Rogue waves and entropy consumption: a key to full statistical description of rare events in non-linear deterministic systems?

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We present evidence that extreme events result from processes through a hierarchy of time scales that consume entropy from the surroundings. In particular, when applied to sea-surface heights we are able to predict the occurrence of rogue waves in the sea from the distribution of entropy variations. Our analysis is based on a recent method for extracting entropy variations in non-equilibrium single trajectories. We also describe how to use this method for predicting if an extreme event is likely to occur or not, even if in a given set of data no extreme events have been sampled. Finally, we discuss the possible equivalence between deterministic approaches in which rogue waves are particular solutions of model equations, and a stochastic approach based on a Fokker-Planck equation that can be derived directly from sets of sea surface height measurements. Our findings point towards a quantitative connection between the statistical description of a system out of equilibrium and its deterministic non-linear behaviour. Such a connection may be of valuable interest not only in the present context of oceanic rogue waves, but in the general context of turbulence, bridging the gap between statistical approaches to turbulent data and the Navier-Stokes equations.

Background and motivation

Oceanic rogue waves are extremely large waves that occur suddenly and unexpectedly, even in situations where the ocean appears relatively calm and quiet. Because of their size rogue waves can be extremely dangerous, even to the large ocean liners. While there are numerous reports from sailors claiming to have observed a rogue wave in the open ocean[1], rogue waves are very rare, which makes researching or forecasting them very difficult. See Fig. 1. As a prototypical example of extreme events emerging in a stochastic "background", rogue waves have been investigated from various perspectives, and a lot of progress has recently been achieved by using tools from non-linear waves and soliton theory[2,3].

Though there is no unique mathematical definition, a rogue wave can be defined as [4–6] a large amplitude wave that appears randomly and very rarely. In the context of ocean waves, rogue waves are known to appear in different forms of rare large amplitude events[7,8].

Due to the scarcity of observational data, many rather fundamental questions are still under debate. What exactly causes a specific rogue wave? Are there any fundamental features of the ambient sea state that lead to the occurrence of a rogue wave in the ocean? Is it possible to predict a rogue wave[9–13] or, at least, to provide quantitative insight into how probable it is to observe a rogue wave in specific regions of the oceans and within given time intervals?

Often investigations into rogue waves are based on models for wave packet evolution in non-linear dispersive media. The lowest order model equation for this class of system is the so-called non-linear Schrödinger equation[4,5,7,8,14]. Studies based on it have been successful in demonstrating the existence of rogue waves and also allowed classifying them into different classes. Still, the approach is fundamentally deterministic, while, as the definition of rogue waves itself suggests, a probabilistic description seems more natural to account for their low frequency of occurrence. Moreover, to improve understanding and grasp the physical causes underlying the emergence of such rare events, one would expect additional insight through disciplines different from deterministic non-linear pattern forming dynamics, i.e. for example from fields like thermodynamics or non-equilibrium statistical physics.

In this paper we provide what is, to our knowledge, the first evidence for thermodynamical processes underlying the occurrence of rogue waves in nature. Our findings do not contradict the findings from previous deterministic approaches to investigate rogue waves, but instead complement the present understanding by an additional thermodynamic perspective.

Two main findings are reported here. First, we show evidence for the hypothesis that rogue wave events are only possible in non-linear dispersive media involving the interplay of more than one time scale. Because of this necessary co-existence of different scales rogue waves are typically observed in systems behaving in a turbulent-like manner. Second, the emergence of a rogue wave results from an exchange of entropy between the wave environment and the rogue wave itself, where the rogue wave experiences a negative entropy variation through a hierarchy of time scales, in a way similar to the picture of the energy flux in Kolmogorov's turbulence cascade[15].

From stochastic time processes to stochastic scale processes

The typical rogue wave behaviour shown schematically in Fig. 1b is further illustrated in Fig. 2 for two data sets from two very different ocean regions. One set originates from the Japan Sea, where rogue waves are frequently observed (Fig. 2a), and another one from the North Sea, where rogue waves occur only rather rarely (Fig. 2b)[16–20]. The central point of our analysis is based on the finding that the entropy variation during sea level fluctuations is substantially different for the two cases, as shown in the panels on the right of Fig. 2a-b. While the total entropy variation associated with the time series of the Japan Sea is typically negative, i.e. the systems tends to consume entropy, the total entropy variation observed for the North Sea data rather tends to increase. Details concerning the computation of the total entropy variation from the sea level time series are described in the next section and in the section on Methods. Moreover, the distribution of total entropy variations can also be modelled consistently by a Laplace distribution, as shown in Fig. 2c. This makes it possible to establish a quantitative approach for predicting the probability for a rogue to occur in the system.

All the findings can be understood and modelled with the help of the fundamental concept of a scale process, which we introduce next.

