\beta}{N}. \end{aligned}$$

*Proof.* The result follows by using [15, Lemma C.3], the  $C_2$ -inequality, the boundedness of certain quantities, and Proposition A.1. The proof is omitted as it is similar to the calculations in [15].  $\square$

**3.4. A return to multilevel considerations.** Returning to section 3.2, we assume that  $h_l = 2^{-l}$  and introduce the further assumption

(A4) The cost to simulate  $E_l^{N_l}$  in (13) is controlled by  $\mathcal{C}(E_l^{N_l}) \leq CN_l h_l^{-\gamma}$ , and the bias is controlled by

$$|\mathbb{E}_{\Pi_{h_L}}(\varphi(\theta, X_{0:n})) - \mathbb{E}_{\Pi_0}(\varphi(\theta, X_{0:n}))| \leq Ch_L^\alpha$$

for  $\gamma, \alpha, C > 0$ .

Following assumption (A2),  $\alpha = \beta/2$  satisfies the above, but it may be larger, e.g., for Euler–Maruyama in which  $\alpha = \beta$ .

Given  $\epsilon > 0$ , in order to ensure the MSE is  $\mathcal{O}(\epsilon^2)$ , the term (3) must be  $\mathcal{O}(\epsilon^2)$ . Following from Assumption (A2), it suffices to let  $L \propto 2|\log(\epsilon)|/\beta$  so that  $h_L = \epsilon$ . Following from Theorem 3.1,

$$\sum_{l=0}^L \mathbb{E} \left[ \bar{E}_l^{N_l}(\varphi)^2 \right] \leq C \sum_{l=0}^L \frac{h_l^\beta}{N_l},$$

and note that the constant  $C$  may depend upon the time parameter  $n$ , which has been suppressed from the notation; we return to this point below.

Suppose we minimize  $\text{COST} = \sum_{l=0}^L h_l^{-\gamma} N_l$  subject to  $\sum_{l=0}^L \frac{h_l^\beta}{N_l} = \mathcal{O}(\epsilon^2)$  as a function of  $N_0, \dots, N_L$ . This is exactly considered in [8] for  $\gamma = 1$  and later in [3] for  $\gamma \neq 1$  and yields that

$$(15) \quad N_l \propto \epsilon^{-2} K_L h_l^{(\beta+\gamma)/2},$$

where  $K_L = \sum_{l=0}^L h_l^{(\beta-\gamma)/2}$  (see also [15, 6]). This gives a cost of  $\mathcal{O}(\epsilon^{-2} K_L^2)$  per time step. Hence the following corollary is immediate.

**COROLLARY 3.1.** *Assume (A1)–(A4). For any  $n > 0$  and any  $\varphi \in \mathcal{B}_b(\Theta \times \mathbb{S}^{n+1}) \cap \text{Lip}(\Theta \times \mathbb{S}^{n+1})$  there exists a  $C < +\infty$  such that for any  $\epsilon > 0$  one can choose  $(L, \{N_l\}_{l=1}^L)$  such that*

$$\mathbb{E} \left[ \left| \sum_{l=0}^L E_l^{N_l}(\varphi) - \mathbb{E}_{\Pi_0}(\varphi(\theta, X_{0:n})) \right|^2 \right] \leq C\epsilon^2$$

with a total cost

$$(16) \quad \text{COST} \leq C \begin{cases} \epsilon^{-2} & \text{if } \beta > \gamma, \\ \epsilon^{-2} |\log(\epsilon)|^2 & \text{if } \beta = \gamma, \\ \epsilon^{-(2+\frac{\gamma-\beta}{\alpha})} & \text{if } \beta < \gamma. \end{cases}$$

In contrast, for the same scenario, the computational cost of PMCMC is  $\mathcal{O}(\epsilon^{-2-\gamma/\alpha})$  per time step, which is asymptotically greater than the method developed here.

