Semisimple filtrations of tilting modules for algebraic groups

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Abstract

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The indecomposable tilting modules $\{T(\lambda)\}$ for G, which are labeled by highest weight, form an important class of self-dual representations over $\Bbbk$. In this thesis we investigate semisimple filtrations of minimal length (Loewy series) of tilting modules.

We first demonstrate a criterion for determining when tilting modules for arbitrary quasi-hereditary algebras are rigid, i.e. have a unique Loewy series. Our criterion involves checking that $T(\lambda)$ does not have certain subquotients whose composition factors extend more than one layer in the radical or socle series. We apply this criterion to show that the restricted tilting modules for SL_4 are rigid when $p \geq 5$, something beyond the scope of previous work on this topic by Andersen and Kaneda.

Even when $T(\lambda)$ is not rigid, in many cases it has a particularly structured Loewy series which we call a *balanced semisimple filtration*, whose semisimple subquotients or "layers" are symmetric about some middle layer. Balanced semisimple filtrations also suggest a remarkably straightforward algorithm for calculating tilting characters from the irreducible characters. Applying Lusztig's character formula for the simple modules, we show that the algorithm agrees with Soergel's character formula for the regular indecomposable tilting modules for quantum groups at roots of unity. We then show that these filtrations really do exist for these tilting modules.

In the modular case, high weight tilting modules exhibit self-similarity in their characters at p-power scales. This is due to what we call *higher-order linkage*, an old character-theoretic result relating modular tilting characters and quantum tilting characters at p-power roots of unity. To better understand this behavior we describe an explicit categorification of higher-order linkage using the language of Soergel bimodules. Along the way we also develop the algebra and combinatorics of higher-order linkage at the de-categorified level. We hope that this will provide a foundation for a tilting character formula valid for all weights in the modular case when p is sufficiently large.

For Hannah

Declaration of originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as detailed below and specified in the text. It is not substantially the same as any that I have submitted or is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared here and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma, or other qualification at the University of Cambridge or any other University or similar institution except as declared here and specified in the text.

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Contents

Introduction

Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. A rational G-module over $\Bbbk$ is a tilting module if it has a filtration by Weyl modules and a filtration by dual Weyl modules. In this thesis we show some consequences of this interaction on Loewy series (i.e. semisimple filtrations of minimal length) of tilting modules. Our results and the methods used to prove them vary considerably. The overarching theme is that in many ways, the Loewy structure of the tilting modules for G is easier to understand and more natural to study than that of the Weyl modules for G or the projective/injective modules for the corresponding Schur algebra.

Rigidity. Our first major result is a general condition for determining when a tilting module for a quasi-hereditary algebra A is rigid. We work directly in the category $A-$radfiltmod of finite-dimensional A-modules with fixed semisimple filtrations. In this category it is important to distinguish between ordinary isomorphisms between modules and filtered isomorphisms which additionally preserve the filtered structure. With the help of model structures, we easily transfer homological tools to this category and provide connections to graded modules via the Rees functor.

Our rigidity criterion in Theorem 2.2.7 states that under reasonable conditions, a tilting module for A is rigid if and only if it does not contain what we call *stretched subquotients*. A stretched subquotient is a subquotient of a tilting module which is isomorphic but not filtered isomorphic to a certain extension of a simple module by a quotient (resp. submodule) of a standard (resp. costandard) module. As might be expected, these subquotients are difficult to construct and necessarily require repetitions of composition factors in a Loewy layer. We apply this criterion in order to calculate the radical series for the restricted weight tilting modules for SL_4.

Previous work by Andersen and Kaneda established the rigidity of a large class of tilting modules for algebraic groups in sufficiently large characteristic [7]. In particular, they showed that tilting modules with highest weights in the fundamental p^2-alcove which are both above the Steinberg weight and not "too close" to the walls of the dominant chamber are rigid. This had already been observed in the work of Bowman, Doty, and Martin [13, 14] and the earlier work of Doty and Henke [24] on the Loewy structure of the indecomposable summands of $L(\lambda) \otimes L(\mu)$ when λ, μ are restricted, for the cases $G = \mathrm{SL}_3$ and $G = \mathrm{SL}_2$ respectively. In fact in their examples all but one of the tilting modules which appear as summands are rigid, including several with highest weight lying outside the region described by Andersen and Kaneda. The new rigidity criterion is flexible enough to deal with such cases, including restricted weight tilting modules, which get more complicated in higher rank.

Balanced semisimple filtrations. The representation theory of quantum groups at lth roots of unity is in many ways analogous to that of reductive groups in positive characteristic. We show in Theorem 3.2.6 that for most values of l, there are self-dual semisimple filtrations for quantum tilting modules, which we call *balanced semisimple filtrations*. This means that even when these tilting modules are not rigid, they still have canonical semisimple filtrations. Balanced semisimple filtrations lead directly to a remarkably simple algorithm for calculating the indecomposable tilting characters given the irreducible characters. Key to our approach are Lusztig's character formula for the simple modules and Soergel's character formula for the indecomposable tilting modules. In fact, our methods also work in the modular case whenever these two character formulas are valid, and possibly even in other settings such as category $\mathcal{O}$ for a complex semisimple Lie algebra.

Lusztig's character formula is the main incarnation of Kazhdan–Lusztig theory in the modular representation theory of reductive groups. Like the original Kazhdan–Lusztig conjecture, it gives the characters of the simple modules in terms of known characters (in this case, the Weyl characters) and certain Kazhdan–Lusztig polynomials evaluated at 1. Lusztig originally conjectured in [45] that his character formula should hold for reductive groups when p is about as large as the Coxeter number of the corresponding root system. The conjecture was later extended to quantum groups, and the quantum version was proven first in a series of papers by Kazhdan and Lusztig [38, 39, 40, 46] and Kashiwara and Tanisaki [35, 36]. The quantum result was later extended to the modular case for p extremely large [3, 27]. However, Williamson recently constructed a series of counterexamples in [56] which show that Lusztig's conjectured lower bounds on p (and indeed any linear function of these bounds) do not hold in general! Williamson's methods also provide counterexamples for the James conjecture, a similar conjectural character formula for modular representations of the symmetric group. These surprising revelations shattered people's expectations, showing that there is still much work to be done in modular representation theory.

For tilting modules, Soergel conjectured and proved a character formula for the indecomposable quantum tilting modules $T_\ell(\lambda)$ in terms of parabolic anti-spherical Kazhdan–Lusztig polynomials [51, 52]. In the modular case it was broadly conjectured by Andersen that $T(\lambda)$ has the same character when p is very large (i.e. large enough for Lusztig's character formula to hold), in particular when λ is in the fundamental p^2-alcove [5]. This is reassuring, but says nothing about higher weight tilting modules. Donkin's tilting tensor product theorem [22, (2.1) Proposition], analogous to Steinberg's tensor product theorem for simple modules, helps somewhat, but there still remain infinitely many unknown modular tilting characters.

Linkage. Generalizing balanced semisimple filtrations to the modular case could be the key to a tilting character formula valid for all weights for p sufficiently large. An obstacle to this is the fact that Lusztig's character formula does not directly give the simple characters for all possible highest weights. For larger weights, it is necessary to apply Steinberg's tensor product theorem, which leads to messy Kazhdan–Lusztig combinatorics due to the presence of the half-root-sum shift in the dot action, versus the lack of such a shift in Steinberg's tensor product theorem. One way to work around this is via what we call *higher-order linkage*.

Higher-order linkage is the known fact (see e.g. [33, Proposition 4.1(ii)] or [6, 4.2]) that every tilting character for a reductive group G is also a tilting character for the corresponding quantum group U_{p^r} at a p^rth root of unity, for all powers of the characteristic p. This connects the behavior of tilting modules at "scale 1" to that at "scale p". So far all known indecomposable tilting characters can be shown to be indecomposable using higher-order linkage. Intuitively higher-order linkage should behave well with respect to Kazhdan–Lusztig combinatorics, because it can be described in terms of a subgroup W_p of the affine Weyl group W which underlies Lusztig's character formula.

To better understand higher-order linkage for tilting modules, we formulate a version of higher-order linkage for Soergel bimodules, which we simply call linkage. We leave the precise definition of Soergel bimodules to Chapter 4, but we will say here that the category $\mathcal{D}$ of Soergel bimodules is a diagrammatic category, i.e. a category whose morphisms are linear combinations of pictures resembling string diagrams. Soergel bimodules have been at the heart of many new discoveries in the modular representation theory of reductive groups, including Williamson's aforementioned counterexamples. More recent work has established direct connections between Soergel bimodules and tilting modules, including the geometric Satake equivalence and the Riche–Williamson correspondence. The fact that these two correspondences work at different scales is strongly suggestive of our conception of linkage. We will discuss this in more detail in Chapter 5.

The primary ingredient of linkage for Soergel bimodules is a functor we call the *linkage functor*, whose properties are summarized in Theorem 5.4.3. The action of the linkage functor on Soergel bimodules closely resembles higher-order linkage of tilting modules, while the action on morphisms gives a higher-level interpretation. Thus in the parlance of higher representation theory, linkage for Soergel bimodules is a categorification of higher-order linkage. Of independent interest is the algebra and combinatorics of linkage, which we develop alongside the linkage functor in the hope that it will provide a framework for understanding the higher-order behavior of both tilting modules for G and Soergel bimodules.

CHAPTER 1

Preliminaries

This chapter contains the background material and notation which will be used throughout this thesis. By necessity the topics covered are varied. We note that although all the results here are used multiple times in later chapters, no later chapter requires full knowledge of everything here.

Most of the results in this chapter are well known and can be found in the references listed below. However, in some places the presentation may seem unfamiliar due to novel notation chosen to emphasize combinatorial aspects of Kazhdan–Lusztig theory. In particular, the reader should be aware that our notion of character sets in Section 1.1.4 is original, although none of the results written using them are particularly new.

The main references for Section 1.1 are [31] and [51], with some notation borrowed from [26]. For Section 1.2 we mostly follow [23]. There are several good references for the group-theoretic material in Section 1.3 including [30], [54], and [12], but the most comprehensive reference for the representation theory is [32, II].

1.1. Hecke algebras of affine Weyl groups

1.1.1. Affine Weyl groups. Let Φ be an irreducible root system for a Euclidean space E, with a choice of simple roots Σ. In this thesis the affine Weyl group W corresponding to Φ is the reflection group on E generated by reflections of the form

$$\begin{aligned} s_{\alpha,k} : E &\longrightarrow E \\ \lambda &\longmapsto \lambda - (\langle \lambda, \alpha^\vee \rangle - k)\alpha \end{aligned}$$

for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$.[1] One can show that W is isomorphic as a reflection group to $W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi$, where W_{f} denotes the (finite) Weyl group of Φ, and the root lattice $\mathbb{Z}\Phi = \mathbb{Z}\Sigma$ acts on E by translation.

To understand W better it is helpful to introduce some fundamental regions in E called alcoves. This is analogous to using Weyl chambers to understand the behavior of the finite Weyl group W_{f}. An alcove is a connected component of

$$E \setminus \bigcup_{\substack{\alpha \in \Phi \\ k \in \mathbb{Z}}} \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle = k\},$$

[1] Many sources, including [31], call this the affine Weyl group corresponding to the dual root system $\Phi^\vee$, but our convention is more useful for many representation-theoretic applications.

which is the complement of the hyperplanes fixed by the reflections above. The closure of an alcove is a simplex of dimension $|\Sigma|$. The affine Weyl group W acts simply transitively on the set of all alcoves $\mathcal{A}$, so fixing an alcove A_0 gives a bijection $x \mapsto xA_0$ between W and $\mathcal{A}$. We will set A_0 to be the fundamental alcove, which is

$$A_0 = \{\lambda \in E : 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\},$$

where Φ^+ is the set of positive roots induced by the simple roots Σ. The alcove A_0 is the unique dominant alcove containing 0 in its closure.

From the alcove geometry one can show that W has a presentation as a Coxeter group, which we describe below. Let S be the set of reflections in the walls of the closed fundamental alcove $\overline{A_0}$. For all distinct $s, t \in S$ let $m_{st} \in \mathbb{Z} \cup \{\infty\}$ such that the angle between the reflection hyperplanes of s and t is π/m_{st}. Then W is isomorphic to the free group on S subject to the relations

$$s^2 = 1 \qquad \text{for all } s \in S, \tag{1.1}$$

$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \qquad \text{for all distinct } s, t \in S, \tag{1.2}$$

where the final relation is omitted when $m_{st} = \infty$.

As W is a Coxeter group, it is additionally equipped with a partial order $\leq$ called the Bruhat order, and a length function $\ell : W \to \mathbb{Z}_{\geq 0}$. We call a finite sequence of generators $\underline{x} = (s_1, s_2, \ldots, s_m)$ an expression in S. The set of all expressions in S is denoted $\underline{S}$. In most cases we will use underlines instead of parentheses to write expressions, e.g. $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$. The non-underlined counterpart then denotes the product in W, i.e. $x = s_1 s_2 \cdots s_m \in W$. The length of $\underline{x}$ is $\ell(\underline{x}) = m$. Note that $\ell(\underline{x}) \neq \ell(x)$ in general (e.g. $\ell(\underline{ss}) = 2 \neq 0 = \ell(ss)$), but when equality holds we call $\underline{x}$ a reduced expression (or rex) for x.

In terms of Φ, we have $S = S_{\mathrm{f}} \cup \{s_{-\tilde{\alpha},1}\}$, where $S_{\mathrm{f}} = \{s_{\alpha,0} : \alpha \in \Sigma\}$ is the set of reflections in the simple roots and $\tilde{\alpha}$ is the highest root in Φ. For brevity we write $\tilde{s} = s_{-\tilde{\alpha},1}$. We call the generators in $S_{\mathrm{f}} \subset S$ the finite generators and $\tilde{s}$ the affine generator.

1.1.2. Hecke algebras. Let $\mathbb{L}$ denote the ring $\mathbb{Z}[v^{\pm 1}]$ of Laurent polynomials with integer coefficients. The Hecke algebra $\mathbb{H} = \mathbb{H}(W, S)$ of the affine Weyl group W is the $\mathbb{L}$-algebra with generators $\{H_s\}_{s \in S}$ and relations

$$H_s^2 = 1 + (v^{-1} - v)H_s \qquad \text{for all } s \in S, \tag{1.3}$$

$$\overbrace{H_s H_t H_s \cdots}^{m_{st} \text{ terms}} = \overbrace{H_t H_s H_t \cdots}^{m_{st} \text{ terms}} \qquad \text{for all distinct } s, t \in S \text{ when } m_{st} \neq \infty, \tag{1.4}$$

where m_{st} is defined as above. The notation throughout this section is mostly taken from [51].

If $w \in W$ and $\underline{w} = \underline{s_1 s_2 \cdots s_m}$ is a rex for w, the element $H_w = H_{s_1} H_{s_2} \cdots H_{s_m}$ is well-defined, and the set $\{H_w\}_{w \in W}$ forms an $\mathbb{L}$-basis for $\mathbb{H}$. Each generator H_s is invertible, with $H_s^{-1} = H_s + v - v^{-1}$, so each basis element H_w is also invertible.

The bar involution or dualization map $(\overline{}) : \mathbb{H} \longrightarrow \mathbb{H}$ is the algebra homomorphism defined by the following action

$$\overline{v} = v^{-1}$$
$$\overline{H_w} = (H_{w^{-1}})^{-1}$$

on the basis. For $s \in S$ we define $\underline{H}_s = H_s + v$, which is self-dual. The set $\{\underline{H}_s\}_{s\in S}$ forms another set of generators for $\mathbb{H}$ as an $\mathbb{L}$-algebra. The action of these generators on the basis $\{H_x\}$ is

$$H_x\underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } xs > x, \\ H_{xs} + v^{-1}H_x & \text{if } xs < x. \end{cases} \tag{1.5}$$

We can also define a self-dual $\mathbb{L}$-basis using these generators, which is called the Kazhdan–Lusztig basis. We include a proof of this fact for later use of the notation therein.

THEOREM 1.1.1 ([41, Theorem 1.1]). *There is a unique* $\mathbb{L}$*-basis* $\{\underline{H}_x\}_{x\in W}$ *for* $\mathbb{H}$ *such that for each* $x \in W$*,*

(i) $\overline{\underline{H}_x} = \underline{H}_x$ *(self-duality);*
(ii) $\underline{H}_x = H_x + \sum_{y<x} h_{y,x}H_y$*, and for all* $y < x$ *we have* $h_{y,x} \in v\mathbb{Z}[v]$*.*

PROOF. Induct on the length of x. Suppose for some $x \in W$ we have already defined $\underline{H}_x$ and all $\underline{H}_y$ with $\ell(y) < \ell(x)$. Suppose $s \in S$ satisfies $xs > x$. Write

$$\underline{H}_x\underline{H}_s = H_{xs} + \sum_{y<xs} h^s_{y,x}H_y.$$

From the action of $\underline{H}_s$ on the basis above we have (for $x, y \in W$)

$$h^s_{y,x} = \begin{cases} h_{ys,x} + vm_{y,x} & \text{if } ys > y, \\ h_{ys,x} + v^{-1}m_{y,x} & \text{if } ys < y. \end{cases}$$

Clearly $\underline{H}_x\underline{H}_s$ is self-dual, so the element

$$\underline{H}_{xs} = \underline{H}_x\underline{H}_s - \sum_{y<xs} h^s_{y,x}(0)\underline{H}_y = H_{xs} + \sum_{y<xs} h_{y,xs}H_y,$$

whose coefficients we have labeled $h_{y,xs}$, is also self-dual with the property that $h_{y,xs}$ has zero constant coefficient. □

Now let $\mathbb{H}_{W_\mathrm{f}} = \mathbb{H}(W_\mathrm{f}, S_\mathrm{f}) \leq \mathbb{H}$ be the Hecke algebra obtained from the finite Weyl group $W_\mathrm{f} < W$. Since $(H_s - v^{-1})(H_s + v) = 0$ for each generator $s \in S_\mathrm{f}$, for each $u \in \{-v, v^{-1}\}$ there is a homomorphism of $\mathbb{L}$-algebras $\varphi_u : \mathbb{H}_{W_\mathrm{f}} \to \mathbb{L}$, defined by mapping $H_s \mapsto u$. This turns $\mathbb{L}$ into a right $\mathbb{H}_{W_\mathrm{f}}$-module which we call $\mathbb{L}(u)$. These modules are analogues of the sign and trivial representations for W_f respectively. Now define two right $\mathbb{H}$-modules

$$\mathbb{M} = \mathbb{L}(v^{-1}) \otimes_{\mathbb{H}_{W_\mathrm{f}}} \mathbb{H},$$
$$\mathbb{N} = \mathbb{L}(-v) \otimes_{\mathbb{H}_{W_\mathrm{f}}} \mathbb{H}.$$

These modules are called the spherical module and the anti-spherical module respectively. They are examples of parabolic modules for $\mathbb{H}$ (see e.g. [51, Section 3]). We can obtain an $\mathbb{L}$-basis for $\mathbb{M}$ via a set of representatives for the right cosets $W_{\mathrm{f}}\backslash W$. A natural choice for such representatives comes from the dominant alcoves, namely, the set

$$^{\mathrm{f}}W = \{x \in W : (x \cdot A_0) \subset C_0\}$$

where

$$C_0 = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Sigma\}$$

is the dominant Weyl chamber. The elements in $^{\mathrm{f}}W$ are in fact precisely the minimal length representatives for the cosets $W_{\mathrm{f}}\backslash W$. Defining M_x to be $1 \otimes H_x$ in $\mathbb{M}$, we get the $\mathbb{L}$-basis $\{M_x\}_{x \in {}^{\mathrm{f}}W}$ (and similarly for $\mathbb{N}$). The action of $\underline{H}_s$ on these bases is

$$M_x\underline{H}_s = \begin{cases} M_{xs} + vM_x & \text{if } xs \in {}^{\mathrm{f}}W \text{ and } xs > x, \\ M_{xs} + v^{-1}M_x & \text{if } xs \in {}^{\mathrm{f}}W \text{ and } xs < x, \\ (v+v^{-1})M_x & \text{if } xs \notin {}^{\mathrm{f}}W, \end{cases} \tag{1.6}$$

$$N_x\underline{H}_s = \begin{cases} N_{xs} + vN_x & \text{if } xs \in {}^{\mathrm{f}}W \text{ and } xs > x, \\ N_{xs} + v^{-1}N_x & \text{if } xs \in {}^{\mathrm{f}}W \text{ and } xs < x, \\ 0 & \text{if } xs \notin {}^{\mathrm{f}}W. \end{cases} \tag{1.7}$$

The dualization map on $\mathbb{H}$ extends to dualization maps on $\mathbb{M}$ and $\mathbb{N}$ by mapping $a \otimes H \mapsto \overline{a} \otimes \overline{H}$. To see this, note that for all $s \in S_{\mathrm{f}}$

$$\varphi_u(\underline{H}_s) = \begin{cases} v + v^{-1} & \text{if } u = v^{-1}, \\ 0 & \text{if } u = -v, \end{cases}$$

so $\varphi_u(\underline{H}_s)$ is self-dual. This means that for $s \in S_{\mathrm{f}}$

$$\begin{aligned} \overline{a \otimes (\underline{H}_s H)} &= \overline{a} \otimes \overline{\underline{H}_s H} \\ &= \overline{a} \otimes \underline{H}_s \overline{H} \\ &= \overline{a}\varphi_u(\underline{H}_s) \otimes \overline{H} \\ &= \overline{a\varphi_u(\underline{H}_s)} \otimes \overline{H} \\ &= \overline{a\varphi_u(\underline{H}_s \otimes H)} \end{aligned}$$

As $\{\underline{H}_s\}_{s \in S_{\mathrm{f}}}$ generates $\mathbb{H}_{W_{\mathrm{f}}}$ this shows that the map above is well-defined.

For the (anti-)spherical module, there is a similar notion of a Kazhdan–Lusztig basis (see e.g. [51, Theorem 3.1]).

Theorem 1.1.2. *There is a unique $\mathbb{L}$-basis $\{\underline{M}_x\}_{x \in {}^{\mathrm{f}}W}$ for $\mathbb{M}$ such that for each $x \in {}^{\mathrm{f}}W$,*

(i) $\overline{\underline{M}_x} = \underline{M}_x$ *(self-duality);*
(ii) $\underline{M}_x = M_x + \sum_{y<x} m_{y,x} H_y$*, and for all $y < x$ we have $m_{y,x} \in v\mathbb{Z}[v]$.*

There is an analogous basis $\{\underline{N}_x\}_{x\in {}^{\mathrm{f}}W}$ *for* $\mathbb{N}$.

The construction of this basis is almost exactly the same as that of the ordinary Kazhdan–Lusztig basis. For later use we define the Laurent polynomials

$$m^s_{y,x} = \begin{cases} m_{ys,x} + v m_{y,x} & \text{if } ys > y \text{ and } ys \in {}^{\mathrm{f}}W, \\ m_{ys,x} + v^{-1} m_{y,x} & \text{if } ys < y \text{ and } ys \in {}^{\mathrm{f}}W, \\ (v+v^{-1}) m_{y,x} & \text{if } ys \notin {}^{\mathrm{f}}W. \end{cases}$$

which play a role similar to $h^s_{y,x}$.

The following theorem provides an analogous basis when the coefficients are restricted to being Laurent polynomials in negative degree instead of positive degree. We state the form for $\mathbb{N}$ as it is the only one we will need later.

THEOREM 1.1.3 ([19, Remark 2.6]). *There is a unique* $\mathbb{L}$*-basis* $\{\underline{\tilde{N}}_x\}_{x\in {}^{\mathrm{f}}W}$ *for* $\mathbb{N}$ *such that for each* $x \in {}^{\mathrm{f}}W$,

(i) $\overline{\underline{\tilde{N}}_x} = \underline{\tilde{N}}_x$ *(self-duality);*

(ii) $\underline{N}_x = N_x + \sum_{y<x} \tilde{n}_{y,x} N_y$*, and for all* $y < x$ *we have* $\tilde{n}_{y,x} \in v^{-1}\mathbb{Z}[v^{-1}]$.

Moreover, we have $\tilde{n}_{y,x} = (-1)^{\ell(x)+\ell(y)}\overline{m_{y,x}}$.

PROOF. The proof of existence and uniqueness is entirely analogous to the previous case, using $\underline{\tilde{H}}_s$ instead of $\underline{H}_s$. For the final result, see e.g. [51, Theorem 3.5]. $\square$

We can now define the inverse polynomials $\{m^{y,x}\}$ for $y, x \in {}^{\mathrm{f}}W$ and $y \geq x$ such that the following formula holds:

$$\sum_z (-1)^{\ell(z)+\ell(x)} m^{z,x} m_{z,y} = \delta_{x,y}. \tag{1.8}$$

These polynomials arise as the coefficients of some element of a module related to $\mathbb{M}$ with respect to a certain basis [51, Theorem 3.6].

1.1.3. Subsequences. Let $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$ be an expression. A subsequence for $\underline{x}$ is a sequence of the form $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_m)$, where each term $\mathbf{e}_i$ is an ordered pair (s_i, t_i) with $t_i \in \{0, 1\}$ denoting an omitted or included generator respectively. We say that $\mathbf{e}_i$ is a term with generator s_i of type t_i, and we refer to the type of $\mathbf{e}$ to mean the sequence of types of the $\mathbf{e}_i$. We denote the set of all subsequences for $\underline{x}$ by $[\underline{x}]$. We write e to denote the group element $s_1^{t_1} s_2^{t_2} \cdots s_m^{t_m} \in W$.

Suppose $\mathbf{e}$ is a subsequence for $\underline{x}$. We assign an integer $d(\mathbf{e})$ to $\mathbf{e}$ called the defect. To calculate $d(\mathbf{e})$, we first construct a sequence of elements in W called the Bruhat stroll. Let $\underline{x}_{\leq i}$ denote the expression containing the first i terms, and let $\mathbf{e}_{\leq i}$ be the similarly truncated subsequence for $\underline{x}_{\leq i}$. The Bruhat stroll is a sequence $w_0, w_1, \ldots, w_m$ defined by

$$w_i = e_{\leq i} = s_1^{t_1} s_2^{t_2} \cdots s_i^{t_i}.$$

Clearly $w_0 = 1$, $w_m = e$, and at i we have $w_i = w_{i-1}$ or $w_i = w_{i-1}s_i$ if $\mathbf{e}_i$ is of type 0 or 1 respectively. Now we add a decoration U (for Up) or D (for Down) to each term in the subsequence in the following manner. At index i, if $w_{i-1}s_i > w_{i-1}$ in the Bruhat order then we add the decoration U to $\mathbf{e}_i$, whereas if $w_{i-1}s_i < w_{i-1}$ we add the decoration D instead. In other words, at each step in the Bruhat stroll we look to see whether the generator of the next term increases or decreases the length, regardless of whether the generator is actually omitted or included in the subsequence $\mathbf{e}$. The defect $d(\mathbf{e})$ is defined to be the number of terms with decorated type U0 minus the number of terms with decorated type D0.

Example 1.1.4. Suppose $s, t \in S$ and $st \neq 1$. The Bruhat stroll for the subsequence $\mathbf{e} = ((s, 1), (t, 0), (s, 0))$ is

$$1, s, s, s$$

so the decorated subsequence is $((s, \mathrm{U}1), (t, \mathrm{U}0), (s, \mathrm{D}0))$, giving a defect of $1-1 = 0$.

Notation 1.1.5. In later examples, we will use the "Tiberian" convention to write subsequences, where we write the terms $\mathbf{e}_i = (s_i, t_i)$ of a subsequence vertically in the form $\underset{s_i}{t_i}$. For example,

$$\underset{s\,t\,s}{100}$$

corresponds to the subsequence $((s, 1), (t, 0), (s, 0))$. If $\mathbf{e} \in [\underline{x}]$ and $\underline{x}$ is known from context we can omit the generators and simply write the type of $\mathbf{e}$ as a sequence of 0's and 1's.

To write a decorated subsequence, simply add the decoration above the basic type, i.e.

$$\overset{\mathrm{UUD}}{\underset{s\,t\,s}{100}}$$

corresponds to the decorated subsequence $((s, \mathrm{U}1), (t, \mathrm{U}0), (s, \mathrm{D}0))$.

The following lemma, which first appeared as [18, Proposition 3.5], is fundamental in understanding Soergel bimodules. It gives a combinatorial interpretation of the product of several Kazhdan–Lusztig generators $\underline{H}_s$ in terms of the standard basis.

Lemma 1.1.6 (Deodhar's defect formula). *Let* $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in S$. *Then*

$$\underline{H}_{\underline{x}} = \underline{H}_{s_1} \underline{H}_{s_2} \cdots \underline{H}_{s_m} = \sum_{\mathbf{e} \in [\underline{x}]} v^{d(\mathbf{e})} H_e.$$

Proof. Induct on m. The lemma clearly holds when $\underline{x}$ is the empty expression. Suppose the lemma holds for expressions of length $m-1$. Write $\underline{y}$ for $\underline{s_1 \cdots s_{m-1}}$. Then we have

$$\underline{H}_{\underline{x}} = \underline{H}_{\underline{y}} \underline{H}_{s_m} = \left(\sum_{\mathbf{f} \in [\underline{y}]} v^{d(\mathbf{f})} H_f \right) \underline{H}_{s_m}$$

by induction. Expanding this out yields

$$\left(\sum_{\mathbf{f}\in[\underline{y}]} v^{d(\mathbf{f})} H_f\right) \underline{H}_{s_m} = \sum_{\substack{\mathbf{f}\in[\underline{y}] \\ fs_m>f}} v^{d(\mathbf{f})}(H_{fs_m} + vH_f) + \sum_{\substack{\mathbf{f}\in[\underline{y}] \\ fs_m<f}} v^{d(\mathbf{f})}(H_{fs_m} + v^{-1}H_f)$$
$$= \sum_{\substack{\mathbf{e}\in[\underline{x}] \\ \mathbf{e}_m \text{ has decoration U}}} v^{d(\mathbf{e})} H_e + \sum_{\substack{\mathbf{e}\in[\underline{x}] \\ \mathbf{e}_m \text{ has decoration D}}} v^{d(\mathbf{e})} H_e$$
$$= \sum_{\mathbf{e}\in[\underline{x}]} v^{d(\mathbf{e})} H_e$$

which completes the proof. □

We can extend this result to the anti-spherical module $\mathbb{N}$ as follows. For an expression $\underline{x}$ let ${}^{\mathrm{f}}[\underline{x}]$ denote the subsequences $\mathbf{e}$ with a Bruhat stroll $\{w_i\}$ such that for all i, both w_i and $w_{i-1}s_i$ never stray outside ${}^{\mathrm{f}}W$. In other words, ${}^{\mathrm{f}}[\underline{x}]$ consists of subsequences for which we can calculate the defect entirely using elements of ${}^{\mathrm{f}}W$. We call these subsequences dominant.

Lemma 1.1.7. *Let $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in S$. Then in $\mathbb{N}$*

$$1 \otimes \underline{H}_{\underline{x}} = 1 \otimes \underline{H}_{s_1}\underline{H}_{s_2}\cdots\underline{H}_{s_m} = \sum_{\mathbf{e}\in{}^{\mathrm{f}}[\underline{x}]} v^{d(\mathbf{e})} N_e.$$

1.1.4. Character sets. An abelian group is the same thing as a $\mathbb{Z}$-module; by analogy, we call a commutative monoid a $\mathbb{Z}_{\geq 0}$-module. Similarly, we call a semiring a $\mathbb{Z}_{\geq 0}$-algebra. Recall that a semiring is an algebraic structure with two binary operations called addition and multiplication, which satisfy all the axioms defining a ring (i.e. a $\mathbb{Z}$-algebra) except for those concerning the existence of additive inverses. In other words, a $\mathbb{Z}_{\geq 0}$-algebra is just a ring without subtraction. For any $\mathbb{Z}_{\geq 0}$-algebra $\mathcal{R}$ we can construct the Grothendieck ring $[\mathcal{R}]$ by introducing subtraction, entirely analogously to the construction of a fraction field from a ring. The Grothendieck ring is equipped with a $\mathbb{Z}_{\geq 0}$-algebra homomorphism $[\mathcal{R}] \to \mathcal{R}$ and is characterized by the obvious universal property. The set $\mathbb{L}_{\geq 0} = \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ is a commutative $\mathbb{Z}_{\geq 0}$-algebra, which gives rise to the entirely similar notions of $\mathbb{L}_{\geq 0}$-modules and $\mathbb{L}_{\geq 0}$-algebras, which we will use extensively.

Nearly all of the $\mathbb{Z}_{\geq 0}$-structures in this thesis are built up from equivalence classes of sets, as in the following examples.

Example 1.1.8.

(i) The collection **FinSet** of all finite sets up to bijective equivalence has the structure of a $\mathbb{Z}_{\geq 0}$-algebra. Addition is given by taking disjoint unions and multiplication by taking direct products. This $\mathbb{Z}_{\geq 0}$-algebra is clearly isomorphic to $\mathbb{Z}_{\geq 0}$.

(ii) The collection of all objects in **FinSet**$/\mathbb{Z}$ (i.e. the slice category of finite sets over $\mathbb{Z}$) up to equivalence has the structure of an $\mathbb{L}_{\geq 0}$-algebra. More concretely, a finite set over $\mathbb{Z}$ is a set A along with a map $a : A \to \mathbb{Z}$.

Two finite sets A, B over $\mathbb{Z}$ with maps a, b to $\mathbb{Z}$ are considered equivalent if there is a bijection $f : A \to B$ such that $b \circ f = a$. As before, for two sets A, B over $\mathbb{Z}$ their sum is defined as $A + B = A \coprod B$, their disjoint union, with map $a + b = (a \coprod b) : A \coprod B \to \mathbb{Z}$. The product AB is just $A \times B$ with map $ab : A \times B \to \mathbb{Z}$ equal to the following composition

$$A \times B \xrightarrow{a \times b} \mathbb{Z} \times \mathbb{Z} \xrightarrow{+} \mathbb{Z},$$

where $+$ denotes the sum map $(x, y) \mapsto x + y$. Finally vA is defined to be A as a set, but with new map $va = v \circ a$, where $v : \mathbb{Z} \to \mathbb{Z}$ is the map $x \mapsto x + 1$.

(iii) The collection $[\underline{S}]$ of all finite sets of subsequences for expressions in $\underline{S}$ forms a $\mathbb{Z}_{\geq 0}$-algebra. The sum of two sets is again the disjoint union, while the product is defined to be the linear extension of the natural concatenation product on expressions; so for two sets $A, B \in [\underline{S}]$ the product is

$$AB = \{\mathbf{ef} : \mathbf{e} \in A,\ \mathbf{f} \in B\}.$$

We would like to extend the last example above to create an $\mathbb{L}_{\geq 0}$-algebra. To do this it will be necessary to extend subsequence generator types beyond 0 and 1.

Notation 1.1.9. We introduce a new symbol $\emptyset$ and two new decorated terms

$$(\emptyset, \mathrm{U}\emptyset) \text{ or } \overset{\mathrm{U}}{\underset{\emptyset}{\emptyset}}, \qquad\qquad (\emptyset, \mathrm{D}\emptyset) \text{ or } \overset{\mathrm{D}}{\underset{\emptyset}{\emptyset}}$$

which use this symbol. These terms do not have a generator and strictly speaking are not of type 0 or 1 and thus do not affect the Bruhat stroll directly. For the purposes of calculating defect, they count as $+1$ and -1 respectively. We call subsequences which include these new terms $01\emptyset$-subsequences.

Definition 1.1.10. The Hecke $\mathbb{L}_{\geq 0}$-algebra $\mathcal{H}$ is a collection of equivalence classes of sets of $01\emptyset$-subsequences of expressions in $\underline{S}$ with the structure of an $\mathbb{L}_{\geq 0}$-algebra. It has the following generators and relations.

- For each $\underline{x} \in \underline{S}$, the equivalence class of the set $[\underline{x}]$ is in $\mathcal{H}$. These sets generate $\mathcal{H}$ as an $\mathbb{L}_{\geq 0}$-module (but they do not usually form a basis!).
- Addition and multiplication are defined as in $[\underline{S}]$ (Example 1.1.8(iii)).
- The singleton sets

$$v = \left\{ \overset{\mathrm{U}}{\underset{\emptyset}{\emptyset}} \right\}, \qquad\qquad v^{-1} = \left\{ \overset{\mathrm{D}}{\underset{\emptyset}{\emptyset}} \right\}$$

are in $\mathcal{H}$. This gives an embedding of $\mathbb{L}_{\geq 0}$ into $\mathcal{H}$ and thus an $\mathbb{L}_{\geq 0}$-action on $\mathcal{H}$ via multiplication.
- Each set of subsequences in $\mathcal{H}$ gives an object in $\mathbf{FinSet}/(W \times \mathbb{Z})$ via the map $\mathbf{e} \mapsto (e, d(\mathbf{e}))$. Two sets of subsequences in $\mathcal{H}$ are considered equivalent if they are equivalent as sets over $W \times \mathbb{Z}$.

Note that by definition $[\underline{x}][\underline{y}] = [\underline{xy}]$ in $\mathcal{H}$ for any expressions $\underline{x}, \underline{y} \in \underline{S}$. The fact that multiplication in $\mathcal{H}$ is well-defined is essentially a consequence of Deodhar's defect formula (Lemma 1.1.6) and the corollary below. We call (equivalence classes of) sets in $\mathcal{H}$ character sets, and sets of the form $[\underline{x}]$ Bott–Samelson character sets.

PROPOSITION 1.1.11. *Multiplication in the Hecke $\mathbb{L}_{\geq 0}$-algebra $\mathcal{H}$ is well defined. Moreover, the mapping*

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathbb{H} \\ C &\longmapsto \sum_{\mathbf{e} \in C} v^{d(\mathbf{e})} H_e \end{aligned}$$

is an $\mathbb{L}_{\geq 0}$-algebra homomorphism. It induces an $\mathbb{L}$-algebra isomorphism $[\mathcal{H}] \xrightarrow{\sim} \mathbb{H}$.

PROOF. Let $\mathcal{H}^0$ denote the free $\mathbb{L}_{\geq 0}$-algebra defined by the generators above, but without the relation of equivalence. Consider the map $\mathcal{H}^0 \to \mathbb{H}$ defined as in the statement of the Proposition. By Lemma 1.1.6, for $\underline{x}, \underline{y} \in \underline{S}$ we have

$$[\underline{x}][\underline{y}] = [\underline{xy}] \longmapsto H_{\underline{xy}} = H_{\underline{x}} H_{\underline{y}}.$$

Combining this with $\mathbb{L}_{\geq 0}$-linearity implies that the map is an $\mathbb{L}_{\geq 0}$-algebra homomorphism. Now note that two sets in $\mathcal{H}^0$ are equivalent over $W \times \mathbb{Z}$ if and only if they map to the same element of $\mathbb{H}$. This implies the following in turn:

(i) multiplication in $\mathcal{H}$ is well defined,
(ii) the homomorphism $\mathcal{H}^0 \to \mathbb{H}$ factors through $\mathcal{H}$,
(iii) the induced homomorphism $[\mathcal{H}] \to \mathbb{H}$ is injective.

To prove the final claim, note that the Bott–Samelson character sets map onto a $\mathbb{L}$-spanning set for $\mathbb{H}$, so the homomorphism $[\mathcal{H}] \to \mathbb{H}$ is an isomorphism. $\square$

We can extend these ideas to $\mathbb{N}$ in a natural way.

DEFINITION 1.1.12. The anti-spherical Hecke $\mathbb{L}_{\geq 0}$-module $\mathcal{N}$ is a collection of equivalence classes of sets of dominant $01\emptyset$-subsequences of expressions in $\underline{S}$ with the structure of a module over the Hecke $\mathbb{L}_{\geq 0}$-algebra. It has the following generators and relations.

- For each $\underline{x} \in \underline{S}$, the equivalence class of the set ${}^{\mathrm{f}}[\underline{x}]$ is in $\mathcal{N}$. These sets generate $\mathcal{N}$ as an $\mathbb{L}_{\geq 0}$-module (but they do not usually form a basis!).
- Addition and $\mathbb{L}_{\geq 0}$-scalar multiplication are defined as in $[\underline{S}]$ (Example 1.1.8(iii)).
- There is a right $\mathcal{H}$-action on $\mathcal{N}$, defined in the following manner. For $N \in \mathcal{N}$ and $C \in \mathcal{H}$, we set

$$NC = \{\mathbf{ef} : \mathbf{e} \in N,\ \mathbf{f} \in C,\ \mathbf{ef} \text{ dominant}\}.$$

- Each set of subsequences in $\mathcal{N}$ gives an object in $\mathbf{FinSet}/({}^{\mathrm{f}}W \times \mathbb{Z})$ via the map $\mathbf{e} \mapsto (e, d(\mathbf{e}))$. Two sets of subsequences in $\mathcal{N}$ are considered equivalent if they are equivalent as sets over ${}^{\mathrm{f}}W \times \mathbb{Z}$.

The next result shows that the right $\mathcal{H}$-action is well defined. By definition ${}^{\mathrm{f}}[\underline{x}][\underline{y}] = {}^{\mathrm{f}}[\underline{xy}]$ in $\mathcal{N}$ for any expressions $\underline{x}, \underline{y} \in \underline{S}$. We call (equivalence classes of) sets in $\mathcal{N}$ anti-spherical character sets.

PROPOSITION 1.1.13. *The right $\mathcal{H}$-action in the anti-spherical $\mathbb{L}_{\geq 0}$-module $\mathcal{N}$ is well defined. Moreover, the mapping*

$$\begin{aligned} \mathcal{N} &\longrightarrow \mathbb{N} \\ N &\longmapsto \sum_{\mathbf{e} \in N} v^{d(\mathbf{e})} N_e \end{aligned}$$

is an $\mathcal{H}$-module homomorphism, where the right $\mathcal{H}$-module structure on the codomain arises from the isomorphism $[\mathcal{H}] \cong \mathbb{H}$. It induces an $\mathbb{H}$-module isomorphism $[\mathcal{N}] \xrightarrow{\sim} \mathbb{N}$.

1.2. Finite-dimensional algebras

Let $\Bbbk$ be a field, and let A be a finite-dimensional $\Bbbk$-algebra. In this thesis all algebras except Lie algebras are associative and unital. We write $A{-}\mathrm{mod}$ for the category of finite-dimensional left A-modules, $[A{-}\mathrm{mod}]$ for the $\mathbb{Z}_{\geq 0}$-module of isomorphism classes of modules in $A{-}\mathrm{mod}$, and $[[A{-}\mathrm{mod}]]$ for the Grothendieck group of $[A{-}\mathrm{mod}]$. For a finite-dimensional A-module M we write $[M]$ and $[[M]]$ to denote the images of M in $[A{-}\mathrm{mod}]$ and $[[A{-}\mathrm{mod}]]$ respectively. Write $[[A{-}\mathrm{mod}]]_{/}$ for the quotient of $[[A{-}\mathrm{mod}]]$ with respect to the ideal generated by elements of the form $[[A]] - [[B]] + [[C]]$ for all short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $A{-}\mathrm{mod}$. Other sources call $[[A{-}\mathrm{mod}]]$ and $[[A{-}\mathrm{mod}]]_{/}$ the "split Grothendieck group of A-modules" and the "ordinary Grothendieck group of A-modules" respectively, and we will sometimes abuse notation and use this terminology.

1.2.1. Filtered algebras.

A generalized filtration on A is a collection of $\Bbbk$-subspaces $\{A^i\}$ (indexed by integers i) such that the $\Bbbk$-linear span of $\{A^i\}$ is A, $1 \in A^0$, and $(A^i)(A^j) \subseteq A^{i+j}$ for all i, j. This is similar to the notion of an ascending or descending filtration on A, but without the containment condition. If A has a generalized filtration $A^\bullet$ we call A a generalized filtered algebra. We will often omit "generalized" for brevity.

A filtered module over a filtered algebra A is an A-module M equipped with a collection of $\Bbbk$-subspaces M^i indexed over the integers such that the $\Bbbk$-linear span of $\{M^i\}$ is M and $(A^i)(M^j) \subseteq M^{i+j}$ for all i, j. A homomorphism between filtered A-modules M and N with filtrations $M^\bullet$ and $N^\bullet$ is an A-module homomorphism $f : M \to N$ such that $f(M^i) \subseteq N^i$ for all i.

If M is a filtered A-module and $N \leq M$ is an A-module, then there are natural filtrations on N and M/N making them into filtered modules, namely by setting $N^i = M^i \cap N$ and $(M/N)^i = (M^i + N)/N$. They ensure that the natural maps $N \to M$ and $M \to M/N$ are homomorphisms of filtered A-modules. Combining these two constructions, we can give any subquotient L/N of M the filtration

$$(L/N)^i = (M^i \cap L + N)/N$$

by first considering L as a submodule of M and then considering L/N as a quotient of L. This is well-defined, for if we apply these processes in the opposite order, we get

$$\begin{aligned}(M/N)^i &= (M^i + N)/N \\ (L/N)^i &= ((M^i + N)/N) \cap L/N \\ &= ((M^i + N) \cap L)/N \\ &= (M^i \cap L + N)/N\end{aligned}$$

which gives the same filtration.

For $i \in \mathbb{Z}$ we denote the degree i filtration shift by $\langle i \rangle$, where $M\langle i \rangle^j = M^{i+j}$. For any two filtered modules M and N, we define a filtered vector space

$$\mathrm{Hom}_A(M, N)^\bullet = \sum_i \mathrm{Hom}_A(M, N)^i = \sum_i \mathrm{Hom}_A(M, N\langle i \rangle),$$

where the sum is taken in $\mathrm{Hom}_{A^{\mathrm{unfilt}}}(M, N)$, the space of all unfiltered A-module homomorphisms.

In the special case of descending (or ascending) filtrations, we provide some notation for subquotients. If $M^\bullet$ is a descending (ascending) filtration, then we write $M_i = M^i/M^{i+1}$ ($M_i = M^{i+1}/M^i$) for the successive subquotients, which are called layers.

We write $A-$filtmod for the category of filtered modules over a filtered algebra A. This category is always additive and in fact pre-abelian, yet even in the case of ascending/descending filtrations, $A-$filtmod is not necessarily abelian.

Example 1.2.1. Suppose $A = \bigoplus_i A_i$ is $\mathbb{Z}$-graded. Then the A_i define a generalized filtration on A, and the category of filtered A-modules with respect to this filtration is just $A-$grmod, the category of graded A-modules. The category $A-$grmod is abelian and as such behaves much better than $A-$filtmod for general filtered A. As a convention we will use subscripts to denote gradings, in order to distinguish them from more general filtrations, and use (i) to denote the degree i grade shift, with $M(i)_j = M_{i+j}$ for a graded module M as before. It is also useful to define the graded dimension of M, which is

$$\dim_\bullet M_\bullet = \sum_i (\dim M_i) v^i,$$

a Laurent polynomial with integer coefficients.

Example 1.2.2. Suppose A is an arbitrary finite-dimensional algebra. Recall that the Jacobson radical $J(A)$ of A is the intersection of the maximal left (or right) ideals of A. It is the minimal two sided ideal for which $A/J(A)$ is a semisimple algebra. For an A-module M the submodule $\mathrm{rad}\, M$ is similarly defined to be the minimal submodule for which $M/\operatorname{rad} M$ is semisimple. It is a general fact that $\operatorname{rad} M = J(A)M$.

Define the filtration $A^i = J(A)^i$ for $i \geq 0$ and $A^i = A$ for $i < 0$. This gives A a descending filtered structure called the radical series, and any A-module M can be given a filtration $M^i = J(A)^i M = \operatorname{rad}^i M$ for $i \geq 0$ and $M^i = M$ for $i < 0$, which is compatible with the filtration on A. In this case, we write $A-$radfiltmod for the radical filtered module category.

Filtered modules in $A-$radfiltmod are essentially just A-modules equipped with a descending filtration whose layers are semisimple. For an ascending filtration with the same property, we can re-index by negating each filtration degree to obtain a descending filtration. We call such filtrations semisimple. A Loewy series is a minimal length semisimple filtration, where "length" here refers to the number of non-zero layers of the filtration. The length of any Loewy series is unique and is called the Loewy length. The radical series is an example of a descending Loewy series. The socle series, which is defined inductively by $\operatorname{soc}^0 M = 0$ and

$$\operatorname{soc}^{i+1} M/\operatorname{soc}^i M = \operatorname{soc}(M/\operatorname{soc}^i M),$$

where $\operatorname{soc} U$ is the maximal semisimple submodule of U, is an ascending Loewy series.