When aiming at reconstructing stochastic time-series such as the ones shown in Fig. 2a-b, it is one of the key objectives to be able to derive a predictor for the next value $h(t+1)$, based on past measurements of the series $\{h(t), h(t-1), \dots, h(0)\}$. If the process is Markovian[21] such a predictor is a function of the present state h_t only, and the time series can be statistically reproduced using the 2-point statistics $p(h(t+1); h(t))$ which completely defines the propagator as a conditional probability density function $p(h(t+\Delta t)|h(t))$, since $p(h(t+\Delta t)|h(t)) = p(h(t+1); h(t))/p(h(t))$ and $p(h(t)) = \int p(h(t+1); h(t)) dh(t+1)$. When the process is not Markovian the value in the series depends on a large set of previous values and consequently we need to extract a N-point statistics which, for the typically large N, is in practice too cumbersome if not impossible to obtain. Ocean surface level time series turn out not to be Markovian.[22]

To overcome this shortcoming we notice that the N-point statistics can be derived in an alternative way, as illustrated in Fig. 2d. Instead of using heights $h(t-\tau_k)$ at previous time lags τ_k, the corresponding height increments $\Delta h_k := h(t-\tau_k) - h(t)$ and the present state $h(t)$ can be used. Indeed, the 3-point statistics $p(h_1(t_1), h_2(t_2), h_3(t_3))$ incorporates the same information as the statistical distribution $p(\Delta h_{12}(\tau_{12}), \Delta h_{23}(\tau_{23}), h_3(t_3))$, where $\Delta h_{ij}(\tau_{ij}) = h_i(t_i) - h_j(t_j)$ and $\tau_{ij} = t_i - t_j$.

The full analysis of the data is based on the computation of N-point statistics derived from the probability density function (PDF) $p(h(t), h(t-\tau_1), ..., h(t-\tau_{N-1}))$ at N different time instants.

The derivation of N-point statistics from the height increments is not as cumbersome as deriving it from the height values directly, because taking the height increments Δh_k through an ordered succession of time lags τ_k yields a Markovian process. In other words, the N-point statistics can be decomposed into $N-1$ increment propagators $p(\Delta h_{k-1}|\Delta h_k, h)$ for each scale $k = 1, \dots, N-1$, and each propagator can be extracted separately from the time series[21,23], thus defining a family of Fokker-Planck equations:

$$-\frac{\partial}{\partial \tau} p(\Delta h_{k-1}|\Delta h_k, h) = \left(-\frac{\partial}{\partial \Delta h} D^{(1)}_{k,k-1}(\Delta h, \tau, h) + \frac{\partial^2}{\partial (\Delta h)^2} D^{(2)}_{k,k-1}(\Delta h, \tau, h) \right) p(\Delta h_{k-1}|\Delta h_k, h). \tag{1}$$

Here it should be noted that we assume stationarity in h, so that the conditional probabilities depend only on the values of $h(t)$ and not explicitly on the time t.

The Fokker-Planck equations in (1) are defined through the extraction of the family of functions $D^{(1)}_{k,k-1}(\Delta h, \tau, h)$ and $D^{(2)}_{k,k-1}(\Delta h, \tau, h)$ for $k = 1, \dots, N-1$ as described in the section on Methods. Notice that each of these Fokker-Planck equations is formally equivalent to the standard one[24], but here the dependent variable is an increment Δh instead of the value of the variable h, and the independent variables is the time lag $\tau \equiv \Delta t$, or time scale, instead of time t. The surface elevation h itself comes in as a second independent variable.

To summarise the procedure, while a time process describes how a variable, here the water surface level h, changes from one time instant t to another t', the associated scale process describes how the variable's increments Δh change from a situation where one measures it during one time lag τ to one where it is measured for another time lag τ'. Important to keep in mind is that the N-point propagator $p(h(t)|h(t-\tau_1), ..., h(t-\tau_{N-1}))$ that predicts the time series of heights is fully determined by a set of Markovian propagators through successive time scales of the corresponding height increments. This Markovain property in scale is the key to how a three point closure can be achieved which enables us in the end to statistically reconstruct the time series from the set of functions $D^{(1)}_{k,k-1}$ and $D^{(2)}_{k,k-1}$. As described in the next section, by knowing these sets of functions one can derive entropy variations associated to the time series.

Entropy-consuming trajectories and rogue waves

Having mapped the physical process of the heights h of a wave into a more abstract process that describes the variations of its relative heights Δh within time lags or time scales τ, we now proceed to compute the total entropy variation in this scale process. The total entropy variation is given by the sum of two contributions,

$$\Delta S_{\mathrm{tot}} = \Delta S_{\mathrm{med}} + \Delta S_{\mathrm{traj}}, \tag{2}$$

with ΔS_{med} being the total entropy variation of the surrounding medium and ΔS_{traj} being the entropy variation along a specific trajectory through the hierarchy of time scales. Our framework is based on previous work[25] which introduces the entropy of individual stochastic trajectories. An integral fluctuation theorem for non-equilibrium systems is fulfilled and the approach has recently been applied[26] successfully to characterize data from a free air-jet experiment of developed turbulence. In particular, the existence of entropy-consuming trajectories generated by small-scale intermittency was shown. Here, we will apply the framework to scale processes and relate it with the occurrence of rogue waves.

The entropy variation for an individual trajectory between two time scales τ_k and τ_{k-1} is given by[25,26]

$$(\Delta S_{\mathrm{traj}})_{k,k-1} = \int_{\tau_k}^{\tau_{k-1}} \frac{\partial}{\partial \tau}\Delta h(\tau) \frac{\partial}{\partial \Delta h} \log\left(p^{\mathrm{stat}}_{k,k-1}(\Delta h, \tau, h)\right) d\tau, \tag{3}$$

where $p^{\rm stat}_{k,k-1}$ is the stationary solution of the Fokker-Planck equation (1), defined as

$$p^{\rm stat}_{k,k-1}(\Delta h,\tau,h) = \frac{1}{D^{(2)}_{k,k-1}(\Delta h,\tau,h)} \exp\left(\int_{-\infty}^{\Delta h} \frac{D^{(1)}_{k,k-1}(\Delta h,\tau,h)}{D^{(2)}_{k,k-1}(\Delta h,\tau,h)} d(\Delta h)\right). \tag{4}$$

The entropy produced by the surrounding medium during the process between the same two time scales is

$$(\Delta S_{\rm med})_{k,k-1} = -\log\left(\frac{p_{k-1}(\Delta h_{k-1})}{p_k(\Delta h_k)}\right). \tag{5}$$

Notice that the entropy variations of the medium are additive with respect to the time-scales, consequently we only need to compute entropy variations between the largest and shortest time scales in Eq (5). However, the entropy variation for the individual trajectory, defined by the path integral in Eq (3), is path dependent.