It is remarked that all of our constants depend upon the time parameter (number of data points) and this element has been ignored. This is due to the technical complexity of the approach. We expect that the constants can be made time-uniform, and hence we conjecture that the results hold true uniformly in time. Then  $N_l$  can be chosen as above, and for Euler–Maruyama ( $\beta = \gamma = 1$  [11]) the cost for a given  $n$  will be  $\mathcal{O}(n^2 |\log(\epsilon)|^2 \epsilon^{-2})$ , with similar results for  $\beta \neq 1$ , according to (16). This results because one needs to take  $M = \mathcal{O}(n)$  for the particle filter in PMMH [1] and the cost to obtain a single sample particle filter trajectory is  $\mathcal{O}(n)$ . A verification of this is left for future work.

#### 4. Numerical simulations.

**4.1. Ornstein–Uhlenbeck process.** First, we consider the following Ornstein–Uhlenbeck process:

$$\begin{aligned} dX_t &= \theta(\mu - X_t)dt + \sigma dW_t, & X_0 &= x_0, \\ Y_k | X_k &\sim \mathcal{N}(X_k, \tau^2), \end{aligned}$$

where  $\mathcal{N}(m, \tau^2)$  denotes the Normal distribution with mean  $m$  and variance  $\tau^2$ . Further, the parameters  $(\theta, \sigma)$  are unknown and are given the following priors:

$$\theta \sim \mathcal{G}(1, 1), \quad \sigma \sim \mathcal{G}(1, 0.5),$$

where  $\mathcal{G}(a, b)$  denotes the Gamma distribution with shape  $a$  and scale  $b$ . The remaining parameters are defined as constants,  $x_0 = 0$ ,  $\mu = 0$ ,  $\delta = 0.5$ , and  $\tau^2 = 0.2$ . A data set with 100 observations is simulated with  $\theta = 1$  and  $\sigma = 0.5$ .

**4.2. Langevin SDE.** Consider the following Langevin SDE:

$$\begin{aligned} dX_t &= \frac{1}{2} \nabla \log \pi(X_t) dt + \sigma dW_t, & X_0 &= x_0, \\ Y_k | X_k &\sim \mathcal{N}(0, \tau^2 \exp X_k), \end{aligned}$$

where  $\pi(x)$  denotes the probability density function of a student’s  $t$ -distribution with  $\theta$  degrees of freedom. The parameters of interest are  $(\theta, \sigma)$ , and these are given prior,

$$\theta \sim \mathcal{G}(1, 1), \quad \sigma \sim \mathcal{G}(1, 1).$$

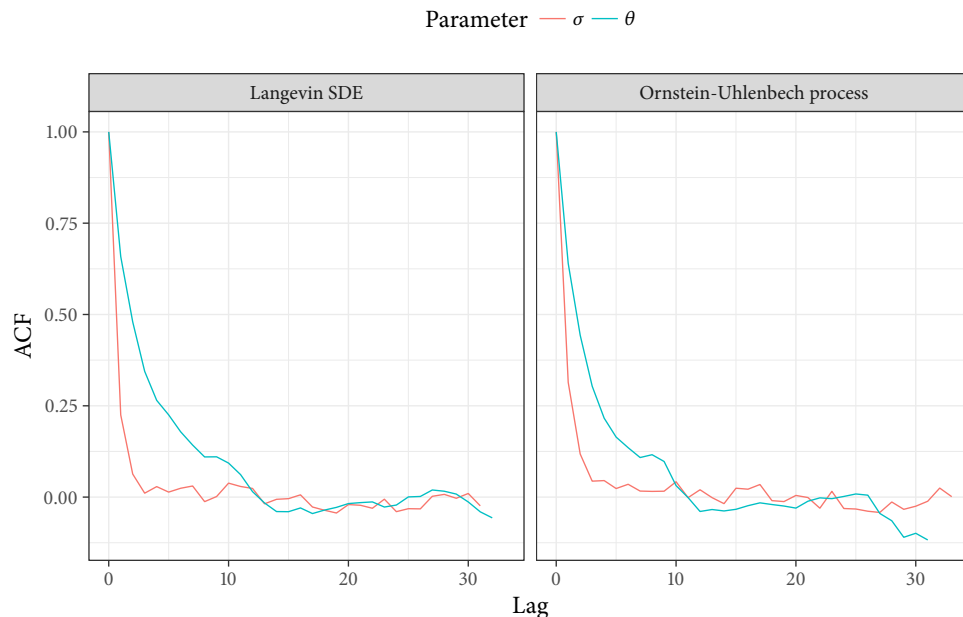


FIG. 1. Autocorrelation of a typical PMCMC chain.