1.2.2. Quasi-hereditary algebras. We recall the notion of a quasi-hereditary algebra. Suppose the simple A-modules $\{L(\lambda) : \lambda \in \Lambda\}$ are indexed by a poset $(\Lambda, \leq)$. Let $P(\lambda)$ denote the projective cover of $L(\lambda)$, and let $\Delta(\lambda)$ be the maximal quotient of $P(\lambda)$ whose composition factors are among $\{L(\mu) : \mu \leq \lambda\}$. We call $\Delta(\lambda)$ a Weyl module or a standard module. A Δ-filtration of a module M is a series of submodules

$$0 = M^0 < M^1 < M^2 < \cdots < M^n = M$$

such that for each $k > 0$, M^k/M^{k-1} is isomorphic to a standard module. If A is graded, we allow grade shifting in these isomorphisms. We write $A(\Delta)-$mod for the full subcategory of Δ-filtered modules.

We say that A is quasi-hereditary if for each $\lambda \in \Lambda$,

(i) the composition factor multiplicity $[\Delta(\lambda) : L(\lambda)]$ is exactly 1; and
(ii) the projective module $P(\lambda)$ has a Δ-filtration.

In this situation, the images $\{[[\Delta(\lambda)]]_{/}\}_{\lambda\in\Lambda}$ of the standard modules in the ordinary Grothendieck group of $A-$mod form a $\mathbb{Z}$-basis. As a result, if M has a Δ-filtration, then the number of subquotients isomorphic to a given standard module $\Delta(\lambda)$ doesn't depend on the choice of Δ-filtration. We denote this number by $(M : \Delta(\lambda))$.

We can also supply an alternative definition of quasi-hereditary using injective modules. Let $I(\lambda)$ be the injective hull of $L(\lambda)$, and let $\nabla(\lambda)$ be the maximal submodule of $I(\lambda)$ whose composition factors are among $\{L(\mu) : \mu \leq \lambda\}$. We call $\nabla(\lambda)$ a dual Weyl module or a costandard module. The dual definition, in which each Δ above is replaced with ∇ and projective covers are replaced with injective hulls turns out to be equivalent to the original definition.

The next few theorems can be found in any reference on quasi-hereditary algebras e.g. [16] or [21], but to ensure consistency we provide proofs. First of all, the following homological property of standard/costandard modules can be used as the basis of a more self-dual definition of a quasi-hereditary algebra.

THEOREM 1.2.3. *Let A be a quasi-hereditary algebra with poset Λ, and suppose $\lambda, \mu \in \Lambda$. Then*

$$\mathrm{Ext}^i_A(\Delta(\lambda), \nabla(\mu)) = \begin{cases} \Bbbk & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We produce the "easy dimension shifting argument" omitted in [23, Theorem 1.3]. Write the short exact sequence

$$0 \longrightarrow M \longrightarrow P(\lambda) \longrightarrow \Delta(\lambda) \longrightarrow 0$$

which induces the following long exact sequence

$$\cdots \to \mathrm{Ext}^i_A(\Delta(\lambda), \nabla(\mu)) \to \mathrm{Ext}^i_A(P(\lambda), \nabla(\mu)) \to \mathrm{Ext}^i_A(M, \nabla(\lambda)) \to \cdots.$$

We first prove the result for $i = 0$. The first few terms of the long exact sequence are

$$0 \to \mathrm{Hom}_A(\Delta(\lambda), \nabla(\mu)) \to \mathrm{Hom}_A(P(\lambda), \nabla(\mu)).$$

The third term above has dimension

$$\dim \mathrm{Hom}_A(P(\lambda), \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$$

which is non-zero only if $\lambda \leq \mu$. By symmetry we must have $\mu \leq \lambda$, so without loss of generality suppose $\lambda = \mu$. But in this case we know that the first term is at least 1-dimensional because the of the non-zero composite $\Delta(\lambda) \to L(\lambda) \to \nabla(\lambda)$, so we are done. Note that in either case the second homomorphism above is an isomorphism, so the next term in the sequence is 0 by exactness.

For $i > 0$, assume that we have proved the result for $i - 1$ and shown that $\mathrm{Ext}^{i-1}(M, \nabla(\mu)) = 0$. Then a portion of the long exact sequence reads

$$\cdots \to 0 = \mathrm{Ext}^{i-1}(M, \nabla(\mu)) \to \mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) \to \mathrm{Ext}^i(P(\lambda), \nabla(\mu)) = 0 \to \cdots$$

which immediately shows the result for i. In addition, the fact that the map from the $\Delta(\lambda)$-Hom-space to the $P(\lambda)$-Hom-space is an isomorphism shows that the next term $\mathrm{Ext}^i(M, \nabla(\mu))$ vanishes, so we are done by induction. □

Restricting for the moment to the case where $i = 0$, one important consequence is the following method for calculating Δ-multiplicities.

THEOREM 1.2.4. *Let A be a quasi-hereditary algebra with poset Λ. If M is a Δ-filtered A-module, then for every $\lambda \in \Lambda$,*

$$[M : \Delta(\lambda)] = \dim \mathrm{Hom}_A(M, \nabla(\lambda)).$$

Similarly if M is a ∇-filtered module, then for every $\lambda \in \Lambda$,

$$[M : \nabla(\lambda)] = \dim \operatorname{Hom}_A(\Delta(\lambda), M).$$

Proof. We will prove the first statement only; the second is completely dual. Suppose

$$0 \longrightarrow M' \longrightarrow M \longrightarrow \Delta(\mu) \longrightarrow 0$$

where M' also has a Δ-filtration. The induced long exact sequence gives

$$\begin{tikzcd}
 & \Bbbk \arrow[d, equal] & & \\
0 \arrow[r] & \operatorname{Hom}_A(\Delta(\mu), \nabla(\lambda)) \arrow[r] & \operatorname{Hom}_A(M, \nabla(\lambda)) \arrow[r] & \operatorname{Hom}_A(M', \nabla(\lambda)) \arrow[dll, rounded corners] \\
 & \operatorname{Ext}^1(\Delta(\mu), \nabla(\lambda)) = 0 \arrow[r] \arrow[d, equal] & \cdots & \\
 & 0 & &
\end{tikzcd}$$

from which the result follows by induction. $\square$

Applying this to $P(\mu)$, we obtain

$$[P(\mu) : \Delta(\lambda)] = \dim \operatorname{Hom}_A(P(\mu), \nabla(\lambda)) = [\nabla(\lambda) : L(\mu)], \tag{1.9}$$

a result sometimes called Brauer–Humphreys reciprocity. Another important consequence is a homological criterion for Δ-filtered modules.

Theorem 1.2.5. *Let A be a quasi-hereditary algebra with poset Λ. An A-module M has a Δ-filtration if and only if $\operatorname{Ext}^1_A(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$.*

Proof. One direction follows immediately from the previous theorem. If M has a Δ-filtration, then the long exact sequence induced by the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow \Delta(\mu) \longrightarrow 0$$

for some Δ-filtered submodule M' immediately implies $\operatorname{Ext}^1_A(M, \nabla(\lambda)) = 0$ by induction on the length of the Δ-filtration.

Conversely, suppose $\operatorname{Ext}^1_A(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$. Choose μ minimal such that $\operatorname{Hom}(M, L(\mu)) \neq 0$. For any $\lambda < \mu$, the short exact sequence

$$0 \longrightarrow L(\lambda) \longrightarrow \nabla(\lambda) \longrightarrow U \longrightarrow 0$$

induces the long exact sequence

$$\cdots \to 0 = \operatorname{Hom}_A(M, U) \to \operatorname{Ext}^1_A(M, L(\lambda)) \to \operatorname{Ext}^1(M, \nabla(\lambda)) = 0 \to \cdots$$

which shows that $\operatorname{Ext}^1_A(M, L(\lambda)) = 0$. Writing V for the kernel of $\Delta(\mu) \to L(\mu)$ it follows that $\operatorname{Ext}^1_A(M, V) = 0$ via the same argument above, since all the composition factors of V are less than μ. But then the long exact sequence induced by

$$0 \longrightarrow V \longrightarrow \Delta(\mu) \longrightarrow L(\mu) \longrightarrow 0$$

gives

$$0 = \mathrm{Hom}_A(M, V) \to \mathrm{Hom}_A(M, \Delta(\mu)) \to \mathrm{Hom}_A(M, L(\mu)) \to 0 = \mathrm{Ext}^1_A(M, V)$$

so $\mathrm{Hom}_A(M, \Delta(\mu)) \neq 0$ and any homomorphism is surjective as it is surjective on the head. By induction we obtain a Δ-filtration. □

In [49] Ringel defined and classified the tilting modules for a quasi-hereditary algebra A. There are several notions of tilting and cotilting modules throughout representation theory, but in the special case of quasi-hereditary algebras there is an elementary description, which we summarize below.

THEOREM 1.2.6. *Let A be a quasi-hereditary algebra with poset Λ. For each weight $\lambda \in \Lambda$, there exists a unique indecomposable module $T(\lambda)$ such that*

(i) *$T(\lambda)$ has both a Δ-filtration and a ∇-filtration;*
(ii) *there is a unique embedding of $\Delta(\lambda)$ as a submodule of $T(\lambda)$ and a unique quotient of $T(\lambda)$ isomorphic to $\nabla(\lambda)$; and*
(iii) *if $L(\mu)$ is a composition factor of $T(\lambda)$ then $\mu \leq \lambda$.*

Note that in some sources, the term tilting module is reserved for "full" tilting modules, i.e. $T \cong \bigoplus_{\lambda\in\Lambda} T(\lambda)$ and what we call tilting modules are called "partial tilting modules". This distinction is more useful for tilting-theoretic applications (such as Ringel duality) which we do not cover here. As in Lie theory the elements of Λ are often called weights. We say that $L(\lambda)$, $\Delta(\lambda)$, $\nabla(\lambda)$, and $T(\lambda)$ are modules with highest weight λ.

1.2.3. Coalgebras. We recall the categorical dual notion of an algebra, which is called a coalgebra. A coalgebra is a vector space C over a field $\Bbbk$, together with two linear maps $\delta : C \to C \otimes C$ and $\epsilon : C \to \Bbbk$ (called comultiplication and counit respectively) such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\delta"] \\ C \otimes C \arrow[r, "\delta\otimes\mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

(coassociativity)

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ C \otimes C \arrow[r, "\epsilon\otimes\mathrm{id}"] & C \end{tikzcd}$$

(coidentity)

Here all tensor products are over $\Bbbk$. Note that we have identified $\Bbbk \otimes C \cong C \cong C \otimes \Bbbk$ via the canonical isomorphisms.

The $\Bbbk$-space dual C^* of a coalgebra C has an algebra structure, with product $\xi\eta = (\xi \otimes \eta) \circ \delta$ and unit ϵ. Moreover, the dual of a finite-dimensional algebra has a coalgebra structure in a similar way. However, one should note that the $\Bbbk$-space dual of an infinite-dimensional algebra is not a coalgebra in this manner. This is due to the fact that if A is infinite-dimensional, $(A \otimes A)^*$ is strictly larger than $A^* \otimes A^*$. This provides the first hint that coalgebras enjoy finiteness properties in comparison with algebras.

The dual notion of a module for a coalgebra is called a comodule. A (right) comodule over a coalgebra C is a $\Bbbk$-vector space V, with a linear map $\tau : V \to V \otimes C$ such that the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "\tau"] & V \otimes C \arrow[d, "\tau\otimes\mathrm{id}"] \\ V \otimes C \arrow[r, "\delta\otimes\mathrm{id}"] & V \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\tau"] \arrow[dr, "\mathrm{id}"] & V \otimes C \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ & V \end{tikzcd}$$

A right C-comodule V is naturally a *left* C^*-module, with C^*-linear action defined by $\xi v = ((\mathrm{id}\otimes\xi)\circ\tau)(v)$. We denote the category of right C-comodules by comod$-C$. It is an abelian category which has enough injectives. When C is finite-dimensional, the correspondence between comod$-C$ and C^*-mod is an equivalence.

As in the previous section, suppose the simple C-comodules $\{L(\lambda) : \lambda \in \Lambda\}$ are indexed by a poset Λ. For a subset $\pi \subseteq \Lambda$ let $C(\pi)$ denote the maximal right subcomodule of C whose composition factors lie in $\{L(\lambda) : \lambda \in \pi\}$. It can be shown that $C(\pi)$ is in fact a subcoalgebra of C. There is a natural correspondence between $C(\pi)$-comodules and C-comodules whose composition factors are labeled by weights in π.

Write $\langle\lambda\rangle$ for the principal poset ideal $\{\mu : \lambda \leq \mu\}$ of Λ (in this thesis a poset ideal is a downwardly closed subset of a poset). Let us further suppose that the poset Λ has the property that for any λ the principal poset ideal $\langle\lambda\rangle$ is finite. This implies that any finitely generated poset ideal $\pi \subseteq \Lambda$ is finite. We call C a quasi-hereditary coalgebra if for every finitely generated poset ideal $\pi \subseteq \Lambda$, the coalgebra $C(\pi)$ is finite-dimensional and the dual algebra $C(\pi)^*$ is a quasi-hereditary algebra with respect to π viewed as a poset.

This last condition can be rephrased without reference to the dual algebra. For $\lambda \in \pi$ let $I_\pi(\lambda)$ denote the injective hull of $L(\lambda)$ as a $C(\pi)$-comodule, and let $\nabla(\lambda)$ be the maximal subcomodule of $I_\pi(\lambda)$ whose composition factors are among $\{L(\mu) : \mu \leq \lambda\}$. From the equivalence between comod$-C(\pi)$ and $C(\pi)^*-$mod, the finite-dimensional coalgebra $C(\pi)$ is quasi-hereditary if and only if for each $\lambda \in \Lambda$,

(i) the composition factor multiplicity $[\nabla(\lambda) : L(\lambda)]$ is exactly 1; and
(ii) the injective module $I_\pi(\lambda)$ has a ∇-filtration.

The comodule $\nabla(\lambda)$ does not depend on the ideal π containing λ, and $I_\pi(\lambda)$ is the maximal subcomodule of the C-comodule injective hull $I(\lambda)$ of $L(\lambda)$ whose composition factors are labeled by weights in π. If Λ is further assumed to be countable, then by taking the union of $\{I_\pi(\lambda)\}$ over all finitely generated poset ideals π, we obtain a ∇-filtration on $I(\lambda)$ with the same indexing properties. Up to finiteness conditions[2] the existence of such a ∇-filtration of $I(\lambda)$ is equivalent to C being quasi-hereditary.

[2]Such as assuming that $\mathrm{End}(\nabla(\lambda), \nabla(\mu))$ is always finite-dimensional, and that composition factor multiplicities for $\nabla(\lambda)$ are always finite, etc.

1.3. Representations of reductive algebraic groups

1.3.1. Algebraic groups. Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$. For us, an algebraic group G is an *affine* variety over $\Bbbk$ with a group structure compatible with the variety structure. We denote the ring of functions of G by $\Bbbk[G]$. Multiplication in G gives $\Bbbk[G]$ the structure of a coalgebra, with

$$\delta(f)(g \otimes h) = f(gh), \qquad \epsilon(f) = f(1)$$

for $f \in \Bbbk[G]$ and $g, h \in G$. If V is a right $\Bbbk[G]$-comodule with coaction map $\tau : V \to V \otimes \Bbbk[G]$, then we can obtain a left representation of G on V by setting $g \cdot v = ((\mathrm{id} \otimes \mathrm{ev}_g) \circ \tau)(v)$, where $g \in G$, $v \in V$, and $\mathrm{ev}_g : \Bbbk[G] \to \Bbbk$ is the evaluation map at g. Representations which arise in this fashion are called rational representations.

If V is a finite-dimensional representation of G, then V is rational if and only if the map $G \times V \to V$ defining the left G-action is a morphism of varieties. More generally, a representation V of G is rational if and only if it is locally finite (i.e. each $v \in V$ is contained inside a finite-dimensional subrepresentation) and all finite-dimensional subrepresentations are rational. In this thesis we will only consider rational representations of G, which we will simply call G-modules. The category of left G-modules is denoted $G-\mathrm{Mod}$, while the category of finite-dimensional left G-modules is denoted $G-\mathrm{mod}$. From coalgebraic considerations $G-\mathrm{Mod}$ has enough injectives, which will be useful later for defining derived functors and Ext-groups.

Example 1.3.1. The left/right regular representations of G, defined by

$$\begin{aligned} \rho_l : G \times \Bbbk[G] &\longrightarrow \Bbbk[G] \\ (g, f) &\longmapsto (x \mapsto f(g^{-1}x)) \end{aligned} \qquad \begin{aligned} \rho_r : G \times \Bbbk[G] &\longrightarrow \Bbbk[G] \\ (g, f) &\longmapsto (x \mapsto f(xg)) \end{aligned}$$

respectively, are both *left* G-modules.

If $H \leq G$ is an algebraic subgroup and V is an H-module, we define the induced module $\mathrm{ind}_H^G V$ to be the space of H-equivariant morphisms for the right regular action of H on G, i.e.

$$\mathrm{ind}_H^G V = \mathrm{Mor}_H(G, V) = \{f : G \to V : f(gh^{-1}) = hf(g)\}.$$

This space inherits a left G-action from the left G-action on $\mathrm{Mor}(G, V)$ arising from left multiplication on G. In other words, for $g, x \in G$ and $f \in \mathrm{ind}_H^G V$ we have $(gf)(x) = f(g^{-1}x)$. An alternative construction uses the $\Bbbk$-space isomorphism $V \otimes \Bbbk[G] \cong \mathrm{Mor}(G, V)$ via the mapping $(v, f) \mapsto (g \mapsto f(g)v)$. Under this correspondence, $\mathrm{ind}_H^G V \cong (V \otimes \Bbbk[G])^H$ where $\Bbbk[G]$ is an H-module via the restriction of the right regular representation. The left G-action on $(V \otimes \Bbbk[G])^H$ comes from the left regular representation, whose action commutes with that of the right regular representation. This construction is functorial and also shows that $\mathrm{ind}_H^G V$ is rational. Induction is a left-exact functor $\mathrm{ind}_H^G : H-\mathrm{Mod} \to G-\mathrm{Mod}$, so we can take the right-derived functors $R^i \mathrm{ind}_H^G$.

We can factor any element $g \in G$ as a product $g = g_s g_u$ of two commuting elements $g_s, g_u \in G$, such that g_s acts diagonally and g_u acts unipotently on any G-module. Moreover, this factorization is essentially unique. Thus we can call elements of G "diagonalizable" or "unipotent" in a well-defined way without fixing an embedding of G into GL_n.

A torus is an algebraic group isomorphic to the group of diagonal matrices $D_n \leq \mathrm{GL}_n$ for some positive integer n. Suppose T is a torus, and let $X = X(T)$ and $Y = Y(T)$ denote the character and cocharacter groups of T. These groups are usually written additively. The elements of X and Y are called weights and coweights respectively. The dual pairing between X and Y is denoted $\langle -, - \rangle$, and is a perfect pairing.

The representation theory of T is particularly straightforward. The irreducible T-modules are all 1-dimensional and every T-module is completely reducible. More precisely, for a T-module V we have the weight space decomposition $V = \bigoplus_{\alpha \in X} V_\alpha$, where for $\alpha \in X$ we define the weight space corresponding to α

$$V_\alpha = \{v \in V : \text{for all } t \in T,\, tv = \alpha(t)v\}.$$

Tori are examples of algebraic groups whose elements are all semisimple. By contrast, an algebraic group whose elements are all unipotent is called a unipotent group. There is an analogue of Engel's theorem for unipotent groups

THEOREM 1.3.2 ([30, Theorem 17.5]). *Let G be a unipotent group, and let $V \neq 0$ be a finite-dimensional G-module. Then there is some non-zero $v \in V$ which is fixed by G.*

This means that a unipotent subgroup of GL_n is conjugate to a subgroup of the group of upper triangular matrices with 1's on the diagonal. Moreover, there is also an analogue of Lie's theorem for solvable groups called the Lie–Kolchin theorem.

THEOREM 1.3.3 ([30, Theorem 17.6]). *Let G be a connected solvable group, and let $V \neq 0$ be a finite-dimensional G-module. Then G has a common eigenvector in V; i.e. there exists a non-zero $v \in V$ such that $Gv \subseteq \Bbbk v$.*

This implies that a connected solvable subgroup of GL_n is conjugate to a subgroup of the group of upper triangular matrices. We call a maximal connected solvable subgroup a Borel subgroup. It can be shown that all Borel subgroups and all maximal tori of an algebraic group are conjugate.

1.3.2. Classification. Let G be a connected reductive algebraic group over $\Bbbk$; in other words, G is a connected algebraic group with no non-trivial closed connected unipotent normal subgroups. If the centre $Z(G)$ is finite, G is called semisimple. Let T be a maximal torus of G. The dimension of T is called the rank of G, and is well defined since all maximal tori are conjugate. The torus T is connected and solvable, so it is contained in some Borel subgroup B of G.

The group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action Ad. Applying the weight space decomposition to $\mathrm{Ad}|_T$ gives the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi = \{\alpha \in X : \alpha \neq 0,\ \mathfrak{g}_\alpha \neq 0\} \subset X$. We call weights in Φ roots, and their corresponding weight spaces in $\mathfrak{g}$ root spaces. The remaining weight space $\mathfrak{g}_0$ is equal to the Lie algebra of T. By taking inverses we see that if $\alpha \in \Phi$ then $-\alpha \in \Phi$.

For each root $\alpha \in \Phi$, there is a rank 1 semisimple subgroup $S_\alpha \leq G$, whose root spaces are $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. There is a unique coweight $\alpha^\vee \in Y$ whose image is the maximal torus of S_α, normalized so that $\langle \alpha, \alpha^\vee \rangle = 2$. The cocharacter $\alpha^\vee$ is called the coroot corresponding to α. Let $\Phi^\vee$ be the set of all coroots. The tuple $(X, \Phi, Y, \Phi^\vee)$ can be shown to be a root datum. The classification of reductive algebraic groups, analogous to the Serre classification of complex semisimple Lie algebras, states that there is an equivalence between root data and isomorphism classes of reductive groups (see e.g. [32, Proposition II.1.15]).

Let E be the subspace of $\mathbb{R} \otimes_{\mathbb{Z}} X$ spanned by Φ. The real vector space E has an inner product $(-,-)$ defined (up to rescaling) by setting

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = \langle \alpha, \beta^\vee \rangle$$

for $\alpha, \beta \in \Phi$ and extending linearly. Then $\Phi \subset E$ defines a root system. Most of the time we will work with root systems instead of root data for simplicity, using the same notation as in Section 1.1. We choose signs for the roots so that the Lie algebra of the Borel subgroup B is the direct sum of the negative root spaces. This choice of signs gives X a partial order by declaring $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Phi^+$. A weight $\lambda \in X$ is called dominant if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Sigma$. We denote the set of dominant weights by X^+.

1.3.3. Weyl modules and simple modules. Let $\lambda \in X^+$. The character λ defines a T-module. The Borel subgroup B contains a maximal closed connected unipotent normal subgroup U, with quotient group B/U isomorphic to the maximal torus T. We take the inflation of the T-module λ via the quotient homomorphism $B \to B/U \cong T$ to get a B-module, which we also denote λ. The induced module or dual Weyl module of highest weight λ is the finite-dimensional module

$$\nabla(\lambda) = \mathrm{ind}_B^G \lambda.$$

This module is a highest weight module, in the sense that $\nabla(\lambda)$ is generated by the λ-weight space $\nabla(\lambda)_\lambda = \Bbbk v_\lambda$, and if $\nabla(\lambda)_\mu \neq 0$, then $\mu \leq \lambda$. The socle $L(\lambda)$ of $\nabla(\lambda)$ is simple and also has highest weight λ. Every finite-dimensional simple G-module is of this form.

Taking duals, we have $L(\lambda)^* \cong L(-w_{\mathrm{f}}\lambda)$, where $w_{\mathrm{f}} \in W_{\mathrm{f}}$ is the longest element of the Weyl group. As a result it is often more convenient to use the so-called contravariant dual ${}^\tau(-)$, defined by twisting via the transpose antiautomorphism instead of taking inverses. Using this dual we have ${}^{\tau}L(\lambda) \cong L(\lambda)$. The Weyl module with highest weight λ is defined to be $\Delta(\lambda) = {}^{\tau}\nabla(\lambda)$.

The following result, known as Kempf's vanishing theorem, can be used to explain some of the most fundamental aspects of the representation theory of G.

THEOREM 1.3.4 ([32, Proposition II.4.5]). *Let G be a connected reductive group, and let $\lambda \in X^+$. Then $R^i \operatorname{ind}_B^G \lambda = 0$ for all $i > 0$, where $R^i \operatorname{ind}_B^G$ denotes the ith right derived functor of induction from B to G.*

One consequence of this theorem is that the Lie-theoretic characters (i.e. the weight space decompositions) of the Weyl modules are given by Weyl's character formula (see e.g. [32, Proposition II.5.10]). We omit the details of this, as Lie-theoretic characters will not be used in this thesis. For us, a "character formula" is just an equation in the Grothendieck ring of $G-$mod. It is still useful to know that Weyl's character formula exists, in order to resolve our Grothendieck ring character formulas involving Weyl modules into Lie-theoretic characters.

In characteristic $p = 0$, a consequence of the Borel–Bott–Weil Theorem is that $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$ for all $\lambda \in X^+$ and that all finite-dimensional G-modules are semisimple. In this situation the representation theory of G is very similar to the representation theory of the Lie algebra $\mathfrak{g}$.

In positive characteristic $p > 0$ it can be shown that $\nabla(\lambda)$ and $\Delta(\lambda)$ are reductions mod p of $\mathbb{Z}$-forms of the simple module in characteristic 0. It is no longer the case that $G-$mod is semisimple, but Kempf's vanishing theorem does imply that $\Bbbk[G]$ is quasi-hereditary as a coalgebra, as defined in Section 1.2.3. This means that for a finitely generated poset ideal $\pi \subseteq X^+$ of dominant weights we can define the finite-dimensional coalgebra $\Bbbk[G](\pi)$ and its dual algebra $S(\pi)$, called a (generalized) Schur algebra. The category $S(\pi)-$mod captures the representation theory of finite-dimensional G-modules whose composition factors have highest weights in π.

As in the previous section we say that a finite-dimensional G-module is tilting if it has both a Δ-filtration and a ∇-filtration. From the classification theorem, the indecomposable $S(\pi)$-tilting modules are labeled $T(\lambda)$ for $\lambda \in \pi$, and these are also indecomposable tilting modules for G. By taking π sufficiently large, this implies that for each $\lambda \in X^+$ there is an indecomposable tilting module $T(\lambda)$ of highest weight λ, and all the indecomposable tilting modules are of this form. Clearly the contravariant dual ${}^{\tau}T(\lambda)$ of the indecomposable tilting module $T(\lambda)$ is indecomposable tilting with highest weight λ, so it must be isomorphic to $T(\lambda)$. This implies that all tilting modules are self-dual.

The following (difficult) theorem due to Wang, Donkin, and Mathieu gives one of the most important properties of tilting modules for reductive algebraic groups.

THEOREM 1.3.5 ([23, 4.10]). *Let G be a reductive algebraic group, and let M and N be G-modules. Suppose M and N both have Δ-filtrations. Then $M \otimes N$ has a Δ-filtration.*

This theorem along with its dual ensure that the tensor product of two tilting modules is also tilting.

1.3.4. Linkage and translation. From now on we assume that the characteristic p is positive. We briefly describe the linkage principle and the translation principle. Let W denote the affine Weyl group corresponding to the root system Φ, as defined in Section 1.1. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, and define the p-dilated or p-scaled dot action

$$w \cdot_p \lambda = pw(p^{-1}\lambda + p^{-1}\rho) - \rho$$

for $w \in W$ and $\lambda \in X$. The linkage principle gives a condition for the composition factors of the Weyl modules in terms of this new action.

THEOREM 1.3.6 ([32, II.6.13]). *Let $\lambda \in X^+$. If $\Delta(\lambda)$ has $L(\mu)$ as a composition factor, then either $\lambda = \mu$ and $L(\lambda)$ occurs in a composition series exactly once or $\lambda > \mu$ and $\mu \in W \cdot_p \lambda$.*

COROLLARY 1.3.7. *For $\lambda, \mu \in X^+$, we have*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0 \Longrightarrow \mu \in W \cdot_p \lambda$$

The p-scaled dot action has fundamental regions called p-alcoves, which are just ordinary alcoves scaled up by a factor of p and shifted by $-\rho$. We call the weights which lie inside a p-alcove (and not on its boundary) regular. Thus each dominant regular weight is "linked" via this action to a unique regular weight in each p-alcove. The corollary shows that the W-orbits or linkage classes form block partitions. We write $\mathcal{B}_\lambda$ for the full subcategory of modules whose composition factors have highest weights lying in $W \cdot_p \lambda$, and $\mathrm{pr}_\lambda : G\text{−mod} \to \mathcal{B}_\lambda$ for the projection functor onto this subcategory. For a dominant p-alcove A and $\lambda, \mu \in \overline{A}$ the translation functor is defined by

$$T^\mu_\lambda(V) = \mathrm{pr}_\mu\left(\mathrm{pr}_\lambda(V) \otimes L(w(\mu - \lambda))\right),$$

where $w \in W_{\mathrm{f}}$ is chosen so that $w(\mu - \lambda) \in X^+$. Note that T^μ_λ is always exact because it is the composition of several exact functors. It can also be shown that T^μ_λ maps tilting modules to tilting modules. The translation principle states that $T^\mu_\lambda, T^\lambda_\mu$ form an equivalence under certain conditions

THEOREM 1.3.8 ([32, II.7.9]). *Let $\lambda, \mu \in X^+$. Suppose λ and μ belong to the same set of p-alcove closures, e.g. if λ and μ are regular then λ and μ lie in the same p-alcove. Then the functors $T^\mu_\lambda, T^\lambda_\mu : \mathcal{B}_\lambda \leftrightarrows \mathcal{B}_\mu$ are adjoint and mutually inverse.*

Suppose $\lambda, \lambda' \in X^+$ are regular weights in the same linkage class which belong to adjacent p-alcoves A, A' with $\lambda < \lambda'$. The p-alcoves A and A' are each in bijection with the fundamental p-alcove $A_{0,p}$. More precisely, there exist unique $x, x' \in {}^{\mathrm{f}}W$ with $A = x \cdot A_{0,p}$ and $A' = x' \cdot A_{0,p}$. Recall that the walls of $\overline{A_{0,p}}$ are in bijection with S. Suppose the wall between A and A' corresponds to the wall of $\overline{A_{0,p}}$ labeled by some $s \in S$. This implies that $x' = xs$. We write $A' = As$ and $\lambda' = \lambda s$ in this situation.

Fix a weight μ on the wall between λ and λ'. The wall-crossing functor θ_s is defined to be $T_\mu^{\lambda'} \circ T_\lambda^\mu$, which is self-adjoint and exact on $\mathcal{B}_\lambda$. It can be shown that $\theta_s\Delta(\lambda) \cong \theta_s\Delta(\lambda')$, and that there is a short exact sequence

$$0 \to \Delta(\lambda') \to \theta_s\Delta(\lambda) \to \Delta(\lambda) \to 0. \tag{1.10}$$

If $\lambda'' \in W \cdot_p \lambda$ is another dominant weight in the same linkage class, but $\lambda'' s$ is not a dominant weight, then $\theta_s\Delta(\lambda'') = 0$. There are similar short exact sequences for dual Weyl modules. This implies that θ_s maps tilting modules to tilting modules.

Now suppose λ is a dominant regular weight in the fundamental p-alcove $A_{0,p}$. Let $\mathcal{T}_{\lambda,\mathrm{BS}}$ denote the full subcategory of tilting modules which are direct sums of those of the form $\theta_{\underline{x}}T(\lambda)$ for all expressions $\underline{x} \in \underline{S}$, where $\theta_{\underline{x}}$ is just a composition of wall-crossing functors. Following the convention for finite-dimensional algebras, let $[\mathcal{T}_{\lambda,\mathrm{BS}}]$ denote the $\mathbb{Z}_{\geq 0}$-module of isomorphism classes in this subcategory. We define a $\mathbb{Z}_{\geq 0}$-module homomorphism called the character homomorphism by

$$\begin{aligned} \mathrm{ch} : [\mathcal{T}_{\lambda,\mathrm{BS}}] &\longrightarrow {}_{v=1}\mathcal{N} \\ [\theta_{\underline{x}}T(\lambda)] &\longmapsto {}_{v=1}[\underline{x}^{-1}] \end{aligned}$$

where ${}_{v=1}\mathcal{N}$ denotes the $\mathbb{Z}_{\geq 0}$-module arising from $\mathcal{N}$ by trivializing the action of $v \in \mathbb{L}_{\geq 0}$.

It is not immediately obvious that ch is well defined. To check this, we compose with the map ${}_{v=1}\mathcal{N} \to {}_{v=1}\mathbb{N}$ to get

$$\begin{aligned} [\mathcal{T}_{\lambda,\mathrm{BS}}] &\longrightarrow \mathcal{H} \longrightarrow {}_{v=1}\mathbb{N} \\ [\theta_{\underline{x}}T(\lambda)] &\longmapsto \sum_{w \in {}^{\mathrm{f}}W} (\theta_{\underline{x}}T(\lambda) : \Delta(w \cdot_p \lambda))N_w \end{aligned}$$

as a consequence of (1.10). Since tilting modules are uniquely determined up to isomorphism by their Δ-multiplicities, the sum only depends on the isomorphism class of $\theta_{\underline{x}}T(\lambda)$, so ch is indeed well defined. Moreover, any two tilting modules in $\mathcal{T}_{\lambda,\mathrm{BS}}$ have the same image only if they are isomorphic. Yet ch is clearly surjective. We summarize our findings in the following proposition.

Proposition 1.3.9. *The action of θ_s mapping $\theta_{\underline{x}}T(\lambda)$ to $\theta_{\underline{sx}}T(\lambda)$ defines a* right *${}_{v=1}\mathcal{H}$-action on $[\mathcal{T}_{\lambda,\mathrm{BS}}]$. The map* ch *is a right ${}_{v=1}\mathcal{H}$-module isomorphism.*

Note that for $\underline{x}$ a rex for $x \in {}^{\mathrm{f}}W$, we have $\theta_{\underline{x}}(T(\lambda)) = T(x^{-1} \cdot \lambda) \oplus T'$ where T' is a tilting module whose indecomposable summands have highest weights smaller than $x \cdot \lambda$. This shows that the Karoubi envelope (i.e. the completion with respect to all direct summands) of the subcategory $\mathcal{T}_{\lambda,\mathrm{BS}}$ is the full subcategory $\mathcal{T}_\lambda$ of all tilting modules in $\mathcal{B}_\lambda$. As a result their split Grothendieck groups coincide. This can also be identified with the ordinary Grothendieck group of $G-$mod.

Corollary 1.3.10. *The split Grothendieck group of $\mathcal{T}_\lambda$ has the structure of a right $[{}_{v=1}\mathcal{H}] \cong {}_{v=1}\mathbb{H} \cong \mathbb{Z}W$-module via the action of the wall-crossing functors. It* *can be identified with*

$$[[G\text{−mod}]]_{/} = [[\mathcal{T}_\lambda]] = [[\mathcal{T}_{\lambda,\mathrm{BS}}]] \cong [{}_{v=1}\mathcal{N}] \cong {}_{v=1}\mathbb{N}.$$

We say that $\mathcal{T}_\lambda$ is a categorification of the anti-spherical module ${}_{v=1}\mathbb{N}$. In Chapter 5 we will briefly describe the Riche–Williamson correspondence, which extends this categorification to all of $\mathbb{N}$ without setting $v = 1$, by attaching a grading to $\mathcal{T}_\lambda$.

1.3.5. Quantum groups at roots of unity. The representation theory of quantum groups at roots of unity is remarkably similar to the representation theory of semisimple algebraic groups, with most of the results of this section carrying over in a completely analogous way. We quickly summarize how to construct these objects here.

Let Φ be a root system for a Euclidean space E of dimension n, and let A_Φ be the Cartan matrix associated to this root system. Write $U_\mathbb{L}$ for the Lusztig integral form quantum group associated to the Cartan matrix A_Φ, as described in [32, H.5]. This quantum group is a Hopf algebra over $\mathbb{L}$ with algebra generators $E_i^{(r)}, F_i^{(r)}, K_i^{\pm 1}$ ranging over positive integers r and $i = 1, \ldots, n$.

Now let l be an odd positive integer (with l coprime to 3 if Φ has a G_2-component). Set $\zeta = e^{2\pi i/l} \in \mathbb{C}$, a primitive lth root of unity. We can make $\mathbb{L}$ into a commutative $\mathbb{C}$-algebra by specializing the indeterminate v to ζ. This leads to a specialization $U_l = \mathbb{C} \otimes_\mathbb{L} U_\mathbb{L}$ of our quantum group at ζ.

We will restrict ourselves to the study of finite-dimensional U_l-modules of type $\mathbf{1}$ (see [32, H.10] for a precise definition). When l is prime, the representation theory of U_l-modules is analogous to the representation theory of an algebraic group G with root system Φ over a field of characteristic l. In particular, if Φ^+ denotes the set of positive roots, and we define

$$X = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi^+\},$$
$$X^+ = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Phi^+\}$$

to be the sets of integral and dominant integral weights respectively, then for each $\lambda \in X^+$ we have U_l-modules $\nabla_l(\lambda)$ and $\Delta_l(\lambda)$, defined in a similar way to the eponymous constructions for G (in [32, H.11-H.12], these are referred to as $H_q^0(\lambda)$ and $H_q^n(w_0 \cdot \lambda)$ respectively). The module $L_l(\lambda) = \operatorname{soc} \nabla_l(\lambda) \cong \Delta_l(\lambda)/\operatorname{rad} \Delta_l(\lambda)$ is simple, and all simple modules are of this form. Moreover, familiar results from this section (including Kempf's vanishing theorem) carry over for these U_l-modules. This means that we can define the indecomposable tilting module $T_l(\lambda)$ in a manner completely analogous to the G-module case.

1.3.6. Tensor product formulas. Assume now that G is a semisimple algebraic group whose corresponding root system Φ is indecomposable. Let h denote the Coxeter number of Φ. We discuss some tensor product formulas for G and U_l, the analogous quantum group at a primitive lth root of unity. We will state results in terms of algebraic groups first and then summarize the changes in the analogous result for quantum groups.

Define the following finite subset of dominant weights

$$X_1 = \{\lambda \in X : \forall \alpha \in \Pi,\ 0 \leq \langle \lambda, \alpha^\vee \rangle < p\}. \tag{1.11}$$

Weights in X_1 are called restricted. The following result (or one of its corollaries) is usually called Steinberg's tensor product theorem.

THEOREM 1.3.11 ([32, Proposition II.3.16]). *Let $\lambda \in X^+$. Write $\lambda = \lambda_0 + p\lambda_{(1)}$ where $\lambda_0 \in X_1$ and $\lambda_{(1)} \in X^+$. Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(p\lambda_{(1)}).$$

By iterating this theorem we can write $L(\lambda)$ as a large tensor product based on the p-adic expansion of λ. In the algebraic groups case, there is a homomorphism of algebraic groups called the Frobenius endomorphism $F : G \to G$ for which $L(p\lambda)$ is isomorphic to $L(\lambda)^F$, the F-twist of the G-module $L(\lambda)$ for all $\lambda \in X^+$. If G embeds into GL_n then F maps matrices (g_{ij}) to (g_{ij}^p), which makes the Lie-theoretic character of $L(p\lambda)$ particularly easy to calculate from that of $L(\lambda)$. This means that to calculate the simple Lie-theoretic characters, it is enough to know the Lie-theoretic characters of the simple modules with restricted highest weight. This reduces the problem from a potentially infinite problem over all dominant weights to a finite one. For a G-module V we write $V^{[r]}$ for V^{F^r}.

The situation for quantum groups is quite different: although Steinberg's tensor product theorem holds and there is an analogue of the Frobenius endomorphism, we have $L_l(l\lambda) \cong L_1(\lambda)^{[1]}$ where $L_1(\lambda)$ is the simple module for the quantum group at 1 (i.e. a 1st root of unity). Since our quantum groups are defined over $\mathbb{C}$, which is of characteristic 0, all modules over $U_1 \cong U(\mathfrak{g})$ are semisimple and $L_1(\lambda)$ is isomorphic to $\Delta_1(\lambda)$, a Weyl module!

Similarly there is a tensor product theorem for tilting modules, due to Donkin.

THEOREM 1.3.12 ([22, Proposition 2.1]). *Suppose G is semisimple and simply connected, and that $p \geq 2h-2$. Let $\lambda = \lambda_0 + p\lambda_{(1)}$ where $\lambda_0 \in (p-1)\rho + X_1$ and $\lambda_{(1)} \in X^+$. Then*

$$T(\lambda) \cong T(\lambda_0) \otimes T(\lambda_{(1)})^{[1]}.$$

As in Steinberg's theorem, we can reduce any $T(\lambda)$ to a tensor product of (possibly Frobenius twisted) tilting modules, but the set of weights which cannot be reduced is infinite (it includes all the alcoves which are adjacent to the walls of the dominant Weyl chamber). The quantum generalization is almost the same, except the last tensor multiplicand has to be replaced with $L_l(l\lambda_{(1)})$. Note that this module is isomorphic to $L_1(\lambda_{(1)})^{[1]} = T_1(\lambda_{(1)})^{[1]}$, since all finite-dimensional $U(\mathfrak{g})$-tilting modules are semisimple.

1.3.7. Character formulas. The most well-known potential character formula for simple G-modules is a generalization of the Kazhdan–Lusztig conjecture in category $\mathcal{O}$. We will call it Lusztig's character formula, or LCF for short, throughout this thesis.

THEOREM 1.3.13 (LCF). *Suppose $p \gg h$. Let $x \in {}^{\mathrm{f}}W$. Then the following character formula*

$$[[\Delta(x \cdot_p 0)]]_{/} = \sum_{y \in {}^{\mathrm{f}}W} m^{x,y}(1)[[L(y \cdot_p 0)]]_{/}$$

holds.

This result was first conjectured for reductive algebraic groups in [45] whenever $p > 2h-2$. The analogous result for quantum groups was shown to hold when $l > h$ in a series of papers by Kazhdan and Lusztig [38, 39, 40, 46] and Kashiwara and Tanisaki [35, 36]. By comparison with the quantum case, the original modular conjecture was proven for extremely large p in [3] and [27]. In [56] Williamson showed that Lusztig's original bound, and more generally any polynomial bound in h, is not large enough to guarantee the validity of LCF in the modular case. When LCF does hold, an important corollary (which is in fact equivalent) is Vogan's conjecture, named after the analogous statement in the setting of category $\mathcal{O}$.

COROLLARY 1.3.14 ([4, Conjecture 2.7]). *Suppose LCF holds. Let $x \in {}^{\mathrm{f}}W$ and $s \in S$ such that $sx > x$. Then $\theta_s L(x \cdot 0)$ has socle and head isomorphic to $L(x \cdot 0)$, and the module*

$$\beta_s L(x \cdot 0) = \operatorname{rad} \theta_s L(x \cdot 0) / \operatorname{soc} \theta_s L(x \cdot 0)$$

is semisimple.

If the corollary holds one can show that $[\beta_s(L(x \cdot 0)) : L(y \cdot 0)] = m^s_{y,x}(0)$. In addition it follows that for any module M, if M has Loewy length m then $\theta_s M$ has Loewy length at most $m+2$ (for a proof see [32, Appendix D.2]).

For indecomposable quantum tilting modules, Soergel proved the following character formula [51, 52].

THEOREM 1.3.15 (SCF). *Suppose $l > h$. Let $x \in {}^{\mathrm{f}}W$. Then the following character formula*

$$[[T_l(x \cdot 0)]]_{/} = \sum_{y \in {}^{\mathrm{f}}W} n_{y,x}(1)[[\Delta_l(y \cdot 0)]]_{/} \tag{1.12}$$

holds.

CHAPTER 2

Rigidity of tilting modules

Let A be a finite-dimensional quasi-hereditary algebra, with standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$. The goal of this chapter is to describe some general conditions for when tilting modules for A are rigid (i.e. have identical radical and socle series). The main results are Theorem 2.2.7 and its corollaries. Although many of these results can be phrased simply, the proofs depend on homological machinery for filtered algebras, which we develop using the language of model categories in Section 2.1. We also describe how to understand the behavior of subquotients using coefficient quivers. As an application, we show in Section 2.3 that the restricted tilting modules for SL_4 over an algebraically closed field of characteristic $p \geq 5$ are rigid, and we calculate their radical series.

2.1. Homological techniques for filtered algebras

Suppose A is a filtered algebra. In order to define a functor analogous to Ext on A−filtmod it will be necessary to use some technology from homotopy theory, which we describe below. The primary reference for this section is [29, Chapter 1]. Throughout this section, $\mathcal{A}$ and $\mathcal{B}$ denote arbitrary categories.

2.1.1. Model structures.

Definition 2.1.1. Suppose $i : U \to V$ and $p : X \to Y$ are maps in a category $\mathcal{A}$. Then i has the left lifting property with respect to p and p has the right lifting property with respect to i if for every commutative diagram of the following form

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r, "g"'] & Y \end{tikzcd}$$

there exists a map $h : V \to X$ such that two triangles introduced in the above diagram commute, i.e. $hi = f$ and $ph = g$.

In this situation we write $i \boxslash p$. A map h fitting into such a commutative square is called a lift.

Definition 2.1.2. A model structure on a category $\mathcal{A}$ is a collection of three subclasses $\mathcal{W}, \mathcal{C}, \mathcal{F}$ of $\operatorname{Mor} \mathcal{A}$ which satisfy the following properties:

(i) (2-out-of-3) Suppose $u, v \in \operatorname{Mor} \mathcal{A}$ such that vu is defined. If two of u, v, and vu are in $\mathcal{W}$ then so is the third.

(ii) (Retracts) Given a commutative diagram of the following form

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "u"'] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r, two heads] \arrow[d, "v"] & A \arrow[d, "u"] \\
V \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r, two heads] & B
\end{tikzcd}$$

if v is in $\mathcal{W}$, $\mathcal{C}$, or $\mathcal{F}$ then so is u.

(iii) (Lifting) Using the obvious setwise extension of the symbol $\boxslash$, we have $(\mathcal{W} \cap \mathcal{C}) \boxslash \mathcal{F}$ and $\mathcal{C} \boxslash (\mathcal{W} \cap \mathcal{F})$.

(iv) (Factorization) For every $f \in \operatorname{Mor} \mathcal{A}$, there exist two (functorial) factorizations:
- $f = pi$ where $i \in \mathcal{W} \cap \mathcal{C}$ and $p \in \mathcal{F}$,
- $f = qj$ where $j \in \mathcal{C}$ and $q \in \mathcal{W} \cap \mathcal{F}$.

A map in one of $\mathcal{W}$, $\mathcal{C}$, or $\mathcal{F}$ is called a weak equivalence, cofibration, or fibration respectively. A map in $\mathcal{W} \cap \mathcal{C}$ or $\mathcal{W} \cap \mathcal{F}$ is called a trivial cofibration or a trivial fibration respectively. In categories with initial and terminal objects (denoted 0 and 1 respectively), an object X of $\mathcal{A}$ is called cofibrant if $0 \to X$ is a cofibration or fibrant if $X \to 1$ is a fibration.

Sometimes a distinction is made between a "category with model structure" and a so-called "model category." A model category is simply a category with a model structure which contains all finite limits and colimits. A closed model category is a model category which additionally contains all small limits and colimits. Since the categories we will be using later have all such limits, we will freely use the phrase "model category" instead of "category with model structure."