Figures 3a and 3b show the evolution of height increments at the largest and smallest scales, respectively. The respective probability distributions are shown in the left panels and are plotted in Fig. 3c in logarithmic scale.

As one can observe in Figs. 3a and 3b the time instant marked with a vertical dotted line, which corresponds to the rogue event occurring for the wave height itself, is characterized by a small increment at the largest scale and a large increment at the smallest scale. Comparing with the probability distributions in Fig. 3c one sees that the small increment at the largest scale occurs with a high probability, while the large increment at the smallest scale occurs with low probability. This feature of evolving from high to low probability when traversing through the time scales is in fact what marks the occurrence of a rogue wave (see Fig. 3d).

From the physical perspective, any wave results from the superposition of amplitude increments Δh_i corresponding to time lags τ_i. Ordering these increments from large to small time lags yields a hierarchy of time scales through which the height increments change. Following such scale processes the total entropy variation $\Delta S_{\rm tot}$ of the event can be positive (entropy production) or negative (entropy consumption).

Now, comparing the increment time series and the height time series with the series of the corresponding total entropy variations (Fig. 3e) one identifies an abrupt entropy consumption at the time of the occurrence of the rogue wave. This is not mere coincidence: rogue waves are always associated with large variations Δh_s within short time lags τ_s together with small variations Δh_0 within the largest time scales τ_0, and thus they result from an abrupt entropy consumption, i.e. large negative values of the entropy variation, during the associated scale process.

The association of rogue waves with small Δh_0 and large Δh_s is also quite intuitive, noting that rogue waves can be regarded as abrupt fluctuations strongly localized in short times. The association of rogue waves with large entropy consumption events can thus be inferred directly from Eq. (3).

Going back to Fig. 2, we can also see that the distribution of total entropy variations depends on the region or the state of the ocean where and when the measurements were taken. The set of height measurements for the Japan Sea, Fig. 2a, has a distribution of the entropy variation shifted to negative values when compared with the measurements taken for the North Sea, Fig. 2b. Note that the mean value for both distributions in Fig. 2c is positive, but the median for the Japan sea shows a negative value.

From the data we can conclude that there are trajectories, as parts of the time series of increments, that produce entropy, and trajectories that consume entropy. We conjecture that the fundamental physical characteristic of extreme rogue waves is that they emerge from a process through a hierarchy of time scales that follows a trajectory from high probability states to low probability states consuming entropy.

By extracting the functions $D^{(1)}(\Delta h, \tau, h)$ and $D^{(2)}(\Delta h, \tau, h)$ for different time lags one can also use the propagators that solve the corresponding Fokker-Planck equation for generating a simulation of the sea surface elevation. In sets of such simulated data one can actually observe events that are very similar to rogue waves already found elsewhere, as can be seen in Fig. 4. We have obtained he same types of rogue wave patterns also in our earlier simulations for the Japan Sea[22]. Interestingly, the patterns obtained seem qualitatively similar to patterns obtained from deterministic modelling, like e.g. from solving the non-linear Schrödinger equation, which is today considered a lowest order deterministic model for rogue waves in non-linear media[27].

Moreover, the thermodynamical approach presented allows us to interpret the emergence of rogue waves in the context of the statistics of recent rigorous results on non-equilibirum systems subject to fluctuations. Although the theory has been developed for microscopic systems, where the free-energy changes are of the order of kT, recent findings suggest the applicability to macrosystems[26] too. E.g. the integral fluctuation theorem (IFT), $\langle e^{-\Delta S} \rangle = 1$ for non-equilibrium systems[25] is fulfilled in our case, both for the measurement data, as well as for the corresponding simulation data based on our stochastic approach. Details concerning the IFT are given in the Methods section. Figures 5a and 5b show both these cases for the Japan sea. Similar results on the IFT are obtained for the North Sea data. It is important to note that to get convergence of the average $\langle e^{-\Delta S} \rangle$ a sufficiently large data set is necessary, as the exponential function puts much weight on rare events. Furthermore, as shown in Fig. 6, we find that it is not the entropy of the medium but the entropy of the trajectory which becomes highly negative when the extreme events occurs.

Finally, returning to the distribution of the entropy variations in both the Japan Sea and the North Sea, we observe that both have the same functional shape, and are well fitted by a Laplacian distribution (see Fig. 2c). Fitting empirical distributions with the corresponding Laplacian shape enables one to predict how likely it is to observe extreme rogue waves: a tendency for a negative mean value of the distribution together with a large variance are signatures of a system where extreme events may occur more often. Using in this sense the entropy value as a measure for rogue waves, from the distributions $p(\Delta S)$ it can be deduced that for the north Sea such rogue waves are by an factor 10^{-4} less likely.

Discussion and conclusion

Using empirical data from the Japan Sea and the North Sea we have presented evidence that rogue waves are extreme events resulting from entropy consumption of local wave heights along a scale process through a hierarchy of decreasing time scales.