The constants are  $x_0 = 0$  and  $\delta = 1$ . A data set with 1,000 observations is simulated with  $\theta = 10$ ,  $\sigma = 1$ , and  $\tau^2 = 1$ .

**4.3. Simulation settings.** The simulations proceed as the following. Let  $h = 2^{-l}$  be the accuracy parameter. At each level  $l$ , we set the number of particles in the PMCMC kernel to be  $M = \mathcal{O}(n)$  fixed, and set the number of PMCMC samples for estimation according to the multilevel analysis. Let  $N_l^L$  denote the number of samples at level  $l$  within a simulation that targets  $L$ -level error,  $L = 1, \dots$ . The value of  $N_0^1$  is determined empirically with variance estimated from 100 samplers. For comparison, a single-level PMCMC sampler is also considered for each  $L$ . Its number of samples  $N^L$  is determined empirically by running 100 simulations simultaneously. And these chains are run until the estimated error of the 100 estimates matches that of the multilevel sampler. In all situations, a fixed burn-in period of 10,000 iterations is used. This is reasonable given the fast decorrelation of the chains, as illustrated by the estimated autocorrelation of the single level PMCMC sampler for  $L = 8$  in Figure 1. The autocorrelation functions look similar for all  $l$  for the multilevel sampler.

**4.4. Results.** We consider the choice of  $M = \mathcal{O}(n)$ . The main results of the cost vs. error are shown in Figure 2. The estimated cost rates are listed in Table 1. It is shown in Appendix C that for Euler discretization the method satisfies the assumptions (A1)–(A3), under the mild additional assumption C.1 (i.e., the assumption is described there). The rate  $\beta = 2$  in (A2), since the diffusion term  $b_\theta$  is constant in  $x$  [11]. Furthermore, Assumption 4 holds with  $\gamma = \alpha = 1$ . Therefore, the theoretical results of Theorem 3.1 and Corollary 3.1 predict the rate  $\mathcal{O}(\epsilon^{-2})$ . Standard PMMH will incur a cost of  $\mathcal{O}(\epsilon^{-3})$ . The numerical results confirm this.

We note that the implementation here of the PMCMC, both in its single level and coupled multilevel instances, is not optimal, and the results may be improved. However, it is of paramount importance to understand that any such improvement can

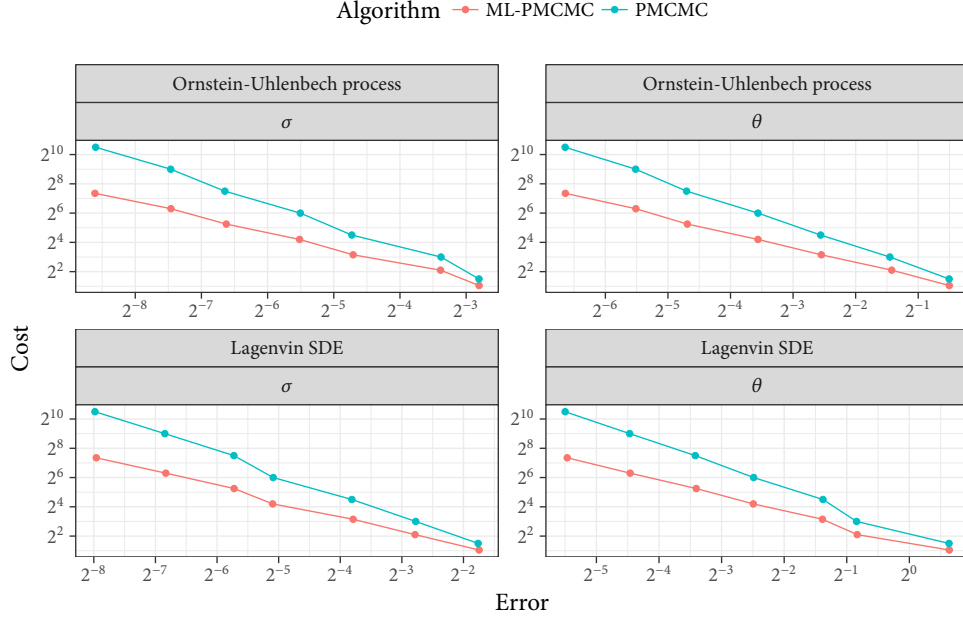


FIG. 2. Cost vs. MSE for the 2 parameters for each of the 2 SDEs.