2.1.2. Homotopy categories and derived functors. The primary motivation for model structures is the homotopy category (sometimes also called the derived category). The homotopy category of a model category is a generalization of the classical derived category $D(A\text{–mod})$ obtained from the category of cochain complexes $\mathbf{Ch}(A\text{–mod})$ for an arbitrary algebra A. Namely, the homotopy category is obtained by adding the inverses of certain "equivalences" to the original category. One can think of model categories as categories with just enough structure to enable calculations in homotopy categories.

Definition 2.1.3. Let $\mathcal{A}$ be a category with a model structure given by $\mathcal{W}, \mathcal{C}, \mathcal{F}$. The homotopy category (or derived category) of $\mathcal{A}$ is a category $\operatorname{Ho} \mathcal{A}$ and a functor $\gamma_{\mathcal{A}} : \mathcal{A} \to \operatorname{Ho} \mathcal{A}$ which is the localization of $\mathcal{A}$ at $\mathcal{W}$.

In other words, $\gamma_{\mathcal{A}}$ maps $\mathcal{W}$ to isomorphisms, and $\operatorname{Ho} \mathcal{A}$ is universal with this property in the sense that if another functor $F : \mathcal{A} \to \mathcal{B}$ maps $\mathcal{W}$ to isomorphisms, there is a unique factorization $F = (\operatorname{Ho} F)\gamma_{\mathcal{A}}$ for some functor $\operatorname{Ho} F : \operatorname{Ho} \mathcal{A} \to \mathcal{B}$.

Definition 2.1.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between two model categories. The left derived functor of F is a functor $\mathbf{L}F : \operatorname{Ho} \mathcal{A} \to \operatorname{Ho} \mathcal{B}$ with a natural transformation $\varepsilon : (\mathbf{L}F)\gamma_{\mathcal{A}} \Rightarrow \gamma_{\mathcal{B}}F$ called the counit which is universal in the following sense: for any other functor $G : \operatorname{Ho}\mathcal{A} \to \operatorname{Ho}\mathcal{B}$ with a natural transformation $\zeta : G\gamma_{\mathcal{A}} \Rightarrow \gamma_{\mathcal{B}}F$, there is a unique $\lambda : G \Rightarrow \mathbf{L}F$ such that $\zeta = \varepsilon \circ \lambda\gamma_{\mathcal{A}}$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\gamma_{\mathcal{B}}"] \\
\operatorname{Ho}\mathcal{A} \arrow[r, "G"'] \arrow[ur, Rightarrow, "\zeta"'] & \operatorname{Ho}\mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{A} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{A}}"'] \arrow[dl, "\gamma_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\gamma_{\mathcal{B}}"] \\
& \operatorname{Ho}\mathcal{A} \arrow[r, "\mathbf{L}F"'] \arrow[ur, Rightarrow, "\varepsilon"'] & \operatorname{Ho}\mathcal{B} \\
\operatorname{Ho}\mathcal{A} \arrow[urr, "G"'] \arrow[ur, Rightarrow, dashed, "\lambda\gamma_{\mathcal{A}}"] & &
\end{tikzcd}
$$

Similarly, the right derived functor of F is a functor $\mathbf{R}F : \operatorname{Ho}\mathcal{A} \to \operatorname{Ho}\mathcal{B}$ with a natural transformation $\eta : \gamma_{\mathcal{B}}F \Rightarrow (\mathbf{R}F)\gamma_{\mathcal{A}}$ called the unit which has the following universal property: for any other functor $G : \operatorname{Ho}\mathcal{A} \to \operatorname{Ho}\mathcal{B}$ with a natural transformation $\theta : \gamma_{\mathcal{B}}F \Rightarrow G\gamma_{\mathcal{A}}$, there exists a unique $\mu : \mathbf{R}F \Rightarrow G$ such that $\theta = \mu\gamma_{\mathcal{A}} \circ \eta$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\gamma_{\mathcal{B}}"] \arrow[dl, Rightarrow, "\theta"'] \\
\operatorname{Ho}\mathcal{A} \arrow[r, "G"'] & \operatorname{Ho}\mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{A} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{A}}"'] \arrow[dl, "\gamma_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\gamma_{\mathcal{B}}"] \arrow[dl, Rightarrow, "\eta"'] \\
& \operatorname{Ho}\mathcal{A} \arrow[r, "\mathbf{R}F"'] \arrow[dl, Rightarrow, dashed, "\mu\gamma_{\mathcal{A}}"'] & \operatorname{Ho}\mathcal{B} \\
\operatorname{Ho}\mathcal{A} \arrow[urr, "G"'] & &
\end{tikzcd}
$$

In general, calculating derived functors can be difficult if no extra information about the functor is given. Thus we will restrict ourselves to taking derived functors of functors which preserve some aspects of the model structure.

Definition 2.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two model categories.

- A left Quillen functor $F : \mathcal{A} \to \mathcal{B}$ is a functor that is left adjoint and preserves cofibrations and trivial cofibrations.
- A right Quillen functor $G : \mathcal{B} \to \mathcal{A}$ is a functor that is right adjoint and preserves fibrations and trivial fibrations.
- A Quillen adjunction $F \dashv G : \mathcal{A} \leftrightarrows \mathcal{B}$ is an adjunction where F is a left Quillen functor and G is a right Quillen functor.

The following proposition shows that these definitions are overdetermined.

Proposition 2.1.6 ([28, Proposition 8.5.3]). *Let $F \dashv G : \mathcal{A} \leftrightarrows \mathcal{B}$ be an adjunction between two model categories. The following are equivalent.*

(i) *$F \dashv G$ is a Quillen adjunction.*
(ii) *F preserves cofibrations and trivial cofibrations.*
(iii) *G preserves fibrations and trivial fibrations.*
(iv) *F preserves cofibrations and G preserves fibrations.*

If F is a left Quillen functor, then the derived functor of F can be calculated via a process called cofibrant replacement. Suppose a category $\mathcal{A}$ with model structure has initial and terminal objects $0, 1$. For any object X, we can factor the map $0 \to X$ as a map $0 \to QX \xrightarrow{q_X} X$, where $0 \to QX$ is a cofibration (and thus QX is cofibrant) and $QX \xrightarrow{q_X} X$ is a trivial fibration. This mapping $X \mapsto QX$ defines a functor called the cofibrant replacement functor, and q_X defines the components for a natural transformation. Similarly there is a fibrant replacement functor R and a natural trivial cofibration with components $X \xrightarrow{r_X} RX$.

Proposition 2.1.7 ([28, Lemma 8.5.9]). *If $F : \mathcal{A} \to \mathcal{B}$ is a left Quillen functor, the left derived functor of F exists, and can be calculated as the following composition:*

$$\operatorname{Ho}\mathcal{A} \xrightarrow{\operatorname{Ho}\gamma_{\mathcal{A}}Q} \operatorname{Ho}\mathcal{A}_c \xrightarrow{\operatorname{Ho}\gamma_{\mathcal{B}}F} \operatorname{Ho}\mathcal{B}$$

where $\operatorname{Ho}\mathcal{A}_c$ denotes the full subcategory of cofibrant objects in $\operatorname{Ho}\mathcal{A}$.

For calculating the right derived functor of a right Quillen functor, we use the fibrant replacement functor in a similar way. Quillen adjunctions have the property that they induce adjunctions in the derived categories, as described below.

Theorem 2.1.8 ([29, Lemma 1.3.10]). *If $F \dashv G : \mathcal{A} \leftrightarrows \mathcal{B}$ is a Quillen adjunction, then $\mathbf{L}F, \mathbf{R}G : \operatorname{Ho}\mathcal{A} \leftrightarrows \operatorname{Ho}\mathcal{B}$ are also adjoint functors. This adjunction is called the derived adjunction of $F \dashv G$.*

2.1.3. Some examples. We will first describe perhaps the most well-known model category, the category of cochain complexes of an abelian category. Let $\mathcal{A}$ denote the abelian category $A-$mod for an algebra A over some field $\Bbbk$, and $\mathbf{Ch}\,\mathcal{A}$ the category of cochain complexes over $\mathcal{A}$. The first step is describing what projective or injective relative to a class of morphisms means.

Definition 2.1.9. Let I be a subclass of maps in some category $\mathcal{A}$. Define

- $I-\mathrm{inj} = \{f \in \operatorname{Mor}\mathcal{A} \mid I \boxslash f\}$;
- $I-\mathrm{proj} = \{f \in \operatorname{Mor}\mathcal{A} \mid f \boxslash I\}$;
- $I-\mathrm{cof} = (I-\mathrm{inj})-\mathrm{proj}$;
- $I-\mathrm{fib} = (I-\mathrm{proj})-\mathrm{inj}$.

Example 2.1.10. Define the following complexes S^n and D^n in $\mathbf{Ch}\,\mathcal{A}$

$$(S^n)^k = \begin{cases} A & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}, \qquad (D^n)^k = \begin{cases} A & \text{if } k = n, n+1 \\ 0 & \text{otherwise} \end{cases},$$

where all differentials of S^n are 0, and the only non-trivial differential map of D^n is $d^n : A \xrightarrow{\mathrm{id}} A$. For each $n \in \mathbb{Z}$ we have an injection $S^{n+1} \to D^n$ given by the identity in (homological) degree $n+1$ and 0 elsewhere. Let

$$\begin{aligned} I &= \{S^{n+1} \to D^n \mid n \in \mathbb{Z}\}, \\ J &= \{0 \to D^n \mid n \in \mathbb{Z}\}, \\ \mathcal{W} &= \{f : X \to Y \mid H^n(f) \text{ is an isomorphism for all } n \in \mathbb{Z}\}. \end{aligned}$$

Here $H^n(f)$ denotes the homomorphism on cohomology groups induced by a cochain map. In other words, $\mathcal{W}$ consists of the set of quasi-isomorphisms in $\mathbf{Ch}\,\mathcal{A}$.

THEOREM 2.1.11. *Let* $\mathcal{C} = I{-}\mathrm{cof}$ *and* $\mathcal{F} = J{-}\mathrm{inj}$. *Then the three sets* $\mathcal{W}, \mathcal{C}, \mathcal{F}$ *define a model structure called the projective model structure on* $\mathbf{Ch}\,A$.

PROOF. See, for example, [29, Section 2.3] or [20, Section 2.2]. □

The fibrations in this model structure are the degreewise surjective cochain maps, and all complexes are fibrant. A cofibrant complex X has the property that for each n, X^n is a projective A-module. For bounded above complexes, the converse is also true, but unbounded cofibrant complexes are trickier to understand. The cofibrations are the degreewise split injective cochain maps with cofibrant cokernels. Throughout this paper we will use the abbreviation $D(\mathcal{A})$ for $\mathrm{Ho}\,\mathbf{Ch}\,\mathcal{A}$.

Here is another example of how one can extend this model structure to similar-looking categories.

EXAMPLE 2.1.12. Suppose B is a graded $\Bbbk$-algebra, and let $\mathcal{B} = B{-}\mathrm{grmod}$, the category of graded B-modules. The category $\mathbf{Ch}\,\mathcal{B}$ of cochain complexes of graded modules has a projective model structure very similar to the one above.

Let S^n and D^n take the obvious gradings from B:

$$((S^n)^k)_i = \begin{cases} B_i & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}, \qquad ((D^n)^k)_i = \begin{cases} B_i & \text{if } k = n, n+1 \\ 0 & \text{otherwise} \end{cases}.$$

The differentials are all graded homomorphisms as they are all 0 or id. Now we define

$$\begin{aligned} I_{\mathrm{gr}} &= \{S^{n+1}(r) \to D^n(r) \mid n, r \in \mathbb{Z}\}, \\ J_{\mathrm{gr}} &= \{0 \to D^n(r) \mid n, r \in \mathbb{Z}\}, \\ \mathcal{W}_{\mathrm{gr}} &= \{f : X \to Y \mid H^n(f) \text{ is an isomorphism for all } n \in \mathbb{Z}\}, \end{aligned}$$

where (r) denotes the degree r grade shift, viewed as a functor on $\mathcal{B}$ and $\mathbf{Ch}\,\mathcal{B}$.

THEOREM 2.1.13. *Let* $\mathcal{C}_{\mathrm{gr}} = I_{\mathrm{gr}}{-}\mathrm{cof}$ *and* $\mathcal{F}_{\mathrm{gr}} = J_{\mathrm{gr}}{-}\mathrm{inj}$. *Then the three sets* $\mathcal{W}_{\mathrm{gr}}, \mathcal{C}_{\mathrm{gr}}, \mathcal{F}_{\mathrm{gr}}$ *define a model structure called the projective model structure on* $\mathbf{Ch}\,\mathcal{B}$.

PROOF. Adapt the proof of Theorem 2.1.11 to the graded case. This is especially easy because $B{-}\mathrm{grmod}$ is an abelian category like $A{-}\mathrm{mod}$ so kernels, images, cokernels, etc. all make sense. □

Again the fibrations in this model structure are the homological degreewise surjective cochain maps, and all complexes are fibrant. A bounded above complex X is cofibrant if and only if X^n is projective as a graded B-module for all n. The cofibrations are the degreewise split injective cochain maps with cofibrant cokernels.

2.1.4. Filtered cochain complexes. Suppose A is a filtered algebra, and let $\mathcal{A} = A-$filtmod. Using the examples from the previous section, we construct a model structure on $\mathbf{Ch}\,\mathcal{A}$ following [20, Section 2.3]. Define the following filtrations on S^n and D^n defined above:

$$((S^n)^k)^i = \begin{cases} A^i & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}, \qquad ((D^n)^k)^i = \begin{cases} A^i & \text{if } k = n, n+1 \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to verify that the differentials are all homomorphisms of filtered modules. In this vein we define

$$\begin{aligned} I_{\text{filt}} &= \{S^{n+1}\langle r\rangle \to D^n\langle r\rangle \mid n, r \in \mathbb{Z}\}, \\ J_{\text{filt}} &= \{0 \to D^n\langle r\rangle \mid n, r \in \mathbb{Z}\}, \\ \mathcal{W}_{\text{filt}} &= \{f : X \to Y \mid H^n((f)^i) \text{ is an isomorphism for all } n \in \mathbb{Z}\}, \end{aligned}$$

where $\langle r\rangle$ denotes the degree r filtration shift, viewed as a functor on $\mathcal{A}$ and $\mathbf{Ch}\,\mathcal{A}$, and $(f)^i$ is the restriction of f to the ith filtration degree. In particular, $\mathcal{W}_{\text{filt}}$ consists of the set of quasi-isomorphisms in $\mathbf{Ch}\,\mathcal{A}$ which restrict to vector space isomorphisms in all filtration degrees.

THEOREM 2.1.14. *Let $\mathcal{C}_{\text{filt}} = I_{\text{filt}}-\text{cof}$ and $\mathcal{F} = J_{\text{filt}}-\text{inj}$. Then the three sets $\mathcal{W}_{\text{filt}}, \mathcal{C}_{\text{filt}}, \mathcal{F}_{\text{filt}}$ define a model structure called the projective model structure on $\mathbf{Ch}\,\mathcal{A}$.*

PROOF. See [20, Theorem 2.18] for a full proof in the case when A has the trivial filtration ($A^i = A$ for $i \geq 0$). This is an adaptation of the proof of Theorem 2.1.11 but with extra care for filtration degrees. The general proof is essentially identical. □

As expected, the fibrations in this model structure are the (homological and filtration) degreewise surjective cochain maps, and all complexes are fibrant. A bounded below complex X is cofibrant if and only if X^n is projective as a filtered A-module for all n (we explain what this means in greater detail in Section 2.1.6). The cofibrations are the degreewise split injective cochain maps with cofibrant cokernels.

2.1.5. The Rees algebra. Now we consider connections to the algebra

$$B = \operatorname{Rees} A = \bigoplus_{i\in\mathbb{Z}} (A^i)t^i,$$

which is a subalgebra of $A[t]$. It has a grading induced both by the grading on $A[t]$ and the filtration structure on A. Functionally the indeterminate t does nothing but record the grading, so that at^i is distinct from at^j in $\operatorname{Rees} A$ for any $a \in A^i \cap A^j$. Let $\mathcal{B} = B-$grmod $= (\operatorname{Rees} A)-$grmod. It is clear that the Rees construction is functorial, i.e. $\operatorname{Rees} : \mathcal{A} \to \mathcal{B}$ is a functor mapping a filtered module M to the graded B-module

$$\operatorname{Rees} M = \bigoplus_i (M^i)t^i.$$

THEOREM 2.1.15. *The functor* Rees *has a left adjoint* $\varphi : \mathcal{B} \to \mathcal{A}$. *The module structure on* $\varphi(M)$ *is the quotient* M/LM *where* L *is the two-sided ideal of* B *generated by*

$$\left\{ \sum_i a_i t^i \;\middle|\; a_i \in A^i,\ \sum_i a_i = 0 \right\}.$$

The filtration on $\varphi(M)$ *is given by defining* M^i *to be the image of* M_i *in this quotient.*

PROOF. First we show that φ is a well-defined functor. This amounts to showing that $B/L \cong A$ so that M/LM has a natural A-module structure. There is a natural homomorphism of ordinary modules

$$\begin{aligned} B &\longrightarrow A \\ a_i t^i &\longmapsto a_i \end{aligned}$$

and the kernel is clearly L. Also, it is surjective because the span of $\{A^i\}$ is A. For the filtration, note that the span of the images of M_i in the quotient M/LM clearly span the quotient. Also, if $a_i \in A^i$ and $m_j \in M_j$, then

$$a_i(m_j + LM) = a_i t^i(m_j + LM) \in M_{i+j} + LM,$$

so this truly gives a filtered A-module structure.

To show the adjunction, we show that

$$\operatorname{Hom}_A(\varphi(M), N) \cong \operatorname{Hom}_B(M, \operatorname{Rees} N)$$

for M a graded B-module and N a filtered A-module. For $f \in \operatorname{Hom}_A(\varphi(M), N)$, we will define a corresponding $g \in \operatorname{Hom}_B(M, \operatorname{Rees} N)$ degreewise in M. Suppose $m_i \in M_i$. By the filtration on $\varphi(M)$, $f(m_i + LM) \in f(\varphi(M)^i) \subseteq N^i$. So define $g(m_i) = f(m_i + LM)t^i$ and extend linearly. This defines a graded homomorphism as required.

To go the other way, suppose $g \in \operatorname{Hom}_B(M, \operatorname{Rees} N)$. For $\overline{m_i} \in \varphi(M)^i$, pick some $m_i \in M_i$ such that $m_i + LM = \overline{m_i}$. Define $f \in \operatorname{Hom}_A(\varphi(M), N)$ by setting $f(\overline{m_i}) = n_i$ if $g(m_i) = n_i t^i$ and extending linearly. To see that this is well defined, we need to show that $g(LM) = 0$. Yet this is clearly true because

$$g(LM) = Lg(M) \subseteq L \operatorname{Rees} N = 0$$

by action of B on $\operatorname{Rees} N$. It is clear that this homomorphism is filtered, and these correspondences are inverse to each other. $\square$

LEMMA 2.1.16. *The adjunction* $\varphi \dashv$ Rees *is a Quillen adjunction of model categories, i.e.* φ *preserves cofibrations and trivial cofibrations while* Rees *preserves fibrations and trivial fibrations.*

PROOF. First we show that $\operatorname{Rees}(\varphi(I){-}\mathrm{inj}) \subseteq I{-}\mathrm{inj}$ and $\varphi(I{-}\mathrm{cof}) \subseteq \varphi(I){-}\mathrm{cof}$ for an arbitrary class of maps I. Suppose $f \in \varphi(I){-}\mathrm{inj}$ and $g \in I$ such that there is a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & \operatorname{Rees} X \arrow[d, "\operatorname{Rees} f"] \\ B \arrow[r] & \operatorname{Rees} Y \end{tikzcd}$$

We need to show this diagram has a lift. By adjointness, we may form the following diagram

$$\begin{tikzcd} \varphi(A) \arrow[r] \arrow[d, "\varphi(g)"'] & X \arrow[d, "f"] \\ \varphi(B) \arrow[r] & Y \end{tikzcd}$$

which has a lift $h : \varphi(B) \to X$. It is easy to see that the corresponding map $h' : B \to \operatorname{Rees} X$ is a lift for the first diagram. We can abbreviate this argument to one line by abuse of notation and remembering that adjointness works similarly with the symbol $\boxslash$ as it does with Hom, i.e. $\varphi(I) \boxslash \varphi(I){-}\mathrm{inj}$ implies $I \boxslash \operatorname{Rees}(\varphi(I){-}\mathrm{inj})$. Similarly, we have

$$\begin{aligned} I{-}\mathrm{cof} \boxslash I{-}\mathrm{inj} &\Rightarrow I{-}\mathrm{cof} \boxslash \operatorname{Rees}(\varphi(I){-}\mathrm{inj}) \\ &\Rightarrow \varphi(I{-}\mathrm{cof}) \boxslash \varphi(I){-}\mathrm{inj} \\ &\Rightarrow \varphi(I{-}\mathrm{cof}) \subseteq \varphi(I){-}\mathrm{cof} \end{aligned}$$

Now we apply the above to the model categories $\mathcal{A}$ and $\mathcal{B}$. First note that $\varphi(J_{\mathrm{gr}}) = J_{\mathrm{filt}}$ and $\varphi(I_{\mathrm{gr}}) = I_{\mathrm{filt}}$. Now we have

$$\operatorname{Rees}(\varphi(J_{\mathrm{gr}}){-}\mathrm{inj}) = \operatorname{Rees}(J_{\mathrm{filt}}{-}\mathrm{inj}) \subseteq J_{\mathrm{gr}}{-}\mathrm{inj},$$

showing that Rees maps fibrations to fibrations. Similarly,

$$\varphi(I_{\mathrm{gr}}{-}\mathrm{cof}) \subseteq \varphi(I_{\mathrm{gr}}){-}\mathrm{cof} = I_{\mathrm{filt}}{-}\mathrm{cof}$$

so φ maps cofibrations to cofibrations. By Proposition 2.1.6, the adjunction is a Quillen adjunction. □

2.1.6. Filtered projective modules.

DEFINITION 2.1.17. Let A be a filtered algebra. A filtered module P is called (filtered) projective if for any filtration surjective homomorphism $p : M \to N$ and any homomorphism $g : P \to N$, there exists a homomorphism $h : P \to M$ such that $ph = g$.

There are many reasons for this to be the correct definition of projective in this context, including the following two lemmas.

LEMMA 2.1.18. *An A-module P is filtered projective if and only if it is a summand of a direct sum of (possibly filtration shifted) copies of A.*

Proof. Suppose P is a direct summand of $L = A\langle -r_1\rangle \oplus \cdots \oplus A\langle -r_k\rangle$. Let $p : M \to N$ be a filtration surjective homomorphism and let $g : P \to N$ be any homomorphism. Write $q : L \to P$ for the projection map and $i : P \to L$ for the inclusion map. Let $n_1, \ldots, n_k \in N$ be the images of 1 (in each copy of A) under the composite map gq. Since the copies of A are filtration shifted we have $n_i \in N^{r_i}$ for each i. Let $m_i \in M^{r_i}$ such that $p(m_i) = n_i$ for each i. There is a unique homomorphism $h' : L \to M$ which maps the ith copy of 1 to m_i, so the map $h = h'i$ is a lift and P is projective.

Conversely, suppose P is projective. The module P has a generating set $\{p_i\}$. By writing each generator as the sum of different filtration components, we may assume that each generator p_i is contained in some filtered part P^{r_i} for integers r_i. As above, there is a unique homomorphism $q : L \to A$ where $L = \oplus_i A\langle -r_i\rangle$ mapping the ith copy of 1 to p_i. Clearly this map is surjective. If it isn't filtration surjective, suppose there is some $p' \in P^r$ such that $p' \notin q(L^r)$. Then we can add p' to the list of generators, replace L with $L \oplus A\langle -r\rangle$, and try again. Thus we have a filtration surjective homomorphism $q : L \to P$. Using projectivity, we show that q has a right inverse $i : P \to L$ with $pi = \mathrm{id}_P$. □

Remark 2.1.19. It doesn't matter if P is a summand as a filtered module or not. If P is a summand of a module $L = \oplus_i A\langle -r_i\rangle$ as a module over an ordinary algebra A, then P can be given a filtration compatible with the filtration on L. Namely, define $P^i = p(L^i)$ where p the canonical projection $p : L \to P$.

Lemma 2.1.20. *If X is a cofibrant cochain complex in $\mathbf{Ch}\, A$ then for each $n \in \mathbb{Z}$, X^n is filtered projective. Conversely, if X is a complex which is bounded above such that X^n is filtered projective, then X is cofibrant.*

Proof. Adapt the proof of the similar fact in [29, Lemma 2.3.6]. The key fact here is that fibrations in this model structure are filtration surjective, not just surjective. □

Definition 2.1.21. Let M be a filtered A-module. A filtered projective resolution of M consists of a complex P (indexed following the chain complex convention, with $P_n = 0$ for $n < 0$) and a homomorphism $P_0 \to M$ such that

(i) the complex P is filtered exact at each $n > 0$, i.e. $H_n(P^i) = 0$ for all i; and

(ii) the homomorphism $P_0 \to M$ is filtered surjective.

It is easy to see using the previous lemmas that filtered projective resolutions exist and are cofibrant replacements for complexes concentrated in one homological degree.

Definition 2.1.22. For two filtered modules M, N, define

$$\mathrm{Ext}^i_A(M, N) = \mathrm{Hom}_{D(\mathcal{A})}(\gamma M, \gamma N[i]),$$

where $N[i]$ is the complex concentrated in cohomological degree $-i$.

PROPOSITION 2.1.23. *For any two filtered A-modules M and N, we have*

$$\operatorname{Ext}^i_A(M,N) \cong \operatorname{Ext}^i_B(\operatorname{Rees} M, \operatorname{Rees} N).$$

PROOF. As $\mathcal{B}$ is an abelian category, we know that

$$\operatorname{Ext}^i_B(\operatorname{Rees} M, \operatorname{Rees} N) \cong \operatorname{Hom}_{D(\mathcal{B})}(\gamma \operatorname{Rees} M, \gamma \operatorname{Rees} N[i]).$$

Now use the derived adjunction:

$$\begin{aligned}
\operatorname{Hom}_{D(\mathcal{B})}(\gamma \operatorname{Rees} M, \gamma \operatorname{Rees} N[i]) &\cong \operatorname{Hom}_{D(\mathcal{B})}(\gamma \operatorname{Rees} M, \mathbf{R} \operatorname{Rees} \gamma N[i]) \\
&\cong \operatorname{Hom}_{D(\mathcal{A})}(\mathbf{L}\varphi\gamma \operatorname{Rees} M, \gamma N[i]) \\
&\cong \operatorname{Hom}_{D(\mathcal{A})}((\operatorname{Ho} \gamma\varphi) \circ (\operatorname{Ho} \gamma Q) \circ \gamma \operatorname{Rees} M, \gamma N[i]) \\
&= \operatorname{Hom}_{D(\mathcal{A})}((\gamma\varphi Q \operatorname{Rees} M, \gamma N[i])
\end{aligned}$$

Now suppose we have a projective resolution P for M. As Rees is clearly an additive functor, it maps projective modules to projective modules, since in both cases these are (possibly shifted) summands of the algebra. The map $P_0 \to M$ induces a trivial fibration $P \to M$, and as Rees is a right Quillen functor, so is $\operatorname{Rees} P \to \operatorname{Rees} M$. Thus a cofibrant replacement for $\operatorname{Rees} M$ is given by $\operatorname{Rees} P$. Yet $\varphi(B) \cong A$, and the same is true for any summand of A, so $\varphi(\operatorname{Rees} P) \cong P$ and the final Hom-space is really just

$$\operatorname{Hom}_{D(\mathcal{A})}(\gamma P, \gamma N[i]) \cong \operatorname{Hom}_{D(\mathcal{A})}(\gamma M, \gamma N[i]) = \operatorname{Ext}^i_A(M,N).$$

□

REMARK 2.1.24. The category $\mathcal{A} = A-$filtmod is not abelian, but it is in fact what Schneiders calls quasi-abelian [50]. Quasi-abelian categories are so close to being abelian categories that nearly all of the tools from homological algebra carry through, not just derived functors. In this context, the essential change in which exact sequences are replaced with strict exact sequences (i.e. those in which the isomorphism between image and coimage is a filtered isomorphism) considerably predates Schneiders' work and can be found in [17]. As we only need the Ext-groups in $\mathcal{A}$ for what follows, we decided to recharacterize this work in terms of model categories in order to motivate the definitions and keep the number of prerequisites down.

2.2. Rigidity of tilting modules

2.2.1. Radically filtered quasi-hereditary algebras. Now let A be a finite-dimensional quasi-hereditary algebra over a field $\Bbbk$ with poset Λ. We give A a filtration structure using the radical series, as seen in Example 1.2.2. As in the previous section write $B = \operatorname{Rees} A$.

Suppose M is an A-module with a Δ-filtration

$$0 = M^{(0)} < M^{(1)} < \cdots < M^{(n)} = M.$$

Following [15] let $[\operatorname{rad}_s M : \operatorname{head}\Delta(\lambda)]$ denote the number of successive subquotients $M^{(n_{s,i})}/M^{(n_{s,i}-1)}$ isomorphic to $\Delta(\lambda)$ such that $M^{(n_{s,i})} \leq \operatorname{rad}^s M$ and such that there is a map $\operatorname{rad}^s M \to \Delta(\lambda)$ extending the quotient map $M^{(n_{s,i})} \to \Delta(\lambda)$. We note that the value of $[\operatorname{rad}_s M : \operatorname{head}\Delta(\lambda)]$ does not depend on the choice of Δ-filtration.

DEFINITION 2.2.1. Let M be an A-module with a Δ-filtration. We say that the Δ-filtration is radical-respecting if the homomorphisms $\operatorname{rad}^s M \to \Delta(\lambda)$ used to calculate $[\operatorname{rad}_s M : \operatorname{head}\Delta(\lambda)]$ induce isomorphisms

$$(\operatorname{rad}^{s+t} M \cap M^{(n_{s,i})} + M^{(n_{s,i})})/M^{(n_{s,i}-1)} \cong \operatorname{rad}^t \Delta(\lambda)$$

for all i and all $t \geq 0$.

Varying s and i, consider each $M^{(n_{s,i})}/M^{(n_{s,i}-1)}$ as a subquotient of $\operatorname{rad}^s M$, which should be viewed as a module in its own right (i.e. $(\operatorname{rad}^s M)^m = \operatorname{rad}^{s+m} M$). The definition above is equivalent to saying that the isomorphisms carrying the subquotient $M^{(n_{s,i})}/M^{(n_{s,i}-1)}$ to $\Delta(\lambda)$ are actually filtered isomorphisms. This implies that the layers of the radical series of M can be determined from the Δ-filtration and the radical series of the modules $\Delta(\lambda)$ using the following formula:

$$[\operatorname{rad}_s M : L(\mu)] = \sum_{\substack{t \leq s \\ \lambda \in \Lambda}} [\operatorname{rad}_t M : \operatorname{head}\Delta(\lambda)][\operatorname{rad}_{s-t}\Delta(\lambda) : L(\mu)]. \tag{2.1}$$

LEMMA 2.2.2. *If a module M has at least one radical-respecting Δ-filtration, then all Δ-filtrations are radical-respecting.*

PROOF. Let $0 = M^{(0)} < M^{(1)} < \cdots < M^{(n)} = M$ be a Δ-filtration. Say a subquotient $M^{(k)}/M^{(k-1)}$ isomorphic to $\Delta(\lambda_k)$ has a head on the s_kth radical layer of M, i.e. the surjective quotient map $M^{(k)} \to \Delta(\lambda_k)$ extends to a map $\operatorname{rad}^{s_k} M \to \Delta(\lambda_k)$. Then for any $t \geq 0$, the restriction $\operatorname{rad}^{s_k+t} M \to \operatorname{rad}^t \Delta(\lambda_k)$ is still surjective. This shows that the composition factors from the tth radical layer of $\Delta(\lambda_k)$ occur at radical layer $h_{k,t} \geq s_k + t$. The Δ-filtration is radical-respecting if $h_{k,t} = s_k + t$ in all such cases.

So suppose not, and pick k and t such that $s_k + t$ is minimal among those subquotients with $h_{k,t} > s_k + t$. By minimality the multiset of composition factors in the $(s_k + t)$th layer of M must be subset of the multiset given by (2.1). Since at least one of these factors is missing from the $(s_k + t)$th layer, it must be a strict subset. But we already know that the radical series is given by (2.1), so this is impossible. □

PROPOSITION 2.2.3. *If the projective modules of A have radical-respecting Δ-filtrations, then B is quasi-hereditary (as a graded algebra) with poset Λ and standard and costandard modules* $\operatorname{Rees}\Delta(\lambda)$ *and* $\operatorname{Rees}\nabla(\lambda)$ *respectively.*

In this situation we say that B is quasi-hereditary via the Rees functor.

Proof. The projective modules for B are all of the form $\operatorname{Rees} P(\lambda)$. The quotient map $P(\lambda) \to L(\lambda)$ is filtered surjective, so it is a fibration. As Rees preserves fibrations we obtain a fibration of B-modules, so $\operatorname{Rees} L(\lambda)$ is a quotient of $\operatorname{Rees} P(\lambda)$. It is clear that $\operatorname{Rees} L(\lambda)$ is still irreducible as a B-module, so this gives us both the irreducible B-modules and their projective covers (up to grade shifting).

Let $0 = P^{(0)} < P^{(1)} < \cdots < P^{(n)} = P(\lambda)$ be a radical-respecting Δ-filtration of $P(\lambda)$. As A is quasi-hereditary, we have $P^{(n)}/P^{(n-1)} \cong \Delta(\lambda)$ while for $k < n$, $P^{(k)}/P^{(k-1)} \cong \Delta(\mu_k)$ and $\mu_k > \lambda$. For each subquotient $P^{(k)}/P^{(k-1)}$ there exists some s_k such that as a filtered module $P^{(k)}/P^{(k-1)} \cong \Delta(\mu_k)$ when $P^{(k)}/P^{(k-1)}$ is viewed as a subquotient of $\operatorname{rad}^{s_k} P(\lambda)$. This means that when viewed as a subquotient of $P(\lambda)$, $P^{(k)}/P^{(k-1)} \cong \Delta(\mu_k)\langle -s_k\rangle$.

The Rees functor induces a chain of submodules

$$0 = \operatorname{Rees} P^{(0)} < \operatorname{Rees} P^{(1)} < \cdots < \operatorname{Rees} P^{(n)} = \operatorname{Rees} P(\lambda).$$

In fact the subquotients in this filtration are isomorphic to $\operatorname{Rees} \Delta(\mu)(-s)$ for various μ and s, because

$$\frac{\operatorname{Rees} P^{(k)}}{\operatorname{Rees} P^{(k-1)}} \cong \operatorname{Rees} P^{(k)}/P^{(k-1)} \cong \operatorname{Rees}(\Delta(\mu_k)\langle -s_k\rangle) \cong \operatorname{Rees} \Delta(\mu_k)(-s_k)$$

Thus B is graded quasi-hereditary via Rees. $\square$

Definition 2.2.4. A Δ-L subquotient of a module M is a subquotient M'/M'' isomorphic to a non-trivial extension of a module U by $L(\mu)$, for some quotient U of $\Delta(\lambda)$ and some weights λ, μ with $\mu > \lambda$. We call the subquotient M'/M'' a stretched subquotient if M' is not isomorphic as a filtered module to a (possibly shifted) quotient of $P(\lambda)$.

An L-∇ subquotient of a module M is a subquotient M'/M'' isomorphic to a non-trivial extension of $L(\mu)$ by V, for some submodule V of $\nabla(\lambda)$ and some weights λ, μ with $\mu > \lambda$. The subquotient M'/M'' is called a stretched subquotient if M' is not isomorphic as a filtered module to a (possibly shifted) submodule of $I(\lambda)$.

Example 2.2.5 ([13, Appendix]). The following example is due to Ringel and was discovered when investigating the rigidity of certain tilting modules for SL_3 in characteristic 3. Let Q denote the following quiver

$$Q = \begin{tikzcd} & 05 \arrow[dl, shift left, "\alpha'"] \\ 10 \arrow[ur, shift left, "\alpha"] \arrow[r, shift left, "\gamma"] \arrow[dr, shift left, "\beta"] & 43 \arrow[l, shift left, "\gamma'"] \\ & 51 \arrow[ul, shift left, "\beta'"] \end{tikzcd}$$

and define the algebra A to be $\Bbbk Q/I$, where $\Bbbk Q$ is the path algebra of Q (with path concatenation from left to right) and I is the ideal generated by

$$\begin{array}{cccc} \alpha'\alpha, & \alpha'\beta, & \beta'\alpha, & \beta'(1-\gamma\gamma')\beta, \\ \gamma'\gamma, & \gamma'(\alpha\alpha'-\beta\beta'), & (\alpha\alpha'-\beta\beta')\gamma, & \gamma'\alpha\alpha'\gamma. \end{array}$$

The category of *right* A-modules is quasi-hereditary with respect to the partial order $10 < 05, 51 < 43$, giving the following standard modules

$$\begin{array}{ll} \Delta(10) \cong e_{10}A/(\alpha,\beta,\gamma), & \Delta(05) \cong e_{05}A/\alpha'\gamma A, \\ \Delta(51) \cong e_{51}A/\beta'\gamma A, & \Delta(43) \cong e_{43}A, \end{array}$$

where e_i denotes the primitive idempotent corresponding to the vertex i. As in any path algebra modulo relations the radical filtration coincides with the path length filtration [10, Section 4.1]; in other words $J(A)^n = A_{(n)}$ where $A_{(n)}$ denotes the span of paths of length at least n.

One can show that the tilting module $T(43)$ is isomorphic to $e_{10}A/\gamma A$. Consider $T(43)$ as a filtered module with the radical filtration, and consider the subquotient

$$X = M'/M'' = \frac{(\alpha\alpha'-\beta\beta')A/\gamma A}{(\alpha\alpha'\gamma\gamma'\alpha\alpha')A/\gamma A}.$$

By counting paths one can show that X is 2-dimensional and isomorphic to

$$\begin{tikzcd} & & 0 \arrow[dll, shift left] \\ \Bbbk(\alpha\alpha'-\beta\beta') \arrow[urr, shift left] \arrow[rr, shift left] \arrow[dr, shift left, "-1"] & & 0 \arrow[ll, shift left] \\ & \Bbbk\alpha\alpha'\gamma\gamma'\beta \arrow[ul, shift left, "0"] & \end{tikzcd}$$

so it is a Δ-L subquotient, as it is an extension of $\Delta(10) = L(10)$ by $L(51)$. More importantly X inherits the following filtration from the radical filtration on $T(43)$:

$$X = X^0 = X^1 = X^2 \geq X^3 = \Bbbk\alpha\alpha'\gamma\gamma'\beta = X^4 = X^5 \geq X^6 = 0.$$

The only quotient of $P(10) = e_{10}A$ isomorphic to this extension is

$$Y = P(10)/Q = e_{10}A/(\alpha,\gamma,\beta\beta')$$

which has filtration

$$Y = Y^0 \geq Y^1 = \Bbbk\beta \geq Y^2 = 0.$$

It is immediately clear that X is not isomorphic to any shifted version of Y, so X is a stretched subquotient. It is the only stretched Δ-L subquotient in $T(43)$.

THEOREM 2.2.6. *Suppose B is quasi-hereditary via* Rees*. If a tilting module T for A has no stretched subquotients, then* $\operatorname{Rees} T$ *is a tilting module for B.*

PROOF. Let $\lambda \in \Lambda$ be a weight. Consider a minimal filtered projective resolution for $\Delta(\lambda)$:

$$\cdots \to P'' \to P' \to P(\lambda) \to \Delta(\lambda) \to 0$$

In particular P' is the direct sum of $P(\mu)\langle -m\rangle$ ranging over μ, m such that $L(\mu)$ appears in the mth radical layer of $P(\lambda)$ and $\mathrm{Ext}^1(\Delta(\lambda), L(\mu)) \neq 0$. For $r \in \mathbb{Z}$ we will show that $\mathrm{Ext}^1(\Delta(\lambda), T\langle r\rangle) = 0$. We know that as an unfiltered module $\mathrm{Ext}^1(\Delta(\lambda), T) = 0$ because T is a tilting module. Let $f \in \mathrm{Hom}_A(P', T\langle r\rangle)$ be a non-zero cycle. The cycle f corresponds to an unfiltered homomorphism $\Omega(\Delta(\lambda)) \to T$, where

$$\Omega(\Delta(\lambda)) = \ker(P(\lambda) \to \Delta(\lambda)).$$

By the unfiltered Ext-vanishing condition f is the boundary of some unfiltered boundary $g \in \mathrm{Hom}(P(\lambda), T)$.

We claim that g actually respects the filtrations. First, if $r < 0$ there is nothing to prove, as

$$g(P(\lambda)^i) = g(\mathrm{rad}^i P(\lambda)) \subseteq \mathrm{rad}^i T \subseteq \mathrm{rad}^{i+r} T = T\langle r\rangle^i.$$

So suppose $r \geq 0$. Choose $r' \geq r$ maximal such that $f \in \mathrm{Hom}_A(P', T\langle r'\rangle)$. Let $M = \mathrm{im}\, g$ and $N = \mathrm{im}\, f = \mathrm{im}\, g|_{\Omega(\Delta(\lambda))}$. The submodule M is a quotient of $P(\lambda)$ and N is a submodule which is a quotient of $\Omega(\Delta(\lambda))$. So g induces a surjective homomorphism between the quotients, as shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega(\Delta(\lambda)) \arrow[r] \arrow[d, "g|_{\Omega(\Delta(\lambda))}"] & P(\lambda) \arrow[r] \arrow[d, "g"] & \Delta(\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Thus $W = M/N$ is a quotient of $\Delta(\lambda)$. Let $0 \leq s \leq r'$ be maximal such that $M \subseteq \mathrm{rad}^s T$. In other words, the image of the head $L(\lambda)$ of $\Delta(\lambda)$ occurs in the sth radical layer of T. Pick an irreducible $L(\mu)$ appearing in $N/\mathrm{rad}\, N$ which is lowest in the radical series of T and take a maximal submodule $N' \leq N$ such that $N/N' \cong L(\mu)$. Then M/N' is a Δ-L subquotient of T.

Since N is also the image of f, it must be the case that the $L(\mu)$ factor is the head of some summand $P(\mu)\langle -m\rangle$ of P', corresponding to a composition factor in the mth radical layer of $P(\lambda)$, with m maximal. So $L(\mu)$ is in the $(r+m')$th radical layer of T, for some $m' \geq m$. If $s < r'$, then the filtration length of this subquotient is $r' + m' - s > m$, which is impossible as m was chosen to be maximal and T has no stretched subquotients. So $s = r'$, and thus

$$g(P(\lambda)^i) = g(\mathrm{rad}^i P(\lambda)) = \mathrm{rad}^i g(P(\lambda)) \subseteq \mathrm{rad}^{r'+i} T \subseteq \mathrm{rad}^{r+i} T = T\langle r\rangle^i.$$

This shows that $\operatorname{Ext}^1_A(\Delta(\lambda), T\langle r\rangle) = 0$, so by applying the shift functor we have $\operatorname{Ext}^1_A(\Delta(\lambda)\langle -r\rangle, T) = 0$. By Proposition 2.1.23 this means that

$$\operatorname{Ext}^1_B(\operatorname{Rees}\Delta(\lambda)(-r), \operatorname{Rees} T) = 0$$

As B is quasi-hereditary, this shows that $\operatorname{Rees} T$ has a Rees(∇)-filtration. A similar method shows that $\operatorname{Ext}^1_A(T, \nabla(\lambda)\langle r\rangle) = 0$ so $\operatorname{Rees} T$ also has a Rees(Δ)-filtration, and hence it is a tilting module for $\operatorname{Rees} A$. □

In particular when the above situation occurs $\operatorname{Rees} T(\lambda)$ is the indecomposable $\operatorname{Rees} A$ tilting module corresponding to λ, because Rees preserves the multiplicities of Δ-filtrations.

Another natural filtration that can be applied to modules is the socle filtration. For an A-module M, we can define a filtration $M^{\vee\bullet}$ by setting $M^{\vee(-i)} = \operatorname{soc}^i M$ for $i \geq 0$ and $M^{\vee(-i)} = 0$ for $i < 0$. It is easy to see that M is a filtered A-module in this sense as well. Let $\operatorname{Rees}^\vee$ denote the use of the Rees functor using this alternative filtration.

THEOREM 2.2.7. *Suppose B is quasi-hereditary via* Rees. *If an indecomposable tilting module $T = T(\lambda)$ for A has no stretched subquotients for either the radical or the socle filtration, then T is rigid.*

PROOF. Suppose $T = T(\lambda)$ is an indecomposable tilting module for A. If T has no stretched subquotients, then by applying Theorem 2.2.6 we know that $\operatorname{Rees} T$ and $\operatorname{Rees}^\vee T$ are both tilting modules for B corresponding to λ. But in a graded quasi-hereditary algebra there is only one such tilting module up to isomorphism and grade shifting. Since the gradings of $\operatorname{Rees} T$ and $\operatorname{Rees}^\vee T$ correspond to the radical and socle layers of T, this shows that T has identical radical and socle layers. □

There is a partial converse to the above theorem.

COROLLARY 2.2.8. *Suppose B is quasi-hereditary via* Rees. *If $T = T(\lambda)$ is a rigid indecomposable tilting module for A with radical-respecting Δ- and ∇-filtrations, then T has no stretched subquotients.*

PROOF. From the proof of Proposition 2.2.3 $\operatorname{Rees} T$ has Rees(Δ)- and Rees(∇)-filtrations. So $\operatorname{Rees} T$ is a tilting module, and from the proof of Theorem 2.2.6 any stretched subquotients would give rise to a non-vanishing $\operatorname{Ext}^1(\Delta(\lambda)\langle -r\rangle, T)$ or $\operatorname{Ext}^1(T, \nabla(\lambda)\langle r\rangle)$. □

2.2.2. Duality of stretched subquotients. The hypotheses of Theorems 2.2.6 and 2.2.7 are rather difficult to check in all but the most basic cases. In many applications A has additional properties which can reduce this checking significantly.

COROLLARY 2.2.9. *Suppose B is quasi-hereditary via* Rees. *Let T be a tilting module for A. If T has a radical-respecting Δ-filtration and has no stretched Δ-L subquotients, then* $\operatorname{Rees} T$ *is a tilting module for B.*

PROOF. From the proof of Theorem 2.2.6, $\operatorname{Rees} T$ has a $\operatorname{Rees}(\nabla)$-filtration. From the proof of Proposition 2.2.3, $\operatorname{Rees} T$ also has a $\operatorname{Rees}(\Delta)$-filtration. Therefore $\operatorname{Rees} T$ is tilting. □

The easiest way to show that T has a radical-respecting Δ-filtration is to show that T has simple socle. For then $\operatorname{head} T \cong L(\lambda)$ for some λ, so T is a quotient $P(\lambda)/U$ of $P(\lambda)$, which we assume already has a radical-respecting Δ-filtration. As T has a Δ-filtration so does U [49, Theorem 3]. Thus Δ-filtrations of T and U give a Δ-filtration of $P(\lambda)$, which is radical-respecting by Lemma 2.2.2. But the radical series of T does not change from that of $P(\lambda)$, so T also has a radical-respecting Δ-filtration.

Another way to reduce the number of cases to check is to use duality. A duality functor on $A-\mathrm{mod}$ is a contravariant, additive, $\Bbbk$-linear, exact endofunctor $\delta : A-\mathrm{mod} \to A-\mathrm{mod}$ such that $\delta \circ \delta$ is naturally isomorphic to the identity. A BGG algebra is a quasi-hereditary algebra A equipped with a duality functor δ which fixes simple modules, i.e. $\delta(L(\lambda)) \cong L(\lambda)$ for all $\lambda \in \Lambda$. In a BGG algebra we have $\delta(P(\lambda)) \cong I(\lambda)$ and $\delta(\Delta(\lambda)) \cong \nabla(\lambda)$.