Since our analysis is solely based in statistical features of the set of measurements, our findings can be straightforwardly generalized to other physical properties in non-equilibrium systems: entropy-consuming trajectories of the increments of a given property through a hierarchy of time scales is a fundamental feature underlying the occurrence of an extreme value of that property.

Moreover, through a simple statistical analysis of sets of measurements we can use these findings to show that it is possible to quantify the likelihood of one extreme event. Indeed, having the Laplace distribution as a model for the values of the total entropy variation in a given set of data, it is possible to estimate or predict the occurrence of rogue ways, namely by ascertaining if the model of the corresponding Laplace distribution has its median at a negative value of the entropy production.

Finally, a note concerning deterministic and stochastic descriptions of natural phenomena. It is known that rogue waves or extreme events can often be derived as solutions of non-linear deterministic evolution equations, like e.g. non-linear Schrödinger equations. In this study we have shown that they also emerge from a purely stochastic modelling of empirical sets of measurements. In the more general context of turbulence, one also finds this duality between determinism and statistics: while the irregular fluctuations observed in turbulent flows can be treated statistically through proper averaging of the (deterministic) Navier-Stokes equations together with closure assumptions[28]. The pioneering work of Lorenz[29] showed that the Navier-Stokes equations together with a temperature gradient equation for describing Rayleigh-Bernard convection in the atmosphere can be simplified into a non-linear deterministic set of equations yielding a chaotic and purely deterministic solution. Turbulence is indeed one paradigmatic example showing the ambiguity between deterministic chaotic behaviour resulting from non-linearities and the stochastic (non-chaotic) manifestation of the solution to the non-linear fluid flow problem at high Reynolds numbers[30].

In what concerns the occurrence of extreme events, a concept that similarly to turbulence has no exact mathematical definition, we have shown that the same seems to be case: while resulting from a purely deterministic system, non-linearity generates extreme events and rogue waves. Although chaotic phenomena, they can be described using purely statistical procedures, which we hope will offer a new approach to forecasting.

Methods

While the extreme rogue wave is a localized structure, and thus emerges in our framework as a small-scale event, it is embedded in a surrounding sea state that must be characterized by more than one scale, and a general N-point statistics is needed.

For this we introduce a cascade model based on the surface height increments, $\Delta h_j \equiv \Delta h_{\tau_j} = h(t_i) - h(t_i - \tau_j)$, to discuss the statistical properties of the water wave system. A general approach is the $(N+1)$-point characterization of the surface height cascade which is given by the joint probability $p(h(t); h(t-\tau_1); \ldots; h(t-\tau_N))$. For any $\tau \geq \tau_{EM}$ the stochastic process can be expressed as a Markov-chain. The joint probability factorizes and derives from the conditional PDFs :

$$\begin{aligned} p(h(t); h(t-\tau_1); \ldots; h(t-\tau_N)) &= p(h(t)-h(t-\tau_1); \ldots; h(t)-h(t-\tau_N); h(t)) \\ &= p(\Delta h_1; \Delta h_2; \ldots; \Delta h_N; h(t)) \\ &= p(\Delta h_1; \Delta h_2; \ldots; \Delta h_N | h(t)) \cdot p(h(t)) \end{aligned} \tag{6}$$

By considering the dependency of the wave height $h(t)$ and its increments, Δh_i , $i = 1, \ldots, N$ and using the earlier result[31] of the Markovian property

$$p(\Delta h_1 | \Delta h_2; \cdots \Delta h_N; h(t)) = p(\Delta h_1 | \Delta h_2; h(t)), \tag{7}$$

the $(N+1)$ -point statistics can be expressed as

$$p(h(t); h(t-\tau_1); \ldots; h(t-\tau_N)) = p(\Delta h_1 | \Delta h_2; h(t)) \cdots p(\Delta h_{N-1} | \Delta h_N; h(t)) \cdot p(\Delta h_N | h(t)) \cdot p(h(t)). \tag{8}$$

The transition probability, $p(\Delta h_j | \Delta h_k, h(t))$, can be described by a Kramers-Moyal expansion. Due to the fact that the first two terms of the Kramers-Moyal expansion strongly dominate the expansion[31], the evolution of the height increments for decreasing τ can be expressed by the following Fokker-Planck equation

$$\begin{aligned} -\tau_j \frac{\partial}{\partial \tau_j} p(\Delta h_j | \Delta h_k, h(t)) &= -\frac{\partial}{\partial \Delta h_j} \left[D^{(1)}(\Delta h_j, \tau_j, h(t)) p(\Delta h_j | \Delta h_k, h(t)) \right] \\ &+ \frac{\partial^2}{\partial \Delta h_j^2} \left[D^{(2)}(\Delta h_j, \tau_j, h(t)) p(\Delta h_j | \Delta h_k, h(t)) \right] . \end{aligned} \tag{9}$$

The remaining two Kramers-Moyal coefficients $D^{(1)}(\Delta h_j, \tau_j, h(t))$ and $D^{(2)}(\Delta h_j, \tau_j, h(t))$, called drift and diffusion, together with the initial distribution, $p(\Delta h_0, \tau_0, h(t))$, contain complete stochastic information of the cascade. Drift and diffusion functions are estimated from conditional moments,

$$D^{(n)}(\Delta h_j, \tau_j, h(t)) = \lim_{\delta_\tau \to 0} \frac{\tau_j}{n!\delta_\tau} \langle [\Delta h'_j(\tau_j - \delta_\tau, h(t)) - \Delta h_j(\tau_j, h(t))]^n \rangle_{\Delta h'_j}. \tag{10}$$

The drift and diffusion coefficients define our Markov cascade process which is an essential point for stochastic thermodynamics [32].