TABLE 1

Estimated rates of convergence of MSE with respect to cost for various parameters, fitted to the curves in Figure 2.

Model	Parameter	ML-PMCMC	PMCMC
Ornstein-Uhlenbeck process	$\theta$	-1.022	-1.463
	$\sigma$	-1.065	-1.522
Langevin SDE	$\theta$	-1.060	-1.508
	$\sigma$	-1.023	-1.481

only hope to improve the constant, and the *rates of convergence* will not change. The results here are only intended to numerically verify the theoretical results regarding the rates of convergence.

**Appendix A. Technical results.** A Markov kernel  $K$  can be viewed as a linear operator  $(Kf)(w) = \int K(w, dw^*)f(w^*)$  for  $f : \mathbb{W} \rightarrow \mathbb{R}$  on a Hilbert space

$$L_0^2(\eta) := \left\{ f : \mathbb{W} \rightarrow \mathbb{R}; \int |f(w)|\eta(dw) < \infty, \int f(w)\eta(dw) = 0 \right\}$$

with an inner product  $\langle f, g \rangle = \int f(w)g(w)\eta(dw)$  and norm  $\|f\|_2 = \sqrt{\langle f, f \rangle}$ . Let  $\|K\|_2 = \sup_{f \in L_0^2(\eta), f \neq 0} \|Kf\|_2 / \|f\|_2$  be the operator norm.

By the Döblin condition (A3), we have the total variation distance bound  $\|K(w, \cdot) - \eta\|_{\text{TV}} = \sup_{A \in \mathcal{W}} |K(w, A) - \eta(A)| \leq 1 - \xi$  (for all  $w \in \mathbb{W}$ ) for some  $\xi \in (0, 1)$ . Since  $K$  is  $\eta$ -reversible, the total variation bound implies  $L^2$ -spectral gap

$$\|K^m\|_2 \leq (1 - \xi)^m,$$

by [19, Theorem 2.1].

For  $\mu$  a finite measure on a measurable space  $(\mathbf{E}, \mathcal{E})$  and  $\varphi \in \mathcal{B}_b(\mathbf{E})$

$$\mu(\varphi) = \int_{\mathbf{E}} \varphi(x) \mu(dx).$$

Defining  $v^i = (\theta^i, z_{0:n}^i)$  as the relevant variables of  $w^i$  from the MCMC kernel, and defining

$$\tilde{\varphi}_h(v^i) := \left\{ \varphi(\theta^i, x_{0:n}^i) H_{1,\theta}(\theta^i, z_{0:n}^i) - \varphi(\theta^i, x_{0:n}^i) H_{2,\theta}(\theta^i, z_{0:n}^i) \right\},$$

we are interested in estimates of the form

$$\frac{1}{N} \sum_{i=1}^N \tilde{\varphi}_h(v^i).$$

**PROPOSITION A.1.** *Assume (A1)–(A3). Suppose that  $\{W^i\}_i$  is a Markov chain with the Markov kernel  $K$ , and  $W^1 \sim \eta$ . Then for any  $n > 0$ , there exists a  $\beta > 0$  such that for any  $\varphi \in \mathcal{B}_b(\Theta \times \mathbf{S}^{n+1}) \cap \text{Lip}(\Theta \times \mathbf{S}^{n+1})$  there exists a  $C < +\infty$  such that*

$$\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N \tilde{\varphi}_h(V^i) - \Pi_{h,h'}(\tilde{\varphi}_h) \right)^2 \right] \leq \frac{C(h')^\beta}{N},$$

where  $V^i = (\theta, Z_{0:n}^i)$  is the relevant variables of  $W^i$ .