COROLLARY 2.2.10. *Suppose A is a BGG algebra and B is quasi-hereditary via* Rees. *If $T = T(\lambda)$ is an indecomposable tilting module for A such that* $\operatorname{Rees} T$ *is a tilting module for B then T is rigid.*

PROOF. If $\operatorname{Rees} T$ is a tilting module for B, then T has radical-respecting Δ- and ∇-filtrations. Thus $\delta(T)$ has socle-respecting-respecting ∇- and Δ-filtrations, so $\operatorname{Rees}^\vee \delta(T)$ is also an indecomposable tilting module for B. Yet $\delta(T) \cong T$, so $\operatorname{Rees}^\vee \delta(T) \cong \operatorname{Rees}^\vee T$. Proceed as in the proof of Theorem 2.2.7. □

Finally, there is a slightly simpler version of Corollary 2.2.8 in the case of a BGG algebra.

COROLLARY 2.2.11. *Suppose A is a BGG algebra and B is quasi-hereditary via* Rees. *If $T = T(\lambda)$ is a rigid indecomposable tilting module for A with radical-respecting Δ-filtration, then T has no stretched subquotients.*

PROOF. By duality $\delta(T) \cong T$ has a socle-respecting ∇-filtration. Yet T is rigid, so T actually has a radical-respecting ∇-filtration. Now use Corollary 2.2.8. □

EXAMPLE 2.2.12. The rigid tilting modules in [7] satisfy the hypotheses above. In this case, these tilting modules are all projective-injective and have radical-respecting Δ-filtrations which arise from the inverse spherical Kazhdan–Lusztig polynomials $m^{x,y}$. As each projective is in fact such a tilting module, this shows that the projectives have radical-respecting Δ-filtrations and thus are quasi-hereditary, so the tilting modules have no stretched subquotients by the previous result.

2.2.3. Coefficient quivers. Finding and eliminating possible stretched subquotients in a module is in general extremely difficult. In addition to calculating the radical series of a module, one must also know enough about the submodule structure to figure out which subquotients exist. We describe some techniques for doing this, which we apply in the next section.

Tilting modules corresponding to high weights tend to have complicated structure, with several composition factors interacting in intricate ways. One common method to depict the structure of a finite-length module is to use Alperin diagrams [2]. However, often the necessary axioms for Alperin diagrams described in [11] do not hold in practice. As a result, the approach in [13, Appendix] using coefficient quivers must be used instead. Coefficient quivers can be viewed as a generalization of Alperin diagrams which always exist.

Definition 2.2.13. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let $X = (X_i)_{i \in Q_0}$ be a representation of Q over a field K. Suppose $\mathcal{B}$ is a basis for X as a quiver representation, i.e. $\mathcal{B}$ is a union of bases for each vector space X_i. The coefficient quiver of X with respect to $\mathcal{B}$ is denoted $\Gamma(X, \mathcal{B})$. It has vertices indexed by $\mathcal{B}$. For $b \in \mathcal{B} \cap X_i$, $b' \in \mathcal{B} \cap X_j$ there is an arrow $b \to b'$ in $\Gamma(X, \mathcal{B})$ if and only if there is an arrow $\rho : i \to j$ such that the corresponding matrix entry $(X_\rho)_{bb'}$ is non-zero.

Drawing a coefficient quiver can be thought of as "unlacing" the representation X into its 1-dimensional irreducible composition factors. For a general module M over some finite-dimensional algebra A, Gabriel's theorem (see e.g. [10, Proposition 4.1.7]) can be used to replace A with a Morita equivalent quotient of $\Bbbk Q$, where Q is the Ext-quiver of A. Thus the coefficient quiver of M depends on the particular quotient and on the chosen basis. Like Alperin diagrams, coefficient quivers are conventionally drawn such that all arrows point downwards so that the arrowheads may be omitted. Another convention is that if Λ is a labeling set for irreducibles $L(\lambda)$, we write λ instead of $L(\lambda)$ in the coefficient quiver.

Arrow-closed subsets of a coefficient quiver Γ for M give submodules of M, and their complements give quotients. This describes much (but not all) of the submodule and quotient structure of M. For other submodules $M' \leq M$, it will be useful to describe which composition factors in Γ correspond to composition factors of M'. Recall from linear algebra that we say a vector v involves a basis vector b if when v is written as a linear combination of basis vectors, the coefficient corresponding to b is non-zero. Since vertices of the coefficient quiver correspond to basis elements, we will say that a submodule M' of M involves a certain composition factor in Γ if M' contains a vector which involves the corresponding basis vector.

An Alperin diagram is called "strong" if both the radical series and the socle series can be calculated from the diagram [2]. This concept can be extended to coefficient quivers as well. Although there exist modules which do not have strong coefficient quivers (e.g. $T(4, 3)$ in [13, Appendix]), for every module M there exists a coefficient quiver which accurately depicts the radical series. In fact, for any subquotient there exists a coefficient quiver which will accurately depict the subquotient's radical series.

Stretched subquotients by necessity require "stretched" arrows connecting composition factors more than one radical layer apart. In most examples it will be impossible to draw a full coefficient quiver for a module. However, even knowing that certain arrows exist can be extremely helpful for eliminating stretched subquotients within tilting modules. We distinguish between two different kinds of arrows in a coefficient quiver.

- Solid lines (λ——μ) denote arrows which definitely exist for the chosen basis.
- Dotted lines ($\lambda \cdots\cdots \mu$) denote arrows which may exist given certain values of the representing matrices X_ρ.

The following lemma shows that in many cases this requires multiple copies of a composition factor.

LEMMA 2.2.14. *Let M be a module with a radical-depicting coefficient quiver Γ. Suppose $\mu > \lambda$ are weights such that $L(\mu) \leq \mathrm{rad}_1 P(\lambda)$. Suppose further that some copy of $L(\lambda)$ in M connects downward in Γ to some factor $L(\lambda')$ which subsequently connects downward to a factor $L(\mu)$ with $\lambda' \not< \lambda$. Then $L(\lambda)$ is not involved in a stretched subquotient with this copy of $L(\mu)$ unless there is another copy of $L(\lambda')$ which connects downward from $L(\lambda)$ and downward to $L(\mu)$ or there is another copy of $L(\lambda)$ (possibly connected to $L(\mu)$) which connects downward to $L(\lambda')$ (see Figure 2.1).*

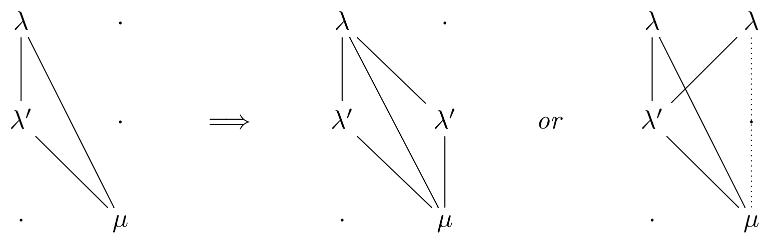


FIGURE 2.1. A portion of a radical-depicting coefficient quiver Γ for some module, where $\mu > \lambda$, $L(\mu) \leq \mathrm{rad}_1 P(\lambda)$ and $\lambda' \not< \lambda$.

PROOF. As $\lambda' \not< \lambda$, there is no composition factor $L(\lambda')$ within $\Delta(\lambda)$. If the given copy of $L(\lambda)$ connects to two copies of $L(\lambda')$, then we can change the basis for the $L(\lambda')$ vectors so that $L(\lambda)$ connects to one copy of $L(\lambda')$. In other words, we draw a new coefficient quiver as in Figure 2.2.

If both copies of $L(\lambda')$ connect downward to $L(\mu)$, then the proposed stretched subquotient is impossible. Thus the dotted line must not exist, so in particular in the original coefficient quiver both copies of $L(\lambda')$ must connect to $L(\mu)$, giving the first case.

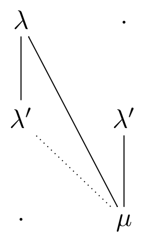


FIGURE 2.2. A new coefficient quiver after changing basis.

Now assume that $L(\lambda)$ connects to exactly one copy of $L(\lambda')$ which connects to $L(\mu)$. This copy of $L(\lambda)$ alone cannot be the head of a stretched subquotient, because there is no way to quotient out $L(\lambda')$ without losing $L(\mu)$ as well. So there must be another copy of $L(\lambda)$ connected to $L(\lambda')$, giving the second case. □

2.2.4. Calculating radical series. The following results of Bowman and Martin on BGG algebras are extremely useful for calculating the radical series of projective modules. They will be used frequently in the following section. The first is a version of Landrock's Lemma ([42, Lemma 1.9.10] or [10, Theorem 1.7.8] for a neater proof).

PROPOSITION 2.2.15 ([15, Theorem 6]). *Let A be a BGG algebra with poset Λ. For $\lambda, \mu \in \Lambda$ we have the following reciprocity:*

$$[\operatorname{rad}_s P(\mu) : L(\lambda)] = [\operatorname{rad}_s P(\lambda) : L(\mu)].$$

The second states that BGG reciprocity (i.e. Brauer–Humphreys reciprocity in a BGG algebra) is compatible with the radical series.

PROPOSITION 2.2.16 ([15, Corollary 7]). *Let A be a BGG algebra with poset Λ. For weights $\lambda, \mu \in \Lambda$ we have*

$$[\operatorname{rad}_s P(\mu) : \operatorname{head} \Delta(\lambda)] = [\operatorname{rad}_s \Delta(\lambda) : L(\mu)].$$

Finally we will use Theorem 1.2.4 frequently to calculate socles of tilting modules from their characters.

2.3. Restricted tilting modules for SL_4

2.3.1. Notation. Let $G = \mathrm{SL}_4$ over an algebraically closed field $\Bbbk$ characteristic $p > 0$. For any finitely generated poset ideal π of dominant weights, let $S(\pi)$ denote the generalized Schur algebra corresponding to those weights as defined in the previous chapter. We showed in Chapter 1 that $S(\pi)$ is quasi-hereditary. It is also a BGG algebra, with the duality functor in $S(\pi)$ coming from contravariant duality of G-modules. When necessary we will deal with $S(\pi)$-modules instead of G-modules for a sufficiently large poset ideal π.

We fix a notation for the weights. The root system corresponding to SL_4 is A_3. Let $\alpha_1, \alpha_2, \alpha_3$ be the simple roots (with $\langle \alpha_1, \alpha_3^\vee \rangle = 0$), and let $\omega_1, \omega_2, \omega_3$ be the corresponding fundamental weights, which span the weight lattice X of A_3. We will use the notation $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ to refer to the weight $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$. In this notation, we have $\alpha_1 = (2, -1, 0)$, $\alpha_2 = (-1, 2, -1)$, and $\alpha_3 = (0, -1, 2)$. The set of dominant weights is therefore $X^+ = \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1, \lambda_2, \lambda_3 \geq 0\}$, which can be given a partial order via the dominance ordering.

Recall that the affine Weyl group W acts on the vector space $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ via the p-scaled dot action, dividing E into p-alcoves. There are 6 alcoves in the restricted region X_1, which we label A_i for i one of 0, 1, 2, 2′, 3, or 4 (see Figure 2.3). The two alcoves 2 and 2′ are related 'by symmetry' in a similar fashion to the SL_3 case. We also consider two alcoves called fl and fl′ which are not in the restricted region but "flank" it. The generators of W are denoted $\tilde{s}, s_1, s_2, s_3$ where s_i is the reflection in α_i and $\tilde{s}$ is the affine reflection.

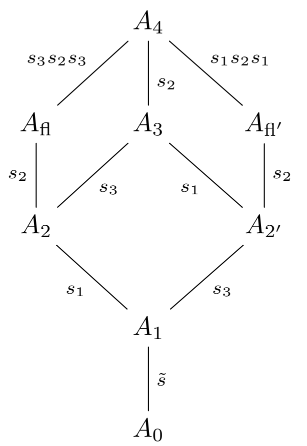


FIGURE 2.3. The dominance lattice for the labeled alcoves. The label on an edge between two alcoves is the affine Weyl group element which maps one alcove to the other.

Let A_i be one of the named p-alcoves listed above. The structure of the module $\Delta(\lambda)$ for any regular λ contained in A_i only depends on i and not on the exact weight λ by the translation principle. So we may abuse notation and write $L(i)$, $\Delta(i)$ etc. instead of $L(\lambda)$, $\Delta(\lambda)$. We can also reconstruct character formulas written in this way using the linkage principle.

Throughout this section we will use the notation $[L_0, L_1 \ldots, L_s]$ to depict the structure of the unique uniserial module M with composition factors $L_0, \ldots, L_s$ such that $\mathrm{rad}_i\, M \cong L_i$.

2.3.2. The result. From [32, II.8.20], the character formulas of the labeled simple modules for type A_3 in terms of Weyl characters can be written independently of the characteristic using our alcove labeling under the assumption that $p \geq 5$. Alternatively, this fact can be viewed as a consequence of LCF. We list these character formulas below:

$[[\Delta(0)]]_{/} = [[L(0)]]_{/}$

$[[\Delta(1)]]_{/} = [[L(1)]]_{/} + [[L(0)]]_{/}$

$[[\Delta(2)]]_{/} = [[L(2)]]_{/} + [[L(1)]]_{/}$

$[[\Delta(\text{fl})]]_{/} = [[L(\text{fl})]]_{/} + [[L(2)]]_{/}$

$[[\Delta(3)]]_{/} = [[L(3)]]_{/} + [[L(2)]]_{/} + [[L(2')]]_{/} + [[L(1)]]_{/} + [[L(0)]]_{/}$

$[[\Delta(4)]]_{/} = [[L(4)]]_{/} + [[L(3)]]_{/} + [[L(\text{fl})]]_{/} + [[L(\text{fl}')]]_{/} + [[L(2)]]_{/} + [[L(2')]]_{/} + [[L(1)]]_{/}$

The characters of $\Delta(2')$ and $\Delta(\text{fl}')$ can be obtained via "symmetry" from the characters of $\Delta(2)$ and $\Delta(\text{fl})$ (i.e swap $2 \leftrightarrow 2'$ and $\text{fl} \leftrightarrow \text{fl}'$). Our goal in this section is to prove the following theorem.

THEOREM 2.3.1. *The regular restricted tilting modules for G are all rigid. They have the following radical series and partial structure:*

$$T(0) = [0], \qquad T(1) = [0,1,0],$$

$$T(2) = \begin{array}{ccc} & 1 & \\ \diagup & & \diagdown \\ 2 & & 0 \\ \diagdown & & \diagup \\ & 1 & \end{array}, \qquad T(\text{fl}) = \begin{array}{ccc} & 2 & \\ \diagup & & \diagdown \\ \text{fl} & & 1 \\ \diagdown & & \diagup \\ & 2 & \end{array},$$

$$T(3) = \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 2 & & 0 & & 2' \\ | & & & & | \\ 1 & & 3 & & 1 \\ & \diagup & | & \diagdown & \\ 2 & & 0 & & 2' \\ & \diagdown & | & \diagup & \\ & & 1 & & \end{array}, \qquad T(4) = \begin{array}{cccccc} & & 2 & 2' & & \\ & & | & | & & \\ \text{fl}' & \text{fl} & 1 & 1 & 3 & \\ | & | & & & \diagup \ | \ \diagdown & \\ 2' & 2 & 4 & 2 & 0 & 2' \\ & \diagup\diagup & | \ \diagdown & & \diagdown \ | \ \diagup & \\ \text{fl} & 1 & 3 & \text{fl}' & 1 & \\ & \diagdown \ | & \times & | \ \diagup & & \\ & 2 & 2' & & & \end{array}.$$

The remainder of this section is devoted to the proof of this theorem.

2.3.3. Weyl modules. First we calculate the structure of the Weyl modules. We claim that the labeled Weyl modules have the following structure:

$$\Delta(0) = [0], \qquad \Delta(1) = [1,0], \qquad \Delta(2) = [2,1],$$

$$\Delta(\text{fl}) = [\text{fl}, 2], \qquad \Delta(3) = \begin{array}{ccccc} & & 3 & & \\ & \diagup & | & \diagdown & \\ 2 & & 0 & & 2' \\ & \diagdown & | & \diagup & \\ & & 1 & & \end{array}, \qquad \Delta(4) = \begin{array}{ccccc} & & 4 & & \\ & \diagup & \diagup \quad \diagdown & \diagdown & \\ \text{fl} & 1 & & 3 & \text{fl}' \\ \diagdown & | & \times & | & \diagup \\ & 2 & & 2' & \end{array}.$$

The cases for $0, 1, 2, \text{fl}$ are obvious from the character formulas. We proceed to cases 3 and 4.

If L is a simple G-module, then from (1.10) we have

$$\operatorname{Hom}_G(L, \Delta(3)) \leq \operatorname{Hom}_G(L, \theta_{s_3}\Delta(2)) \cong \operatorname{Hom}_G(\theta_{s_3}(L), \Delta(2)),$$

and similarly for $\theta_{s_1}(L)$ and $\Delta(2')$. As $\theta_{s_3}L(0)$, $\theta_{s_3}L(2')$, and $\theta_{s_1}L(2)$ are all 0, we must have $\operatorname{soc}\Delta(3) = L(1)$. LCF imposes a parity condition on the vanishing of the Ext^1-groups, namely, $\mathrm{Ext}^1(L(\lambda), L(\mu)) = 0$ for regular weights λ, μ if λ and μ have the same parity, where λ and μ have the same parity if $\mu = x \cdot_p \lambda$ for $x \in W$ with $\ell(x)$ even (see e.g. [32, Lemma C.3]). Thus the weights in the p-alcoves A_0, A_2, $A_{2'}$, and A_4 have the same parity, which we call "even", and weights in the other labeled p-alcoves have "odd" parity. As the remaining composition factors $L(2)$, $L(2')$, and $L(0)$ have the same parity, the structure of $\Delta(3)$ must be the one depicted above.

Similarly, for L a simple G-module we have

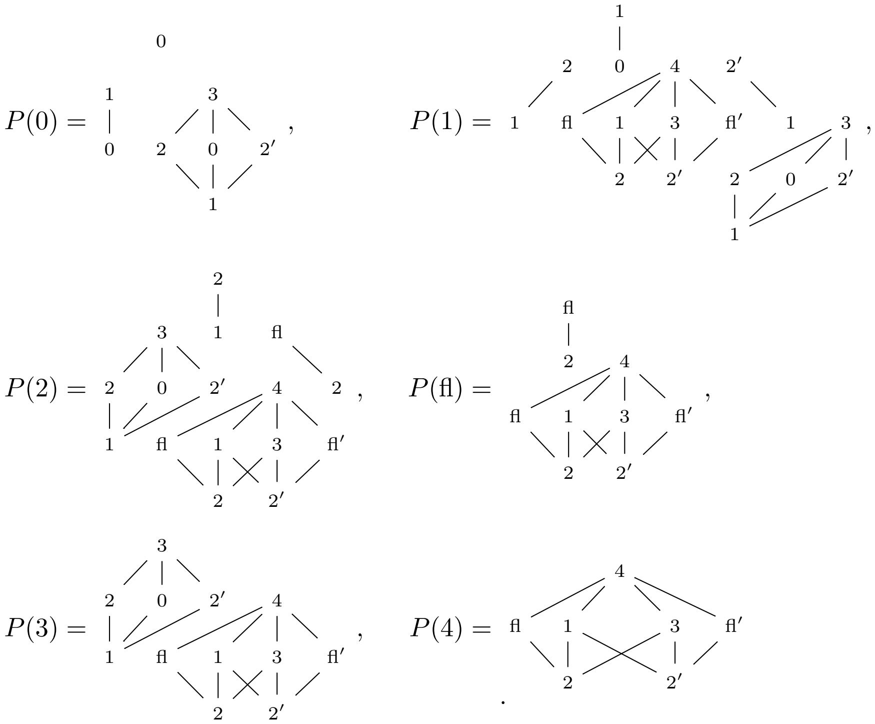
$$\mathrm{Hom}_G(L, \Delta(4)) \leq \mathrm{Hom}_G(L, \theta_{s_2}\Delta(3)) \cong \mathrm{Hom}_G(\theta_{s_2}L, \Delta(3)).$$

As $\theta_{s_2}L(\mathrm{fl})$, $\theta_{s_2}L(\mathrm{fl}')$, and $\theta_{s_2}L(1)$ are all 0 they cannot be summands of $\mathrm{soc}\,\Delta(4)$. From (1.10) we calculate

$$\begin{aligned}
[\theta_{s_2}L(2)] &= [\theta_{s_2}\Delta(2)] - [\theta_{s_2}L(1)] \\
&= [\Delta(\mathrm{fl})] + [\Delta(2)] \\
&= [L(\mathrm{fl})] + 2[L(2)] + [L(1)] \\
[\theta_{s_2}L(2')] &= [L(\mathrm{fl}')] + 2[L(2')] + [L(1)] \\
[\theta_{s_2}L(3)] &= [\theta_{s_2}\Delta(3)] - [\theta_{s_2}L(2)] - [\theta_{s_2}L(2')] - [\theta_{s_2}L(1)] - [\theta_{s_2}L(0)] \\
&= [\Delta(4)] + [\Delta(3)] - [\theta_{s_2}L(2)] - [\theta_{s_2}L(2')] \\
&= [L(4)] + 2[L(3)] + [L(0)]
\end{aligned}$$

By considering the structure of $\Delta(3)$, $L(3)$ also is not contained in $\mathrm{soc}\,\Delta(4)$. So $\mathrm{soc}\,\Delta(4)$ contains at least one of $L(2)$ and $L(2')$, but by symmetry if it contains one it contains both, so $\mathrm{soc}\,\Delta(4) = L(2) \oplus L(2')$. Again, the remaining composition factors have the same parity so $\Delta(4)$ must have the structure depicted above.

2.3.4. Projective modules. The radical series and partial structures of the projective modules now follows using Propositions 2.2.15 and 2.2.16:



It should be noted that Proposition 2.2.16 only specifies where the heads of Weyl modules are located in the radical series. Any other composition factor in a Weyl subquotient must be located at least as far down in the radical series relative to the head of the subquotient as in the Weyl module itself. If none of the composition factors appear any further down, then (2.1) holds for the radical series and the projectives have radical-respecting Δ-filtrations, so B is a quasi-hereditary algebra by Proposition 2.2.3.

There are several ways to show that (2.1) holds. First of all, many possibilities can be ruled out using parity. For example, consider $P(0)$ and the factors $L(0)$, $L(2)$, and $L(2')$ inside $\Delta(3)$. These factors cannot occur any lower down the radical series, for this would require a connection (i.e. a non-zero Ext^1) between the $L(0)$ in $\Delta(1)$ and one of these modules, which is impossible by parity.

Secondly, we can use the fact that the projectives of the Schur algebra corresponding to a saturated subset of the weights are quotients of the projectives above. For example, consider $P(0)$ and the factor $L(0)$ inside $\Delta(1)$. We know that the projective cover of $L(0)$ for the Schur algebra corresponding to the weight set $\{0,1\}$ is a quotient of $P(0)$ by $\Delta(3)$. Therefore $\Delta(3)$ must be a submodule of $P(0)$, so in particular $L(0)$ cannot occur lower down in the radical series. This shows that $P(0)$ has the depicted radical series.

Finally, we can use Proposition 2.2.15 for any other cases which remain. For example, consider $P(1)$ and the factor $L(1)$ inside $\Delta(4)$. If $L(1)$ is lower down in the radical series, then it must be in the 4th layer by parity. This would push $L(2)$ and $L(2')$ down to the 5th layer, so $[\mathrm{rad}_5 P(1) : L(2)] > 0$. This implies that $[\mathrm{rad}_5 P(2) : L(1)] > 0$. But this is impossible (for the reasons above). Thus $L(1)$ (and similarly $L(3)$, $L(\mathrm{fl})$, and $L(\mathrm{fl}')$) are actually in the 3rd layer as depicted above.

2.3.5. Tilting modules. Now we proceed to prove the rigidity of the labeled tilting modules. First we briefly calculate the characters of the wall-crossed tilting modules using Proposition 1.3.9 and its corollary. First, the dominant character sets for the relevant p-alcoves are

$$
\begin{aligned}
{}^{\mathrm{f}}[\underline{\tilde{s}}] &= \left\{ \overset{\mathrm{U}}{0}, \overset{\mathrm{U}}{1} \right\} \\
{}^{\mathrm{f}}[\underline{\tilde{s}s_1}] &= \left\{ \overset{\mathrm{UU}}{10}, \overset{\mathrm{UU}}{11} \right\} \\
{}^{\mathrm{f}}[\underline{\tilde{s}s_1s_2}] &= \left\{ \overset{\mathrm{UUU}}{110}, \overset{\mathrm{UUU}}{111} \right\} \\
{}^{\mathrm{f}}[\underline{\tilde{s}s_1s_3}] &= \left\{ \overset{\mathrm{UUU}}{100}, \overset{\mathrm{UUU}}{101}, \overset{\mathrm{UUU}}{110}, \overset{\mathrm{UUU}}{111} \right\} \\
{}^{\mathrm{f}}[\underline{\tilde{s}s_1s_3s_2}] &= \left\{ \overset{\mathrm{UUUU}}{1010}, \overset{\mathrm{UUUU}}{1011}, \overset{\mathrm{UUUU}}{1100}, \overset{\mathrm{UUUU}}{1101}, \overset{\mathrm{UUUU}}{1110}, \overset{\mathrm{UUUU}}{1111} \right\}
\end{aligned}
$$

Using the isomorphism ch, the characters of the corresponding tilting modules $\theta_{\underline{x}^{-1}} T(0)$ for each expression $\underline{x}$ above are the same as the proposed characters of the indecomposable tilting modules from the main theorem. Yet $T(x \cdot_p 0) \leq \theta_{\underline{x}^{-1}} T(0)$. We will show equality in each case by showing that any other possible tilting character is impossible.

Clearly $T(0) = L(0)$. If $T(1) \neq \theta_{\tilde{s}} T(0)$, then $\theta_{\tilde{s}} T(0) = T(1) \oplus T(0)$ and we must have $T(1) = \Delta(1)$, which contradicts the known structure of $\Delta(1)$. The same argument works for $T(2)$ and $T(\mathrm{fl})$. Moreover if $T(3) \neq \theta_{\underline{s_3 s_1 \tilde{s}}} T(0)$, then $\theta_{\underline{s_3 s_1 \tilde{s}}} T(0)$ is either $T(3) \oplus T(2)$ or $T(3) \oplus T(2')$. Neither of these cases can occur; e.g. in the first case we obtain the contradiction

$$1 = \dim \mathrm{Hom}_G(\Delta(2), \Delta(3)) \leq \dim \mathrm{Hom}_G(\Delta(2), T(3)) = 0$$

using Theorem 1.2.4. Finally for $T(4)$ a similar argument with $\Delta(\mathrm{fl})$ works to rule out $\theta_{\underline{s_2 s_3 s_1 \tilde{s}}} T(0)$ being $T(4) \oplus T(\mathrm{fl})$ or $T(4) \oplus T(\mathrm{fl}')$ instead of just $T(4)$.

Thus we may assume that the characters of the indecomposable tilting modules are as stated in the main theorem. These characters and the known Weyl module structures give the socles of the tilting modules using Theorem 1.2.4. In fact for all the labeled tilting modules we have $\mathrm{soc}\, T(\lambda) = \mathrm{soc}\, \Delta(\lambda)$.

Obviously $T(0) = [0]$, and $T(1)$ is $P_\pi(0)$ for $\pi = \{0, 1\}$. If the socle of $T(2)$ coincides with $\mathrm{soc}\, \Delta(2) \cong L(1)$ then $\mathrm{head}\, T(2) \cong L(1)$, so $T(2)$ is a quotient of $P_\pi(1)$ for $\pi = \{0, 1, 2\}$. The only quotient which possibly contains $\Delta(2)$ as a submodule is all of $P_\pi(1)$, and in order for it to have a ∇-filtration there must be a connection between the $L(1)$ in $\Delta(2)$ and the $L(0)$ in $\Delta(1)$. The case for $T(\mathrm{fl})$ is similar.

The case for $T(3)$ is more complicated. Using Theorem 1.2.4, the socle of $T(3)$ is $\operatorname{soc}\Delta(3) \cong L(1)$ from the character of $T(3)$ and the structure of $\Delta(3)$. So we must have $T(3)$ as a quotient of $P_\pi(1)$, where $\pi = \{0, 1, 2, 2', 3\}$. As $P_\pi(1)$ has a radical-respecting Δ-filtration, $T(3)$ also has one, so we can apply Corollaries 2.2.9 and 2.2.10 if we can show $P_\pi(1)$ (and therefore $T(3)$) has no stretched Δ-L subquotients. The only possible stretched Δ-L subquotient is between the $L(0)$ in $\Delta(1)$ and the $L(1)$ in $\Delta(3)$. By Lemma 2.2.14 this can only happen if there is no connection between this copy of $L(0)$ and $L(3)$. But in that case, $P_\pi(1)$ would not have a quotient isomorphic to $\nabla(3)$, which must be the case using the structure of $\nabla(3)$ and Theorem 1.2.4. Thus $T(3)$ is rigid.

Again from Theorem 1.2.4, $\operatorname{soc} T(4) \cong \operatorname{soc}\Delta(4) \cong L(2) \oplus L(2')$. Thus $T(4)$ is a quotient of $P(2) \oplus P(2')$. The only possible stretched Δ-L subquotient in $P(2) \oplus P(2')$ is between a copy of $L(1)$ in radical layer 1 and $L(2)$ in the bottom radical layer (or the symmetric counterpart between $L(1)$ and $L(2')$). First, if $L(2)$ inside $\Delta(\text{fl})$ does not connect downwards to anything, then $\operatorname{soc}(P(2) \oplus P(2'))$ is too large, and any quotient which eliminates this socle does not have a quotient isomorphic to a submodule of $\nabla(4)$. Similarly the $L(1)$ inside $\Delta(3)$ must connect downwards to some factor.

We know that $L(1)$ is connected to this $L(2)$ by the structure of $T(\text{fl})$. Thus we are in the situation of Lemma 2.2.14. The only other copy of $L(1)$ is not attached to this copy of $L(2)$. Thus $L(1)$ must also connect to the $L(2)$ inside $\Delta(3)$, which connects downwards to another $L(1)$. But we know that the first copy of $L(2)$ doesn't attach to this $L(1)$, because $\Delta(\text{fl})$ is a submodule of $P_\pi(2)$ for $\pi = \{0, 1, 2, 2', 3, \text{fl}\}$. Thus we do not have a stretched subquotient. This shows that $T(4)$ must be rigid, and so it must have the radical series given above as $P(2) \oplus P(2')$ doesn't have any other non-trivial rigid quotients.

CHAPTER 3

Balanced semisimple filtrations for tilting modules

In this chapter we investigate some consequences of Kazhdan–Lusztig theory with regards to Loewy series of tilting modules. We give a "balancing algorithm" (Algorithm 3.1.1) for calculating indecomposable tilting characters and prove the existence of balanced semisimple filtrations in Theorem 3.2.6. For convenience we work in the context of quantum groups at roots of unity, where LCF and SCF almost always hold, but our methods easily generalize to algebraic groups after adding the relevant character-theoretic hypotheses.

3.1. A balancing algorithm

Let U_l be the Lusztig form of a quantized universal enveloping algebra at an lth root of unity, corresponding to some complex semisimple Lie algebra (as described in Section 1.3.5) and let us assume that $l > h$. As described in the Introduction, for many dominant weights λ Andersen and Kaneda showed that the indecomposable tilting module $T_l(\lambda)$ is rigid [7]. If the Loewy length of $T_l(\lambda)$ is $2N + 1$, then by self-duality $\operatorname{rad}_{N+i} T_l(\lambda) \cong \operatorname{rad}_{N-i} T_l(\lambda)$ for integers $0 \leq i \leq N$. In other words, the radical series is symmetric about the middle layer containing $L_l(\lambda)$. We call such Loewy series *balanced*. From [7] we know that not all indecomposable quantum tilting modules are rigid, but even the non-rigid examples in that paper exhibit a Loewy series which is balanced.

In addition, these Loewy series are compatible with a certain Loewy series of the Weyl module which we call the dual parity filtration (cf. the parity filtration in [7]). Since LCF holds in this situation, the characters of the Weyl modules in terms of the simple modules are given by the spherical Kazhdan–Lusztig polynomials $m^{x,y}$. For ease of notation, we will extensively use the module labeling convention in the previous chapter, where we refer to $L_l(A)$, $\Delta_l(A)$, etc. for a dominant p-alcove A, and we will also extend this notation to Kazhdan–Lusztig polynomials using the bijection between dominant p-alcoves and ${}^{\mathrm{f}}W$ via the p-scaled dot action $x \mapsto x \cdot_p A_{0,p}$.

For A a p-dominant alcove, the dual parity filtration of $\Delta_l(A)$ is an increasing filtration $\Delta_l(A)^i$, indexed by *non-positive* integers, such that the successive subquotients $\Delta_l(A)_i = \Delta_l(A)^i / \Delta_l(A)^{i-1}$ are all semisimple, with character

$$[[\Delta_l(A)_i]] = \sum_B (\overline{m^{A,B}})_i [L_l(B)], \tag{3.1}$$

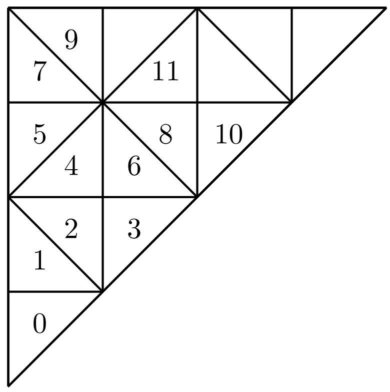


FIGURE 3.1. Some p-alcoves for the quantum group corresponding to the root system B_2.

where $(\overline{m^{A,B}})_i$ denotes the coefficient of v^i in the (negative degree) polynomial $\overline{m^{A,B}}$. We note that if the quantum analogue of Jantzen's conjecture (written as $(F, w, s)^+$ in [32, II.C.9]) holds then this filtration coincides with the Jantzen filtration described in [8, Section 10].

Suppose we know the dual parity filtrations for all Weyl modules $\Delta_l(B)$ for all dominant p-alcoves $B \leq A$ (the ordering here is the Bruhat order, transfered to p-alcoves). Assuming $T_l(A)$ has a Loewy series with the above properties, we can calculate its minimal possible character and Loewy series using the following algorithm.

ALGORITHM 3.1.1.

(1) Write the dual parity filtration of the Weyl module $\Delta_l(A)$. We view this as the bottom layers of a partial Loewy series for $T_l(A)$. We will reflect Loewy layers about the "middle" Loewy layer in which $L_l(A)$ appears.
(2) Find the highest "unbalanced" weight; that is, the largest $B < A$ such that $L_l(B)$ appears in the series below $L_l(A)$ but there is no corresponding factor $L_l(B)$ in the reflected layer above $L_l(A)$.
(3) Add the dual parity filtration of $\Delta_l(B)$ to the partial Loewy series so that the head of $\Delta_l(B)$ is in the reflected Loewy layer above $L_l(A)$.
(4) Repeat from Step (2) until the Loewy series is balanced.

EXAMPLE 3.1.2. We will apply the algorithm to an indecomposable tilting module for the quantum group corresponding to the root system B_2 (this was originally done in [7] using different methods). We label the first few B_2 p-alcoves, following the same module labeling conventions as in the previous chapter (see Figure 3.1). Applying the algorithm to $T_l(8)$ yields the partial Loewy series in Figure 3.2.

In fact, this naïve algorithm gives the characters of all the regular indecomposable tilting modules! To prove this, we will translate Algorithm 3.1.1 into the

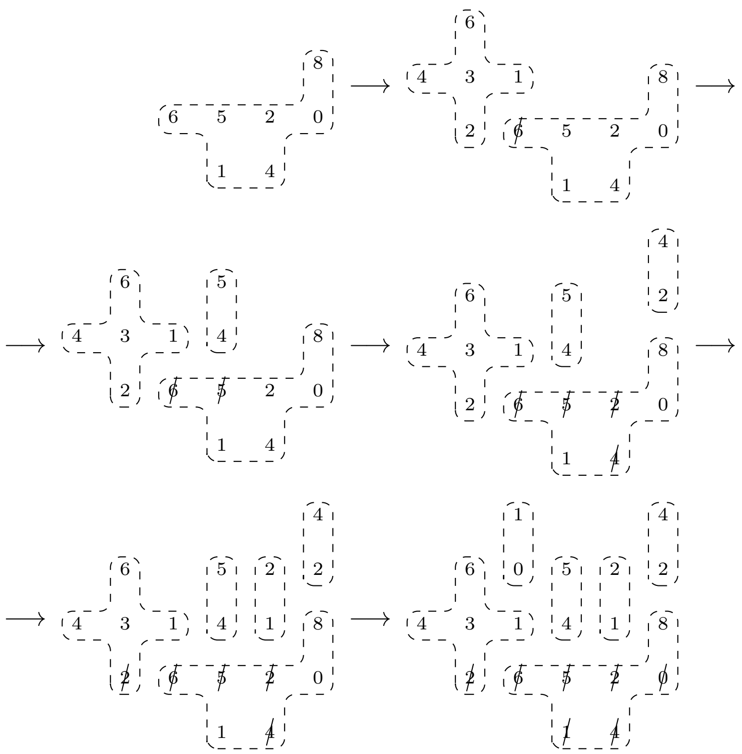


FIGURE 3.2. The partial Loewy series obtained by applying the algorithm to $T_l(8)$.

language of Kazhdan–Lusztig polynomials and prove the result using known properties of the Kazhdan–Lusztig basis. Later we will show that the balanced semisimple filtrations above also exist for all regular indecomposable tilting modules.

First we recall the isomorphism ch between $[[U_l\text{−mod}]]_/$ and ${}_{v=1}\mathbb{N}$ from Corollary 1.3.10. Under this isomorphism $[[\Delta_l(A)]]_/ = [[\nabla_l(A)]]_/$ corresponds to ${}_{v=1}N_A$ and $[[T_l(A)]]_/$ corresponds to ${}_{v=1}\underline{N}_A$. From LCF we have

$$[[\Delta_l(A)]]_/ = \sum_B m^{A,B}(1)[[L_l(B)]]_/,$$

which is equivalent to

$$[[L_l(A)]]_/ = \sum_B (-1)^{\ell(A)+\ell(B)} m_{B,A}(1)[[\Delta_l(B)]]_/ \tag{3.2}$$

by the definition of $m^{A,B}$. Since $m_{B,A}(1) = \overline{m_{B,A}}(1)$ and the negative-degree Kazhdan–Lusztig basis element is

$$\underline{\tilde{N}}_A = \sum_B (-1)^{\ell(A)+\ell(B)} \overline{m_{B,A}} N_B,$$

this means that $[[L_l(A)]]_/$ corresponds to ${}_{v=1}\underline{\tilde{N}}_A$. Now we are ready to prove the polynomial equivalent of the algorithm.

LEMMA 3.1.3. *The Laurent polynomial*

$$t_{B,A} = \sum_C n_{C,A} \overline{m^{C,B}}$$

is self-dual.

PROOF. By inversion,

$$N_A = \sum_B \overline{m^{A,B}} \underline{\tilde{N}}_B.$$

Write

$$\underline{N}_A = \sum_C n_{C,A} N_C = \sum_{B,C} n_{C,A} \overline{m^{C,B}} \underline{\tilde{N}}_B.$$

Clearly the coefficient of $\underline{\tilde{N}}_B$ is $t_{B,A}$, but both $\underline{N}_A$ and $\underline{\tilde{N}}_B$ are self-dual, so $t_{B,A}$ must be self-dual. □

The proof shows that $t_{B,A}(1)$ is simply $[T_l(A) : L_l(B)]$, but it also shows that the degrees of $t_{B,A}$ have some meaning. We can interpret this through the lens of hidden gradings on various module categories. Under this philosophy, whenever there is a "Kazhdan-Lusztig-like" character formula expressing a character by evaluating certain polynomials at 1, there should be a similar graded category for which whose graded characters are given by the polynomials themselves. There have been many investigations of this behavior with respect to tilting modules, see for example [51, 9, 48]. After we have shown that the requisite filtrations exist, we can transfer behavior to a graded category using the Rees functor as in Chapter 2.

For later use we define the following polynomials, which are $t_{B,A}$-analogues of $m^s_{B,A}$:

$$t^s_{B,A} = (v + v^{-1}) \sum_C n_{C,A} \overline{m^{C,B}} + \sum_{C,D} n_{C,A} m^s_{B,D}(0) \overline{m^{C,D}}. \tag{3.3}$$

3.2. Balanced semisimple filtrations

3.2.1. Isotropic filtrations. Let V be a self-dual U_l-module. Fix an isomorphism $\phi : V \to {}^\tau V$. This isomorphism is equivalent to a non-degenerate bilinear form $(-,-)$ on V, with the property that $(xv, v') = (v, \tau(x)v')$ for all $x \in U_l$ and $v, v' \in V$. Forms obeying this property are called contravariant [32, II.8.17]. For any contravariant form on V, we have $(V_\lambda, V_\mu) = 0$ unless $\lambda = \mu$, where V_λ and V_μ are the λ and μ weight subspaces of V. For convenience we will further assume that the form arising from ϕ is symmetric.

For a subspace U of V, recall that the orthogonal subspace is defined to be $U^\perp = \{v \in V : (u, v) = 0 \text{ for all } u \in U\}$. If the form is symmetric, $U \leq U^{\perp\perp}$, and by non-degeneracy the dimensions must match, so $U = U^{\perp\perp}$. If U is a submodule of V then $U^\perp$ is also a submodule of V.

DEFINITION 3.2.1. Suppose U is a submodule of V. Then U is totally isotropic (with respect to $(-,-)$) if $U \leq U^\perp$. Dually U is totally coisotropic if $U^\perp \leq U$.

It is immediately clear that U is totally isotropic if and only if $U^\perp$ is totally coisotropic.

The translation functors T_λ^μ are exact, so the map $T_\lambda^\mu\phi : T_\lambda^\mu(V) \to T_\lambda^\mu({}^\tau V)$ is also an isomorphism. Additionally one can check that $T_\lambda^\mu({}^\tau V) \cong {}^\tau T_\lambda^\mu(V)$, so $T_\lambda^\mu\phi$ defines a non-degenerate contravariant form on $T_\lambda^\mu(V)$.

Lemma 3.2.2. *Let A be a dominant p-alcove, and suppose $\lambda, \mu \in \overline{A}$. If U is a totally isotropic submodule of V, then $T_\lambda^\mu(U)$ is a totally isotropic submodule of $T_\lambda^\mu(V)$.*

Proof. Total isotropy of U can be rephrased in terms of homomorphisms: U is totally isotropic if and only if the inclusion $U \hookrightarrow V$ factors through the inclusion $U^\perp \hookrightarrow V$:

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ U^\perp \arrow[r] & V. \end{tikzcd}$$

Applying T_λ^μ to the above triangle gives

$$\begin{tikzcd} T_\lambda^\mu(U) \arrow[d] \arrow[dr] & \\ T_\lambda^\mu(U^\perp) \arrow[r] & T_\lambda^\mu(V). \end{tikzcd}$$

Since $U^\perp \cong {}^\tau(V/U)$, we have $T_\lambda^\mu(U^\perp) \cong (T_\lambda^\mu U)^\perp$. This implies that $T_\lambda^\mu(U)$ is a totally isotropic submodule of $T_\lambda^\mu(V)$. $\square$

Definition 3.2.3. We call a filtration $V^\bullet$ of V isotropic (with respect to $(-,-)$) if it can be written in the form

$$0 = (V^m)^\perp \leq \cdots \leq (V^1)^\perp \leq V^1 \leq \cdots \leq V^m = V,$$

for some $m \geq 0$. In this situation we typically reindex so that $V^{-i} = (V^i)^\perp$ for $i > 0$. We call V^{-1} and V^1 the lower half and upper half of $V^\bullet$ respectively, denoted $\operatorname{lower} V^\bullet$ and $\operatorname{upper} V^\bullet$. We call $V^\bullet$ maximal isotropic if $\operatorname{lower} V^\bullet$ is maximal, i.e. if there is no other isotropic filtration $V^{\bullet'}$ such that $\operatorname{lower} V^{\bullet'} \geq \operatorname{lower} V^\bullet$. The subquotient $\operatorname{upper} V^\bullet / \operatorname{lower} V^\bullet$ is called the middle and is denoted $\operatorname{mid} V^\bullet$.

We denote the layers of an isotropic filtration by

$$V_i = \begin{cases} V^{i+1}/V^i & \text{if } i > 0, \\ V^i/V^{i-1} & \text{if } i < 0, \\ V^1/V^{-1} & \text{if } i = 0. \end{cases}$$

If $V^\bullet$ is a maximal isotropic filtration, then $\operatorname{mid} V^\bullet$ must be semisimple. To see this, suppose otherwise. We have $(\operatorname{soc} \operatorname{mid} V^\bullet)^\perp = \operatorname{rad} \operatorname{mid} V^\bullet$. For any non-semisimple indecomposable summand U of $\operatorname{mid} V^\bullet$ we have $\operatorname{rad} U \geq \operatorname{soc} U$. From this summand we could construct a larger isotropic filtration, which is a contradiction.

DEFINITION 3.2.4. Suppose T is a tilting module. A semisimple isotropic filtration (with respect to $(-,-)$) $T^\bullet$ of T is called a balanced semisimple filtration if there is a Δ-filtration

$$0 \leq T^{(\lambda_1,1)} \leq T^{(\lambda_1,2)} \leq \cdots \leq T^{(\lambda_1,n_1)} \leq T^{(\lambda_2,1)} \leq \cdots \leq T,$$

indexed over distinct weights and integers, such that the following conditions hold:

- $\lambda_1, \lambda_2, \ldots$ are distinct weights labeled such that if $\lambda_j > \lambda_k$ then $j < k$;
- $n_1, n_2, \ldots$ are positive integers;
- for each k and r, $T^{(\lambda_k,r)}/T^{(\lambda_k,r-1)} \cong \Delta(\lambda_k)$;
- the following induced filtration on the above subquotient (as defined in Section 1.2.1)

  $$(T^{(\lambda_k,r)}/T^{(\lambda_k,r-1)})^i = (T^{(\lambda_k,r)} \cap T^i + T^{(\lambda_k,r-1)})/T^{(\lambda_k,r-1)}$$

  is a shifted version of the dual parity filtration, i.e.

  $$(T^{(\lambda_k,r)}/T^{(\lambda_k,r-1)})^i \cong \Delta_l(\lambda_k)^{i+m(\lambda_k,r)}$$

  for some integer shift $m(\lambda_k,r)$, which for fixed k weakly decreases as r increases.

When using p-alcoves instead of weights as labels, we will use Weyl filtrations labelled like $\{T^{(A_k,r)}\}$ instead of $T^{(\lambda_k,r)}$, where A_k is the p-alcove containing λ_k.

3.2.2. Proof of existence. Before we state and prove the main theorem on the existence of balanced semisimple filtrations, we will need an auxiliary result regarding indecomposable tilting module endomorphisms.

LEMMA 3.2.5. *Let T be an indecomposable tilting module with highest weight vector v. An endomorphism $\phi : T \to T$ is an isomorphism if and only if $\phi(v) \neq 0$.*

PROOF. From the classification of indecomposable tilting modules the highest weight space of T is $\mathbb{C}v$. As T is indecomposable, $\mathrm{End}(T)$ is local. The subspace I of endomorphisms mapping v to 0 is clearly an ideal, and the quotient $\mathrm{End}(T)/I$ is isomorphic to $\mathbb{C}$, so I is the unique maximal ideal of all non-isomorphisms of T. □

Next we develop some language for talking about subquotients of a module. Suppose we have a flag of submodules $W < V < U$. We say that U/V lies above V/W if the extension U/W doesn't split. Otherwise there is a submodule $M \leq U$ with $M + V = U$ and $M \cap V = W$. Then we have $U/V = (M+V)/V \cong M/W$ and also $M \leq U$, so $U/M = (M+V)/M \cong V/W$, and we can switch the order of the subquotients.

Finally we introduce some convenient notation for Laurent polynomials. Suppose $p = \sum_j p_j v^j \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$. For $i \in \mathbb{Z}$ set

- $(p)_i = p_i$;
- $(p)_{\leq i} = \sum_{j \leq i} p_i$;
- $\{p\}_i = v^j$ if $(p)_{\leq j-1} < i \leq (p)_{\leq j}$ and zero otherwise;
- $\{p\}_{\leq i} = \sum_{j \leq i} \{p\}_j$.