A typical example is the motion of colloidal particles applying external force [33]. In this non-equilibrium setting, the particles produce entropy as they move through the fluid. The entropy production ΔS can be defined for individual fluctuating trajectories by which it becomes a fluctuating quantity itself. Due to the nanoscopic setting, also negative values of ΔS are possible. The balance between fluctuations that produce or consume entropy is expressed by the IFT [25]

$$\langle e^{-\Delta S} \rangle = 1, \tag{11}$$

where $\langle \cdots \rangle$ is the expectation value over many fluctuating trajectories. Note that Eq. (11) implies that on average $\langle \Delta S \rangle > 0$, in agreement with the second law of thermodynamics.
For any Markov process the IFT for ΔS in the form of Eq.(11) is known to hold. Since we have shown that the ocean wave system is Markovian in scale, the IFT should hold too and could validate our approximation which was based on Kramers-Moyal coefficients with the initial distribution. The entropy production for a single realization $\Delta h(\bullet)$ can be calculated by the following formula :

$$\Delta S[\Delta h(\bullet)] = -\int_{\tau_0}^{\tau_{EM}} \partial_\tau \Delta h(\tau) \partial_{\Delta h} \varphi(\Delta h(\tau), \tau, h) d\tau - \ln \frac{p(\Delta h(\tau_{EM}), \tau_{EM})}{p(\Delta h(\tau_0), \tau_0)} \tag{12}$$

Here $(\bullet)$ represents τ-evolution from τ_0 to τ_{EM}. Note that τ_{EM} is the smallest time scale for which we have the Markovian properties and $\tau_0 = N\tau_{EM}$ is the largest scale in the cascade with $N = 13$. In this the non-equilibrium potential is defined as follows:

$$\varphi(\Delta h) = \ln D^{(2)}(\Delta h, \tau, h) - \int_{-\infty}^{\Delta h} \frac{D^{(1)}(\Delta h', \tau, h)}{D^{(2)}(\Delta h', \tau, h)} d\Delta h'. \tag{13}$$

To check whether the IFT holds or not, we use Eq.(12) to determine $\Delta S^{(i)} = \Delta S[\Delta h^{(i)}(\bullet)]$ from the estimated drift and diffusion coefficients for different measured realizations $\Delta h^{(i)}(\bullet)$.

The probability densities $p(\Delta h(\tau_{EM}), \tau_{EM})$, $p(\Delta h(\tau_0), \tau_0)$ are taken directly from the measured data. By averaging over different realizations, we have :

$$\langle e^{-\Delta S} \rangle_n = \frac{1}{n} \sum_{i=1}^{n} e^{-\Delta S^{(i)}} \simeq 1. \tag{14}$$

As shown in Fig. 5 this equation holds if we have very well estimated drift and diffusion coefficient, as well as a large enough number of n for different realizations of $\Delta h(\bullet)$ to include rare fluctuations with negative values for ΔS.

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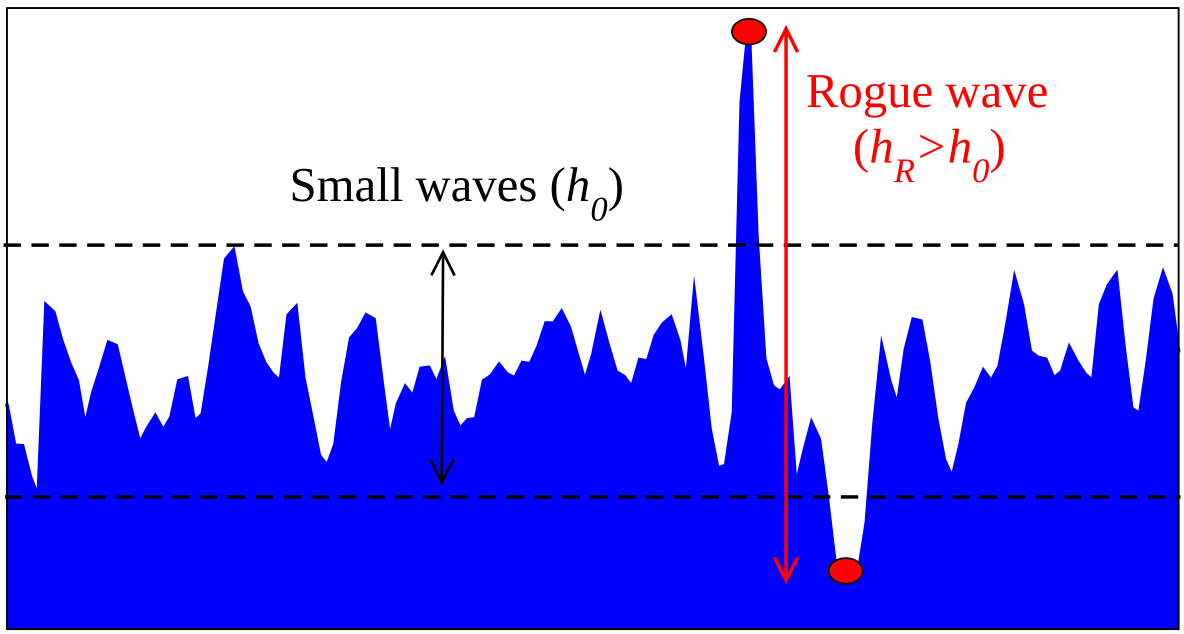

Figure 1: **(a)** The occurrence of a rogue wave: in a relatively calm sea, suddenly a localized wave rises several meters. In **(b)** one sees a real rogue wave measurement observed in the Japan Sea. **(c)** The data was obtained using an oceanographic buoy that measures the wave fluctuations of the buoy.