*Proof.* Denote the map  $w^i \mapsto v^i$  by  $\psi$ . Then

$$\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N \tilde{\varphi}_h(V^i) - \Pi_{h,h'}(\tilde{\varphi}_h) \right)^2 \right] = \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N f(W^i) \right)^2 \right]$$

for  $f(w) = \tilde{\varphi}_h \circ \psi(w) - \eta(\tilde{\varphi}_h \circ \psi) = \tilde{\varphi}_h(v) - \Pi_{h,h'}(\tilde{\varphi}_h)$ . We have

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N f(W^i) \right)^2 \right] &= \frac{1}{N^2} \sum_{i,j=1}^N \langle f, K^{|i-j|} f \rangle \\ &= \frac{1}{N} \|f\|_2^2 + \frac{2}{N^2} \sum_{n=1}^{N-1} (N-n) \langle f, K^n f \rangle \\ &\leq \frac{1}{N} \|f\|_2^2 + \frac{2}{N^2} \sum_{n=1}^{N-1} (N-n) \|K^n\|_2 \|f\|_2^2 \\ &\leq \frac{1}{N} \|f\|_2^2 + \frac{2}{N} \sum_{n=1}^{\infty} \|K^n\|_2 \|f\|_2^2 \\ &\leq \frac{1}{N} \|f\|_2^2 \left( 1 + \frac{1-\xi}{\xi} \right) \leq \frac{2\|f\|_2^2}{N\xi}. \end{aligned}$$

On the other hand, by Lemma A.2,

$$\|f\|_2^2 = \mathbb{E}_{\Pi_{h,h'}} \left[ \left\{ \tilde{\varphi}_h(V^i) - \Pi_{h,h'}(\tilde{\varphi}_h) \right\}^2 \right] \leq C(h')^\beta.$$

Thus, the claim follows.  $\square$

LEMMA A.2. *Assume (A1)–(A2). Then for any  $n > 0$ ,  $q \in \{1, 2\}$  there exists a  $\beta > 0$  such that for any  $\varphi \in \mathcal{B}_b(\Theta \times \mathbb{S}^{n+1}) \cap Lip(\Theta \times \mathbb{S}^{n+1})$  there exists a  $C < +\infty$*

$$\left( \mathbb{E}_{\Pi_{h,h'}} [|\varphi(\theta, X_{0:n})H_{1,\theta}(\theta, Z_{0:n}) - \varphi(\theta, X'_{0:n})H_{2,\theta}(\theta, Z_{0:n})|^q] \right)^{3-q} \leq C(h')^\beta.$$

*Proof.* We prove the result for  $q = 1$ , the case  $q = 2$  being almost the same. The result is proved by induction on  $n$ . Set  $n = 1$ , and then

$$\begin{aligned} & \mathbb{E}_{\Pi_{h,h'}} [|\varphi(\theta, X_{0:1})H_{1,\theta}(\theta, Z_{0:1}) - \varphi(\theta, X'_{0:1})H_{2,\theta}(\theta, Z_{0:1})|] \\ &= \left( \int_{\theta \times \mathbb{Z}^2} |\varphi(\theta, x_{0:1})H_{1,\theta}(\theta, z_{0:1}) - \varphi(\theta, x'_{0:1})H_{2,\theta}(\theta, z_{0:1})| \right. \\ & \quad \left. \Pi(d\theta)\nu_\theta(dz_0)G_{1,\theta}(z_1)Q_{\theta,h,h'}(z_0, dz_1) \right) \\ & \quad \times \left( \int_{\theta \times \mathbb{Z}^2} \Pi(d\theta)\nu_\theta(dz_0)G_{1,\theta}(z_1)Q_{\theta,h,h'}(z_0, dz_1) \right)^{-1}. \end{aligned}$$