In other words, $(-)_i$ and $(-)_{\leq i}$ take coefficients of terms with degree i and sums of coefficients of degree at most i respectively, while $\{-\}_i$ and $\{\}_{\leq i}$ take the ith monomial or the first i monomials respectively, where the monomials are ordered by degree. For example,

$$\begin{aligned}
(v^{-1} + 2v^2 + 3v^3)_{\leq 1} &= 1, & \{v^{-1} + 2v^2 + 3v^3\}_{\leq 1} &= v^{-1}, \\
(v^{-1} + 2v^2 + 3v^3)_{\leq 2} &= 3, & \{v^{-1} + 2v^2 + 3v^3\}_{\leq 2} &= v^{-1} + v^2, \\
(v^{-1} + 2v^2 + 3v^3)_{\leq 3} &= 6, & \{v^{-1} + 2v^2 + 3v^3\}_{\leq 3} &= v^{-1} + 2v^2, \\
(v^{-1} + 2v^2 + 3v^3)_{\leq 4} &= 6, & \{v^{-1} + 2v^2 + 3v^3\}_{\leq 4} &= v^{-1} + 2v^2 + 2v^3.
\end{aligned}$$

THEOREM 3.2.6. *Let $T = T_l(A)$. There exists a balanced semisimple filtration $T^\bullet$ of T with Δ-filtration $\{T^{(A_k,r)}\}$ such that*

$$[T_i : L_l(B)] = (t_{B,A})_i,$$
$$[(T^{(A_k,r)}/T^{(A_k,r-1)})_i : L_l(B)] = (\{n_{A_k,A}\}_r \overline{m^{A_k,B}})_i.$$

PROOF. Write $A = x \cdot A_{0,p}$ and induct on $\ell(x)$. The base case is when $A = A_{0,p}$ is the fundamental p-alcove and we have $T_l(A_{0,p}) \cong L_l(A_{0,p})$. Pick an isomorphism $\phi : L_l(A_{0,p}) \to {}^\tau L_l(A_{0,p})$, which gives a non-degenerate contravariant symmetric form $(-,-)$ [32, II.8.17]. The isotropic filtration $0 = (T^1)^\perp \leq T^1 = T_l(A)$ has the properties we want.

For the inductive step, suppose we have shown that the claim holds for all p-alcoves $y \cot A_{0,p}$ where $y < x$ in the Bruhat order, and that we have chosen isomorphisms between these tilting modules and their duals which induce symmetric contravariant forms. Pick a simple reflection $s \in S$ such that $As > A$ in the dominance ordering. Define $Q = \theta_s(T_l(A))$. Then Q decomposes as $T_l(As) \oplus Q'$ where Q' is a tilting module with highest weights lower than As. Fix an isomorphism $Q \xrightarrow{\sim} T_l(As) \oplus Q'$ once and for all. We will denote $T_l(A)$ by T and $T_l(As)$ by T' for simplicity.

By induction there is a non-degenerate symmetric contravariant form on T and a balanced semisimple filtration $T^\bullet$ satisfying the claim. Applying the functor θ_s to the form on T gives a form with the same properties on Q. By Lemma 3.2.2, $\theta_s(T^\bullet)$ is an isotropic filtration of Q, which we will label $Q^\bullet$.

Suppose the bottom layer of T is $T^m = 0$ for some $m \leq 0$. Consider the submodules $0 = Q^m \leq Q^{m+1} \leq Q^{m+2}$. These describe a filtration for a summand of the module $\theta_s(T^{m+2})$. Clearly T^{m+2} has Loewy length at most 2, so by Corollary 1.3.14 $\theta_s(T^{m+2})$ has a Loewy length of at most $2 + 2 = 4$.

Now define $(Q^{m+1})^+$ and $(Q^{m+1})^-$ such that

$$(Q^{m+1})^+/Q^{m+1} \cong \mathrm{soc}(Q^{m+2}/Q^{m+1}),$$
$$(Q^{m+1})^-/Q^m \cong \mathrm{rad}(Q^{m+1}/Q^m).$$

As $(Q^{m+1})^+/Q^{m+1}$ is semisimple, any composition factor can be written as U/Q^{m+1}, and similarly any composition factor of $Q^{m+1}/(Q^{m+1})^-$ can be written Q^{m+1}/W. If there is a composition factor U/Q^{m+1} which lies above Q^{m+1}/W, then the Loewy length of Q^{m+2} is at least 6, which is a contradiction. Thus all such composition factors can be switched, so there exists a module Y which does this, i.e. $Y + Q^{m+1} = (Q^{m+1})^+$ and $Y \cap Q^{m+1} = (Q^{m+1})^-$ (see Figure 3.3).

This leaves us with a semisimple filtration

$$0 = Q^m \leq (Q^m)^+ \leq (Q^{m+1})^- \leq Y \leq (Q^{m+1})^+ \leq (Q^{m+2})^- \leq Q^{m+2},$$

where we have continued the notation suggested above in the obvious manner. Yet $Y/(Q^{m+1})^- \cong (Q^{m+1})^+/Q^{m+1}$ and $(Q^{m+1})^-/(Q^m)^+$ have the same Kazhdan–Lusztig parity, so in fact $Y/(Q^m)^+$ is semisimple. Similarly $(Q^{m+2})^-/Y$ is semisimple. With this in mind, we redefine the filtration $Q^\bullet$ so that its first few lower layers are $0 \leq (Q^m)^+ \leq Y \leq (Q^{m+2})^- \leq Q^{m+2}$. We continue in this manner up through the lower half of Q, re-indexing as we go along so that all indices are integers. Obviously by taking orthogonal spaces this works for the upper half as well.

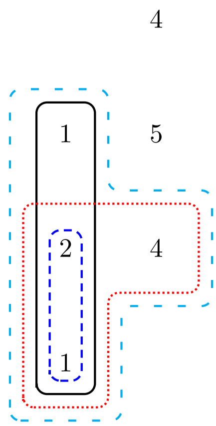


FIGURE 3.3. An illustration of a possible Loewy series for Q^{m+2}. As in the example in the introduction, the numbers are composition factors. The Kazhdan-Lusztig parity of a factor corresponds to the parity of the number labeling it. The submodule Q^{m+1} is circled with a solid line, while $(Q^{m+1})^-$ and $(Q^{m+1})^+$ are circled with dashed lines and the submodule Y is circled with a dotted line.

By induction $\mathrm{mid}\, T^\bullet$ is semisimple. Thus $\mathrm{mid}\, Q^\bullet = \theta_s(\mathrm{mid}\, T^\bullet)$, which is a self-dual module of Loewy length 3 by Corollary 1.3.14. Now define V such that $V/Q^{-1} \cong \mathrm{rad}(Q^1/Q^{-1})$. Then $Q^1/V \cong \mathrm{head}(Q^1/Q^{-1})$, and by taking orthogonal complements $V^\perp/Q^{-1} \cong \mathrm{soc}(Q^1/Q^{-1})$ so $V \geq V^\perp$. Thus $V^\perp$ is a larger totally isotropic submodule of Q, so we can redefine Q^1 and Q^{-1} to be V and $V^{\perp}$ respectively. The resulting filtration after all these changes has layers given by (3.3), i.e. $[Q_i : L_l(B)] = (t^s_{B,A})_i$ for any integer i and any p-alcove B.

The module Q naturally has a Δ-filtration because T does, which we label $Q^{(A_k,r)}$. Recall that if $\mathrm{Ext}^1(\Delta_l(C), \Delta_l(D)) \neq 0$ then $C < D$. This means we can rearrange and relabel the Weyl subquotients (as described in the beginning of this section) so that they have the same ordering properties as in Definition 3.2.4. We claim that $Q^{(A_k,r)} \cap Q^i$ has the following character[1] based on a "partial" version of $t^s_{B,A}$:

$$[Q^{(A_k,r)} \cap Q^i : L_l(B)] = \tag{3.4}$$
$$= \left((v+v^{-1}) \sum_{j \leq k} \{n_{A_j,A}\}_{\leq r} \overline{m^{A_j,B}} + \sum_{\substack{j \leq k \\ C}} m^s_{B,C}(0) \{n_{A_j,A}\}_{\leq r} \overline{m^{A_j,C}} \right)_{\leq i} .$$

To see this, note that a similar result holds for the original filtration on Q, since it was a wall-crossed version of a balanced semisimple filtration on T. The modifications made to this filtration don't change the fact that composition factors in the layers Q_i can be identified as belonging to some Weyl subquotient.

The induced filtration on $Q^{(A_k,r)}/Q^{(A_k,r-1)}$ has ith layer

$$\frac{(Q^i \cap Q^{(A_k,r)} + Q^{(A_k,r-1)})/Q^{(A_k,r-1)}}{(Q^{i-1} \cap Q^{(A_k,r)} + Q^{(A_k,r-1)})/Q^{(A_k,r-1)}} \cong$$
$$\begin{aligned} &\cong \frac{Q^i \cap Q^{(A_k,r)} + Q^{(A_k,r-1)}}{Q^{i-1} \cap Q^{(A_k,r)} + Q^{(A_k,r-1)}} \\ &\cong \frac{Q^i \cap Q^{(A_k,r)}}{(Q^i \cap Q^{(A_k,r)}) \cap (Q^{i-1} \cap Q^{(A_k,r)} + Q^{(A_k,r-1)})} \\ &= \frac{Q^i \cap Q^{(A_k,r)}}{Q^i \cap Q^{(A_k,r)} \cap Q^{(A_k,r-1)} + Q^{i-1} \cap Q^{(A_k,r)}} \\ &= \frac{Q^i \cap Q^{(A_k,r)}}{Q^i \cap Q^{(A_k,r-1)} + Q^{i-1} \cap Q^{(A_k,r)}}. \end{aligned}$$

Now we calculate the character of the denominator in the above quotient:

$$[Q^i \cap Q^{(A_k,r-1)} + Q^{i-1} \cap Q^{(A_k,r)}] =$$
$$\begin{aligned} &= [Q^i \cap Q^{(A_k,r-1)}] + [Q^{i-1} \cap Q^{(A_k,r)}] \\ &\quad - [(Q^i \cap Q^{(A_k,r-1)}) \cap (Q^{i-1} \cap Q^{(A_k,r)})] \\ &= [Q^i \cap Q^{(A_k,r-1)}] + [Q^{i-1} \cap Q^{(A_k,r)}] - [Q^{i-1} \cap Q^{(A_k,r-1)}]. \end{aligned}$$

[1]We have implicitly assumed positivity of various Kazhdan-Lusztig polynomials. For Weyl groups and affine Weyl groups this follows from geometric interpretations of these polynomials first shown in [37, Theorem 1.4].

Using (3.4), the character of this ith layer is

$$\begin{aligned}
(3.5)\quad & \left((v+v^{-1})\sum_{j\le k}\{n_{A_j,A}\}_{\le r}\overline{m^{A_j,B}} + \sum_{\substack{j\le k\\ C}} m^s_{B,C}(0)\{n_{A_j,A}\}_{\le r}\overline{m^{A_j,C}}\right)_{\le i} \\
& - \left((v+v^{-1})\sum_{j\le k}\{n_{A_j,A}\}_{\le r-1}\overline{m^{A_j,B}} + \sum_{\substack{j\le k\\ C}} m^s_{B,C}(0)\{n_{A_j,A}\}_{\le r-1}\overline{m^{A_j,C}}\right)_{\le i} \\
& - \left((v+v^{-1})\sum_{j\le k}\{n_{A_j,A}\}_{\le r}\overline{m^{A_j,B}} + \sum_{\substack{j\le k\\ C}} m^s_{B,C}(0)\{n_{A_j,A}\}_{\le r}\overline{m^{A_j,C}}\right)_{\le i-1} \\
& + \left((v+v^{-1})\sum_{j\le k}\{n_{A_j,A}\}_{\le r-1}\overline{m^{A_j,B}} + \sum_{\substack{j\le k\\ C}} m^s_{B,C}(0)\{n_{A_j,A}\}_{\le r-1}\overline{m^{A_j,C}}\right)_{\le i-1} \\
& = (\{n^s_{A_k,A}\}_r\overline{m^{A_k,B}})_i,
\end{aligned}$$

which is a shifted version of the dual parity filtration.

Now we will obtain analogous results for the direct summand T' of Q. First note that the restriction of the bilinear form on Q to T' is non-degenerate if and only if the map

$$\begin{aligned} T' &\longrightarrow {}^{\tau}T' \\ v &\longmapsto (v,-) \end{aligned}$$

is an isomorphism. In the case of the above map, this is readily apparent: for v_{As} a highest weight vector of T' (and therefore of Q) we have $(v_{As}, Q_\lambda) = 0$ for all λ below the highest weight, so $(v_{As}, v_{As}) \neq 0$ as the form is non-degenerate on Q. As $T' \cap T'^{\perp} = 0$, this implies that Q is isomorphic to $T' \oplus T'^{\perp}$ as a vector space. But $T'^{\perp}$ is a submodule isomorphic to $Q/T' \cong Q'$ so without loss of generality $Q' = T'^{\perp}$ and the form is non-degenerate on Q' too. Let $\pi_{T'}, \pi_{Q'}$ be the projection maps onto the two summands of Q. We say a subquotient U/V lies entirely in T' if $\pi_{T'}(U)/\pi_{T'}(V) \cong U/V$ and $\pi_{Q'}(U) = \pi_{Q'}(V)$.

We will modify each Weyl factor to lie entirely in either T' or Q'. Recall that the filtration shift of the Weyl factor $Q^{(A_k,r)}/Q^{(A_k,r-1)}$ is the smallest i such that $Q^{(A_k,r)} \le Q^i$. From (3.5) this corresponds to the degree of some monomial term in $n^s_{A_k,A}$. These terms can be divided into those which come from $n_{A_k,As}$ and those which don't, corresponding to Weyl factors lying in T' and Q' respectively.

Consider the first Weyl factor $Q^{(A_1,1)}$. It has to be isomorphic to the highest Weyl factor $\Delta_l(As)$. From highest weight theory $\mathrm{Hom}(\Delta_l(As), Q') = 0$, so $\pi_{Q'}(Q^{(A_1,1)}) = 0$ and thus $Q^{(A_1,1)} \le T'$. The quotient $Q/Q^{(A_1,1)}$ still has a Weyl filtration, and we induct on the number of Weyl factors. Suppose the quotient

$Q/Q^{(A_k,r-1)}$ has bottom Weyl factor $Q^{(A_k,r)}/Q^{(A_k,r-1)}$. In general if one of T' or Q' doesn't have $\Delta_l(A_k)$ as a factor, then the same trick still works.

Otherwise, suppose this bottom Weyl factor has filtration shift i, and both T' and Q' contain copies of $\Delta_l(A_k)$ but only one of $n_{A_k,As}$ and $n^s_{A_k,A} - n_{A_k,As}$ has a non-zero coefficient of v^i. Then the Weyl factor lies entirely in T' or Q' respectively. To see this, note that the top simple factor $L_l(A_k)$ in this copy of $\Delta_l(A_k)$ corresponds to a summand in Q_i, and is dual to a summand in Q_{-i}. By induction and using Lemma 3.1.3 this summand in Q_{-i} lies entirely in only one of T'_{-i} or Q'_{-i}, so by non-degeneracy the top summand of the Weyl factor lies entirely in either T'_i or Q'_i, which implies that the whole Weyl factor does too.

Finally suppose both T' and Q' contain copies of $\Delta_l(A_k)$ and both $n_{A_k,As}$ and $n^s_{A_k,A} - n_{A_k,As}$ have non-zero coefficient of v^i. Pick $s > r$ maximal such that the submodule $Q_{(A_k,s)}/Q_{(A_k,r)}$ is isomorphic as a filtered module to a direct sum of copies of $\Delta_l(A_k)$ all shifted by i. Clearly all indecomposable direct summands are filtration isomorphic, so one can choose a new direct sum decomposition of this module so that each summand lies entirely in one of T' or Q'. The number of summands lying in each also corresponds to the coefficient of v^i in each of the above polynomials, using a similar argument to the previous case. Thus T' has a balanced semisimple filtration with the correct filtration layers. $\square$

CHAPTER 4

Soergel bimodules

The remainder of this thesis is devoted to Soergel bimodules and connections to higher-order linkage for tilting modules as discussed in the Introduction. Because the theory of Soergel bimodules is extensive and less well known, this chapter focuses on summarizing some fundamental results, mostly from [26].

Let $\Bbbk$ denote a field of characteristic not equal to 2. Soergel bimodules over $\Bbbk$ are characterized by the following fundamental property, which appears later in this chapter as Corollary 4.2.6: the category $\mathcal{D}$ of Soergel bimodules (over a suitable $\Bbbk$-realization of W) is an entirely algebraic construction of a $\Bbbk$-linear categorification of $\mathbb{H}$. More precisely, $\mathcal{D}$ is an additive, monoidal, $\Bbbk$-linear category, defined in terms of generators and relations, whose split Grothendieck ring $[[\mathcal{D}]]$ is isomorphic to $\mathbb{H}$. Moreover, in Theorem 4.2.3 we show that for each $x \in W$ there is an indecomposable Soergel bimodule B_x labeled by x, and all indecomposable Soergel bimodules are of this form. The elements $\{[[B_x]]\}$ form a basis in the split Grothendieck ring $[[\mathcal{D}]]$ and thus correspond to a basis in $\mathbb{H}$. Beyond categorification, other important results include the light leaves and double leaves bases for various Hom-spaces in Theorems 4.2.1 and 4.2.2, as well as the important technique of localization described in Section 4.2.2.

4.1. Construction

4.1.1. Realizations. A realization of the affine Weyl group (W, S) over $\Bbbk$ consists of a $\Bbbk$-vector space V along with subsets

$$\{a_s : s \in S\} \subset V, \qquad\qquad \{a_s^\vee : s \in S\} \subset V^*$$

such that

(i) for all $s \in S$, we have $\langle a_s, a_s^\vee \rangle = 2$;
(ii) if we set $s(v) = v - \langle v, a_s^\vee \rangle a_s$ for each $s \in S$ and all $v \in V$, then this defines a representation of W on V.

Note that we use Latin letters for vectors inside a realization, to distinguish them from weights or vectors in a root system, which are usually labeled by Greek letters. We call the matrix $a_{st} = \langle a_s, a_t^\vee \rangle$ the Cartan matrix of the realization V. If U and V are two realizations of (W, S) we call a linear map $\phi : U \to V$ a homomorphism of realizations if ϕ is a homomorphism of W-representations and $\phi(a_s) = a_s$ for all $s \in S$.

DEFINITION 4.1.1. The universal realization $V_{\Sigma,-\tilde{\alpha}}$ of (W,S) with respect to the root vectors $\Sigma \cup \{-\tilde{\alpha}\}$ is defined as follows. Let $V_{\Sigma,-\tilde{\alpha}} = \bigoplus_{s\in S} \Bbbk a_s$ and define $\{a_s^\vee\} \subseteq V^*_{\Sigma,-\tilde{\alpha}}$ by

$$\langle a_s, a_t^\vee \rangle = \langle \alpha_s, \alpha_t^\vee \rangle \tag{4.1}$$

where $\alpha_{s_{\beta,0}} = \beta$ for all $\beta \in \Sigma$ and $\alpha_{\tilde{s}} = -\tilde{\alpha}$.

DEFINITION 4.1.2. The dual universal realization $V^\vee_{\Sigma,-\tilde{\alpha}}$ of (W,S) with respect to the root vectors $\Sigma\cup\{-\tilde{\alpha}\}$ is defined as follows. Temporarily abusing notation, let $(V^\vee_{\Sigma,-\tilde{\alpha}})^* = \bigoplus_{s\in S} \Bbbk a_s^\vee$. Now write $V^\vee_{\Sigma,-\tilde{\alpha}} = ((V^\vee_{\Sigma,-\tilde{\alpha}})^*)^*$ and define $\{a_s\} \subset V^\vee_{\Sigma,-\tilde{\alpha}}$ such that (4.1) holds.

By definition the universal and dual universal realizations have the same Cartan matrix, which is the Cartan matrix of the affine root system. The universal realization has the following universal property: for any realization V of (W,S) with the same Cartan matrix, there is a unique homomorphism of realizations $V_{\Sigma,-\tilde{\alpha}} \to V$. In fact for any matrix which is the Cartan matrix of some realization one can construct in exactly the same way the universal realization for that matrix which has the same universal property. In particular, the geometric representation in [26] and other papers is what we would call the dual universal realization for the unique symmetric Cartan matrix.

The (dual) universal realization only depends on its Cartan matrix, so it can also be defined for the finite Weyl group $(W_{\mathrm{f}}, S_{\mathrm{f}})$; in this case, the two realizations V_Σ and $V_\Sigma^\vee$ of $(W_{\mathrm{f}}, S_{\mathrm{f}})$ are isomorphic, and for both realizations the sets $\{a_s\}$ and $\{a_s^\vee\}$ are bases. In the affine case, one of these sets is a basis but the other is linearly dependent. More precisely, suppose $\tilde{\alpha} \in E$ decomposes as a sum

$$\tilde{\alpha} = \sum_{s\in S_{\mathrm{f}}} c_s \alpha_s = \sum_{\alpha\in\Sigma} c_\alpha \alpha \in E$$

of simple roots in E. Then for the dual universal realization we have

$$a_{\tilde{s}} = \sum_{s\in S_{\mathrm{f}}} c_s a_s.$$

Similarly for the universal realization we can define coefficients $c_s^\vee$ similarly so that $a_{\tilde{s}} = \sum_{s\in S_{\mathrm{f}}} c_s^\vee a_s^\vee$. For convenience, we write

$$\tilde{a} = \sum_{s\in S_{\mathrm{f}}} c_s a_s, \qquad \tilde{a}^\vee = \sum_{s\in S_{\mathrm{f}}} c_s^\vee a_s^\vee \tag{4.2}$$

for any realization of (W,S) or $(W_{\mathrm{f}}, S_{\mathrm{f}})$.

DEFINITION 4.1.3. Let V_Σ be the universal realization of $(W_{\mathrm{f}}, S_{\mathrm{f}})$ with respect to Σ. The inflated finite realization V_Σ^π of (W,S) with respect to $\Sigma, -\tilde{\alpha}$ is defined as follows. As a W-representation, V_Σ^π is the inflation of V_Σ via the canonical projection $\pi : W \to W_{\mathrm{f}}$. Moreover, we set $a_{\tilde{s}} = -\tilde{a}$ and $a_{\tilde{s}}^\vee = -\tilde{a}^\vee$.

In the inflated finite realization, the sets $\{a_s\}$ and $\{a_s^\vee\}$ both span but are linearly dependent. By contrast, there is another realization mimicking the construction of Kac-Moody algebras in which both sets are linearly independent, but neither span.

Now we describe the relationship between the universal realization and the affine action of W on E. Let $V_\mathbb{R} = V_{\Sigma,-\tilde{\alpha}}$ be the universal realization of (W, S) over $\mathbb{R}$ with respect to $\Sigma, -\tilde{\alpha}$. Let $v_{\mathrm{stab}} = a_{\tilde{s}} + \tilde{a}$. One can show that $\mathbb{R}v_{\mathrm{stab}}$ is a 1-dimensional subspace of fixed vectors in $V_\mathbb{R}$.

LEMMA 4.1.4 ([31, 6.5]). *Let $\{a_s^*\} \subseteq V_\mathbb{R}^*$ be the dual basis of $\{a_s\} \subseteq V_\mathbb{R}$. Then the affine hyperplane*

$$E' = \{a^* \in V_\mathbb{R}^* : \langle v_{\mathrm{stab}}, a^* \rangle = 1\}$$

is fixed by the action of W. Moreover, the affine map $f : E \to E'$ defined by $f(0) = a_{\tilde{s}}^*$ and $f(\alpha_s^\vee) = a_{\tilde{s}}^* + a_s^\vee$ for $s \in S_\mathrm{f}$ is an isomorphism of W-spaces.

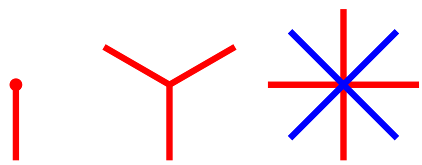
4.1.2. The diagrammatic category. Let V be a realization of (W, S) and $R = \mathrm{Sym}(V)$ the symmetric algebra in V. We view R as a polynomial algebra in the generators $\{a_s\}$ and define a grading on R by setting $\deg(a_s) = 2$. The algebra R inherits a W-action from V, and we define the Demazure operator $\partial_s : R \to R(-2)$ using the formula

$$\partial_s(f) = \frac{f - sf}{a_s}. \tag{4.3}$$

We also identify S with a set of colors for the purposes of drawing pictures.

DEFINITION 4.1.5 ([26, Definition 5.1]). An S-graph (or Soergel graph) is a finite decorated graph with boundary properly embedded into $\mathbb{R} \times [0, 1]$ with the following properties:

- the edges of an S-graph are colored by S;
- the planar regions are labeled with polynomials in R;
- the interior vertices are of the following types



univalent	trivalent	$2m_{st}$-valent
"dot"	"fork"	"braid"
degree $+1$	degree -1	degree 0

The final picture above shows an example of a braid vertex for $s, t \in S$, where s is red, t is blue, and $m_{st} = 4$.

The degree of an S-graph is the sum of the degrees of all the vertices and the degrees of the polynomial labels. By convention we omit any labels $1 \in R$ for planar regions. The sequence of boundary points of an S-graph lying in $\mathbb{R} \times \{0\}$ (resp. $\mathbb{R} \times \{1\}$) give an expression in S, which we call the bottom (resp. top) boundary.

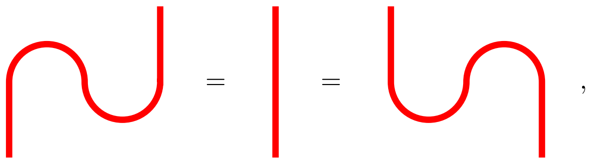
Definition 4.1.6 ([26, Definition 5.2]). The diagrammatic Bott–Samelson category $\mathcal{D}_{\mathrm{BS}}$ is the $\Bbbk$-linear monoidal category defined as follows.

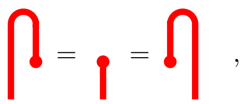
Objects: For each expression $\underline{x} \in \underline{S}$ there is an object $B_{\underline{x}}$ in $\mathcal{D}_{\mathrm{BS}}$ called a Bott–Samelson bimodule. The tensor product of these objects is defined by $B_{\underline{x}} \otimes B_{\underline{y}} = B_{\underline{xy}}$.

Morphisms: The morphism space $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}}(B_{\underline{x}}, B_{\underline{y}})$ is defined to be the set of $\Bbbk$-linear combinations of S-graphs with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, modulo the relations listed below. Composition of morphisms is given by vertical concatenation, while the tensor product of morphisms is given by horizontal concatenation.

Relations: We have the following relations on the morphisms between two Bott–Samelson bimodules. The diagrams below should be viewed as generators for all the relations with respect to composition and tensor products. In other words, any *region* of a diagram can be simplified using these relations.

Isotopy: We only consider S-graphs up to isotopy; informally, this means edges and vertices can be moved continuously, e.g.

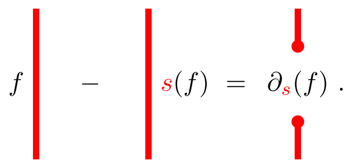




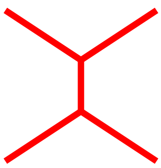
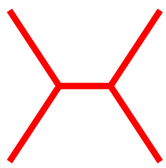
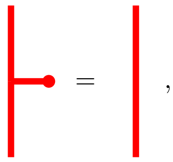
etc.

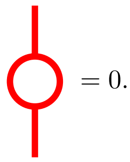
Polynomial relations: For each color (i.e. each generator $s \in S$) we have

(4.4)  $= \; a_s$,

(4.5) 

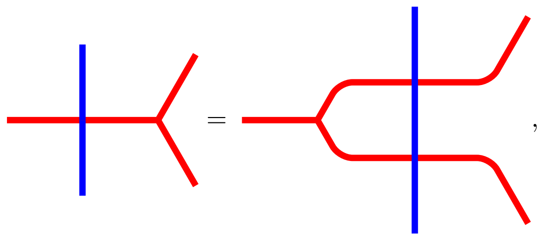
One-color relations: For each color we have

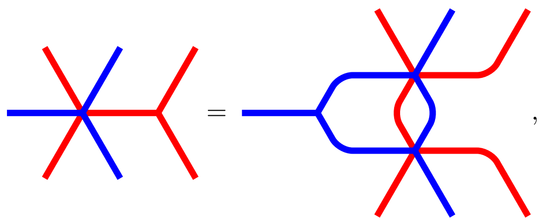
(4.6)  $=$  , (4.7) 

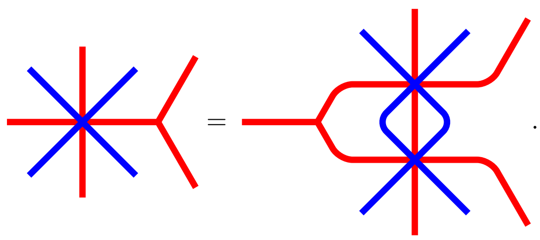
(4.8) 

Two-color relations: For every finite rank 2 parabolic subgroup of W (i.e. for each $s, t \in S$ such that $m_{st} < \infty$) there are two relations called two-color associativity and the Jones–Wenzl relation. In the diagrams below s is colored red and t is colored blue.

- Two-color associativity involves forks and braid vertices and does not depend on the realization, only on the order m_{st}. It has the following form for parabolics of Coxeter types $A_1 \times A_1$, A_2, and BC_2 (i.e. $m_{st} = 2, 3, 4$):

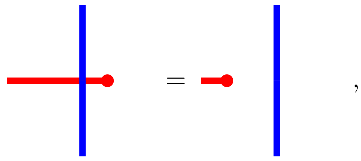
(4.9) 

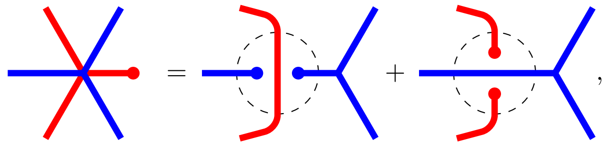
(4.10) 

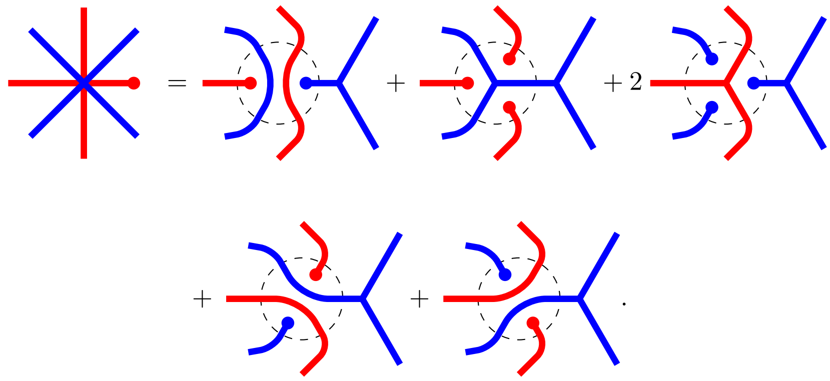
(4.11) 

- The Jones–Wenzl relation involves dots and braid vertices. Unlike two-color associativity it depends on the Cartan matrix of the realization. It has the following form for parabolics of Dynkin types $A_1 \times A_1$, A_2, and B_2 (for the last case, assume

$a_{st} = -2$ and $a_{ts} = -1$, i.e. a_t corresponds to the short root vector):

(4.12) 

(4.13) 

(4.14) 

For the general case see [26, Section 5.2] For each relation, the linear combination of diagrams within the circular region is called a Jones–Wenzl morphism. It is not technically a morphism of Bott–Samelson bimodules, as the diagrams are embedded inside the disk instead of the strip $\mathbb{R} \times [0, 1]$ but they can be embedded into a disk-shaped region inside an S-graph as in the relations.

Three-color relations: For each finite rank 3 parabolic subgroup of W there is a relation called the Zamolodchikov relation. We do not reproduce the diagrams here but instead point the reader to [26, (5.8)–(5.12)].

There are left and right R-actions on each Hom-space induced by multiplication of the leftmost or rightmost label in each diagram. Thus $\mathcal{D}_{\mathrm{BS}}$ has the structure of an R-linear category. As R-modules the Hom-spaces are graded by the degree of the S-diagrams. We will write $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}}(-,-)_\bullet$ for the set of all morphisms considered as a graded vector space, and $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}}(-,-)_0$ to denote the morphisms of degree 0 (homogeneous morphisms).

There is a duality functor $\overline{(\;)} : \mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS}}^{\mathrm{op}}$ on $\mathcal{D}_{\mathrm{BS}}$ defined as follows. For each $\underline{x} \in \underline{S}$ we have $\overline{B_{\underline{x}}} = B_{\underline{x}}$, $\overline{R(1)} = R(-1)$, and for any morphism $\phi : B \to B'$, the morphism $\overline{\phi} : \overline{B'} \to \overline{B}$ corresponds to flipping the diagrams representing ϕ upside-down.

Now we are ready to define the category $\mathcal{D}$ of Soergel bimodules using $\mathcal{D}_{\mathrm{BS}}$.

Definition 4.1.7. The category $\mathcal{D}$ is defined to be the Karoubi envelope of $\mathcal{D}_{\mathrm{BS}}$, that is to say, the completion of $\mathcal{D}_{\mathrm{BS}}$ with respect to all direct sums, all direct summands, and all grade shifts of objects in $\mathcal{D}_{\mathrm{BS}}$.

For an object B and an integer m, the m-degree grade shift of B is denoted $B(m)$. It has the property that

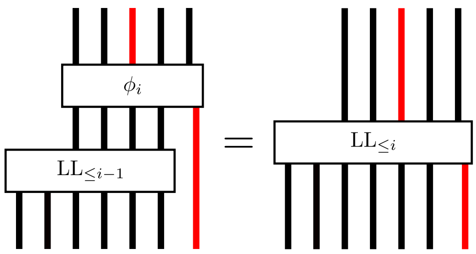
$$\mathrm{Hom}_{\mathcal{D}}(B(m), B')_n = \mathrm{Hom}_{\mathcal{D}}(B, B')_{n-m},$$

just like the grade shift of a module over a graded ring.

4.2. Fundamental results

4.2.1. Light leaves and double leaves. We briefly summarize the diagrammatic construction of light leaves bases for the Hom-spaces in $\mathcal{D}_{\mathrm{BS}}$, as described in [26, Section 6].

Let $\underline{x} = \underline{s_1 \cdots s_m} \in \underline{S}$. For each subsequence $\mathbf{e} \in [\underline{x}]$ we construct the light leaves morphism $\mathrm{LL}_{\mathbf{e},\underline{w}} : B_{\underline{x}} \to B_{\underline{w}}$, where $\underline{w} \in \underline{S}$ is a rex for e. The construction proceeds inductively in the following manner. Let $\underline{x}_{\leq i}$ and $\mathbf{e}_{\leq i}$ be the truncated forms of $\underline{x}$ and $\mathbf{e}$ respectively, and let $\underline{w}_{\leq i}$ be a rex for $e_{\leq i}$. For brevity write $\mathrm{LL}_{\leq i}$ for $\mathrm{LL}_{\mathbf{e}_{\leq i},\underline{w}_{\leq i}}$. We choose a map ϕ_i based on the decorated type of $\mathbf{e}_i$ and define $\mathrm{LL}_{\leq i} = \phi_i \circ (\mathrm{LL}_{\leq i-1} \otimes \mathrm{id}_{B_{s_i}})$:



There are four possibilities for ϕ_i, which are illustrated in Figure 4.1. In these pictures, boxes labeled "rex" correspond to rex moves. A rex move is a diagram between two rexes which does not factor through a non-rex. In other words, a rex move is a diagram whose only vertices are braids, without any "cups" or "caps". The different choices in this construction (e.g. of rexes for e and rex moves at each ϕ_i) give slightly different maps, so this construction is not unique, but this will not matter for our purposes. The degree of $\mathrm{LL}_{\mathbf{e},\underline{w}}$ is equal to the defect $d(\mathbf{e})$ and thus is independent of the choices made in the construction.

Suppose for each $w \in W$ we have chosen a corresponding rex $\underline{w}$. For $\underline{x} \in \underline{S}$, let $\mathrm{LL}_{[\underline{x}]}$ denote a complete set of light leaves maps $\{\mathrm{LL}_{\mathbf{e},\underline{w}}\}$ over all subsequences $\mathbf{e} \in [\underline{x}]$, where for each $\mathbf{e}$ the rex $\underline{w}$ is the fixed rex corresponding to e. In this way,

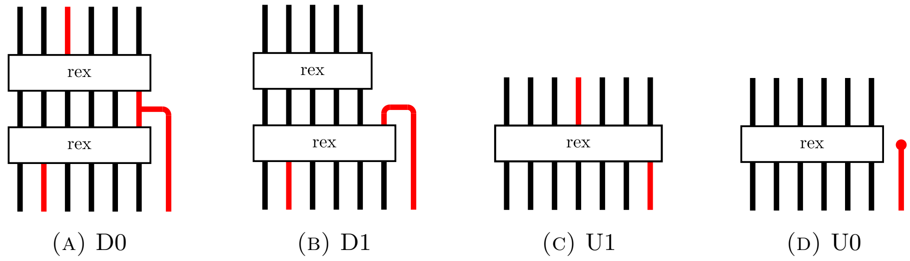


FIGURE 4.1. Four maps for constructing light leaves.

subsequences which evaluate to the same element of W give rise to light leaves maps with the same codomain. The following fundamental result is the most important step towards understanding $\mathcal{D}$.

THEOREM 4.2.1 ([26, Proposition 7.6]). *Let $\underline{x} \in \underline{S}$ and $w \in W$. Suppose $\underline{w}$ is the fixed rexes chosen above. Let*

$$\mathrm{Hom}_{\mathcal{D}^{\geq w}}(B_{\underline{x}}, B_{\underline{w}})_\bullet = \mathrm{Hom}_{\mathcal{D}}(B_{\underline{x}}, B_{\underline{w}})_\bullet / J = \mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}}(B_{\underline{x}}, B_{\underline{w}})_\bullet / J$$

denote space of homomorphisms $B_{\underline{x}} \to B_{\underline{w}}$, modulo the 2-sided ideal J of morphisms which factor through $B_{\underline{y}}$ for some rex $\underline{y}$ such that $y \not\geq w$. Then $\mathrm{LL}_{[\underline{x}]}$ forms a (left/right) graded R-basis for this quotient space, regardless of the realization of W and any choices made during the construction of the light leaves maps.

An extension of this theorem gives a basis for the Hom-spaces in $\mathcal{D}_{\mathrm{BS}}$. Suppose $\underline{x}, \underline{y} \in \underline{S}$. If we have subsequences $\mathbf{e} \in [\underline{x}]$ and $\mathbf{f} \in [\underline{y}]$ such that e and f are the same element $w \in W$, then the double leaves map is defined to be $\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}} = \overline{\mathrm{LL}_{\mathbf{f},\underline{w}}} \circ \mathrm{LL}_{\mathbf{e},\underline{w}}$ which is a morphism $B_{\underline{x}} \to B_{\underline{y}}$. We write $\mathbb{LL}_{[\underline{x}]}^{[\underline{y}]}$ to denote a complete selection of double leaves maps $B_{\underline{x}} \to B_{\underline{y}}$ over all such pairs of subsequences.

THEOREM 4.2.2 ([26, Theorem 6.12]). *Let $\underline{x}, \underline{y} \in \underline{S}$. The set $\mathbb{LL}_{[\underline{x}]}^{[\underline{y}]}$ is a graded R-basis for*

$$\mathrm{Hom}^\bullet_{\mathcal{D}_{\mathrm{BS}}}(B_{\underline{x}}, B_{\underline{y}}) = \mathrm{Hom}^\bullet_{\mathcal{D}}(B_{\underline{x}}, B_{\underline{y}})$$

regardless of the realization.

Finally, from these bases the indecomposables in $\mathcal{D}$ can be classified.

THEOREM 4.2.3 ([26, Theorem 6.26]). *Suppose $w \in W$, and let $\underline{w}$ be a rex for w. There is a unique indecomposable summand B_w of $B_{\underline{w}}$ which is not a summand of $B_{\underline{y}}$ for $\underline{y}$ a rex with $y < w$. Up to isomorphism, the object B_w does not depend on the choice of rex for w. Each indecomposable in $\mathcal{D}$ is isomorphic to a shift of B_w for some $w \in W$.*

As with module categories, for B an object in $\mathcal{D}$ we write $[B]$ to denote the isomorphism class of B, and $[\mathcal{D}]$ for the $\mathbb{L}_{\geq 0}$-algebra of all isomorphism classes of objects. We denote the Grothendieck ring of $[\mathcal{D}]$ by $[[\mathcal{D}]]$, and we write $[[B]]$ for the image of $[B]$ inside $[[\mathcal{D}]]$. We will sometimes abuse notation and say "the (split)

Grothendieck ring of $\mathcal{D}$" following the usual convention for module categories. The quotient Hom-space in Theorem 4.2.1 gives rise to the following homomorphism on $[\mathcal{D}_{\mathrm{BS}}]$.

DEFINITION 4.2.4. The character homomorphism is the unique $\mathbb{L}_{\geq 0}$-algebra homomorphism defined by

$$\begin{aligned} \mathrm{ch} : [\mathcal{D}_{\mathrm{BS}}] &\longrightarrow \mathcal{H} \\ [B_{\underline{x}}] &\longmapsto [\underline{x}] \end{aligned}$$

To check well-definedness, compose with the map $\mathcal{H} \to \mathbb{H}$ to get

$$\begin{aligned} [\mathcal{D}_{\mathrm{BS}}] &\longrightarrow \mathcal{H} \longrightarrow \mathbb{H} \\ [B_{\underline{x}}] &\longmapsto \sum_{w \in W} \dim_{\bullet} \mathrm{Hom}_{\mathcal{D}^{\geq w}}(B_{\underline{x}, B_{\underline{w}}})_{\bullet} H_w \end{aligned}$$

where $\dim_{\bullet}$ denotes the graded dimension of this quotient Hom-space as a graded left/right R-module. Clearly the sum only depends on the isomorphism class of $B_{\underline{x}}$, so the character set map is indeed well defined. Moreover, the converse holds as well by Theorem 4.2.3; that is to say two objects give the same character set only if they are isomorphic. Yet ch is surjective since $[B_{\underline{x}}]$ maps onto the generators $[\underline{x}]$. Thus we have shown

PROPOSITION 4.2.5. *The map* ch *is an isomorphism of* $\mathbb{L}_{\geq 0}$*-algebras.*

As an easy consequence we get

COROLLARY 4.2.6 ([26, Corollary 6.27]). *The Grothendieck ring of* $\mathcal{D}$ *is*

$$[[\mathcal{D}]] = [[\mathcal{D}_{\mathrm{BS}}]] \cong [\mathcal{H}] \cong \mathbb{H}.$$

Thus $\mathcal{D}$ is a categorification of $\mathbb{H}$.

4.2.2. Localization and mixed diagrams. Let Q be the fraction field of R. We denote the localization of $\mathcal{D}$ by $Q \otimes_R \mathcal{D}$. In $Q \otimes_R \mathcal{D}$ diagrams are allowed to have a rational function $f \in Q$ as a left coefficient (since we can "push" polynomials through strings, we can also consider right coefficients as well). Although $\mathcal{D}$ is idempotent complete, the localization $Q \otimes_R \mathcal{D}$ is not. To remedy this we add new indecomposable objects to $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$.

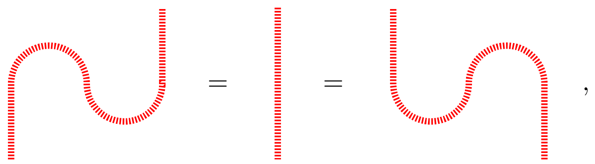
DEFINITION 4.2.7 ([26, Section 5.4]). The diagrammatic Bott–Samelson-standard category $\mathcal{D}_{\mathrm{BS,std}}$, or the mixed category for short, is the following Q-linear monoidal extension of $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$.

Objects: For each $\underline{x} \in \underline{S}$ add the object $Q_{\underline{x}}$, which is called a standard bimodule. As with Bott–Samelson bimodules the tensor product is defined by concatenation, i.e. $Q_{\underline{x}} \otimes Q_{\underline{y}} = Q_{\underline{xy}}$.

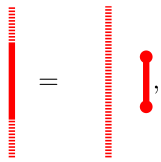
Morphisms: As in $\mathcal{D}_{\mathrm{BS}}$ the Hom-spaces are spanned by diagrams with some fixed bottom and top boundary. Here the diagrams are mixed diagrams, where some of the edges are dashed. Dashed edges on the top or bottom boundaries denote standard bimodules in the domain or codomain. There are two new morphisms between standard bimodules and Bott–Samelson bimodules, which are both of degree $+1$. These are drawn diagrammatically as bivalent vertices:

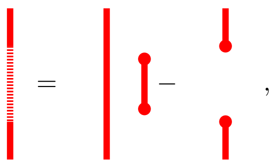
(4.15)


Relations: In addition to isotopy of dashed edges, i.e.

(4.16)


add the following relations involving the bivalent vertices:

(4.17)


(4.18)


(4.19)


REMARK 4.2.8.

(i) Note that (4.19) implies that the bivalent vertex is *not* cyclic! In other words, we can no longer “twist” and pull apart strings in mixed diagrams at will; special care must be taken with bivalent vertices. Thankfully the failure of isotopy is only up to a sign change. In particular our sign convention differs from that in [26] by a sign. This is to ensure that the menorah vertex in Chapter 5 is semi-cyclic.

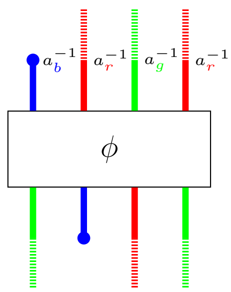
(ii) The mixed category can also be defined as over R, using the same diagrammatic generators and slightly modified relations.

As Q is not graded in an especially useful way, we will ignore the inherited grading on $\mathcal{D}_{\mathrm{BS,std}}$ most of the time. From (4.17) and (4.18) we see that the bivalent vertices are idempotent projectors (up to rescaling), with a complementary idempotent (up to rescaling) given by the "two dots" morphism (the second term on the right-hand side of (4.18)). Thus the Bott–Samelson bimodule B_s decomposes a direct sum $Q_s \oplus Q$. This means that every Bott–Samelson bimodule is isomorphic to a direct sum of standard bimodules, so it suffices to understand morphisms between standard bimodules.

For $s \in S$, the dashed "cap" morphism $Q_{\underline{ss}} \to Q$ between standard bimodules and the analogous "cup" morphism are isomorphisms in $\mathcal{D}_{\mathrm{BS,std}}$ (this is apparent from the first picture in the first part of the proof of [26, Proposition 5.23]). By combining a braid vertex with several bivalent vertices and rescaling, one can construct a dashed version of the braid vertex which is an isomorphism between standard bimodules [26, (5.27)]. Thus if $\underline{x}, \underline{y} \in \underline{S}$ such that $x = y$, then $Q_{\underline{x}} \cong Q_{\underline{y}}$, so we can label standard bimodules Q_x by elements $x \in W$ instead of expressions. In fact it can be shown that $\mathrm{End}_{\mathcal{D}_{\mathrm{BS,std}}}(Q_x) = Q$ for all $x \in W$, so such an isomorphism is unique up to scalars; yet if $x \neq y$ then $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS,std}}}(Q_x, Q_y) = 0$ [26, Proposition 5.23]. According to Elias and Williamson the full subcategory of standard bimodules is called "the 2-groupoid of W over Q". Since the standard bimodules are obviously indecomposable, this gives an alternative construction of $\mathcal{D}_{\mathrm{BS,std}}$.