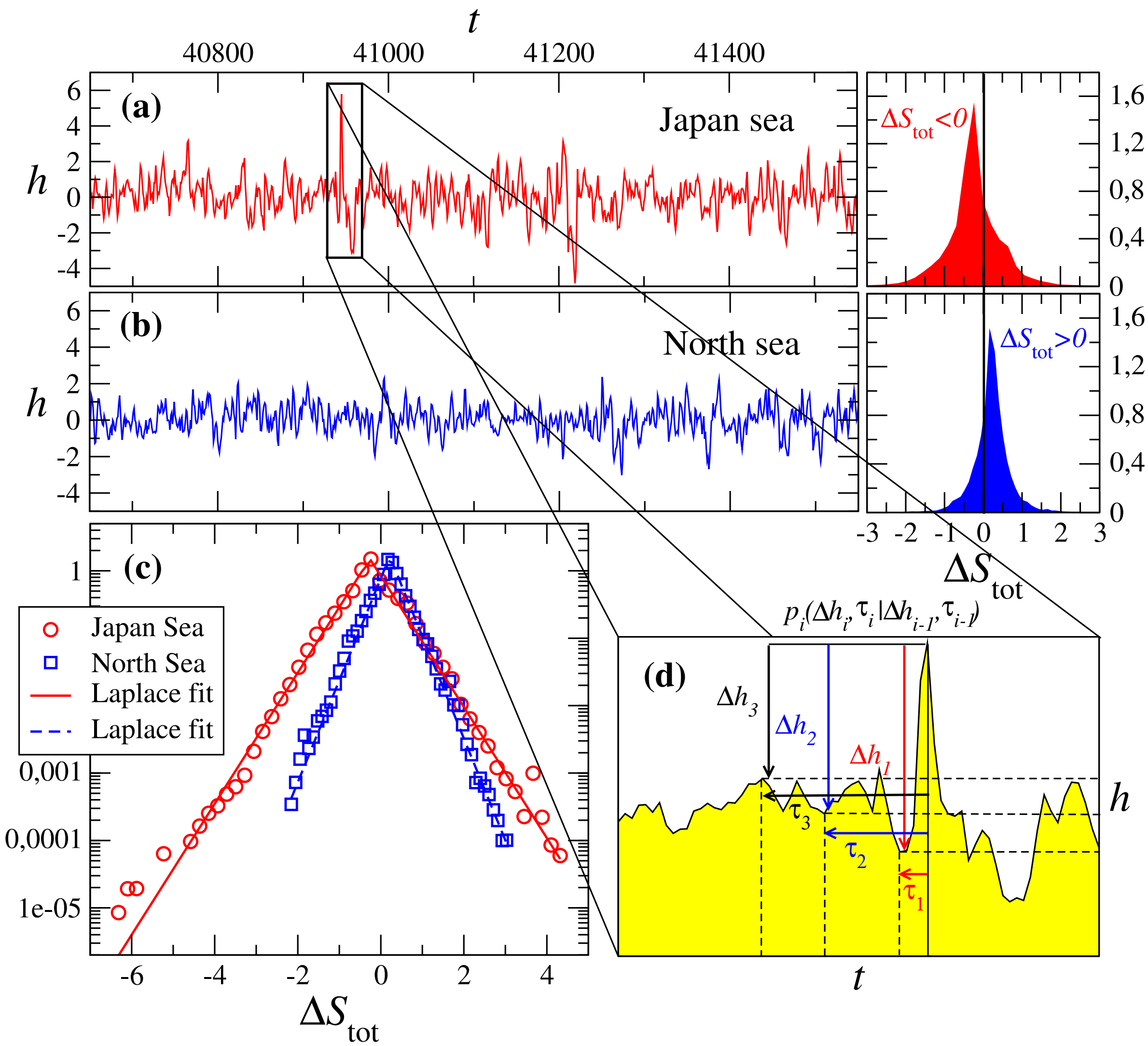

Figure 2: In **(a)** we see the water surface height measurements taken in the Japan Sea, where rogue waves are observed, and in **(b)** one sees a set of water surface height measurements from the North Sea, where rogue waves are not observed. As shown in the right panels, while in the Japan Sea the distribution of the total entropy variation has negative median, for the entropy variations in the North Sea median is positive. As shown in **(c)**, the functional shape of both entropy variation distributions is well approximated by the Laplace distribution with negative median for Japan sea. This finding is important for rogue wave forecasting (see text). **(d)** Illustration of the cascade of fluctuations $\Delta h_1 \to \Delta h_2 \to \Delta h_3$, from large time lags ($\tau_3$) to the smallest ones ($\tau_1$). This cascade describes the dynamics of the variations of a given property when passing from large time lags to a smaller ones (see Figs. 3 and 4).