As  $G_{1,\theta}(z)$  is uniformly (in  $\theta, z$ ) bounded below, the denominator on the right-hand side (R.H.S.) is uniformly lower bounded by a constant that is independent of  $h, h'$ . The numerator on the R.H.S. is

$$\begin{aligned} & \int_{\theta \times \mathbb{Z}^2} |\varphi(\theta, x_{0:1})H_{1,\theta}(\theta, z_{0:1}) - \varphi(\theta, z'_{0:1})H_{2,\theta}(\theta, z_{0:1})| \\ & \quad \times \Pi(d\theta)\nu_\theta(dz_0)G_{1,\theta}(z_1)Q_{\theta,h,h'}(z_0, dz_1) \\ &= \int_{\theta \times \mathbb{Z}^2} |\varphi(\theta, x_{0:1})g_\theta(x_1, y_1) - \varphi(\theta, x'_{0:1})g_\theta(x'_1, y_1)| \Pi(d\theta)\nu_\theta(dz_0)Q_{\theta,h,h'}(z_0, dz_1). \end{aligned}$$

Application of (A2) hence yields

$$\mathbb{E}_{\Pi_{h,h'}} [|\varphi(\theta, X_{0:1})H_{1,\theta}(\theta, Z_{0:1}) - \varphi(\theta, X'_{0:1})H_{2,\theta}(\theta, Z_{0:1})|] \leq C(h')^{\beta/2}.$$

Assuming the result for  $k - 1$ ,  $k > 1$ , by the above argument we only have to consider

$$\begin{aligned} & \int_{\theta \times \mathbb{Z}^{k+1}} |\varphi(\theta, x_{0:k})H_{1,\theta}(\theta, z_{0:k}) - \varphi(\theta, x'_{0:k})H_{2,\theta}(\theta, z_{0:k})| \\ & \quad \Pi(d\theta)\nu_\theta(dz_0) \prod_{p=1}^k G_{p,\theta}(z_p)Q_{\theta,h,h'}(z_{p-1}, dz_p) \\ &= \int_{\theta \times \mathbb{Z}^{k+1}} \left| \varphi(\theta, x_{0:k}) \prod_{p=1}^k g_\theta(x_p, y_p) - \varphi(\theta, x'_{0:k}) \prod_{p=1}^k g_\theta(x'_p, y_p) \right| \\ & \quad \Pi(d\theta)\nu_\theta(dz_0) \prod_{p=1}^k Q_{\theta,h,h'}(z_{p-1}, dz_p). \end{aligned}$$

The R.H.S. can be upper-bounded by

$$\begin{aligned} & \int_{\theta \times \mathbb{Z}^{k+1}} \varphi(\theta, x_{0:k}) g_\theta(x_k, y_k) \left| \prod_{p=1}^{k-1} g_\theta(x_p, y_p) - \prod_{p=1}^{k-1} g_\theta(x'_p, y_p) \right| \\ & \quad \Pi(d\theta) \nu_\theta(dz_0) \prod_{p=1}^k Q_{\theta, h, h'}(z_{p-1}, dz_p) \\ & + \int_{\theta \times \mathbb{Z}^{k+1}} \prod_{p=1}^{k-1} g_\theta(x'_p, y_p) |\varphi(\theta, x_{0:k}) g_\theta(x_k, y_k) - \varphi(\theta, x'_{0:k}) g_\theta(x'_k, y_k)| \\ & \quad \Pi(d\theta) \nu_\theta(dz_0) \prod_{p=1}^k Q_{\theta, h, h'}(z_{p-1}, dz_p). \end{aligned}$$

The first term can be treated by the induction hypothesis and the second term via (A2), which completes the proof.  $\square$

**Appendix B. Coupling Euler approximations.** Consider  $(x, x') \in \mathbb{S}^2$ , the current position of the discretized diffusions. Now we have  $h, h'$  the discretization levels with  $0 < h < h'$  and for simplicity set  $h' = 2h$ . Associated to the discretization level  $h$  (resp.,  $h'$ ), one must sample  $k = \delta/h$  (resp.,  $k' = \delta/h'$ ) points to obtain the sampled position of the diffusion at the next observation time. Set  $X(0) = X'(0) \sim \mu_\theta(dx)$  and then one can sample the fine discretization for  $m \in \{0, \dots, k-1\}$  as

$$X(m+1) = X(m) + ha_\theta(X(m)) + \sqrt{h}b_\theta(X(m))\xi(m),$$

where  $\xi(m) \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, I_d)$ . ( $I_d$  is the  $d \times d$  identity matrix.) For the course discretization, using the same simulated  $\xi(0), \dots, \xi(k-1)$  we set for  $m \in \{0, \dots, k'-1\}$

$$X'(m+1) = X'(m) + 2ha_\theta(X'(m)) + \sqrt{h}b_\theta(X'(m))[\xi(2m) + \xi(2m+1)].$$

**Appendix C. Verifying assumptions.** Now, we want to check conditions (A2) and (A3) under Assumption 2.1(i)–(ii), (A1), and the following assumption on  $\Pi(d\theta) = \pi(\theta)d\theta$  and  $R(d\theta'|\theta) = r(\theta'|\theta)d\theta'$ .