Theorem 4.2.9 ([26, Theorem 5.17]). *The category $\mathcal{D}_{\mathrm{BS,std}}$ is the Karoubi envelope of $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$.*

For $\underline{x} \in \underline{S}$ we can show by induction on $\ell(\underline{x})$ that $B_{\underline{x}} \cong \bigoplus_{\mathbf{e} \in [\underline{x}]} Q_e$. Thus any morphism $\phi : B_{\underline{x}} \to B_{\underline{y}}$ in $\mathcal{D}_{\mathrm{BS}}$ can be decomposed into a matrix of morphisms between standard bimodules. This matrix is called the localization of ϕ. If $\mathbf{e} \in [\underline{x}]$ and $\mathbf{f} \in [\underline{y}]$, the $(\mathbf{f}, \mathbf{e})$-term in this matrix is determined by adding certain vertices to boundary strings in the diagrams representing ϕ in the following manner. For each index of type 0 we add a dot to the corresponding boundary string. For each index of type 1 we add a bivalent vertex. Finally for generator s in the codomain, we put a factor of a_s^{-1} to the right of the dot or the bivalent vertex.



For a left-biased version of this process, see the picture in [26, Section 5.5]. Note that the placement of the scalar factor near the codomain is an arbitrary convention, but one followed from [26].

The localization of a diagram is well-defined, not just up to sign, since isotopy of solid strings is still a relation in $\mathcal{D}_{\mathrm{BS,std}}$. More importantly, localization is faithful, i.e. two morphisms in $\mathcal{D}_{\mathrm{BS}}$ are equal if and only if they have the same localization. Sometimes it will be useful to localize only some of the boundary strings corresponding to certain indices in the domain/codomain expressions. We say that an index or boundary string is standardized if it has been localized using one of the bivalent vertices.

From the double leaves basis, the Hom-spaces of $\mathcal{D}_{\mathrm{BS}}$ and $\mathcal{D}$ are free R-modules. As a consequence, the natural mapping $\mathcal{D}_{\mathrm{BS}} \to Q \otimes_R \mathcal{D}_{\mathrm{BS}}$ is faithful (see Remark 1.4, Section 3.6, and Section 5.5 of [26]). Since $\mathcal{D}_{\mathrm{BS,std}}$ is the Karoubi envelope of $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$, this means that the composition $\mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS,std}}$ is also faithful. Yet localization of morphisms is just another way of writing morphisms in $\mathcal{D}_{\mathrm{BS}}$ in terms of decompositions in $\mathcal{D}_{\mathrm{BS,std}}$, so we have shown the following.

Corollary 4.2.10. *Localization of morphisms in the above sense is faithful.*

Since localization distinguishes between the domain and the codomain, the localization of the dual $\overline{\phi}$ of ϕ is not the same as the dual-transpose of the localization of ϕ. To remedy this, we introduce the following notation.

Notation 4.2.11. Let $f, g \in Q$. We write $\binom{f}{g}$ in a region of an S-graph to indicate a polynomial term which changes depending on whether the S-graph is right-side-up (f) or upside-down (g). Its usage is similar to a $\pm$ sign, which can be used to denote two solutions of an equation at once. By definition $\overline{\binom{f}{g}} = \binom{g}{f}$, and $\binom{f}{g}$ is considered to have degree equal to the average of the degrees of f and g.

This notation is useful for depicting in a single diagram how a morphism and its dual localize. In particular, we can rescale dots or bivalent morphisms with downwards pointing solid strings by $\binom{a_s^{-1}}{1}$. These projectors combined with their duals give the idempotents described above. In this language, localization is just projection to these summands via these projectors.

CHAPTER 5

A linkage principle for Soergel bimodules

In this chapter we will build the machinery of linkage for Soergel bimodules. All notational conventions from the previous chapter continue here. Suppose $\Bbbk$ is a field of characteristic $p > 2$ which does not divide the index of connection of Φ (see e.g. [31, 4.9, Table 1] for a table of these values). Let $\mathcal{D}$ be the category of Soergel bimodules over the universal realization $V_{\Sigma,-\tilde{\alpha}}$ of the affine Weyl group W over $\Bbbk$. The main result in this chapter is the construction of the linkage functor, whose fundamental properties are summarized in Theorem 5.4.3. Very briefly, the linkage functor shows how Soergel bimodules can be understood in terms of Soergel bimodules which are "smaller" by a factor of p.

To be more precise, the linkage functor is a monoidal functor

$$\mathrm{pr} : \mathcal{D}^{\mathrm{ungr}} \longrightarrow \mathrm{End}_{\hat{R}\otimes\mathcal{D}^{\mathrm{ungr},F}}\left(\left(\hat{R}\otimes\mathcal{D}^{\mathrm{ungr},F}\right)^{|{}^pW|}\right)$$

from the category of ungraded Soergel bimodules $\mathcal{D}^{\mathrm{ungr}}$ into a category of endofunctors of the direct sum of $|{}^pW|$ copies of $\hat{R}\otimes\mathcal{D}^{\mathrm{ungr},F}$. Here $\mathcal{D}^{\mathrm{ungr},F}$ is the category of ungraded Soergel bimodules over a twisted realization of W (see Proposition 5.2.4), and $\hat{R}$ is a localization of R (see Section 5.2.2). Roughly speaking, the functor pr maps a Soergel bimodule B_y to a matrix of smaller Soergel bimodules. The entries of this matrix are smaller by a factor of p in the following sense: for each summand B_x of an entry of the matrix, the vector lengths of $x(0)$ and $y(0)$ satisfy the approximate inequality $p|x(0)| \lessapprox |y(0)|$. Informally we say that ordinary Soergel bimodules like B_y are at "scale 1", while those coming from entries of the matrix like B_x are at "scale p", since the natural way to directly compare x and y is to reinterpret x as acting via the p-affine action, in which translations are scaled upwards by a factor of p.

In addition to an explicit construction of the linkage functor for Bott–Samelson bimodules, we also develop the algebra and combinatorics of the linkage functor at the decategorified level in Section 5.1. Decategorification of linkage provides new lower bounds for the p-canonical basis (see Example 5.4.12) using Grothendieck rings and bimodules of categories introduced in this chapter. In particular, we define the algebra $\mathbb{H}_*$ and an $\mathbb{H}$-$\mathbb{H}_*$ bimodule $\mathbb{H}_{p|*}$ in terms of generators and relations and prove that these correspond to the Grothendieck ring and bimodule for the categories $\mathcal{D}^{\langle - \rangle}$ and $\mathcal{D}_{p|*}$ respectively. As far as the author is aware these algebraic structures have not appeared before in the literature. We also reformulate the combinatorics of expressions and subsequences for $\mathbb{H}$ into combinatorial sequences we call patterns and matches, which play a similar role for $\mathbb{H}_*$ and $\mathbb{H}_{p|*}$.

Motivation. We will say a little more about the connections between Soergel bimodules and the modular representation theory of a reductive group G. Recall from the previous chapter that the basis $\{[[B_x]]\}$ of the split Grothendieck ring $[[\mathcal{D}]]$ corresponds to a basis of $\mathbb{H}$. If the corresponding basis coincides with the Kazhdan–Lusztig basis $\{\underline{H}_x\}$, then we say that Soergel's conjecture holds for the underlying realization. Elias and Williamson proved Soergel's conjecture for a wide class of $\mathbb{R}$-realizations [25]. Otherwise, for realizations over a field $\Bbbk$ characteristic $p > 0$, it can be shown that the corresponding basis $\{{}^p\underline{H}_x\}$ for $\mathbb{H}$ only depends on p and the Cartan matrix of the realization. This basis and the similarly constructed basis coming from the category $\mathcal{D}_{\mathrm{f}}$ built from realizations of the finite Weyl group W_{f} appear to play an enormous role in several areas of modular representation theory. For this reason these bases are called p-canonical bases [55]. For example, Soergel showed that a result related to LCF, sometimes called "LCF around the Steinberg weight", is equivalent to showing that Soergel's conjecture holds for $\mathcal{D}_{\mathrm{f}}$ [53]. Williamson generated counterexamples to Lusztig's conjectured lower bounds on p for LCF by finding instances where ${}^p\underline{H}_x \neq \underline{H}_x$ for $x \in W_{\mathrm{f}}$ [56]. Here Soergel's conjecture over $\mathbb{R}$-realizations corresponds to the fact that LCF does hold, but only for p extremely large.

Other connections between Soergel bimodules and modular representation theory of G use tilting modules. The geometric Satake equivalence establishes a correspondence between perverse sheaves on the Langlands dual affine Grassmannian and representations of G [47]. In this setting, when the characteristic is larger than some small bound then questions about perverse sheaves can be reformulated in terms of Soergel bimodules [34]. As a result the character of $T(\lambda)$ can be calculated directly from ${}^p\underline{H}_{w_\lambda}$, a p-canonical basis element for the affine root system, for some w_λ with $|w_\lambda(0)| \approx |\lambda|$. By Soergel's conjecture, for fixed λ there must be some bound on p above which ${}^p\underline{H}_{w_\lambda} = \underline{H}_{w_\lambda}$. When this happens, we have $T(\lambda) \cong \Delta(\lambda) \cong L(\lambda)$, which is obvious since for p sufficiently large λ lies in the fundamental p-alcove.

Much more interesting is the newer conjecture of Riche and Williamson [48], now a theorem for all types when $p > h$ [1]. It establishes an equivalence between the full subcategory of tilting modules for the principal block and a quotient of $\mathcal{D}$ called the anti-spherical category. Under this equivalence, the character of a principal block tilting module $T(\lambda)$ depends on ${}^p\underline{H}_x$, where $x \in W$ such that $\lambda = x \cdot_p 0$. Recall that the p-scaled dot action scales up translations by a factor of p, so in terms of the unscaled action this means that $p|x(0)| \approx \lambda$. Unlike the geometric Satake equivalence, the Riche–Williamson correspondence automatically takes the linkage principle into account, giving the tilting character for a weight inside a fixed p-alcove just like LCF. When p is very large, Soergel's conjecture implies that the character of $T(\lambda) = T(x \cdot_p 0)$ is the same as that of the quantum tilting module $T_\ell(\lambda)$ for $\ell = p$. Yet when p is very large, λ lies in the fundamental p^2-alcove. In other words, we have shown that the indecomposable tilting characters for G in the fundamental p^2-alcove coincide with their quantum counterparts for large p, which was originally conjectured by Andersen (see Introduction).

A surprising feature of the two correspondences above is that they work at different scales! In other words, if $\lambda = x \cdot_p 0$ the character of $T(\lambda)$ can be derived from either ${}^p\underline{H}_{w_\lambda}$ (using geometric Satake) or ${}^p\underline{H}_x$ (using Riche–Williamson), but in terms of vector lengths $p|x(0)| \approx |w_\lambda(0)|$ due to the different actions involved! This leads to self-similarity properties for both tilting modules and Soergel bimodules at scales equal to powers of p. This has already been observed in [55] when $G = \mathrm{SL}_2$ and W is of type $\tilde{A}_1$, where we know all the indecomposable tilting characters using Donkin's tilting tensor product theorem. Linkage for Soergel bimodules is a more precise way of describing this self-similarity in a categorical manner.

Higher-order linkage. Under the Riche–Williamson correspondence the linkage functor has a well-known analogue in the world of tilting modules which we call *higher-order linkage*[1] (e.g. [33, Proposition 4.1(ii)] or [6, 4.2]). Let T be a tilting module for G with character

$$[T]_/ = \sum_i a_i[\Delta(\lambda_i)]_/$$

for some $a_i \in \mathbb{Z}_{\geq 0}$. Higher-order linkage is the fact that for any positive integer r, the formal character

$$\sum_i a_i[\Delta_{p^r}(\lambda_i)]_/$$

for the corresponding quantum group U_{p^r} at a p^rth root of unity is the character of a tilting module for U_{p^r}. We will rewrite this in a more combinatorial form.

The quantum group U_{p^r} has a linkage principle governed by a p^r-scaled dot action, and there is a similar translation principle as well on the level of p^r-alcoves. For any U_{p^r}-tilting module T_{p^r}, we can write the character of T_{p^r} in terms of p^r-linkage, i.e.

$$[T_{p^r}]_/ = \sum_i a_i[\Delta_{p^r}(y_i \cdot_{p^r} \mu_i)]_/$$

for μ_i a dominant regular weight in the fundamental p^r-alcove, and $y_i \in W$. Now for each i let ν_i be a dominant regular weight in the fundamental p-alcove and $w_i \in W$ such that $\mu_i = w_i \cdot_p \nu_i$. Then in terms of the p-scaled dot action the character of T_{p^r} is

$$[T_{p^r}]_/ = \sum_i a_i[\Delta_{p^r}(F^{r-1}(y_i)w_i \cdot_p \nu_i)]_/,$$

where $F : W \to W$ is the Frobenius homomorphism, which has the property that $F^{r-1}(x) \cdot_p \lambda = x \cdot_{p^r} \lambda$. Note that the action of w_i on ν_i corresponds to translating ν_i within a p^r-alcove, which is exactly what the p^r-translation functors do.

[1]We use the term "linkage" here loosely, in the sense of a relationship between the orbits of a Coxeter group action and characters or blocks of G.

For notational convenience fix $r = 2$. Suppose T_{p^2} lies in a single p-linkage component, or in other words that there exists ν such that for all i, $\nu_i = \nu$. As T_{p^2} is the direct sum of indecomposable tilting modules, its character must lie in

$$[T_{p^2}]_/ \in \sum_{x \in W} \mathbb{Z}_{\geq 0}[T_{p^2}(x \cdot_p \nu)]_/ = \sum_{\substack{y \in W_p \\ w \in {}^pW}} \mathbb{Z}_{\geq 0}[T_{p^2}(yw \cdot_p \nu)]_/,$$

where $W_p = \operatorname{im} F$ and $w \in {}^pW$ if and only if $w \cdot_p \nu$ is in the fundamental p^2-alcove. Indecomposable tilting characters for U_{p^2} are given by the following p^2-version of SCF:

$$[T_{p^2}(yw \cdot_p \nu)] = \sum_{z \in W_p} n_{F^{-1}(z),F^{-1}(y)}(1)[\Delta_{p^2}(zw \cdot_p \nu)]_/.$$

Note how w doesn't affect the character directly, since all it does is translate the character within a p^2-alcove. Putting this all together, if T is a tilting module for G in the p-linkage component ν, higher-order linkage is equivalent to the following character-theoretic statement:

$$[T]_/ \in \sum_{\substack{y \in W_p \\ w \in {}^pW}} \mathbb{Z}_{\geq 0} \left(\sum_{z \in W_p} n_{F^{-1}(z),F^{-1}(y)}(1)[\Delta(zw \cdot_p \nu)]_/ \right). \tag{5.1}$$

This result combined with the corresponding simpler statement for $r = 1$, gives a non-trivial lower bound on the character of T. In [33] Jensen used this lower bound as part of a strategy for calculating several indecomposable tilting characters of SL_3 beyond the fundamental p^2-alcove.

The linkage functor extends these ideas to Soergel bimodules. For a Soergel bimodule B, the first row of the matrix $\mathrm{pr}(B)$ is analogous to the decomposition of $[T]_/$ into U_{p^2}-characters, while Theorem 5.4.11 corresponds to (5.1) above. Since $[[\mathcal{D}]] \cong \mathbb{H}$ our result works at the level of the whole Hecke algebra, not just the anti-spherical quotient. In addition the linkage functor provides concrete information about what happens to morphisms with respect to the scale p decomposition. For tilting modules it is not obvious that morphisms between tilting modules for G lift uniquely to morphisms between tilting modules for U_{p^r}, let alone that this lifting is functorial. For this reason we say that the linkage functor is a categorification of higher-order linkage. We hope that linkage for Soergel bimodules will provide a basis for better understanding the higher-order behavior of both Soergel bimodules over realizations of affine Weyl groups and tilting modules for algebraic groups.

5.1. Linkage algebra and patterns

5.1.1. p-affine Weyl groups. The p-affine Weyl group $W_p \leq W$ is the subgroup generated by the reflections $s_{\alpha,pk}$ for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$. We define the Frobenius map on W to be

$$F : W \longrightarrow W$$

$$s_{\alpha,k} \longmapsto s_{\alpha,pk}$$

for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$. The Frobenius map is well-defined because it is just conjugation by the scaling map $\lambda \mapsto p\lambda$. As F is injective it induces an isomorphism $W \xrightarrow{\sim} W_p$, so we can transfer the constructions in Section 1.1 to W_p. Thus as a reflection group $W_p \cong W_{\mathrm{f}} \ltimes p\mathbb{Z}\Phi$, we have a set $\mathcal{A}_p$ of p-alcoves[2] and a fixed fundamental p-alcove $A_{0,p}$, and W_p is a Coxeter group with Coxeter generating set $S_p = S_{\mathrm{f}} \cup \{\tilde{s}_p\}$ which are reflections in the walls of $\overline{A}_{0,p}$. In particular the isomorphism $W \xrightarrow{\sim} W_p$ induced by F is an isomorphism of Coxeter groups, with $F(s) = s$ for all $s \in S_{\mathrm{f}}$ and $F(\tilde{s}) = \tilde{s}_p$. As $W_p \cong W$ as a Coxeter group, the Hecke algebra $\mathbb{H}_p$ of the p-affine Weyl group is isomorphic to $\mathbb{H}$ via an extension of F.

Let ${}^p\mathcal{A}$ denote the set of ordinary alcoves contained inside $A_{0,p}$. The bijection $W \xrightarrow{\sim} \mathcal{A}$ restricts to a bijection ${}^pW \xrightarrow{\sim} {}^p\mathcal{A}$, where ${}^pW$ is the set of minimal length representatives for the right cosets $W_p \backslash W$. This bijection induces a right action of W on ${}^p\mathcal{A}$.

Let ${}^p\mathcal{W}$ be the powerset of ${}^pW$. Then ${}^p\mathcal{W}$ is a set algebra (i.e. a collection of subsets of some universal set closed under finite unions, intersections, and complements) with a compatible right W-action. For each $s \in S$ define the subset

$$s(*) = \{w \in {}^pW : W_p ws = W_p w\}$$

of ${}^pW$. Geometrically, $s(*)$ corresponds to the subset of alcoves in ${}^p\mathcal{A}$ whose s-wall lies on one of the walls of the fundamental p-alcove $A_{0,p}$. Let ${}^p\mathcal{W}(*)$ be the smallest set subalgebra containing $\{s(*)\}_{s\in S}$ which is closed under the action of W.

5.1.2. Linkage Hecke algebras and linkage bimodules.

Definition 5.1.1. The p-linkage Hecke algebra $\mathbb{H}_*$ is the $\mathbb{L}$-algebra with generators

$$u_A \qquad \text{for each } A \in {}^p\mathcal{W}(*),$$
$$H_s \qquad \text{for each } s \in S,$$

and relations

$$u_\emptyset = 1, \qquad u_{{}^pW} = v, \tag{5.2}$$
$$u_A^2 = (v+1)u_A - v \qquad \text{for all } A \in {}^p\mathcal{W}(*), \tag{5.3}$$
$$u_A + u_B = u_{A\cup B} + u_{A\cap B} \qquad \text{for all } A, B \in {}^p\mathcal{W}(*), \tag{5.4}$$
$$u_A u_B = u_{A\cup B} u_{A\cap B} \qquad \text{for all } A, B \in {}^p\mathcal{W}(*), \tag{5.5}$$
$$H_s^2 = 1 + (u_{s(*)}^{-1} - u_{s(*)})H_s \qquad \text{for all } s \in S, \tag{5.6}$$
$$\overbrace{H_sH_tH_s\cdots}^{m_{st}\text{ terms}} = \overbrace{H_tH_sH_t\cdots}^{m_{st}\text{ terms}} \qquad \text{for all } s,t \in S, \tag{5.7}$$
$$H_s u_A = u_{As} H_s \qquad \text{for all } s \in S \text{ and } A \in {}^p\mathcal{W}(*). \tag{5.8}$$

[2]Note that the "p-alcoves" in what follows are *unshifted*, unlike the p-alcoves described in Chapter 1 which are shifted by $-\rho$.

DEFINITION 5.1.2. The p-linkage bimodule $\mathbb{H}_{p|*}$ is the $(\mathbb{H}_p, \mathbb{H}_*)$-bimodule described as follows. As an $\mathbb{L}$-module it has basis

$$H_x H_w \qquad\qquad \text{where } x \in W_p \text{ and } w \in {}^pW.$$

In this basis, the $\mathbb{H}_p$-action is given by $H_s(H_x H_w) = (H_s H_x)H_w$ for all $s \in S_p$, while the $\mathbb{H}_*$-action is given by

$$(H_x H_w)H_s = \begin{cases} (H_x H_{wsw^{-1}})H_w & \text{if } W_p ws = W_p w, \\ H_x H_{ws} & \text{otherwise,} \end{cases} \qquad \text{for all } s \in S, \tag{5.9}$$

$$(H_x H_w)u_A = \begin{cases} vH_x H_w & \text{if } w \in A, \\ H_x H_w & \text{otherwise,} \end{cases} \qquad \text{for all } A \in {}^p\mathcal{W}(*). \tag{5.10}$$

Note that the condition that $W_p ws = W_p w$ is equivalent to $wsw^{-1} \in W_p$.

In later sections when we define $\mathbb{L}_{\geq 0}$-variants of these structures, it will be more convenient to describe $\mathbb{H}_{p|*}$ first as a left $\mathbb{H}_p$-module and then define $\mathbb{H}_*$ as an algebra of $\mathbb{H}_p$-module endomorphisms.

LEMMA 5.1.3. *The right $\mathbb{H}_*$-action on $\mathbb{H}_{p|*}$ is faithful.*

PROOF. Suppose $a \in \mathbb{H}_*$ such that for all $m \in \mathbb{H}_{p|*}$, we have $m \cdot a = 0$. From the relations defining $\mathbb{H}_*$, the set

$$\{u_A H_x : x \in W,\ A \in {}^p\mathcal{W}(*)\}$$

is an $\mathbb{L}$-spanning set for $\mathbb{H}_*$, where $H_x \in \mathbb{H}_*$ is defined in exactly the same way as the corresponding element in $\mathbb{H}$. Now write

$$a = \sum_{i=1}^{n} p_i H_{x_i}$$

where p_i is a non-zero $\mathbb{L}$-linear combination of the u-elements and the x_i are distinct. The action of H_x on the elements $\{H_w : w \in {}^pW\}$ of $\mathbb{H}_{p|*}$ is $H_w \cdot H_x = H_y H_z$ where $y \in W_p$ and $z \in {}^pW$ such that $wx = yz$. Thus we have

$$H_w \cdot a = \sum_{i=1}^{n} p_i(w) H_{y_i} H_{z_i}$$

where $p_i(w) = H_w^{-1}(H_w \cdot p_i) \in \mathbb{L}$, $y_i \in W_p$, and $z_i \in {}^pW$ such that $wx_i = y_i z_i$. But since the elements $H_{y_i}H_{z_i}$ are linearly independent, this means that we must have $p_i(w) = 0$ for each $w \in {}^pW$. In other words, it suffices to show that the $\mathbb{L}$-subalgebra $U(*)$ generated by the u-elements acts faithfully on $\mathbb{H}_{p|*}$.

Now note that the relations defining $\mathbb{H}_*$ ensure that $U(*)$ is isomorphic to a subring of $\mathbb{L}^{{}^pW}$, the algebra of $\mathbb{L}$-valued functions on ${}^pW$. Moreover under this correspondence the action of $U(*)$ on $\mathbb{H}_{p|*}$ is a linearized version of the evaluation action on functions. Since this is clearly faithful, the result follows. $\square$

5.1.3. Patterns. Let $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$ be an expression.

Definition 5.1.4. A pattern for $\underline{x}$ is a sequence $\underline{r} = (\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$ where each $\underline{r}_i$ is an ordered pair (s_i, t_i) with each $t_i \in \{0, 1, *\}$.

The new symbol $*$ is used to denote indices whose type is indeterminate (i.e. not yet fixed as either 0 or 1). We call an index i or the generator at that index indeterminate if $t_i = *$; otherwise we call it fixed. Patterns can be viewed as generalized expressions, where fixed generators are already included or discarded to begin with. In particular, an expression is a pattern whose generators are all indeterminate. We write $\hat{r}$ for the product of all the generators in $\underline{r}$ with type 1.

Definition 5.1.5. Let $\underline{r}$ be a pattern. A matching subsequence or match for the pattern $\underline{r}$ is a sequence $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m)$, where each term $\mathbf{c}_i = (\underline{r}_i, t'_i)$ consists of the pattern term $\underline{r}_i$ and a choice of decoration $t'_i \in \{0, 1\}$ for the indeterminate indices. We conventionally attach the decoration $\emptyset$ to all fixed indices.

The match type of a match $\mathbf{c}$ is the sequence of decorations defining the match. If $\underline{r}$ is a pattern for an expression $\underline{x}$ and $\mathbf{c}$ is a match for $\underline{r}$, then $\mathbf{c}$ corresponds to a subsequence for $\underline{x}$ in an obvious way. We write $\hat{c}$ for the group element $c\hat{r}^{-1}$, where $c \in W$ is obtained by viewing $\mathbf{c}$ as a subsequence. We write $[\underline{r}]$ to denote the set of matching subsequences for $\underline{r}$.

Definition 5.1.6. Let $\mathbf{c}$ be a match for some pattern $\underline{r}$. The Bruhat stroll on the match $\mathbf{c}$ is defined as follows. Let $\underline{r}_{\leq i}$ denote the pattern made up of the first i terms of $\underline{r}$ and let $\mathbf{c}_{\leq i}$ be the match of $\underline{r}_{\leq i}$ made up of the first i terms of $\mathbf{c}$. Now set $\hat{w}_i = \hat{c}_{\leq i}$. For each indeterminate index i we add a decoration U or D to the matching type according to whether $\hat{w}_{i-1}(\hat{r}_{\leq i} s_i \hat{r}_{\leq i}^{-1}) > \hat{w}_{i-1}$ or $\hat{w}_{i-1}(\hat{r}_{\leq i} s_i \hat{r}_{\leq i}^{-1}) < \hat{w}_{i-1}$. For each fixed index, we conventionally add the decoration $\emptyset$. The match defect $\hat{d}(\mathbf{c})$ of $\mathbf{c}$ is equal to the number of terms with decorations U0 minus the number of terms with decorations D0.

Note that the collection of all patterns (resp. matches) has a monoid structure through the concatenation product just like the collection of all expressions (resp. subsequences). For $\underline{r}$ a pattern for some expression in $\underline{S}$ and $w \in W$, let $[\underline{rw}]$ denote the match set corresponding to the pattern product $\underline{rq}$, where $\underline{q}$ is a pattern for a rex $\underline{x}$ for w with all types equal to 1 (this is not quite uniquely defined, but it will be good enough for our purposes later).

Example 5.1.7. Suppose W is of Coxeter type $\tilde{A}_2$, with generators labelled 0, 1, and 2. Let $\underline{x} = \underline{101202122} \in S$. Let $\underline{r}$ be a pattern for $\underline{w}$ of type $*1111****$, and let $\mathbf{c}$ be a match for $\underline{r}$ of type $1\emptyset\emptyset\emptyset\emptyset0110$. Using the Tiberian convention we write this match as

$$\begin{array}{c}\scriptstyle 1\emptyset\emptyset\emptyset\emptyset0110 \\ *1111**** \\ \scriptstyle 101202122\end{array}$$

By replacing the pattern type of the indefinite terms with the match type, we have room to add the decorations coming from the Bruhat stroll:

$$\begin{array}{c}\scriptstyle 1\emptyset\emptyset\emptyset\emptyset0110 \\ *1111**** \\ \scriptstyle 101202122\end{array} \longrightarrow \begin{array}{c}\scriptstyle \emptyset\emptyset\emptyset\emptyset \\ 111110110 \\ \scriptstyle 101202122\end{array} \longrightarrow \begin{array}{c}\scriptstyle \mathrm{U}\emptyset\emptyset\emptyset\emptyset\mathrm{DUUD} \\ 111110110 \\ \scriptstyle 101202122\end{array}$$

Thus the match defect $\hat{d}(\mathbf{c})$ is $0 - 2 = -2$.

5.1.4. Linkage sets. Let $\underline{S}_{p|1} = \underline{S}_p|\underline{S}$ denote the following subset

$$\underline{S}_{p|1} = \underline{S}_p|\underline{S} = \{\underline{xw} : \underline{x} \in \underline{S}_p,\ \underline{w} \in \underline{S}\}$$

of expressions involving S- and S_p-generators. We will sometimes write expressions in $\underline{S}_{p|1}$ with a bar in the form $\underline{x}|\underline{w}$ in order to emphasize that $\underline{x} \in \underline{S}_p$ and $\underline{w} \in \underline{S}$. The set $\underline{S}_{p|1}$ inherits an $(\underline{S}_p, \underline{S})$-biaction structure from the (free) monoid structures on $\underline{S}_p$ and $\underline{S}$.

DEFINITION 5.1.8. For $\underline{x} \in \underline{S}_{p|1}$, let $[\underline{x}]_{p|*}$ be the set of patterns for $\underline{x}$ defined inductively in the following manner. Suppose $\underline{x} = \underline{ys}$ for some $s \in S_p \cup S$, where $[\underline{y}]_{p|*}$ is already known. Then we set

$$[\underline{x}]_{p|*} = \bigcup_{\underline{r} \in [\underline{y}]_{p|*}} [\underline{r}, s]_{p|*},$$

where

$$[\underline{r}, s]_{p|*} = \begin{cases} \left\{\underline{r}\underset{s}{*}\right\} & \text{if } \hat{r}s\hat{r}^{-1} \in S_p, \\ \left\{\underline{r}\underset{s}{0}, \underline{r}\underset{s}{1}\right\} & \text{otherwise.} \end{cases}$$

The match sets $[\underline{r}]$ for $\underline{r} \in [\underline{x}]_{p|*}$ induce a partition of $[\underline{x}]$. We can apply this construction towards a Deodhar-like defect formula for $\mathbb{H}_{p|*}$.

LEMMA 5.1.9. *Let* $\underline{x}|\underline{w} \in \underline{S}_{p|1}$. *Then*

$$\underline{H}_{\underline{x}} \cdot 1 \cdot \underline{H}_{\underline{w}} = \sum_{\substack{\underline{r} \in [\underline{x}|\underline{w}]_{p|*} \\ \mathbf{e} \in [\underline{r}]}} v^{\hat{d}(\mathbf{e})} H_{\hat{e}} H_{\hat{r}}$$

as an element of $\mathbb{H}_{p|*}$.

PROOF. Induct on the length of $\underline{w}$. When $\ell(\underline{w}) = 0$, we have $[\underline{x}] = \{\underline{r}\}$ for $\underline{x} \in \underline{S}_p$, where all the terms of $\underline{r}$ are of type $*$, so the result holds by Lemma 1.1.6 for $\mathbb{H}_p$. Now suppose $\ell(\underline{w}) = m$ and that the lemma holds for expressions with

$\underline{S}$-part of smaller length. Write $\underline{w} = \underline{zs}$ for some $\underline{z} \in \underline{S}$ and $s \in S$. Then we have

$$
\begin{aligned}
\underline{H}_{\underline{x}} \cdot 1 \cdot \underline{H}_{\underline{w}} &= (\underline{H}_{\underline{x}} \cdot 1 \cdot \underline{H}_{\underline{z}})\underline{H}_s \\
&= \left(\sum_{\substack{\underline{q} \in [\underline{x}|\underline{z}]_{p|*} \\ \mathbf{f} \in [\underline{q}]}} v^{\hat{d}(\mathbf{f})} H_{\hat{f}} H_{\hat{q}} \right) \underline{H}_s \\
&= \sum_{\substack{\underline{q} \in [\underline{x}|\underline{z}]_{p|*} \\ \mathbf{f} \in [\underline{q}] \\ W_p \hat{q} s = W_p \hat{q}}} v^{\hat{d}(\mathbf{f})} (H_{\hat{f}} \underline{H}_{\hat{q} s \hat{q}^{-1}}) H_{\hat{q}} + \sum_{\substack{\underline{q} \in [\underline{x}|\underline{z}]_{p|*} \\ \mathbf{f} \in [\underline{q}] \\ W_p \hat{q} s \neq W_p \hat{q}}} v^{\hat{d}(\mathbf{f})} (H_{\hat{f}} H_{\hat{q} s} + H_{\hat{f}} H_{\hat{q}}) \\
&= \sum_{\substack{\underline{r} \in [\underline{x}|\underline{w}]_{p|*} \\ \mathbf{e} \in [\underline{r}] \\ \underline{r}_m \text{ of type } *}} v^{\hat{d}(\mathbf{e})} H_{\hat{e}} H_{\hat{r}} + \sum_{\substack{\underline{r} \in [\underline{x}|\underline{w}]_{p|*} \\ \mathbf{e} \in [\underline{r}] \\ \underline{r}_m \text{ not of type } *}} v^{\hat{d}(\mathbf{e})} H_{\hat{e}} H_{\hat{q}} \\
&= \sum_{\substack{\underline{r} \in [\underline{x}|\underline{w}]_{p|*} \\ \mathbf{e} \in [\underline{r}]}} v^{\hat{d}(\mathbf{e})} H_{\hat{e}} H_{\hat{r}}
\end{aligned}
$$

which proves the result. □

Now we are ready to introduce combinatorial versions of the linkage bimodule and the linkage algebra. We will start with the linkage bimodule as a left $\mathcal{H}_p$-module and defer the definition of $\mathcal{H}_*$ until later.

Definition 5.1.10. The linkage $\mathbb{L}_{\geq 0}$-bimodule $\mathcal{H}_{p|*}$ is a collection of equivalence classes of sets of $01\emptyset*$-patterns for expressions in $\underline{S}_{p|1}$ with the structure of a left $\mathcal{H}_p$-module. It has the following generators and relations.

- For each $\underline{x} \in \underline{S}_{p|1}$, the set $[\underline{x}]_{p|*}$ is in $\mathcal{H}_{p|*}$. These sets generate $\mathcal{H}_{p|*}$ as an $\mathbb{L}_{\geq 0}$-module (but they do not usually form a basis!).
- Addition and scalar multiplication by elements of $\mathbb{L}_{\geq 0}$ are defined as in $\mathcal{H}$ or $\mathcal{H}_p$.
- We interpret the Bott–Samelson character set $[\underline{x}]$ for $\underline{x} \in \underline{S}_p$ as a singleton set of patterns $\{\underline{r}\}$, where $\underline{r}$ is a pattern for $\underline{x}$ with all types equal to $*$. By $\mathbb{L}_{\geq 0}$-linearity we can extend this to all character sets in $\mathcal{H}_p$. The action of $\mathcal{H}_p$ on $\mathcal{H}_{p|*}$ is then defined via multiplication of sets of patterns (analogous to $[\underline{S}]$).
- Each set of patterns in $\mathcal{H}_{p|*}$ gives rise to an object in $\mathbf{FinSet}/({}^pW \times \mathcal{H}_p)$ via the map $\underline{r} \mapsto (\hat{r}, [\underline{r}\hat{r}^{-1}])$. Two sets of patterns in $\mathcal{H}_{p|*}$ are considered equivalent if they are equivalent as sets over ${}^pW \times \mathcal{H}_p$.

It is not immediately obvious that the left $\mathcal{H}_p$-action is well defined; we will defer this proof briefly. We call sets in $\mathcal{H}_{p|*}$ linkage sets, and sets of the form $[\underline{x}]$ for $\underline{x} \in \underline{S}_{p|1}$ Bott–Samelson linkage sets. Assuming that the $\mathbb{H}_p$-action is well defined, we have $[\underline{x}][\underline{y}]_{p|*} = [\underline{xy}]_{p|*}$ in $\mathcal{H}_{p|*}$ for all $\underline{x} \in \underline{S}_p$ and all $\underline{y} \in \underline{S}_{p|1}$.

PROPOSITION 5.1.11. *The left $\mathcal{H}_p$-action is well defined. Moreover, the mapping*

$$\begin{aligned}\mathcal{H}_{p|*} &\longrightarrow \mathbb{H}_{p|*}\\ C &\longmapsto \sum_{\substack{\underline{r}\in C\\ \mathbf{e}\in[\underline{r}]}} v^{\hat{d}(\mathbf{e})} H_{\hat{e}} H_{\hat{r}}\end{aligned}$$

is an $\mathcal{H}_p$-module homomorphism, where the left $\mathcal{H}_p$-module structure on the codomain arises from the isomorphism $[\mathcal{H}_p] \cong \mathbb{H}_p$. This homomorphism induces an $\mathbb{H}_p$-module isomorphism $[\mathcal{H}_{p|*}] \xrightarrow{\sim} \mathbb{H}_{p|*}$.

PROOF. Let $\mathcal{H}^0_{p|*}$ denote the free $\mathcal{H}_p$-module defined by the basis above, but without the relation of equivalence from the pattern sets. Consider the mapping $\mathcal{H}^0_{p|*} \to \mathbb{H}_{p|*}$ defined as above. By Lemma 5.1.9, for $\underline{xy}|\underline{z} \in \underline{S}_{p|1}$ we have

$$[\underline{x}][\underline{y}|\underline{z}] = [\underline{xy}|\underline{z}] \longmapsto \underline{H}_{\underline{xy}} \cdot 1 \cdot \underline{H}_{\underline{z}} = (\underline{H}_{\underline{x}}\underline{H}_{\underline{y}}) \cdot 1 \cdot \underline{H}_{\underline{z}}.$$

Combining this with $\mathbb{L}_{\geq 0}$-linearity implies that the map is an $\mathcal{H}_p$-module homomorphism. Now note that two sets in $\mathcal{H}^0_{p|*}$ are equivalent over ${}^pW \times \mathcal{H}_p$ if and only if they map to the same element of $\mathbb{H}_{p|*}$. This implies the following in turn:

(i) the left $\mathcal{H}_p$-action on $\mathcal{H}_{p|*}$ is well defined,
(ii) the homomorphism $\mathcal{H}^0_{p|*} \to \mathbb{H}_{p|*}$ factors through $\mathcal{H}_p$,
(iii) the induced homomorphism $[\mathcal{H}_{p|*}] \to \mathbb{H}_{p|*}$ is injective.

To prove the final claim, note that the Bott–Samelson linkage sets map onto an $\mathbb{L}$-spanning set for $\mathbb{H}_{p|*}$, so the homomorphism $[\mathcal{H}_{p|*}] \to \mathbb{H}_{p|*}$ is an isomorphism. □

5.1.5. Linkage sections.

DEFINITION 5.1.12. For $\underline{x} \in \underline{S}$, let $[\underline{x}]_*$ be the function mapping each coset representative in ${}^pW$ to a set of patterns for $\underline{x}$ defined inductively as follows. Suppose $\underline{x} = \underline{ys}$ for some $s \in S$, and $[\underline{y}]_*$ is known. Then we define

$$[\underline{x}]_* : w \longmapsto \bigcup_{\underline{r}\in[\underline{y}]_*(w)} \underline{r}[w\hat{r}, s]_*,$$

where

$$[z,s]_* = \begin{cases} \left\{\underset{s}{*}\right\} & \text{if } zsz^{-1} \in W_p,\\ \left\{\underset{s}{0}, \underset{s}{1}\right\} & \text{otherwise.}\end{cases}$$

For each $A \in {}^p\mathcal{W}(*)$ we also define the functions

$$u_A : w \longmapsto \begin{cases} v = \left\{\begin{smallmatrix}\mathrm{U}\\ \emptyset\\ \emptyset\end{smallmatrix}\right\} & \text{if } w \in A,\\ \left\{\begin{smallmatrix}\emptyset\\ \emptyset\\ \emptyset\end{smallmatrix}\right\} & \text{otherwise,}\end{cases}$$

$$u_A^{-1} : w \longmapsto \begin{cases} v^{-1} = \left\{\begin{smallmatrix}\mathrm{D}\\ \emptyset\\ \emptyset\end{smallmatrix}\right\} & \text{if } w \in A,\\ \left\{\begin{smallmatrix}\emptyset\\ \emptyset\\ \emptyset\end{smallmatrix}\right\} & \text{otherwise,}\end{cases}$$

using the $\emptyset$ symbol introduced in Section 1.1.4. Note that the new term $\begin{smallmatrix}\emptyset\\ \emptyset\\ \emptyset\end{smallmatrix}$ has defect 0, unlike the terms marked with U or D.

DEFINITION 5.1.13. The linkage Hecke $\mathbb{L}_{\geq 0}$-algebra $\mathcal{H}_*$ is a collection of equivalence classes of functions mapping coset representatives in ${}^pW$ to sets of $01\emptyset*$-patterns for expressions in S with the structure of an $\mathbb{L}_{\geq 0}$-algebra. It has the following generators and relations.

- For each $\underline{x} \in \underline{S}$ and $A \in {}^pW(*)$, the functions $[\underline{x}]_*$ and u_A are in $\mathcal{H}_*$. These functions, along with products of the form $u_A[\underline{x}]_*$ (defined below), generate $\mathcal{H}_*$ as an $\mathbb{L}_{\geq 0}$-module.
- Addition and scalar multiplication by elements of $\mathbb{L}_{\geq 0}$ are defined pointwise as in $\mathcal{H}$.
- For $b, c \in \mathcal{H}_*$, the product bc is defined to be
$$bc : w \longmapsto \{\underline{qr} : \underline{q} \in b(w),\ \underline{r} \in c(w\hat{q})\}.$$
- There is a right $\mathcal{H}_*$-action on $\mathcal{H}_{p|*}$, defined in the following manner. For $b \in \mathcal{H}_*$ and $C \in \mathcal{H}_{p|*}$, we set
$$Cb = \{\underline{qr} : \underline{q} \in C,\ \underline{r} \in b(\hat{q})\}.$$
Two functions in $\mathcal{H}_*$ are considered equivalent if they have equivalent actions on $\mathcal{H}_{p|*}$.

Again it is not immediately clear that multiplication in $\mathcal{H}_*$ is well defined. We call sets in $\mathcal{H}_{p|*}$ linkage sections, and sets of the form $[\underline{x}]$ for $\underline{x} \in \underline{S}_{p|1}$ Bott–Samelson linkage sections. Assuming that multiplication is well defined, we have $[\underline{x}]_*[\underline{y}]_* = [\underline{xy}]_*$ for all $\underline{x}, \underline{y} \in \underline{S}$.

THEOREM 5.1.14. *Multiplication in $\mathcal{H}_*$ is well defined. Moreover, the mapping*
$$\begin{aligned}\mathcal{H}_* &\longrightarrow \operatorname{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})\\ b &\longmapsto ([C] \mapsto [Cb])\end{aligned}$$
is an $\mathbb{L}_{\geq 0}$-algebra homomorphism. It induces an injective $\mathbb{L}$-algebra homomorphism $[\mathcal{H}_*] \to \operatorname{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$, whose image coincides with the image of $\mathbb{H}_* \to \operatorname{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$.

PROOF. Let $\mathcal{H}_*^0$ denote the free $\mathbb{L}_{\geq 0}$-algebra defined by the generators above, but without the relation of equivalence via the action on $\mathcal{H}_{p|*}$. Consider the map $\mathcal{H}_*^0 \to \operatorname{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$ defined as above. By Lemma 5.1.9, for $\underline{x}, \underline{y} \in \underline{S}$ we have
$$\begin{aligned}[\underline{x}]_*[\underline{y}]_* = [\underline{xy}]_* \longmapsto &([[\underline{z}|\underline{w}]_{p|*}] \mapsto [[\underline{z}|\underline{w}]_{p|*}[\underline{xy}]_*])\\ =&(\underline{H}_{\underline{z}}\underline{H}_{\underline{w}} \mapsto \underline{H}_{\underline{z}}\underline{H}_{\underline{w}}\underline{H}_{\underline{xy}})\\ =&(\underline{H}_{\underline{z}}\underline{H}_{\underline{w}} \mapsto \underline{H}_{\underline{z}}\underline{H}_{\underline{w}}\underline{H}_{\underline{x}}\underline{H}_{\underline{y}})\\ =&([[\underline{z}|\underline{w}]_{p|*}] \mapsto [[\underline{z}|\underline{w}]_{p|*}[\underline{x}]_*] \mapsto [([\underline{z}|\underline{w}]_{p|*}[\underline{x}]_*)[\underline{y}]_*]).\end{aligned}$$

This can be extended to products of elements of the form $u_A[\underline{x}]_*$ for $A \in {}^p\mathcal{W}$. Combining this with $\mathbb{L}_{\geq 0}$-linearity implies that the map is an $\mathbb{L}_{\geq 0}$-algebra homomorphism. Now note that two sections in $\mathcal{H}^0_*$ have equivalent actions on $\mathcal{H}_{p|*}$ if and only if they map to the same endomorphism in $\mathrm{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$. This implies the following in turn:

- multiplication in $\mathcal{H}_*$ is well defined,
- the homomorphism $\mathcal{H}^0_* \to \mathrm{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$ factors through $\mathcal{H}_*$,
- the induced homomorphism $[\mathcal{H}_*] \to \mathrm{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$ is injective.

To prove the final claim, note that the linkage section $u_A[\underline{x}]_*$ maps onto the same endomorphism of $\mathbb{H}_{p|*}$ induced by multiplication by $u_A \underline{H}_{\underline{x}}$. But these elements form an $\mathbb{L}$-spanning set of $\mathbb{H}_*$, so $[\mathcal{H}_*] \to \mathrm{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$ has the same image as the homomorphism $\mathbb{H}_* \to \mathrm{End}_{\mathbb{H}_p}(\mathbb{H}_{p|*})$ from Lemma 5.1.3. $\square$

Corollary 5.1.15. *There is a unique* $\mathbb{L}_{\geq 0}$*-algebra homomorphism which maps*

$$\begin{aligned}\mathcal{H}_* &\longrightarrow \mathbb{H}_*\\ [\underline{s}]_* &\longmapsto \underline{H}_s\\ u_A &\longmapsto u_A\end{aligned}$$

for all $s \in S$ *and* $A \in {}^p\mathcal{W}(*)$*. This homomorphism induces an* $\mathbb{L}$*-algebra isomorphism* $[\mathcal{H}_*] \cong \mathbb{H}_*$*.*

5.2. The linkage category

5.2.1. Positive characteristic realizations. Write $V_{\mathbb{R}}$ and $V^\vee_{\mathbb{R}}$ for the universal and dual universal realizations of (W,S) over $\mathbb{R}$ with respect to $\Sigma, -\tilde{\alpha}$. Define the following lattices

$$\begin{aligned}E_{\mathbb{Z}} &= \{v \in E : \langle v, \alpha^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha^\vee \in \Phi^\vee\},\\ V_{\mathbb{Z}} &= \{v \in V_{\mathbb{R}} : \langle v, a_s^\vee\rangle \in \mathbb{Z} \text{ for all } s \in S\},\\ V^*_{\mathbb{Z}} &= \{v^* \in V^*_{\mathbb{R}} : \langle a_s, v^*\rangle \in \mathbb{Z} \text{ for all } s \in S\},\\ E'_{\mathbb{Z}} &= \{v^* \in V^*_{\mathbb{Z}} : \langle v_{\mathrm{stab}}, v^*\rangle = 1\}.\end{aligned}$$

The lattices $V_{\mathbb{Z}}, V^*_{\mathbb{Z}}$, along with the images of $\{a_s\}$ and $\{a_s^\vee\}$ in these lattices, define what we could call a $\mathbb{Z}$-form of the universal realization, for which Lemma 4.1.4 still holds. Similarly, observe that $V = \Bbbk \otimes V_{\mathbb{Z}}$ and $V^* = \Bbbk \otimes V^*_{\mathbb{Z}}$ give the universal realization of (W,S) over $\Bbbk$. Now set $E_\Bbbk = \Bbbk \otimes E_{\mathbb{Z}}$ and $E'_\Bbbk = \Bbbk \otimes E'_{\mathbb{Z}} \subseteq V^*_{\mathbb{Z}}$. Then by tensoring the $\mathbb{Z}$-isomorphism $E_{\mathbb{Z}} \cong E'_{\mathbb{Z}}$ with $\Bbbk$ we get the corresponding result over $\Bbbk$. This fact will help us obtain some results using facts about V from the affine reflection action of W on $E_\Bbbk$.