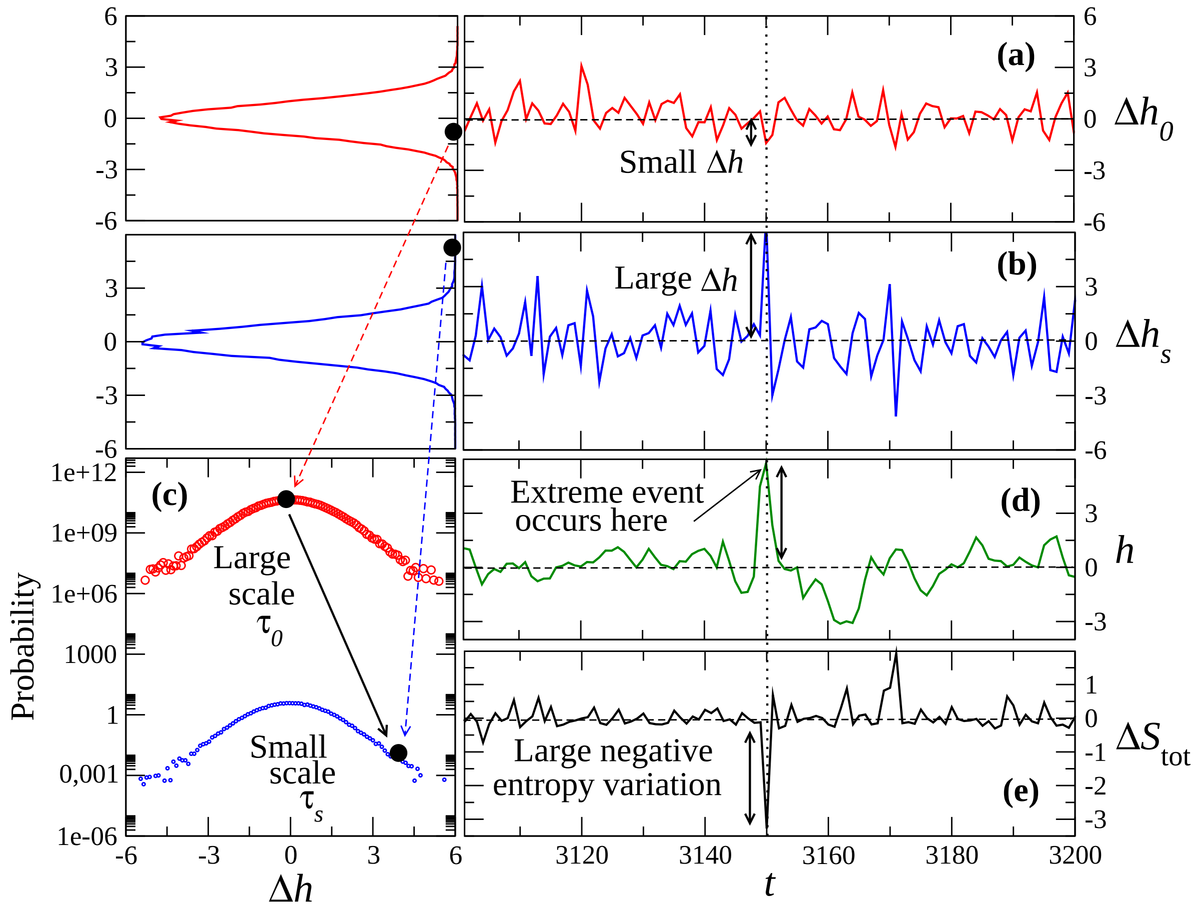

Figure 3: The close relation between an extreme event, i.e. a large fluctuation of a given property, the wave height for rogue waves, within a short time interval, and the entropy production $\Delta S_{\rm tot}$: the occurrence of an extreme event is identified by a *negative* entropy production, $\Delta S_{\rm tot} < 0$. In **(a-b)** we illustrate the time series of the height increments $\Delta h_0(t) = h(t) - h(t - \tau_0)$ (large scales) and $\Delta h_s(t) = h(t) - h(t - \tau_s)$ (small scales) respectively. On the left one sees the corresponding probability distribution for the increments, also plotted (in logarithmic scale) in **(c)** and shifted vertically for better comparison. **(d)** The vertical dotted line marks the instant when a rogue wave event takes place: a large value of h emerges. As one sees, **(e)** the entropy variation is strongly negative, which indicates the statistical feature of a rogue wave or an extreme event in general: it has a small height increment at large time lags τ_0 associated with large height increments at the smallest time lags τ_s.