*Assumption C.1.*  $\Theta$  is a compact set of  $\mathbb{R}^{d_\theta}$ , and  $\pi : \Theta \rightarrow \mathbb{R}_+$  and  $r : \Theta^2 \rightarrow \mathbb{R}_+$  are continuous and strictly positive.

By assumption,  $Q_{\theta, h, h'}(z, dz^*)$  is the conditional distribution of  $Z^* = (X(k), X'(k'))$  given  $Z = (X(0), X'(0))$ . Then, under Assumption 2.1(i)–(ii), the condition (A2) is satisfied with  $\beta = 1$  for any  $q = 1, 2$ , since this is the  $L^q$  bound of the Euler–Maruyama scheme (in fact for constant diffusion coefficient  $b_\theta$  it coincides with the Milstein method and  $\beta = 2$ ) [11].

Next, we want to check the condition (A3). The proposal kernel  $\psi$  on  $W = \Theta \times \{1, \dots, M\}^n \times \mathbb{Z}^{M(n+1)} \times \{1, \dots, M\}$  of PMMH is

$$\psi(w, dw^*) = P \left( da_{0:n-1}^{*,1:M}, dz_{0:n}^{*,1:M} | \theta^* \right) R(d\theta^* | \theta) \frac{G_{n, \theta^*}(z_n^{*,l})}{\sum_{i=1}^M G_{n, \theta^*}(z_n^{*,i})},$$

where  $w = (\theta, a_{0:(n-1)}^{1:M}, z_{1:n}^{1:M}, k)$ ,  $w^* = (\theta^*, a_{0:(n-1)}^{*,1:M}, z_{1:n}^{*,1:M}, l)$ , and  $P(da_{0:n-1}^{*,1:M}, dz_{0:n}^{*,1:M} | \theta^*)$  is defined in (10). The transition kernel  $K$  is

$$K(w, dw^*) = \psi(w, dw^*)\alpha(w, w^*) + \delta_w(dw^*)R(w),$$

where the acceptance probability  $\alpha(w, w^*)$  is

$$\alpha(w, w^*) = \min \left\{ 1, \frac{p_{h,h'}^M(y_{0:n}|\theta^*)\pi(\theta^*)r(\theta|\theta^*)}{p_{h,h'}^M(y_{0:n}|\theta)\pi(\theta)r(\theta^*|\theta)} \right\},$$

and the rejection probability  $R(w)$  is

$$R(w) = 1 - \int_{w^*} \psi(w, dw^*)\alpha(w, w^*).$$

By (A1) together with Assumption C.1,  $C_1 = \inf_{w, w^* \in W} \alpha(w, w^*) > 0$ , and

$$\begin{aligned} \inf_w \psi(w, dw^*) &\geq \left\{ \min_{\theta, \theta^*} r(\theta^*|\theta) \right\} 1_{\Theta}(\theta^*) d\theta^* P \left( da_{0:n-1}^{*,1:M}, dz_{0:n}^{*,1:M} | \theta^* \right) \frac{G_{n,\theta^*}(z_n^{*,l})}{\sum_{i=1}^M G_{n,\theta^*}(z_n^{*,i})} \\ &=: C_2 \psi(dw^*) \end{aligned}$$

for a constant  $C_2 = \text{Leb}(\Theta) \min_{\theta, \theta^*} r(\theta^*|\theta) > 0$  with a probability measure  $\psi(dw^*)$ . Here,  $\text{Leb}$  is the Lebesgue measure on  $\mathbb{R}^{d_\theta}$ . Thus, we have

$$K(w, dw^*) \geq C_1 C_2 \psi(dw^*).$$

In particular, the condition (A3) is satisfied.