Lemma 5.2.1. *The action of* W *on* $E_\Bbbk$ *is faithful modulo the* p*-translation subgroup* $p\mathbb{Z}\Phi$*. As a result the actions of* W_p *on* $E_\Bbbk$ *and* $E_\Bbbk^{\pi_p}$ *(where the latter is inflated via the map* $\pi_p : W_p \to W_{\mathrm{f}}$*) are identical.*

PROOF. If $x, y \in W$ have the same action on $E_{\Bbbk}$, then xy^{-1} must map any $v \in E_{\mathbb{Z}}$ to some element of the coset $v + pE_{\mathbb{Z}}$. But xy^{-1} acts isometrically on $E_{\mathbb{Z}}$ so it must be a translation by some element of the lattice $pE_{\mathbb{Z}}$. The translations in W correspond to the lattice $\mathbb{Z}\Phi$, and the index of $\mathbb{Z}\Phi$ inside $E_{\mathbb{Z}}$ is by definition the index of connection, so the translation must be by an element of $p\mathbb{Z}\Phi$. Since $W_p \cong W_{\mathrm{f}} \ltimes p\mathbb{Z}\Phi$ this means that W_p acts only by the W_{f} component as claimed. □

LEMMA 5.2.2. *Let $w \in {}^pW$ and $s \in S$. Then the coefficient of $a_{\tilde{s}}$ in $wa_s \in V$ is zero if and only if $wsw^{-1} \in W_p$. Moreover, in this case we have $wa_s = \sum_{t \in S_{\mathrm{f}}} r_t a_t$ where $\alpha = \sum_{t \in S_{\mathrm{f}}} r_t \alpha_t$ is some root in Φ.*

PROOF. Let $\{a_t^*\} \subseteq V^*$ denote the dual basis of $\{a_t\} \subseteq V$. For any $\alpha \in E$ write $H_{\alpha,\Bbbk} = \Bbbk \otimes H_\alpha$ for the image in $E_{\Bbbk}$ of the hyperplane orthogonal to α and $H_{s,\Bbbk}$ for the affine hyperplane fixed by s. Suppose the coefficient of $a_{\tilde{s}}$ in wa_s equals zero. This is equivalent to

$$\begin{aligned}
\langle wa_s, a_{\tilde{s}}^* \rangle = 0 \Leftrightarrow \langle a_s, w^{-1} a_{\tilde{s}}^* \rangle = 0 & \\
\Leftrightarrow w^{-1} \text{ maps } 0 \in E_{\Bbbk} \text{ to } H_{s,\Bbbk} & \\
\Leftrightarrow w \text{ maps } H_{s,\Bbbk} \text{ to some } H_{\alpha,\Bbbk} \text{ for some } \alpha \in \Phi & \\
\Leftrightarrow wsw^{-1} \text{ and } s_\alpha \text{ have the same action on } E_{\Bbbk} \text{ for some } \alpha \in \Phi & \\
\Leftrightarrow wsw^{-1} \in W_p &
\end{aligned}$$

where the last equivalence is a consequence of the previous lemma. In this situation, we can choose $\alpha = w(\alpha_s) - w(0) \in E_{\mathbb{Z}}$ (note that $w(\alpha_s) - w(0) = \pi(w)(\alpha_s) \in \Phi$, where $\pi : W \to W_{\mathrm{f}}$ is the canonical projection). If we write $\alpha = \sum_{t \in S_{\mathrm{f}}} r_t \alpha_t$ then for $t \in S_{\mathrm{f}}$ we have

$$\begin{aligned}
\langle wa_s, a_t^\vee \rangle &= \langle wa_s, a_{\tilde{s}}^* + a_t^\vee \rangle \\
&= \langle a_s, w^{-1}(a_{\tilde{s}}^* + a_t^\vee) \rangle \\
&= \langle \alpha_s, w^{-1}(\alpha_t^\vee) \rangle \\
&= \langle \alpha_s, w^{-1}(\alpha_t^\vee) - w^{-1}(0) \rangle \\
&= \langle \alpha_s, \pi(w^{-1})(\alpha_t^\vee) \rangle \\
&= \langle \pi(w)(\alpha_s), \alpha_t^\vee \rangle \\
&= \langle w(\alpha_s) - w(0), \alpha_t^\vee \rangle \\
&= \langle \alpha, \alpha_t^\vee \rangle
\end{aligned}$$

which shows that $wa_s = \sum_{t \in S_{\mathrm{f}}} r_t a_t$. □

COROLLARY 5.2.3. *If $w \in {}^pW$ and $s \in S$ such that $wsw^{-1} = \tilde{s}_p$ then $wa_s = -\tilde{a}$.*

PROOF. The p-affine reflection $\tilde{s}_p$ acts like $s_{-\tilde{\alpha}}$ on $E_{\Bbbk}$, so from the previous result we know that $wa_s = \pm\tilde{a}$, with the sign matching $\pi(w)(\alpha_s) = \pm\tilde{\alpha} \in E_{\mathbb{Z}}$. Now $ws = \tilde{s}_p w \in W_p w$, so $ws > w$ because w is a minimal length coset representative. But ws and w both correspond to dominant alcoves wsA_0 and wA_0, so the vector

$\pi(w)(\alpha_s)$ which is orthogonal to the s-wall of wA_0 and points to the inside of this alcove must be negative. Thus $\pi(w)(\alpha_s) = -\tilde{\alpha}$. □

For this reason, we will define $a_{\tilde{s}_p} = -\tilde{a} \in V$. From this we get

PROPOSITION 5.2.4. *Let V^F denote the F-twist of the realization V; in other words, as a vector space $V = V^F$, but $a_s^F = a_{F(s)}$ for each $s \in S$ and the W-action is $w \cdot_F v = F(w)v$ for all $w \in W$ and $v \in V^F$. Then V^F is isomorphic as a realization to $V^\pi \oplus \Bbbk$; in other words V^F is the inflated finite realization $V^\pi = V^\pi_\Sigma$ over $\Bbbk$ augmented by the trivial representation.*

PROOF. Choose w, s as in the previous corollary. We first show that $\tilde{s}$ acts on V^F as a reflection. For $v \in V^F$, $\tilde{s} \cdot_F v$ is

$$\tilde{s}_p v = wsw^{-1}v = ws(v' + ca_s) = w(v' - ca_s) = w(v' + ca_s - 2ca_s) = v - 2c(-\tilde{a})$$

where v' is some linear combination of $\{a_t\}_{t \neq s}$ and $c = \langle w^{-1}v, a_s^\vee \rangle / 2$. Yet

$$\langle w^{-1}v, a_s^\vee \rangle = \langle v, wa_s^\vee \rangle = \langle v, -\tilde{a}^\vee \rangle$$

which shows that the $\tilde{s}$-action is a reflection in $-\tilde{a}$. Let $U = \sum_{s \in S_{\mathrm{f}}} \Bbbk a_s \leq V^F$. We have shown that U is a subrepresentation of V^F isomorphic to V^π. But we also have the trivial subrepresentation $\Bbbk v_{\mathrm{stab}} \leq V^F$ which is a complement to U as a vector space, so $V^F = V^\pi \oplus \Bbbk$ as realizations. □

5.2.2. Diagrammatics. As above, fix V to be the universal realization of (W, S) over $\Bbbk$ with respect to $\Sigma, -\tilde{\alpha}$. Recall that $R = \mathrm{Sym}(V)$ is the symmetric algebra of V over $\Bbbk$. Now define $\hat{R}$ to be

$$\hat{R} = \widehat{R\left[\frac{a_s}{a_t} : s, t \in S_{\mathrm{f}}\right]_{(a_{S_{\mathrm{f}}})}},$$

the completion of a localized ring, where $(a_{S_{\mathrm{f}}})$ denotes the prime ideal generated by a_s for any $s \in S_{\mathrm{f}}$. The ring $\hat{R}$ is a complete discretely valued extension of R (with valuation ν) whose maximal ideal contains every linear combination of the form $\sum_{s \in S_{\mathrm{f}}} r_s a_s$ but does not contain $a_{\tilde{s}}$. From the results in the previous section $\hat{R}$ is stable under the action of W_p. As with R, we scale ν so that $\nu(a_s) = 2$ for any $s \in S_{\mathrm{f}}$.

Let $\hat{R} \otimes_R \mathcal{D}_{\mathrm{BS,std}}$ denote the extension of scalars to $\hat{R}$ of the R-form of the mixed category $\mathcal{D}_{\mathrm{BS,std}}$ on the left. In general, objects in this extension are $(\hat{R}, R)$-bimodules but in some cases the right action can be enlarged. For convenience we will generally omit the "$\hat{R} \otimes_R (-)$" when describing the image in $\hat{R} \otimes_R \mathcal{D}_{\mathrm{BS,std}}$ of a module in $\mathcal{D}_{\mathrm{BS,std}}$.

We next define an extension of the Frobenius map, a monoidal embedding

$$F : \mathcal{D}^F_{\mathrm{BS}} \to \hat{R} \otimes \mathcal{D}_{\mathrm{BS,std}},$$

where $\mathcal{D}^F_{\mathrm{BS}}$ and $\mathcal{D}^F$ denote the categories of Bott–Samelson bimodules and Soergel bimodules for the F-twisted realization V^F. For each $t \in S_{\mathrm{f}}$ let $F(B_t) = B_t$, the image of B_t in $\hat{R}\otimes\mathcal{D}_{\mathrm{BS,std}}$. Since $\hat{R}$ is stable under W_p, $F(B_t)$ is in fact an $(\hat{R},\hat{R})$-bimodule. Now fix a coset representative $w_p \in {}^pW$, a rex $\underline{w}_p$ for w_p and a generator $s_p \in S$ such that $w_p s_p w_p^{-1} = \tilde{s}_p$. We define $F(B_{\tilde{s}})$ to be

$$F(B_{\tilde{s}}) = B_{\tilde{s}_p} = \hat{R}_{\underline{w}_p} \otimes_R B_{s_p} \otimes_R \hat{R}_{\underline{w}_p^{-1}}$$

where $\hat{R}_{\underline{w}}$ denotes the standard bimodule over $\hat{R}$ and $\underline{w}^{-1}$ is just the reverse of $\underline{w}$. Note that all the bimodules defined so far have been $(\hat{R},\hat{R})$-bimodules, so they have a monoidal tensor product $\otimes_{\hat{R}}$, and F is defined on all other Bott–Samelson bimodules using this tensor product.

On scalar morphisms (i.e. polynomials in R), F is defined to be the embedding $R^F \to \hat{R}$, where $R^F = R$ denotes the symmetric algebra on the F-twisted realization V^F. The functor F further maps all dots, forks, and braids colored by S_{f} to their respective images in $\hat{R}\otimes\mathcal{D}_{\mathrm{BS,std}}$. Finally F maps $\tilde{s}$-colored vertices to what we call the $\tilde{s}_p$-morphisms. In Figure 5.1 we have illustrated these morphisms in the case where $p=3$, $\Phi = A_2$, $s_p = 0$ and $\underline{w}_p = \underline{0121}$, with a placeholder for the $\tilde{s}_p$-braid. The construction generalizes in an obvious way by adding more strings. With some work one can show that all the relations in $\mathcal{D}^F_{\mathrm{BS}}$ involving only dots and forks hold for the $\tilde{s}_p$-dot and $\tilde{s}_p$-fork, including isotopy.

The $\tilde{s}_p$-braid morphism is defined as follows. First decompose the corresponding ordinary braid vertex (involving $\tilde{s}$) using a dashed braid vertex (see Section 4.2.2) plus diagrams with only forks and dots. For example, one such decomposition is depicted in Figure 5.2. One way to construct these decompositions is by applying (4.18) to the all the strings above the braid and using the Jones–Wenzl relation (4.12)–(4.14). The $\tilde{s}_p$-braid morphism is the sum of a dashed morphism combined with some bivalent projectors, corresponding to the summand containing a dashed braid vertex (see Figure 5.3) and the $\tilde{s}_p$-versions of the remaining fork-and-dot terms constructed using the $\tilde{s}_p$-dot and $\tilde{s}_p$-fork previously defined. It can be shown that these morphisms satisfy all the relations defining $\mathcal{D}^F_{\mathrm{BS}}$.

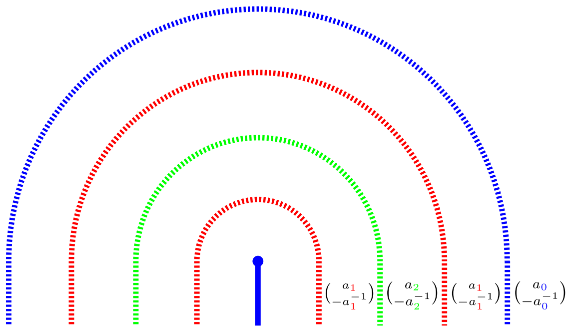
We are now ready to define the linkage category using the Frobenius embedding.

Definition 5.2.5. The diagrammatic Bott–Samelson linkage category $\mathcal{D}_{\mathrm{BS},p|*}$ is the following $\hat{R}$-linear subcategory of the mixed category $\hat{R}\otimes_R\mathcal{D}_{\mathrm{BS,std}}$ which has the structure of $(\mathcal{D}^F_{\mathrm{BS}},\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}})$-bimodule. Here $\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}}$ refers to the ungraded version of $\mathcal{D}_{\mathrm{BS}}$, where we forget the grading completely.

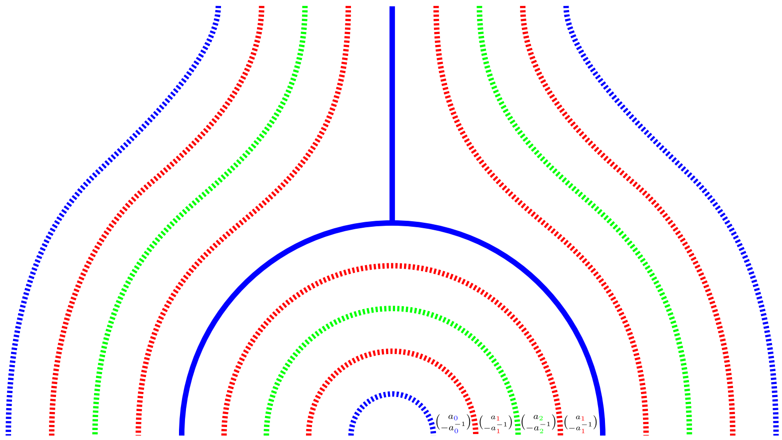
Objects: For each $\underline{x}|\underline{w} \in \underline{S}_{p|1}$ there is an object $B_{\underline{x}|\underline{w}} = F(B_{F^{-1}(\underline{x})}) \otimes_{\hat{R}} B_{\underline{w}}$ called the Bott–Samelson linkage bimodule.The $(\mathcal{D}^F_{\mathrm{BS}},\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}})$-bimodule structure is defined by

$$B_{\underline{x}} \otimes B \otimes B_{\underline{w}} = F(B_{\underline{x}}) \otimes_{\hat{R}} B \otimes_{\hat{R}} B_{\underline{w}}.$$

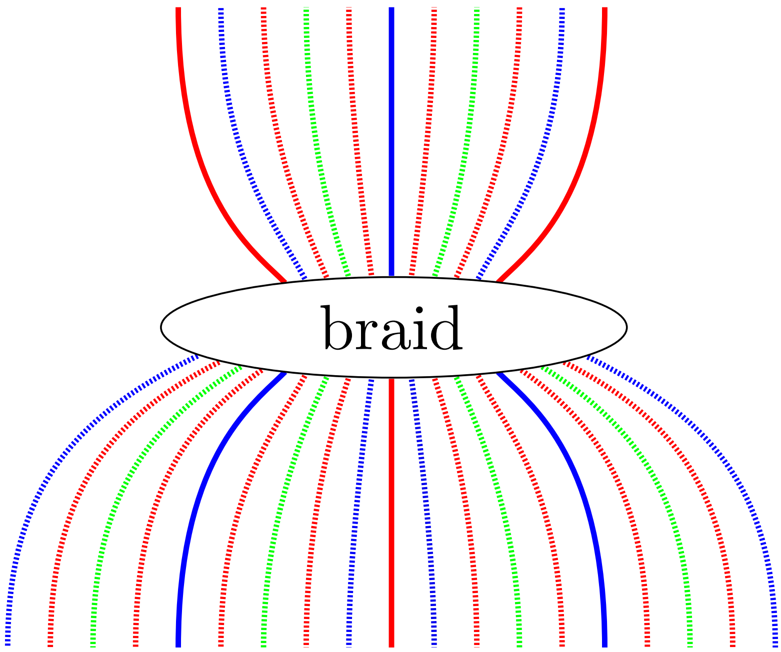
Morphisms: The morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$ are generated using the $(\mathcal{D}^F_{\mathrm{BS}},\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}})$-bimodule structure. In particular, this means that all morphisms in $\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}}$ (i.e. all solid colored morphisms) and the $\tilde{s}_p$-morphisms above are morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$. The remaining morphisms are generated from a new



(A) $\tilde{s}_p$-dot



(B) $\tilde{s}_p$-fork



(C) $\tilde{s}_p$-braid

FIGURE 5.1. The $\tilde{s}_p$-morphisms for $p = 3$ and $\Phi = A_2$.

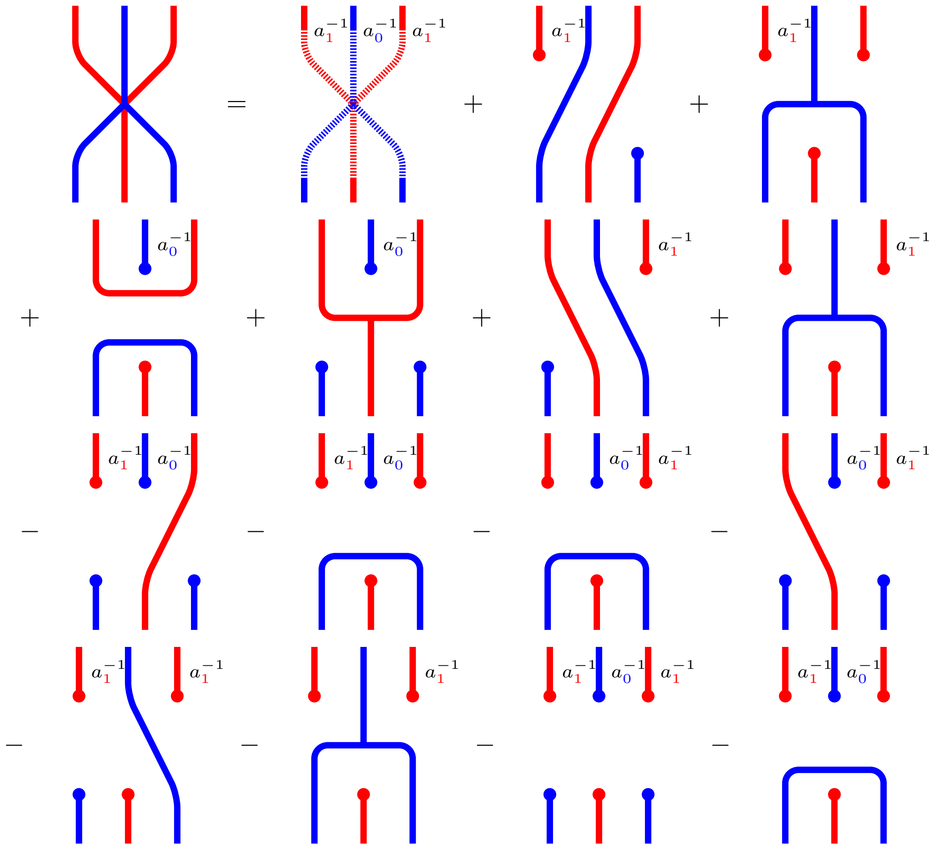


FIGURE 5.2. A braid decomposition for $\Phi = A_2$.

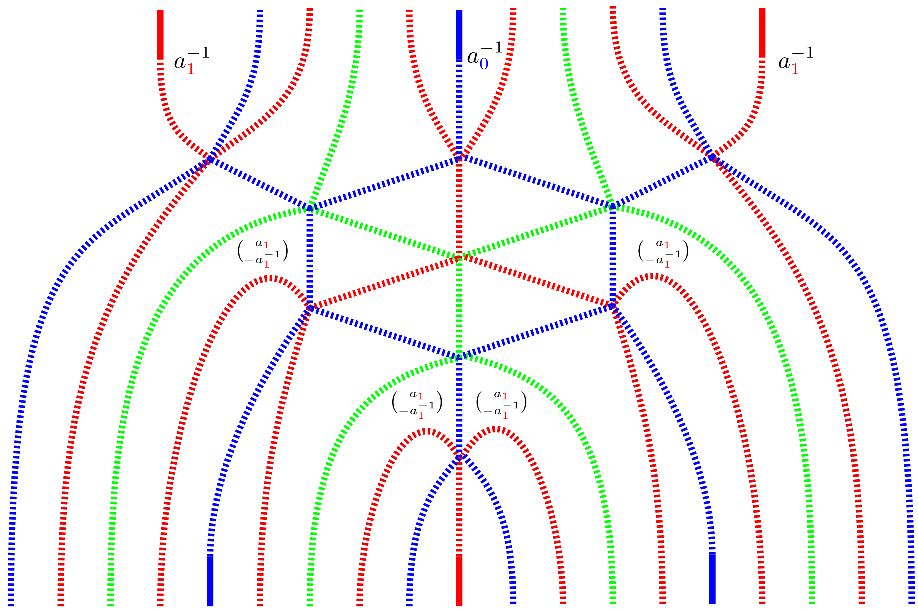


FIGURE 5.3. The bivalent-projected dashed $\tilde{s}_p$-braid morphism, for $p = 3$ and $\Phi = A_2$.

morphism $B_{\tilde{s}_p} \to B_{\underline{w}_p \underline{s}_p \underline{w}_p^{-1}}$ and its upside-down variant which we call the *menorah morphism* (Figure 5.4).

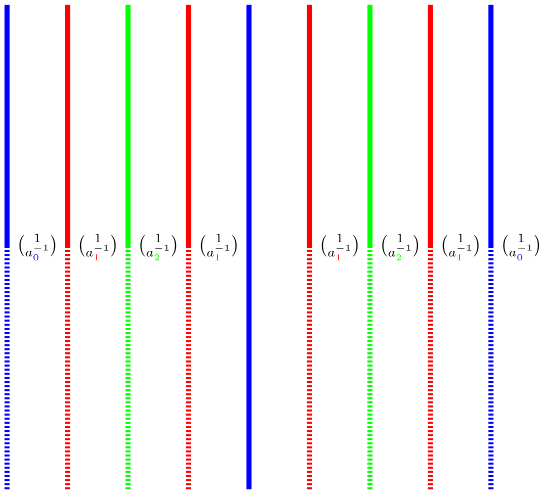


FIGURE 5.4. The menorah morphism for $p = 3$ and $\Phi = A_2$.

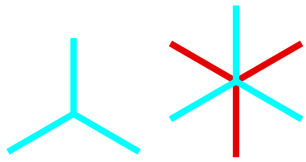
REMARK 5.2.6.

(i) The Frobenius embedding and the category $\mathcal{D}_{\mathrm{BS},p|*}$ do not depend on the choices of $w_p, \underline{w}_p, s_p$; any such choices generate equivalent embeddings and categories. In fact, by combining braid vertices with the menorah morphism we can obtain similar morphisms $B_{\tilde{s}_p} \to B_{\underline{x}}$ for each rex $\underline{x}$ for $\tilde{s}_p$. We will also call these morphisms "menorah morphisms".

(ii) The diagrams defining morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$ are not quite "graphs up to isotopy" since bivalent vertices can change sign under arbitrary isotopies. However, if we restrict to diagrams that never factor through a non-linkage Bott–Samelson bimodule, then isotopy classes of such diagrams do define a unique morphism, not just up to sign.

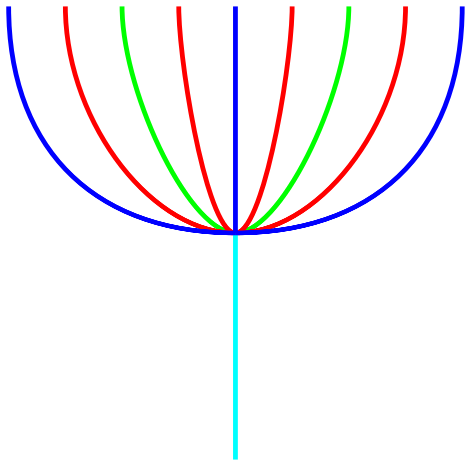
(iii) The menorah morphism is strictly speaking not cyclic, since some rotations of it do not correspond to a morphism in $\mathcal{D}_{\mathrm{BS},p|*}$ but it is what we call semi-cyclic. In other words, if we twist the right-side-up menorah map by 180 degrees clockwise we get the upside-down menorah map, and vice-versa.

NOTATION 5.2.7. We assign the W_p-generator $\tilde{s}_p$ a lighter version of the color corresponding to $\tilde{s}$ (e.g. if $\tilde{s}$ is colored blue then $\tilde{s}_p$ is colored cyan). In the diagrams we use this color to abbreviate morphisms which involve $B_{\tilde{s}_p}$, by using solid $\tilde{s}_p$-colored lines. For example, the morphisms corresponding to S-graph vertices described above abbreviate to



p-dot p-fork p-braid

so that they look exactly the same as their lower scale counterparts. Similarly, the menorah morphism in Figure 5.4 abbreviates to



menorah

For this reason we will also call these morphisms "vertices".

We also have some special terminology for the menorah vertex. The $\tilde{s}_p$-colored edge is called the *handle* or *shaft*, while the middle edge among the S-colored edges (corresponding to s above) is called the *shamash*. The remaining edges are called *candles*.

The grading on $\mathcal{D}_{\mathrm{BS},p|*}$ inherited from $\mathcal{D}_{\mathrm{BS,std}}$ is not a very useful invariant because $\hat{R}$ is no longer meaningfully graded. However we can define a valuation (or "degree function") on morphisms which is compatible with the valuation on $\hat{R}$. Suppose L is a morphism in $\mathcal{D}_{\mathrm{BS},p|*}$. Localizing the solid indices gives a matrix of standard morphisms (i.e. morphisms only using dashed lines), and we can push the coefficients to the left-hand side to ensure that they are in $\hat{R}$. The valuation $\nu(L)$ is defined to be the minimal valuation of all the coefficients in this matrix. This satisfies several nice properties, including:

- for all $f \in \hat{R}$ and morphisms L, $\nu(fL) = \nu(f) + \nu(L)$;
- for any object B, $\nu(\mathrm{id}_B) = 0$;
- $\nu(0) = \infty$;
- for any morphisms L and L', $\nu(L + L') \geq \min(\nu(L), \nu(L'))$;
- for any morphisms L and L', $\nu(L \otimes L') = \nu(L) + \nu(L')$;
- for any composable morphisms L and L', $\nu(L \circ L') \geq \nu(L) + \nu(L')$.

These properties are essentially the axioms defining a non-archimedean norm on non-commutative algebras, restated in terms of a valuation. This can easily be transformed into the language of filtered algebras by assigning the filtration

$$\hat{R}^i = \{f \in \hat{R} : \nu(f) \geq i\}$$

and similarly for the Hom-spaces. This gives $\mathcal{D}_{\mathrm{BS},p|*}$ the structure of a category enriched in $\hat{R}$-filtered modules. As always we use angular brackets $\langle - \rangle$ to denote the filtration shift of an object or morphism.

The basic morphisms represented by the different kinds of vertices have easily determined valuations. Polynomials in $\hat{R}$ have the same valuation as in $\hat{R}$. The braid and menorah morphisms have valuation 0. The t-colored dot morphism (resp. fork morphism) has valuation $+1$ (resp. -1) if $t \in S_p$ and 0 if $t = \tilde{s}$. In particular, this is reasonably compatible with the grading on the $\mathcal{D}_{\mathrm{BS}}^F$ which acts on the left, but not the grading on $\mathcal{D}_{\mathrm{BS}}$ acting on the right. The tensor product property is helpful for calculating valuations of more complicated morphisms, but the inequality with respect to function composition does mean that valuations of general diagrams cannot be computed as simply as degrees in $\mathcal{D}_{\mathrm{BS}}$.

Finally, the category $\mathcal{D}_{p|*}$ is given by taking the Karoubi envelope (i.e. the completion with respect to all direct sums, direct summands and filtration shifts) of $\mathcal{D}_{\mathrm{BS},p|*}$.

5.3. Fundamental results for $\mathcal{D}_{p|*}$

5.3.1. Linkage light leaves. We will construct a basis for the Hom-spaces in $\mathcal{D}_{p|*}$ analogous to the light leaves basis for $\mathcal{D}$. Generalizing rex moves, we call a morphism in $\mathcal{D}_{p|*}$ an mrex move if it can be generated using composition and the tensor product from identity morphisms, braid morphisms and either of the following "braid-like" incarnations of a menorah morphism (see Figure 5.5). In other words, mrex moves correspond to morphisms in $\mathcal{D}_{p|*}$ which do not factor through Bott–Samelson bimodules of shorter length than the domain/codomain.

Let $\underline{x} = \underline{s_1 \cdots s_m} \in \underline{S}_{p|1}$, and suppose $\underline{r} \in [\underline{x}]_{p|*}$. For each match $\mathbf{c} \in [\underline{r}]$ we construct a linkage light leaves map ${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w}|\underline{z}} : B_{\underline{x}} \to B_{\underline{w}} \otimes \hat{R}_{\underline{z}}$, where $\underline{w} \in \underline{S}_p$ is a rex for $\hat{c}$ and $\underline{z} \in \underline{S}$ is a rex for $\hat{r}$. The construction proceeds inductively in the following manner. Let $\underline{x}_{\leq i}$, $\underline{r}_{\leq i}$, and $\mathbf{c}_{\leq i}$ be the truncated forms of $\underline{x}$, $\underline{r}$, and $\mathbf{c}$, and let $\underline{w}_{\leq i}$ and $\underline{z}_{\leq i}$ be rexes for $\hat{c}_{\leq i}$ and $\hat{r}_{\leq i}$ respectively. As with ordinary light leaves we set ${}^{p|*}\mathrm{LL}_i = {}^{p|*}\mathrm{LL}_{\mathbf{c}_{\leq i},\underline{w}_{\leq i}|\underline{z}_{\leq i}}$ and define ${}^{p|*}\mathrm{LL}_i = \phi_i \circ ({}^{p|*}\mathrm{LL}_{i-1} \otimes \mathrm{id}_{B_{s_i}})$, where ϕ_i depends on the decorated type of $\mathbf{c}_i$. There are six possibilities for ϕ_i, which are illustrated in Figure 5.6.

In Figure 5.6, boxes labeled "mrex" are mrex moves, and boxes labeled "std" are standard morphisms to a standard bimodule $\hat{R}_{\underline{z}}$ for some rex $\underline{z}$. We also use $(\cdot)$ to denote the normalizing factor for the nearest bivalent vertex to the left. For the cases of $\emptyset 1$ and $\emptyset 0$ above, we note that by Lemma 5.2.2, the normalizing factor $(\cdot)$ lies in $\hat{R}$ after it is "pushed" to the left side of the diagram. As $\mathcal{D}_{p|*}$ is a Karoubi envelope this means that these ϕ_i really are morphisms in $\mathcal{D}_{p|*}$. Similarly for $\underline{w}$ a rex for some $w \in {}^pW$, $\hat{R}_{\underline{w}}$ is an object in $\mathcal{D}_{p|*}$.

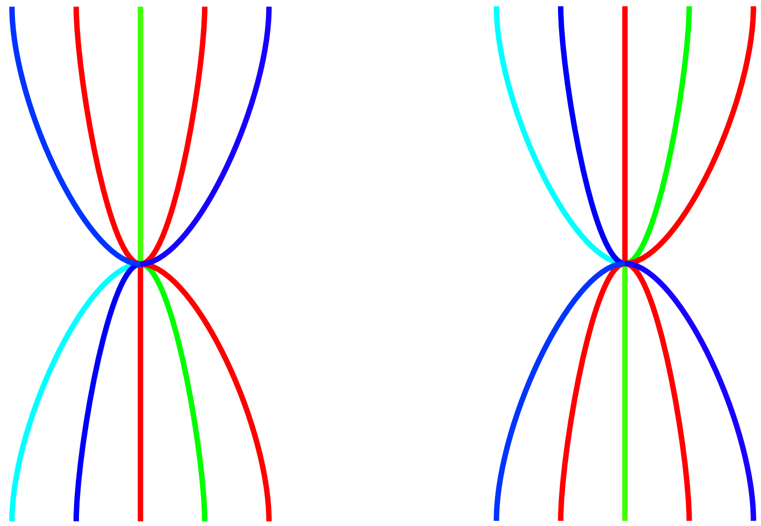
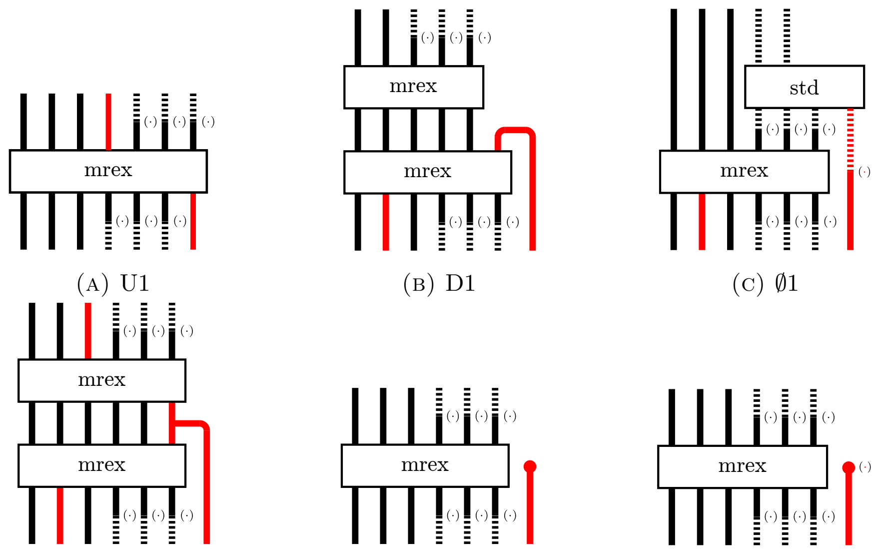


FIGURE 5.5. Braid-like versions of a menorah morphism, for $p = 3$ and $\Phi = A_2$.



(A) U1 (B) D1 (C) ∅1 (D) D0 (E) U0 (F) ∅0

FIGURE 5.6. Six maps for constructing linkage light leaves.

EXAMPLE 5.3.1. We continue Example 5.1.7, where $p = 3$ and $\Phi = A_2$, with $\underline{x} = \underline{0_p 101202122} \in \underline{S}_{p|1}$. We depict a light leaves map for the match of type $11\emptyset\emptyset\emptyset\emptyset0110$ for the pattern $**1111****\in [\underline{x}]_{p|*}$ in Figure 5.7.

Suppose for each $w \in W_p$ and $z \in {}^pW$ we have chosen rexes $\underline{w}, \underline{z}$. Let ${}^{p|*}\mathrm{LL}_{[[x]]}$ denote a complete collection of linkage light leaves maps ${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w}|\underline{z}}$ over all patterns $\underline{r} \in [\underline{x}]_{p|*}$ and all matches $\mathbf{c} \in [\underline{r}]$, where $\underline{w}$ and $\underline{z}$ are the rexes corresponding to $\hat{c}$ and $\hat{r}$ respectively.

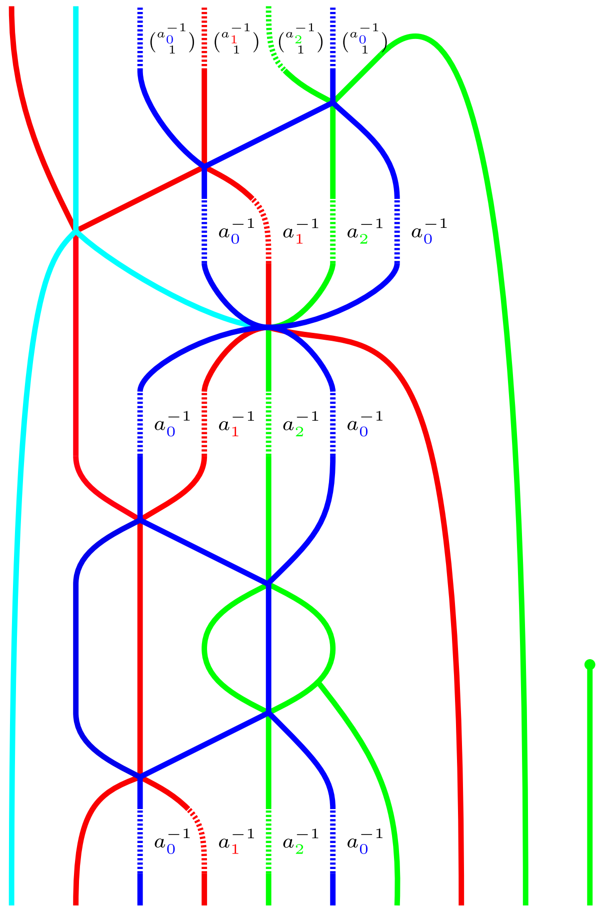


FIGURE 5.7. A linkage light leaves map.

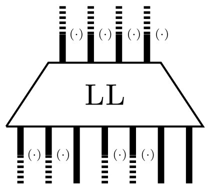
As with ordinary light leaves, for linkage expressions $\underline{x}, \underline{y} \in \underline{S}_{p|1}$ and linkage patterns $\underline{q} \in [\underline{x}]_{p|*}$ and $\underline{r} \in [\underline{y}]_{p|*}$, if we have matches $\mathbf{e} \in [\underline{q}]$ and $\mathbf{f} \in [\underline{r}]$ such that $e = \hat{e}\hat{q}$ and $\hat{f}\hat{r} = f$ are the same element $w \in W$, then we can construct the double leaves map ${}^{p|*}\mathbb{LL}^{\mathbf{f}}_{\mathbf{e}} = \overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{w}}} \circ {}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{w}}$ which is a morphism $B_{\underline{x}} \to B_{\underline{y}}$. We write ${}^{p|*}\mathbb{LL}^{[[\underline{y}]]}_{[[\underline{x}]]}$ to denote a complete selection of linkage double leaves maps $B_{\underline{x}} \to B_{\underline{y}}$.

LEMMA 5.3.2. *The valuation of a linkage light leaves or linkage double leaves map is the same as its degree.*

PROOF. Look carefully at the matrices coming from the localized versions of the vertices used to generate the light leaves maps. All the polynomial entries lie in $\hat{R}_{\mathrm{f}} \cap Q$, where $\hat{R}_{\mathrm{f}}$ denotes the subring of $\hat{R}$ consisting of all elements which don't involve $a_{\tilde{s}}$. This means calculating the valuation is the same as calculating the degree in Q. $\square$

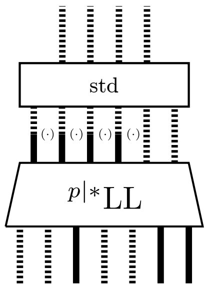
THEOREM 5.3.3. *Let $\underline{x} \in \underline{S}_{p|1}$. Suppose we have chosen a set ${}^{p|*}\mathrm{LL}_{[[\underline{x}]]}$ of linkage light leaves maps. Let $\underline{x}_0 \in \underline{S}$ be the expanded S-generator form of $\underline{x}$, where each p-affine generator is expanded using the same substitution for $\underline{\tilde{s}}_p$ used to define* *the Frobenius embedding F. Then there exists a set of partially localized ordinary light leaves maps* $\mathrm{LL}'_{[\underline{x}_0]}$*, each of the form*

$$\mathrm{LL}'_{\mathbf{e},\underline{w}_0} : B_{\underline{x}} \xrightarrow{\textit{biv. proj.}} B_{\underline{x}_0} \xrightarrow{\mathrm{LL}} B_{\underline{w}_0} \xrightarrow{\textit{biv. proj.}} \hat{R}_{\underline{w}_0}$$



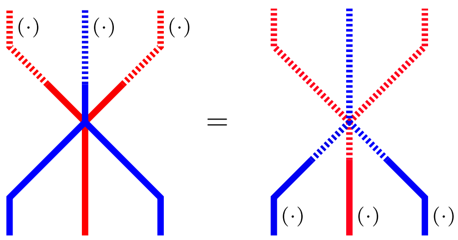
which is spanned by ${}^{p|*}\mathrm{LL}'_{[[\underline{x}]]}$*, the partially localized linkage light leaves maps, each of the form*

$${}^{p|*}\mathrm{LL}'_{\mathbf{c},\underline{w}} : B_{\underline{x}} \xrightarrow{{}^{p|*}\mathrm{LL}} B_{\underline{w}} \xrightarrow{\textit{biv. proj.}} \hat{R}_{\underline{w}} \xrightarrow{\textit{standard}} \hat{R}_{\underline{w}_0}$$

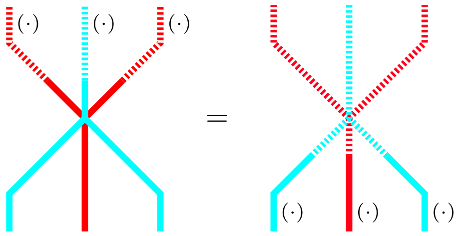


Here $\underline{w} \in \underline{S}_{p|1}$ is a reduced linkage expression, while $\underline{w}_0 \in \underline{S}$ is an ordinary reduced expression for w.

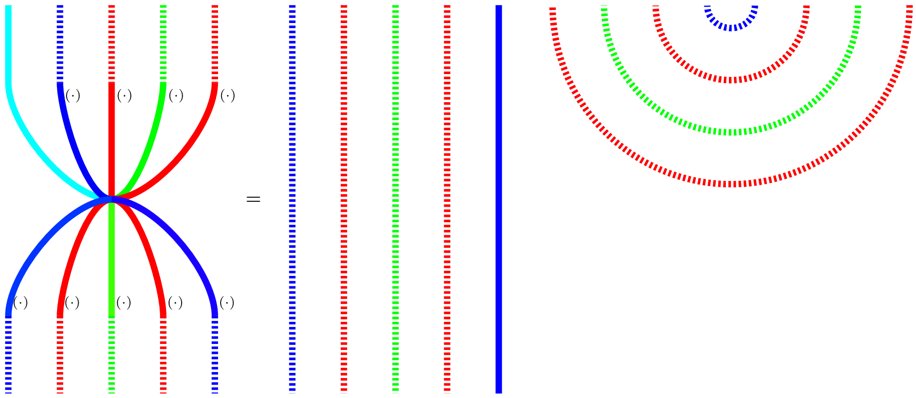
Proof. First we determine the effect of partially standardizing an mrex move. From [26, (5.28)] we know that bivalent projectors placed on the top of an ordinary solid braid (i.e. one only involving S-generators) "propagate" through the braid:



Doing the same thing with a p-affine braid results in a standard morphism, plus some projectors on the bottom:



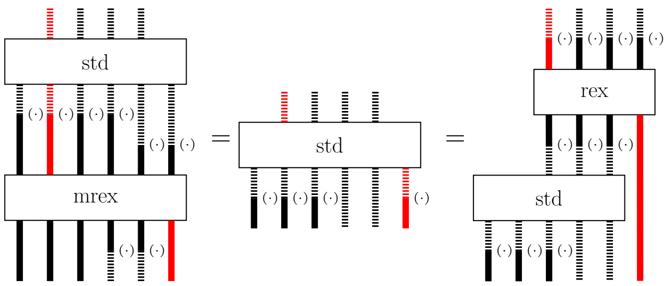
Finally if the candles of a braid-like menorah are standardized then the resulting morphism is just the identity, up to a standard morphism:



Similarly, using the Jones–Wenzl relations we can "pull" a dot placed on the top of an ordinary braid or a p-affine braid through the braid to get a rex move on a smaller expression, plus a dot on the bottom. The same is true for dots on the shamash or the handle of a braid-like menorah, as long as all the candles are standardized.

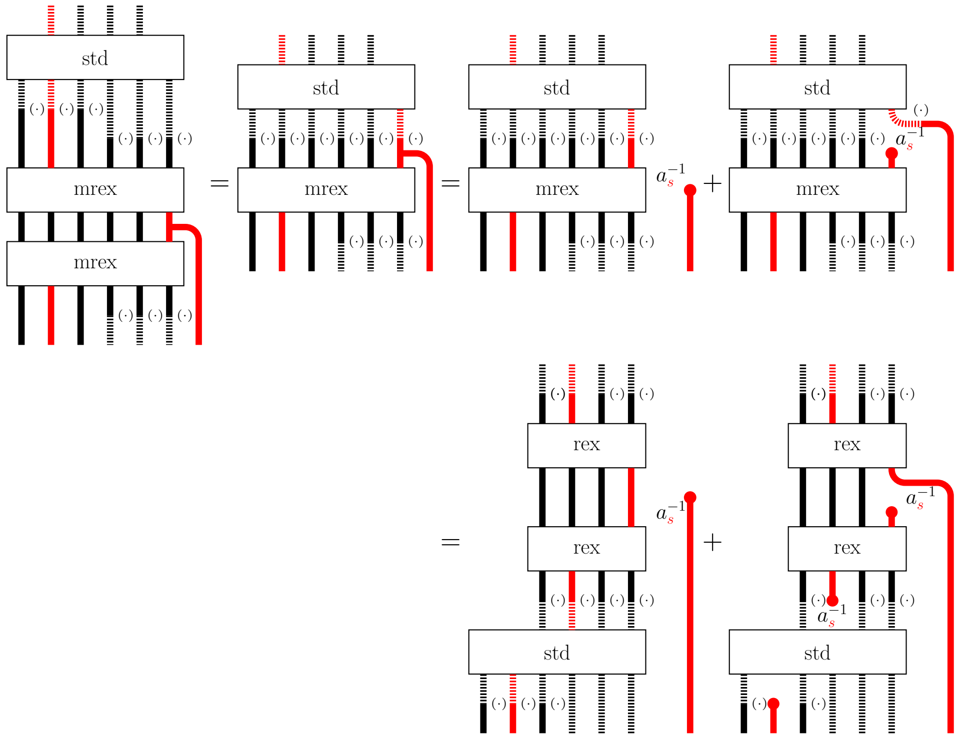
Next we try partially standardizing the maps ϕ_i above. As in Figures 4.1 and 5.6, boxes labeled "rex" are rex moves between two ordinary reduced expressions, boxes labeled "mrex" are mrex moves between two reduced linkage expression, and boxes labeled "std" are standard morphisms to a standard bimodule corresponding to some reduced expression.

When i is an indeterminate index with decoration U, we can easily show that the partially localized version of ϕ_i is nearly the same as that in the ordinary case. For example, when i is of decorated type U1 we have



The calculation for U0 is similar.

When i has decoration D we have to split the diagram into a sum. For example, when i is of decorated type D0 we have

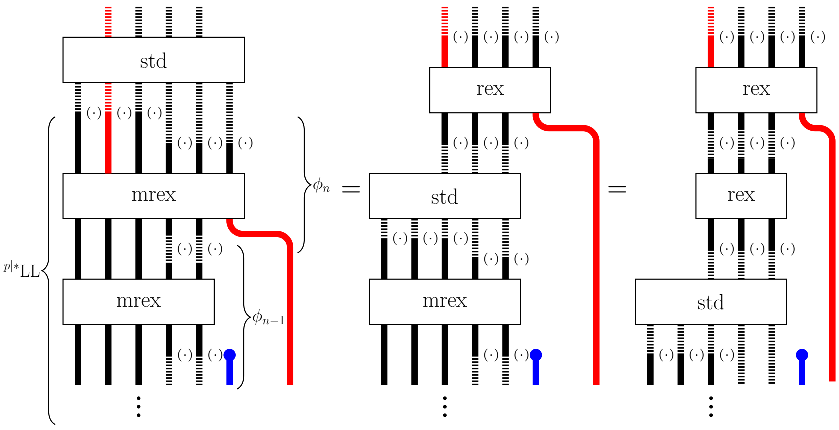


Again, the calculation for D1 is similar. In each of these cases, we get a partially localized version of one of the four maps used for defining ordinary light leaves.