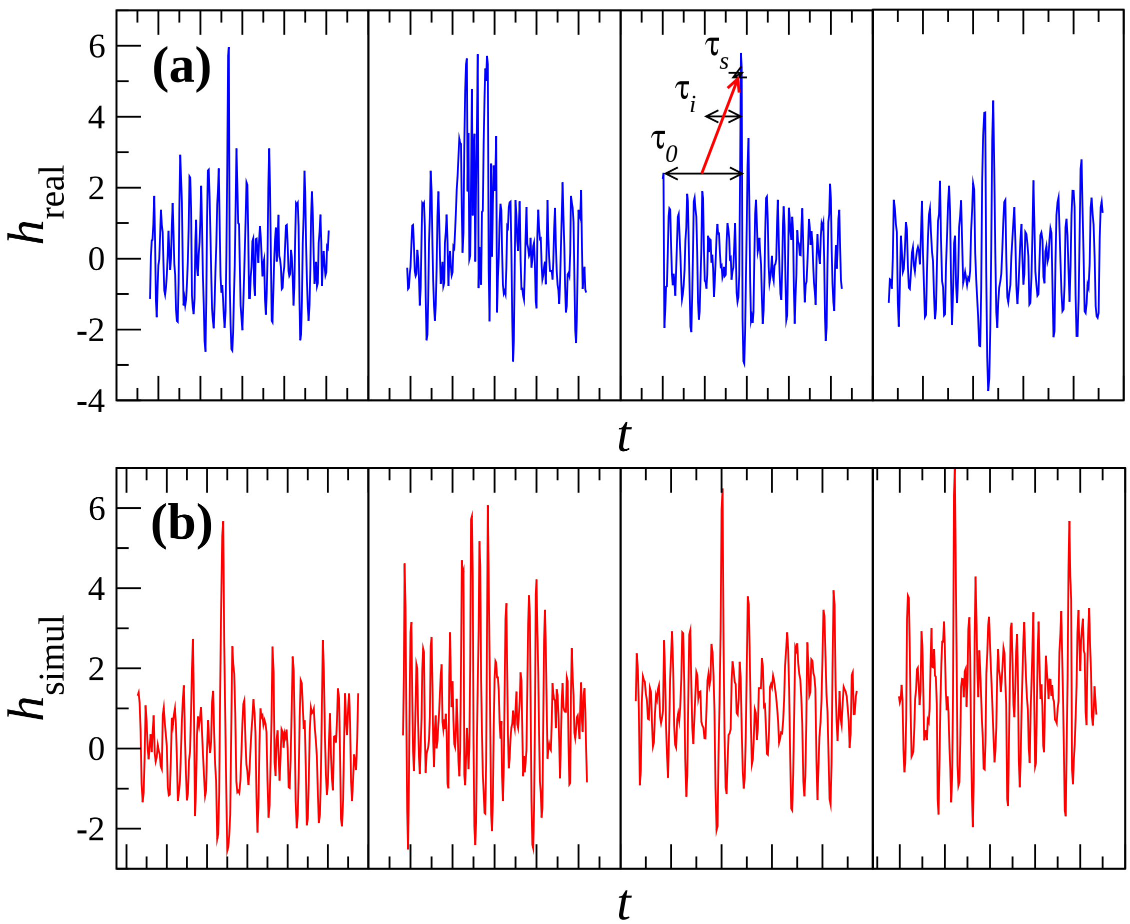

Figure 4: **(a)** Illustration of short samples of sea height from the Japan Sea during which rogue waves are observed with **(b)** the corresponding simulations using the stochastic framework based on a Fokker-Planck equation for the distribution of height increments. The similarity between simulation and measurement suggests an equivalence between the deterministic and the stochastic description based on what we call a *scale*-process of the height increments (see text).

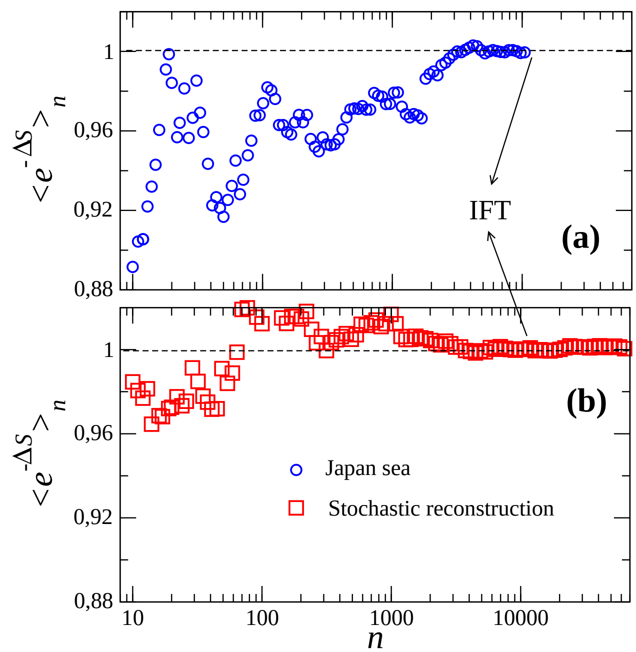

Figure 5: **(a)** Rogue waves and extreme events in general fulfil the integral fluctuation theorem (IFT), which means that statistically the entropy production fluctuates around zero. **(b)** The IFT also holds for the reconstructed data from the stochastic model of Japan Sea heights. For the latter, since the modelling can be performed for arbitrarily large time windows, it is possible to observe the good convergence towards one (see text).

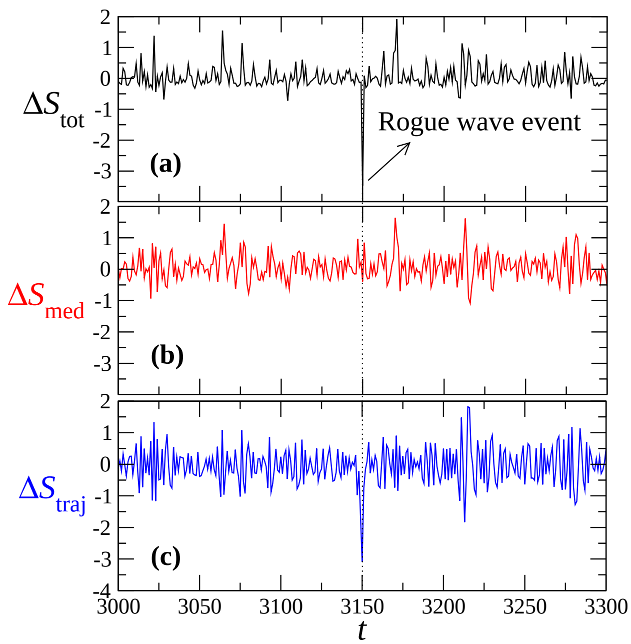

Figure 6: **(a)** With the total entropy production being negative whenever a rogue wave takes place, one sees that such negative variation of the entropy is not present in **(b)** the variation of the entropy of the medium, $\Delta S_{\rm med}$, but rather in **(c)** the variation of the individual trajectory's entropy only, $\Delta S_{\rm traj}$.