**Acknowledgments.** AJ is affiliated with the Risk Management Institute, the Center for Quantitative Finance, and the OR & Analytics cluster at NUS. KK & AJ acknowledge CREST, JST for additionally supporting the research.

#### REFERENCES

- [1] C. ANDRIEU, A. DOUCET, AND R. HOLENSTEIN, *Particle Markov chain Monte Carlo methods*, J. R. Stat. Soc. Ser. B Stat. Methodol., 72 (2010), pp. 269–342.
- [2] O. CAPPE, T. RYDEN, AND E. MOULINES, *Inference in Hidden Markov Models*, Springer, New York, 2005.
- [3] K.A. CLIFFE, M.B. GILES, R. SCHEICHL, AND A.L. TECKENTRUP, *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients*, Comput. Vis. Sci., 14 (2011), pp. 3–15.
- [4] G. DELIGIANNIDIS, A. DOUCET, AND M.K. PITT, *The Correlated Pseudo-Marginal Method*, preprint, arXiv:1511.0492, 2015.
- [5] P. DEL MORAL, *Feynman–Kac Formulae: Genealogical and Interacting Particle Systems with Applications*, Springer, New York, 2004.
- [6] P. DEL MORAL, A. JASRA, K. LAW, AND Y. ZHOU, *Multilevel sequential Monte Carlo samplers for normalizing constants*, Trans. Mod. Comp. Sim., 27 (2017), 20.
- [7] A. DOUCET AND A. JOHANSEN, *A tutorial on particle filtering and smoothing: Fifteen years later*, in Handbook of Nonlinear Filtering, D. Crisan and B. Rozovsky, eds., Oxford University Press, Oxford, 2011.
- [8] M.B. GILES, *Multilevel Monte Carlo path simulation*, Oper. Res., 56 (2008), pp. 607–617.
- [9] M.B. GILES, *Multilevel Monte Carlo methods*, Acta Numer., 24 (2015), pp. 259–328.
- [10] P. GLYNN AND S. MEYN, *A Lyapunov bound for solutions of the Poisson equation*, Ann. Probab., 24 (1996), pp. 916–931.
- [11] C. GRAHAM AND D. TALAY, *Stochastic Simulation and Monte Carlo methods*, Springer, Berlin, 2013.
- [12] S. HEINRICH, *Multilevel Monte Carlo methods*, in Large-Scale Scientific Computing, S. Margenov, J. Wasniewski, and P. Yalamov, eds., Springer, New York, 2001.
- [13] V. HOANG, C. SCHWAB, AND A. STUART, *Complexity analysis of accelerated MCMC methods for Bayesian inversion*, Inverse Problems, 29 (2013), 085010.
- [14] P.E. JACOB, F. LINDSTEN, AND T. SCHONN, *Coupling of Particle Filters*, preprint, arXiv:1606.01156, 2016.



- [15] A. JASRA, K. KAMATANI, K.J.H. LAW, AND Y. ZHOU, *Multilevel particle filters*, SIAM J. Numer. Anal., 55 (2017), pp. 3068–3096.
- [16] A. JASRA, K. KAMATANI, P.P. OSEI, AND Y. ZHOU, *Multilevel particle filters: Normalizing constant estimation*, Statist. Comput., to appear.
- [17] C. KETELSEN, R. SCHEICHL, AND A.L. TECKENTRUP, *A hierarchical multilevel Markov chain Monte Carlo algorithm with applications to uncertainty quantification in subsurface flow*, SIAM/ASA J. Uncertain. Quantif., 3 (2015), pp. 1075–1108.
- [18] K. LAW, A. STUART, AND K. ZYGALAKIS, *Data Assimilation*. Springer, New York, 2015.
- [19] G.O. ROBERTS AND J.S. ROSENTHAL, *Geometric ergodicity and hybrid Markov chains*, Electron. Commun. Probab., 2 (1997), pp. 13–25.
- [20] D. SEN, A. THIERY, AND A. JASRA, *On coupling particle filter trajectories*, Statist. Comput., to appear.