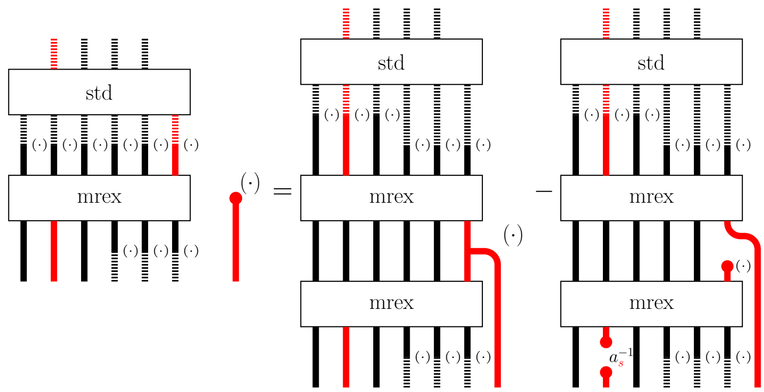
Now let $\mathbf{e} \in [\underline{x}_0]$ be a subsequence expressing w. We will show that $\mathrm{LL}'_{\mathbf{e},\underline{w}_0}$ is spanned by ${}^{p|*}\mathrm{LL}'$ maps using induction. Suppose we have already shown this for $\mathrm{LL}'_{\mathbf{f},\underline{w}_0}$ for all $\mathbf{f} < \mathbf{e}$, where the subsequences are ordered using the path dominance order introduced in [26, Section 2.4].

If any of the standardized indices in the domain of $\mathrm{LL}'_{\mathbf{e},\underline{w}_0}$ (i.e. any generator belonging to some expansion of $\tilde{s}_p$ corresponding to a candle of some menorah vertex) has type 0, then by pulling bivalent projectors and dots through braid moves, any partially localized $\mathrm{LL}'_{\mathbf{e},\underline{w}_0}$ is 0. So without loss of generality all of these indices must have type 1, and there is a unique $\mathbf{c} \in [\underline{r}]$ for some $\underline{r} \in [\underline{x}]_{p|*}$ which as a subsequence equals $\mathbf{e}$.

Now we consider ${}^{p|*}\mathrm{LL}'_{\mathbf{c},\underline{w}}$. We use the above calculations to pull the projectors (and any dots introduced by D-decorated indices) through the ϕ_i down to the bottom of the diagram. The goal is to get the resulting map to look like a light leaves map. The first step might look like



and continue downwards to the bottom of the diagram. For indeterminate indices i of $\mathbf{c}$ the resulting diagram is (possibly a scalar multiple of) a light leaves map. Fixed indices are similar except those corresponding to an index of $\mathbf{e}$ of type D0. In this situation, we use the relation



which is a difference of ordinary light leaves maps. Note that the first term in this difference looks like the corresponding ϕ_i in $\mathrm{LL}_{\mathbf{e},\underline{w}_0}$, while the second term looks like the corresponding ϕ_i in $\mathrm{LL}_{\mathbf{f},\underline{w}_0}$ for some $\mathbf{f} < \mathbf{e}$.

After pulling through ϕ_1 and getting to the bottom we have shown that ${}^{p|*}\mathrm{LL}'_{\mathbf{c},\underline{w}}$ is equal to the partially localized light leaves map $\mathrm{LL}'_{\mathbf{e},\underline{w}_0}$, plus some other partially localized light leaves maps $\mathrm{LL}'_{\mathbf{f},\underline{w}_0}$ for $\mathbf{f} < \mathbf{e}$. By induction we already know all such $\mathrm{LL}'_{\mathbf{f},\underline{w}_0}$ are spanned by linkage light leaves maps, so we are done.

□

As a result of this theorem we have the following basis result for $\mathcal{D}_{\mathrm{BS},p|*}$, analogous to Theorem 4.2.2.

COROLLARY 5.3.4. *Let $\underline{x}, \underline{y} \in \underline{S}_{p|1}$. The double leaves maps ${}^{p|*}\mathbb{LL}_{[[\underline{x}]]}^{[[\underline{y}]]}$ form a filtered $\hat{R}$-basis of* $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}}, B_{\underline{y}})^{\bullet}$.

5.3.2. Indecomposables. We can use Corollary 5.3.4 to tell us several things about the indecomposable objects in $\mathcal{D}_{p|*}$.

LEMMA 5.3.5. *The linkage category $\mathcal{D}_{p|*}$ is Krull-Schmidt.*

PROOF. Recall that any finitely generated algebra over a complete local ring is either local or contains an idempotent. For B indecomposable, the endomorphism ring $E = \mathrm{End}_{\mathcal{D}_{p|*}}(B)^\bullet$ is finitely generated as an $\hat{R}$-algebra by Corollary 5.3.4. As $\mathcal{D}_{p|*}$ is a Karoubi envelope, E cannot contain an idempotent, so it is local and thus B satisfies the Krull-Schmidt property. $\square$

THEOREM 5.3.6. *Let $w \in {}^pW$ and suppose $\underline{w}$ is a rex for w. If B is an indecomposable object in $\mathcal{D}^F$ then $F(B) \otimes \hat{R}_{\underline{w}}$ is indecomposable in $\mathcal{D}_{p|*}$.*

PROOF. Suppose B is an indecomposable summand of $B_{\underline{x}}$, where $\underline{x} \in \underline{S}$. Then $F(B) \otimes \hat{R}_{\underline{w}}$ is a direct summand of $F(B_{\underline{x}}) \otimes \hat{R}_{\underline{w}} = B_{F(\underline{x})} \otimes \hat{R}_{\underline{w}}$. Let $E \leq E'$ denote the endomorphism rings of $F(B) \otimes \hat{R}_{\underline{w}}$ and $B_{F(\underline{x})} \otimes \hat{R}_{\underline{w}}$ respectively. We can determine the generators of E' from Corollary 5.3.4. More precisely, conjugating a linkage double leaves map ${}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$ by the appropriate idempotent results in a non-zero map only when $\mathbf{e}$ and $\mathbf{f}$ match the linkage pattern $* \cdots * |1 \cdots 1 \in [\underline{x}|\underline{w}]_{p|*}$. This shows that E' is generated (as an $\hat{R}$-module) by $F(\mathrm{LL}_{[\underline{x}]}) \otimes \mathrm{id}_{\hat{R}_{\underline{w}}}$, that is to say, the image of ordinary double leaves maps under the Frobenius embedding tensored with the identity on $\hat{R}_{\underline{w}}$. Thus $E' \cong \hat{R} \otimes_R \mathrm{End}_{\mathcal{D}^F}(B_{\underline{x}})^\bullet$ and similarly $E \cong \hat{R} \otimes_R \mathrm{End}_{\mathcal{D}^F}(B)^\bullet$. Thus without loss of generality we may assume that $\underline{w}$ is the empty expression.

Let $E_0 = \hat{R}\,\mathrm{End}_{\mathcal{D}^F}(B)_0 \leq E$ be the $\hat{R}$-subalgebra generated by the degree 0 morphisms in the ordinary diagrammatic category. Note that $a_{\tilde{s}}^{-1} \in \hat{R}$, so if f is a $\mathcal{D}^F$-morphism of non-positive degree $-n$ and $r \in \hat{R}$, then

$$rf = ra_{\tilde{s}}^{-n}(a_{\tilde{s}}^n f) \in \hat{R}\,\mathrm{End}_{\mathcal{D}^F}(B)_0.$$

This shows that $\hat{R}\,\mathrm{End}_{\mathcal{D}^F}(B)_{\leq 0} \leq E_0$. In addition, for an ordinary light leaves map LL in $\mathcal{D}^F$ we have $\deg \mathrm{LL} \leq \nu(\mathrm{LL})$, which implies that

$$\mathrm{End}_{\mathcal{D}_{p|*}}(F(B))^1 \geq \hat{R}\,\mathrm{End}_{\mathcal{D}^F}(B)_{>0},$$

where because the filtrations are descending, $\mathrm{End}_{\mathcal{D}_{p|*}}(-)^1$ consists of all morphisms with positive valuation. Since $E = \sum_i \hat{R}\,\mathrm{End}_{\mathcal{D}^F}(B)_i$, combining these facts gives

$$E = E_0 + \mathrm{End}_{\mathcal{D}_{p|*}}(F(B))^1. \tag{5.11}$$

As B is indecomposable in $\mathcal{D}^F$, the ring $\mathrm{End}_{\mathcal{D}^F}(B)_0$ is local, with unique maximal ideal $\mathfrak{m}$. Let I be the following subset

$$I = \hat{R}\mathfrak{m} + (a_{S_{\mathrm{f}}})\,\mathrm{End}_{\mathcal{D}^F}(B)_0 + \mathrm{End}_{\mathcal{D}_{p|*}}(F(B))^1$$

of E, where $(a_{S_{\mathrm{f}}})$ is the maximal ideal of $\hat{R}$ and the last term is the ideal of all morphisms of positive valuation. The first two terms are ideals in E_0, so from the decomposition (5.11) I is an ideal in E. Clearly $E = E_0 + I$ follows from (5.11) as well.

We will show that all morphisms in $E \setminus I$ are invertible, and thus that E is local with maximal ideal I and that B is indecomposable in $\mathcal{D}_{p|*}$. Suppose $f \in E \setminus I$.

We write
$$f = r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}} + r_{S_{\mathrm{f}}} f_{S_{\mathrm{f}}} + r_1 f_1$$
where we have $r_0, r_{\mathfrak{m}}, r_1 \in \hat{R}$, $r_{S_{\mathrm{f}}} \in (a_{S_{\mathrm{f}}})$, $f_0, f_{S_{\mathrm{f}}} \in \mathrm{End}_{\mathcal{D}^F}(B)_0$, $f_{\mathfrak{m}} \in \mathfrak{m}$, and $f_1 \in \mathrm{End}_{\mathcal{D}_{p|*}}(F(B))^1$.

Clearly $r_0 \notin (a_{S_{\mathrm{f}}})$ and $f_0 \notin \mathfrak{m}$ as $f \notin I$. Thus we can write
$$r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}} = r_0 f_0 \left(1 + \frac{r_{\mathfrak{m}}}{r_0} f_0^{-1} f_{\mathfrak{m}} \right).$$
The subalgebra $\mathrm{End}_{\mathcal{D}^F}(B)_0$ is finite-dimensional, so the maximal ideal $\mathfrak{m}$ is nilpotent. But $f_0^{-1} f_{\mathfrak{m}}$ is contained in $\mathfrak{m}$, so the sum on the right-hand side above is not in $\mathfrak{m}$ and is therefore invertible. Thus $r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}}$ is invertible.

The remaining two terms in the sum for f above are contained in an ideal $J = (a_{S_{\mathrm{f}}}) \mathrm{End}_{\mathcal{D}^F}(B)_0 + \mathrm{End}_{\mathcal{D}_{p|*}}(F(B))^1$. From Theorem 4.2.2 and Corollary 5.3.4 J is generated as an $\hat{R}$-module by morphisms $a_s \mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$ (for $d(\mathbf{e}) + d(\mathbf{f}) = 0$ and any $s \in S_{\mathrm{f}}$) and ${}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$ (for $\hat{d}(\mathbf{e}) + \hat{d}(\mathbf{f}) > 0$). This basis is finite, so for sufficiently large n we have $J^n \leq (a_{S_{\mathrm{f}}})J$. Yet $\hat{R}$ is complete with respect to its maximal ideal $(a_{S_{\mathrm{f}}})$ so $f \in (r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}}) + J$ is invertible using the standard formula $(1+x)^{-1} = 1 + x + x^2 + \cdots$ for the inverse of a nilpotent element x. □

For $x \in W_p$ let $B_x = F(B_{F^{-1}(x)})$ be the indecomposable object in $\mathcal{D}_{p|*}$ induced by the above result. As in $\mathcal{D}^F$, the object B_x is well-defined by x alone — we do not need to specify a rex for $\underline{x}$. In particular it can be constructed indirectly in the following manner.

For $I \subseteq W$ a poset ideal with respect to the Bruhat order, let ${}^{p|*}\mathbb{LL}_I$ be the span of double leaves maps ${}^{p|*}\mathbb{LL}$ which factor through $B_{\underline{x}} \otimes \hat{R}_{\underline{w}}$ for rexes $\underline{x}|\underline{w} \in \underline{S}_{p|1}$ with $xw \in I$. It can be shown that ${}^{p|*}\mathbb{LL}_I$ is in fact a 2-sided ideal of morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$. In a similar way to what happens in $\mathcal{D}_{\mathrm{BS}}$ (see [26, Section 6.4]) this ideal is in fact equal to the ideal of morphisms which, after localization, induce the zero map on every object Q_{xw} for each $xw \notin I$. In any case, for any $w \in W$ we define the quotient category $\mathcal{D}_{p|*}^{\geq w} = \mathcal{D}_{p|*} / {}^{p|*}\mathbb{LL}_I$, where $I = \{z \in W : z \not\geq w\}$. For $x \in W_p$ and $w \in {}^pW$, the object $B_x \otimes \hat{R}_w$ is the unique indecomposable summand of $B_{\underline{x}} \otimes \hat{R}_{\underline{w}}$ (for some rexes $\underline{x}, \underline{w}$ of x, w) which does not vanish in $\mathcal{D}_{p|*}^{\geq xw}$.

THEOREM 5.3.7. *Any indecomposable object in $\mathcal{D}_{p|*}$ is filtered isomorphic to a filtration shift of $B_y \otimes \hat{R}_w$ for some $y \in W_p$ and $w \in {}^pW$.*

PROOF. This is similar to [26, Theorem 6.25]. Let B be an indecomposable object of $\mathcal{D}_{p|*}$. Suppose B is a direct summand of $B_{\underline{x}}$ for some $\underline{x} \in \underline{S}_{p|1}$, and that $e \in \mathrm{End}_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}})^{\bullet}$ is the idempotent corresponding to this summand. We can write
$$e = \sum \lambda_{\mathbf{e},z,\mathbf{f}} \, {}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$$
where $\lambda_{\mathbf{e},z,\mathbf{f}} \in \hat{R}$, summing over matches $\mathbf{e}, \mathbf{f}$ for linkage patterns for $\underline{x}$ corresponding to the same group element $z \in W$. Pick $z' \in W$ maximal in the Bruhat order

such that $\lambda_{\mathbf{e},z',\mathbf{f}} \neq 0$ for some matches $\mathbf{e},\mathbf{f}$. In $\mathcal{D}_{p|*}^{\geq z'}$ we get

$$e = \sum \gamma_{\mathbf{e},\mathbf{f}} (\overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{y}|\underline{w}}} \circ {}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{y}|\underline{w}})$$

for some coefficients $\gamma_{\mathbf{e},\mathbf{f}} \in \hat{R}$, summed over matches $\mathbf{e},\mathbf{f}$ whose corresponding subsequences evaluate to z'. Now assume that for all matches in the sum we have

$${}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{y}|\underline{w}} \circ e \circ \overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{y}|\underline{w}}} \in (a_{S_\mathrm{f}}) \leq \hat{R} = \mathrm{End}_{\mathcal{D}_{p|*}^{\geq z'}}(B_{\underline{y}} \otimes \hat{R}_{\underline{w}})^{\bullet}.$$

Then by expanding out $e^3 = e$ we get $\gamma_{\mathbf{e},\mathbf{f}} \in (a_{S_\mathrm{f}})$. But this implies that

$$e \in (a_{S_\mathrm{f}})\,\mathrm{End}_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}})^{\bullet} \leq J\left(\mathrm{End}_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}})^{\bullet}\right)$$

where $J(-)$ denotes the Jacobson radical. Since e is idempotent, we obtain a contradiction. Hence there must be matches $\mathbf{e},\mathbf{f}$ for which the following composition

$$B_{\underline{y}} \otimes \hat{R}_w \xrightarrow{\overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{y}|\underline{w}}}} B \xrightarrow{{}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{y}|\underline{w}}} B_{\underline{y}} \otimes \hat{R}_w$$

is invertible in $\mathcal{D}_{p|*}^{\geq z'}$. This induces an invertible morphism

$$B_y \otimes \hat{R}_w \xrightarrow{i} B \xrightarrow{p} B_y \otimes \hat{R}_w$$

which proves the result. $\square$

5.3.3. Linkage characters. Let $\mathcal{D}_{\mathrm{BS},p|*}^{\oplus,\langle-\rangle}$ denote the additive, filtered closure of $\mathcal{D}_{\mathrm{BS},p|*}$. As with $\mathcal{D}$ we can define what we call the linkage character homomorphism $\mathrm{ch} : [\mathcal{D}_{\mathrm{BS},p|*}^{\oplus,\langle-\rangle}] \to \mathcal{H}_{p|*}$ as

$$\begin{aligned} \mathrm{ch} : [\mathcal{D}_{\mathrm{BS},p|*}^{\oplus,\langle-\rangle}] &\longrightarrow \mathcal{H}_{p|*} \\ [B_{\underline{x}}] &\longmapsto [\underline{x}]_{p|*} \\ [\hat{R}(1)] &\longmapsto v = \begin{Bmatrix} \mathrm{U} \\ \emptyset \\ \emptyset \end{Bmatrix} \end{aligned}$$

To see that this is well defined, compose with the natural map $\mathcal{H}_{p|*} \to [\mathcal{H}_{p|*}] \cong \mathbb{H}_{p|*}$. Using Theorem 5.3.3, we get

$$\begin{aligned} [\mathcal{D}_{\mathrm{BS},p|*}^{\oplus,\langle-\rangle}] &\xrightarrow{\mathrm{ch}} \mathcal{H}_{p|*} \to \mathbb{H}_{p|*} \\ [B_{\underline{x}}] &\longmapsto \sum \dim_{\bullet} \mathrm{Hom}_{\mathcal{D}_{p|*}^{\oplus,\langle-\rangle,\geq yw}}(B_{\underline{x}}, B_{\underline{y}|\underline{w}})_{\bullet} H_y H_w \end{aligned}$$

where the sum is over all $y \in W_p$ and $w \in {}^pW$ with $\underline{y},\underline{w}$ any rexes for these group elements, and $\dim_{\bullet}$ here gives the *filtered* dimension, or in other words the graded dimension of the associated graded vector space formed from successive subquotients. Clearly the right-hand side only depends on the isomorphism class of $B_{\underline{x}}$, so the character homomorphism is indeed well defined. In addition, our knowledge of the indecomposables from the previous section establishes that two objects have the same character if and only if they are isomorphic. As the homomorphism is obviously surjective on the generators of $\mathcal{H}_{p|*}$, we have shown the following.

PROPOSITION 5.3.8. *The map* ch *is an isomorphism of left $\mathcal{H}_p$-modules, where the left $\mathcal{H}_p$-module structure on $[\mathcal{D}^{\oplus,\langle-\rangle}_{\mathrm{BS},p|*}]$ comes from the left $\mathcal{D}^F_{\mathrm{BS}}$-module structure and the isomorphism $[\mathcal{D}^{F,\oplus,(-)}_{\mathrm{BS}}] \cong \mathcal{H} \overset{F}{\cong} \mathcal{H}_p$.*

An easy corollary is

COROLLARY 5.3.9. *The Grothendieck module of the linkage category is*

$$[[\mathcal{D}_{p|*}]] \cong [[\mathcal{D}^{\oplus,\langle-\rangle}_{\mathrm{BS},p|*}]] \cong [\mathcal{H}_{p|*}] \cong \mathbb{H}_{p|*}.$$

5.4. The linkage functor

5.4.1. Construction. Consider the category $\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}}$ of ungraded Bott–Samelson bimodules in $\underline{S}$. As a monoidal category, by general principles it is isomorphic to the category of endofunctors of the form $(- \otimes B_{\underline{x}})$ for expressions $\underline{x} \in \underline{S}$. We will show that functors of the same form acting instead on $\mathcal{D}^{\oplus,\langle-\rangle}_{\mathrm{BS},p|*}$ give a faithful representation of $\mathcal{D}^{\mathrm{ungr}}_{\mathrm{BS}}$. By rewriting this representation in terms of $\mathcal{D}^F$ we will obtain the linkage functor.

For each $w \in {}^pW$, fix a rex $\underline{w}$. From Theorem 5.3.7 the indecomposables in $\mathcal{D}_{p|*}$ are each of the form $F(B) \otimes \hat{R}_{\underline{w}}$ for B an indecomposable in $\mathcal{D}^F$ and $w \in {}^pW$, so in some sense the category $\mathcal{D}_{p|*}$ decomposes as a left $(\hat{R} \otimes \mathcal{D}^F)$-module as $\bigoplus_{w \in {}^pW} (\hat{R} \otimes \mathcal{D}^F)_w$, a direct sum of copies of scalar extensions of $\mathcal{D}^F$ indexed by ${}^pW$. The functor $(-\otimes B_{\underline{x}})$ acting on $\mathcal{D}_{p|*}$ commutes with this left $\mathcal{D}^F$-structure, so it should have a "matrix form" in terms of this categorical decomposition. Calculating this matrix form is equivalent to finding (for each $w \in {}^pW$) decompositions of $\hat{R}_{\underline{w}} \otimes B_{\underline{x}}$ into direct summands of the form $B_{\underline{y}} \otimes \hat{R}_{\underline{z}}$, where $\underline{y} \in \underline{S}_p$ and $z \in {}^pW$.

In fact, there is a tailor-made method of doing this using the linkage sections $[\underline{x}]_*$. Namely for each such w and $\underline{x}$ we have the isomorphism

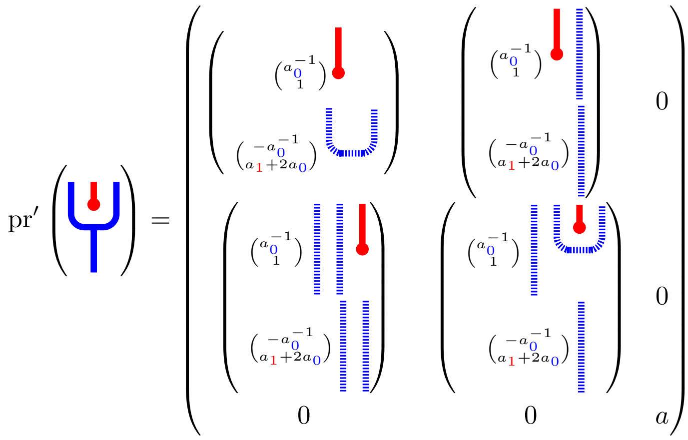
$$\hat{R}_{\underline{w}} \otimes B_{\underline{x}} \overset{\mathrm{std}}{\cong} \bigoplus_{z \in {}^pW} \bigoplus_{\substack{\underline{r} \in [\underline{x}]_*(w) \\ w\hat{r} = z}} B_{w\underline{r}z^{-1}} \otimes \hat{R}_{\underline{z}}, \tag{5.12}$$

where $w\underline{r}z^{-1}$ is viewed as an expression in $\underline{S}_p$. Here the isomorphism is only up to composition with standard morphisms, but this is enough for our purposes. The isomorphism arises by decomposing some of the generators in $\underline{x}$ using (4.18) (i.e. localizing) according to the patterns in $[\underline{x}]_*$. We will explain this below in more detail.

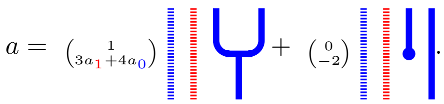
To each morphism $f : B_{\underline{x}} \to B_{\underline{y}}$ in $\mathcal{D}^{\mathrm{ungr},\oplus}_{\mathrm{BS}}$ we associate a ${}^pW \times {}^pW$ array[3] $\mathrm{pr}'(f)$ of morphisms in $\mathcal{D}_{p|*}$ as follows. The (w, z)-entry of $\mathrm{pr}'(f)$ is the partial localization of the morphism $\hat{R}_w \otimes f$ with respect to patterns $\underline{q} \in [\underline{x}]_*(w)$ and $\underline{r} \in [\underline{y}]_*(w)$, such that $w\hat{q} = w\hat{r} = z$.

EXAMPLE 5.4.1. Suppose $p = 3$ and $\Phi = A_1$. Label the unique finite generator 1 (colored red), and the affine generator 0 (colored blue). Here is an example of pr' acting on a morphism $B_{\underline{010}} \to B_{\underline{0}}$:

[3]We use "array" here to denote a matrix without the structure of matrix multiplication.



where

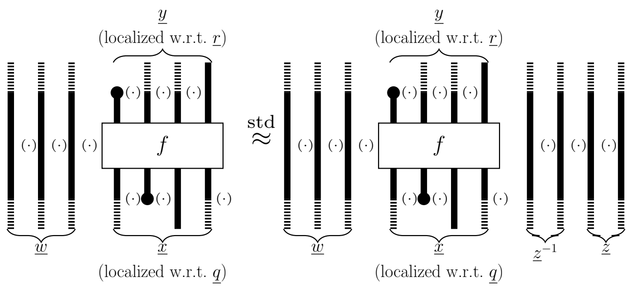


PROPOSITION 5.4.2. *Let f be a morphism in $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus}$. The (w,z)-entry of $\mathrm{pr}'(f)$ is always in $\hat{R} \otimes F(\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}) \otimes \sigma_{\hat{R}_z}$ for $z \in {}^pW$ and $\sigma_{\hat{R}_z}$ a standard morphism. In other words, each entry consists of the Frobenius embedding of a morphism in $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}$ tensored with a standard morphism on $\hat{R}_z$, and with additional leftmost coefficients in $\hat{R}$.*

PROOF. Let $f : B_{\underline{x}} \to B_{\underline{y}}$ be a morphism in $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus}$, and let $w, z \in {}^pW$ with corresponding rexes $\underline{w}, \underline{z} \in \underline{S}$. The map

$$\mathrm{id} \otimes f \otimes \mathrm{id} : B_{\underline{w}} \otimes B_{\underline{x}} \otimes B_{\underline{z}^{-1}} \to B_{\underline{w}} \otimes B_{\underline{y}} \otimes B_{\underline{z}^{-1}}$$

is in the linkage category $\mathcal{D}_{\mathrm{BS},p|*}$, so we can decompose it using the linkage double leaves basis. The (w,z)-entry in $\mathrm{pr}'(f)$ comes from localizing $\hat{R}_w \otimes f$ in a particular way. Since we can write



this shows that the (w,z)-entry can be written in terms of partially localized double leaves maps (with domain/codomain in $F(\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus})$) tensored with the identity on $\hat{R}_z$, which gives the result. □

Now let $\mathrm{pr}(f)$ denote the ${}^pW \times {}^pW$ matrix of morphisms whose (w,z)-entry is the $\hat{R} \otimes F(\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus})$ part of the (w,z)-entry of $\mathrm{pr}'(f)$. For notational simplicity we will usually omit the Frobenius embedding and write $\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}$, although in diagrams we will still use the lighter versions of colors reserved for p-affine generators (e.g. cyan for blue etc.). We first note that $\mathrm{pr}(f)$ gives the decompositions of $\hat{R}_{\underline{w}} \otimes f$ using (5.12). By the properties of localization, for two morphisms f, g we have

$$\mathrm{pr}(f \circ g) = \mathrm{pr}(f) \cdot_{{}^pW} \mathrm{pr}(g)$$

where the operator $\cdot_{{}^pW}$ denotes the Hadamard product, or entrywise multiplication of matrices, with entry multiplication interpreted as function composition. Thus pr defines a functor into $\mathrm{End}_{\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}} \mathcal{D}_{p|*}$, the category of endofunctors of $\mathcal{D}_{p|*}$.

Moreover, this category has monoidal structure from functor composition, which corresponds to matrix multiplication of the matrices arising from pr, with entry multiplication and addition interpreted as tensor product and direct sum inside each copy of $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}$. For $\underline{x}, \underline{y} \in \underline{S}$ it can be shown that the decomposition of $\hat{R}_{\underline{w}} \otimes B_{\underline{x}} \otimes B_{\underline{y}}$ behaves well with respect to linkage set multiplication:

$$\begin{aligned}
\hat{R}_{\underline{w}} \otimes (B_{\underline{x}} \otimes B_{\underline{y}}) &= (\hat{R}_{\underline{w}} \otimes B_{\underline{x}}) \otimes B_{\underline{y}} \\
&\overset{\mathrm{std}}{\cong} \left(\bigoplus_{z \in {}^pW} \bigoplus_{\substack{\underline{q} \in [\underline{x}]_*(w) \\ w\hat{q} = z}} B_{w\underline{q}z^{-1}} \otimes \hat{R}_{\underline{z}} \right) \otimes B_{\underline{y}} \\
&\overset{\mathrm{std}}{\cong} \bigoplus_{z, z' \in {}^pW} \bigoplus_{\substack{\underline{q} \in [\underline{x}]_*(w) \\ w\hat{q} = z}} \bigoplus_{\substack{\underline{r} \in [\underline{y}]_*(z) \\ z\hat{r} = z'}} B_{w\underline{q}z^{-1}} \otimes B_{z\underline{r}z'^{-1}} \otimes \hat{R}_{\underline{z}'} \\
&\overset{\mathrm{std}}{\cong} \bigoplus_{z' \in {}^pW} \bigoplus_{\substack{\underline{q} \in [\underline{x}]_*(w) \\ \underline{r} \in [\underline{y}]_*(w\hat{q}) \\ w\hat{q}\hat{r} = z'}} B_{w\underline{qr}z'^{-1}} \otimes \hat{R}_{\underline{z}'} \\
&\overset{\mathrm{std}}{\cong} \bigoplus_{z \in {}^pW} \bigoplus_{\substack{\underline{q} \in [\underline{x}]_*[\underline{y}]_*(w) \\ w\hat{q}\hat{r} = z}} B_{w\underline{qr}z^{-1}} \otimes \hat{R}_{\underline{z}}
\end{aligned}$$

This is enough to show that pr is monoidal.

We summarize our results in the following theorem.

THEOREM 5.4.3. *The mapping* pr *defines a functor*

$$\mathrm{pr} : \mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus} \longrightarrow \mathrm{End}_{\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}} \left(\bigoplus_{w \in {}^pW} \left(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus} \right)_w \right)$$

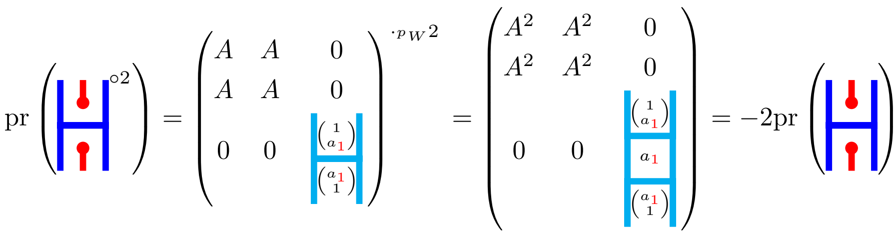
from $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus}$ to left $\left(\hat{R} \otimes \mathcal{D}^{\mathrm{ungr},F,\oplus}\right)$-endofunctors of a direct sum of $|{}^pW|$ copies of $\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},F,\oplus}$. Moreover, this functor preserves the monoidal structure; for morphisms f, g in $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus}$ we have

$$\mathrm{pr}(f \otimes g) = \mathrm{pr}(f) \cdot \mathrm{pr}(g)$$
$$\mathrm{pr}(f \circ g) = \mathrm{pr}(f) \cdot_{{}^pW} \mathrm{pr}(g)$$

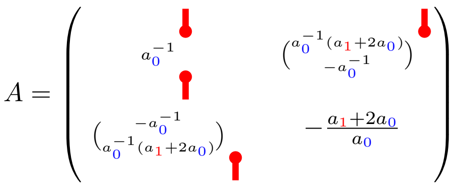
where the first operator $\cdot$ denotes matrix multiplication and the second operator $\cdot_{{}^pW}$ denotes the Hadamard product.

We call pr the linkage functor. As promised it describes a connection between Soergel bimodules in $\mathcal{D}$ at ordinary scales and Soergel bimodules in $\mathcal{D}^F$ at scale p.

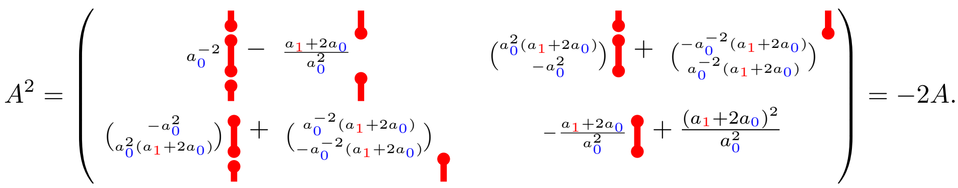
EXAMPLE 5.4.4. As before suppose $p = 3$ and $\Phi = A_1$ with labeling as in Example 5.4.1. Here is an example of the Hadamard product on an idempotent (up to scaling):



where



and



5.4.2. Linkage sections. We decategorify the effects of the linkage functor, using a map similar to the linkage character homomorphism. It will be useful to introduce a new category to extend $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus}$. First we define a selective version of a filtration shift in $\mathcal{D}_{p|*}$. For each $A \in {}^p\mathcal{W}(*)$ let

$$(B_y \otimes \hat{R}_w)\langle 1\rangle_A = \begin{cases} (B_y \otimes \hat{R}_w)\langle 1\rangle & \text{if } w \in A, \\ (B_y \otimes \hat{R}_w) & \text{otherwise.} \end{cases}$$

These filtration shifts can be combined and inverted in all the ways one might expect, giving an action of the multiplicative abelian group

$$\langle u_A : A \in {}^p\mathcal{W}(*)\rangle_{\mathrm{mult}} \subset \mathbb{H}_*$$

on the category of endofunctors of $\mathcal{D}_{p|*}$. Note in particular that although selective filtration shifts commute with each other, they do not necessarily commute with other functors, including functors of the form $(- \otimes B_{\underline{x}})$. Let $\mathcal{D}_{\mathrm{BS}}^{\oplus,\langle - \rangle}$ denote the category of functors generated by $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr},\oplus}$ and all selective filtration shift functors.

The section homomorphism is a map sec : $[\mathcal{D}_{\mathrm{BS}}^{\oplus,\langle - \rangle}] \to \mathcal{H}_*$ given by

$$\begin{aligned} \mathrm{sec} : [\mathcal{D}_{\mathrm{BS}}^{\oplus,\langle - \rangle}] &\longrightarrow \mathcal{H}_* \\ [(- \otimes B_{\underline{x}})] &\longmapsto [\underline{x}]_* \\ [(- \otimes R)\langle 1 \rangle_A] &\longmapsto u_A \end{aligned}$$

This is well defined: if two functors on the left-hand side are isomorphic, then they induce the same action on $\mathcal{D}_{p|*}$, which in turn means that the corresponding linkage sections on the right-hand side induce the same action on $\mathcal{H}_{p|*}$ and are thus equivalent. As with linkage characters we can reverse this reasoning to show that sec is injective. Namely, if two functors F, G in $\mathcal{D}_{\mathrm{BS}}^{\oplus,\langle - \rangle}$ induce the same linkage section, then we can pick isomorphisms $F(\hat{R}_w) \cong G(\hat{R}_w)$ for each $w \in {}^pW$ and extend this to a natural isomorphism $F \cong G$ using our knowledge of the indecomposables in $\mathcal{D}_{p|*}$. Finally the map is obviously surjective, so we have shown the following.

Proposition 5.4.5. *The map* sec *is an isomorphism of left $\mathbb{L}_{\geq 0}$-algebras. Moreover, the previous map* ch *is an isomorphism of right $\mathcal{H}_*$-modules, where the right $\mathcal{H}_*$-module structure on $[\mathcal{D}_{\mathrm{BS},p|*}^{\oplus,\langle - \rangle}]$ comes from the right $\mathcal{D}_{\mathrm{BS}}^{\mathrm{ungr}}$-module structure on $\mathcal{D}_{\mathrm{BS},p|*}$.*

Let $\mathcal{D}^{\langle - \rangle}$ be the Karoubi envelope of $\mathcal{D}_{\mathrm{BS}}^{\oplus,\langle - \rangle}$, i.e. the closure with respect to all direct sums, direct summands, and selective filtration shifts. It is clearly an extension of $\mathcal{D}^{\mathrm{ungr}}$, the de-graded version of $\mathcal{D}$.

Corollary 5.4.6. *The Grothendieck ring of $\mathcal{D}^{\langle - \rangle}$ is*

$$[[\mathcal{D}^{\langle - \rangle}]] \cong [[\mathcal{D}_{\mathrm{BS}}^{\oplus,\langle - \rangle}]] \cong [\mathcal{H}_*] \cong \mathbb{H}_*.$$

5.4.3. Quantized linkage algebra. We now attempt to combine some of the decategorified aspects of $\mathcal{D}$ and $\mathcal{D}^{\langle - \rangle}$. Let $\mathbb{L}_q = \mathbb{L}[q^{\pm 1}] = \mathbb{Z}[q^{\pm 1}, v^{\pm 1}]$, a Laurent polynomial ring in two variables.

Definition 5.4.7. The quantized p-linkage Hecke algebra $\mathbb{H}_q$ is the $\mathbb{L}_q[\frac{1}{2}]$-algebra with generators

u_A	for each $A \in {}^p\mathcal{W}(*)$,
H_s	for each $s \in S$,

and relations

$$u_\emptyset = 1, \qquad u_{{}^pW} = q, \tag{5.13}$$

$$u_A^2 = (q+1)u_A - q \qquad \text{for all } A \in {}^p\mathcal{W}(*), \tag{5.14}$$

$$u_A + u_B = u_{A\cup B} + u_{A\cap B} \qquad \text{for all } A, B \in {}^p\mathcal{W}(*), \tag{5.15}$$

$$u_A u_B = u_{A\cup B} u_{A\cap B} \qquad \text{for all } A, B \in {}^p\mathcal{W}(*), \tag{5.16}$$

$$H_s^2 = 1 + (qv^{-1}u_{s(*)}^{-1} - q^{-1}vu_{s(*)})H_s \qquad \text{for all } s \in S, \tag{5.17}$$

$$\overbrace{H_sH_tH_s\cdots}^{m_{st}\text{ terms}} = \overbrace{H_tH_sH_t\cdots}^{m_{st}\text{ terms}} \qquad \text{for all } s, t \in S, \tag{5.18}$$

$$H_su_A - u_{As}H_s = \frac{qv^{-1} - q^{-1}v}{2}(u_A - u_{As}) \qquad \text{for all } s \in S \text{ and } A \in {}^p\mathcal{W}(*). \tag{5.19}$$

From the relations we see that $\mathbb{H}_q/(q-v) \cong \mathbb{H}_*$, while

$$\mathbb{H}_q/(q-1, u_A : A \in {}^p\mathcal{W}(*)) \cong \mathbb{H}.$$

When evaluating in the second quotient, we will usually abuse notation and say that we are "setting $q = 1$". There is also a similar quantization $\mathbb{H}_{p|q}$ of the linkage bimodule.

Definition 5.4.8. The quantized linkage bimodule $\mathbb{H}_{p|q}$ is the $(\mathbb{H}_p, \mathbb{H}_q)$-bimodule described as follows. As an $\mathbb{L}_q[\frac{1}{2}]$-module it has basis

$$H_xH_w \qquad \text{where } x \in W_p \text{ and } w \in {}^pW.$$

In this basis, the $\mathbb{H}_p$-action is given by $H_s(H_xH_w) = (H_sH_x)H_w$ for all $s \in S_p$, while the $\mathbb{H}_q$-action is given by

$$(H_xH_w)H_s = \begin{cases} (H_xH_{wsw^{-1}})H_w & \text{if } W_pws = W_pw, \\ H_xH_{ws} & \text{if } W_pws \neq W_pw \text{ and } ws > w, \\ H_xH_{ws} + (qv^{-1} - q^{-1}v)H_xH_w & \text{if } W_pws \neq W_pw \text{ and } ws < w, \end{cases} \tag{5.20}$$

and

$$(H_xH_w)u_A = \begin{cases} qH_xH_w & \text{if } w \in A, \\ H_xH_w & \text{otherwise} \end{cases} \tag{5.21}$$

for all $s \in S$ and $A \in {}^p\mathcal{W}(*)$.

When $q = v$ we get the linkage bimodule $\mathbb{H}_{p|*}$ from before, and when $q = 1$ the resulting bimodule is isomorphic (as a right $\mathbb{H}$-module) to the right regular representation.

We define the bar involution $(\overline{}) : \mathbb{H}_q \to \mathbb{H}_q$ on $\mathbb{H}_q$ as the algebra homomorphism with the following action

$$\begin{aligned}
\overline{q} &= q^{-1}, && \\
\overline{v} &= v^{-1}, && \\
\overline{u_A} &= u_A^{-1} && \text{for each } A \in {}^p\mathcal{W}(*), \\
\overline{H_s} &= H_s + q^{-1}vu_{s(*)} - qv^{-1}u_{s(*)}^{-1} && \text{for each } s \in S
\end{aligned} \tag{5.22}$$

on generators. This descends to the familiar bar involution on $\mathbb{H}$.

Similarly, we define the bar involution on $\mathbb{H}_{p|*}$ as the unique bar-linear module homomorphism $(\overline{}) : \mathbb{H}_{p|q} \to \mathbb{H}_{p|q}$ with the following action

$$\overline{H_x H_w} = \overline{H_x} \cdot \overline{H_w} \qquad \text{for all } x \in W_p \text{ and } w \in {}^pW \tag{5.23}$$

on the basis.

For $s \in S$ let ${}^q\underline{H}_s = H_s + q^{-1}vu_{s(*)} \in \mathbb{H}_q$. As in $\mathbb{H}$ we can easily verify that ${}^q\underline{H}_s$ is self-dual, descending to $\underline{H}_s$ in $\mathbb{H}$ when $q = 1$.

LEMMA 5.4.9. *Let $x \in W$. The element $[\sec B_x] \in \mathbb{H}_*$ is self-dual.*

PROOF. Recall that $\mathcal{D}$ has a duality functor $(\overline{}) : \mathcal{D} \to \mathcal{D}^{\mathrm{op}}$ which fixes Bott–Samelson objects, reverses grade shifts, and flips S-diagrams upside-down. This functor can be extended to the linkage category $\mathcal{D}_{p|*}$ and also to the functor category $\mathcal{D}^{\langle - \rangle}$. An induction on $\ell(x)$ then proves the statement in the same way as the corresponding statement in $\mathcal{D}$ (e.g. [55, Proposition 4.2(1)]). $\square$

Combining this result with basic facts about the p-canonical basis [55], we get

COROLLARY 5.4.10. *Let $x \in W$. There exists a self-dual $\underline{\hat{H}}_x \in \mathbb{H}_q$, unique modulo $(q-1)(q-v)\mathbb{H}_q$, such that*

$$\underline{\hat{H}}_x \xmapsto{q=1} {}^p\underline{H}_x$$
$$\underline{\hat{H}}_x \xmapsto{q=v} {}^q\underline{H}_x$$

where ${}^p\underline{H}_x = [\mathrm{ch}\, B_x]$ denotes the p-canonical basis.

We can now prove a consequence of linkage for Soergel bimodules analogous to higher-order linkage of tilting modules, as discussed in the beginning of this chapter.

THEOREM 5.4.11. *Let $x \in W$. The quotient of $\mathbb{H}$ by the ideal generated by $(v-1)$ is the group ring $\mathbb{Z}W$. Let $\{{}_{v=1}^{\;p}\underline{H}_x\}$ denote the image of the p-canonical basis in $\mathbb{Z}W$. Then*

$${}_{v=1}^{\;p}\underline{H}_x \in \sum_{\substack{y \in W_p \\ w \in {}^pW}} \mathbb{Z}_{\geq 0} F\left({}_{v=1}^{\;p}\underline{H}_{F^{-1}(y)}\right) H_w,$$

where F is extended linearly on $\mathbb{Z}W$.

PROOF. Consider $1 \cdot \underline{\hat{H}}_x$, an element of $\mathbb{H}_{p|q}$ modulo $(q-1)(q-v)\mathbb{H}_{p|q}$. Setting $q = v$ gives the linkage character of $[\hat{R} \otimes B_x]$ in $[\mathcal{D}_{p|*}]$, which must be the sum of

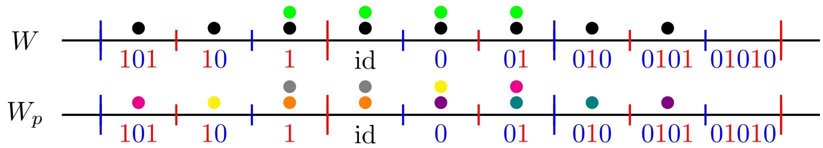


FIGURE 5.8. Weight diagrams for ${}^3\underline{H}_{010}\underline{H}_1$.

linkage characters of indecomposables, i.e.

$$ {}_{q=v}(1 \cdot \underline{\hat{H}}_x) \in \sum_{\substack{y \in W_p \\ w \in {}^pW}} \mathbb{Z}_{\geq 0}\; {}_{q=v}\underline{\hat{H}}_y H_w, $$

where $\underline{\hat{H}}_y$ is interpreted as an element of $\mathbb{H}_{W_p}$, which can be calculated from the corresponding $\mathbb{H}$-element as the F-conjugate $F\left({}_{q=v}\underline{\hat{H}}_{F^{-1}(y)}\right)$. Taking quotients by $v = 1$ gives the result. $\square$

EXAMPLE 5.4.12. Let $\Phi = A_1$ and $p = 3$. We have ${}^3\underline{H}_{010}\underline{H}_1 = \underline{H}_{0101} + \underline{H}_{01}$. By [55, Proposition 4.2(6)] this is a sum of p-canonical basis elements, so we can apply Theorem 5.4.11. Setting $v = 1$ we get

$$ {}^{3}_{v=1}\underline{H}_{010}\underline{H}_1 = 2({}^{3}_{v=1}\underline{H}_1)H_{\text{id}} + ({}^{3}_{v=1}\underline{H}_1 + {}^{3}_{v=1}\underline{H}_{0_3})H_0 + ({}^{3}_{v=1}\underline{H}_1 + {}^{3}_{v=1}\underline{H}_{0_3})H_{01}. $$

We depict this using weight diagrams as in Figure 5.8, where the alcove corresponding to $y \in W$ is marked with a number of dots equal to the coefficient of H_y. One can visualize the two decompositions above by coloring the dots (i.e. standard subquotients) according to which underlined terms (i.e. indecomposable summands) they lie in. Since the decompositions lead to different colorings, we draw a complete colored weight diagram for each decomposition. Theorem 5.4.11 implies that the p-canonical summands partition the colors in the W_p-weight diagram. In particular, it is easy to see that ${}^3\underline{H}_{0101} \neq \underline{H}_{0101}$. Otherwise the green dots and the black dots in the W-weight diagram correspond to different p-canonical basis elements, but it is impossible to partition the colors in the W_p-weight diagram below in the same manner. Weight diagrams are very similar to the diagrams in [33] depicting tilting characters, and the processes of applying Theorem 5.4.11 or higher-order linkage to a potential diagram are essentially identical.

We conclude with some remarks about the quantized linkage Hecke algebra.

REMARK 5.4.13.

(i) The fact that $\mathbb{H}_q/(q-1)$ is not $\mathbb{H}$, but a $2^{|{}^p\mathcal{W}|}$-fold cover of $\mathbb{H}$ is similar to the fact that the quotient $U_q(\mathfrak{sl}_2)/(q-1)$ of the quantized universal enveloping algebra is a double cover of the universal enveloping algebra $U(\mathfrak{sl}_2)$.

(ii) Some examples in the anti-spherical category for $\Phi = A_2$ (i.e. tilting characters for SL_3) [44] seem to suggest that working modulo $(q-1)(q-v)\mathbb{H}_q$ is necessary. More precisely, there are a few examples where calculating

$\underline{\hat{N}}_x$ (the analogous construction in the anti-spherical module) inductively in two different ways give different answers which are the same modulo this ideal.

(iii) We conjecture that there is a Kazhdan-Lusztig-type construction for a self-dual basis of $\mathbb{H}_q$ or $\mathbb{H}_{p|q}$. Unfortunately, precisely characterizing such a construction is tricky; we do not know what should take the place of the degree condition on coefficients of the standard basis elements H_x. Once the correct definition is found, the next step would be to prove a Soergel conjecture-like result, equating this basis with $\underline{\hat{H}}_x$ for p sufficiently large. Their images in $\mathbb{H}$ should correspond to notions of what one might call a "2nd generation Kazhdan–Lusztig basis" analogous to 2nd generation tilting characters [43, 44].

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