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# Floer simple manifolds and L-space intervals

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An oriented three-manifold with torus boundary admits either no L-space Dehn filling, a unique L-space filling, or an interval of L-space fillings. In the latter case, which we call "Floer simple," we construct an invariant which computes the interval of L-space filling slopes from the Turaev torsion and a given slope from the interval's interior. As applications, we give a new proof of the classification of Seifert fibered L-spaces over $S^2$, and prove a special case of a conjecture of Boyer and Clay [6] about L-spaces formed by gluing three-manifolds along a torus.



## 1. Introduction

An oriented rational homology 3-sphere $Y$ is called an L-space if the Heegaard Floer homology $\widehat{HF}(Y)$ satisfies $\widehat{HF}(Y, \mathfrak{s}) \simeq \mathbb{Z}$ for each Spin$^c$ structure $\mathfrak{s}$ on $Y$. Recent interest in the topological meaning of this condition has been stirred by a conjecture of Boyer, Gordon, and Watson [7], which states that a prime oriented three-manifold $Y$ is an

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L-space if and only if $\pi_1(Y)$ is non left-orderable. Subsequently, Boyer and Clay [6] studied a relative version of this problem for manifolds with toroidal boundary.

In this paper, we study the set of L-space fillings of a connected manifold $Y$ with a single torus boundary component. If $Y$ is such a manifold, we let

$$Sl(Y) = \{\alpha \in H_1(\partial Y) \,|\, \alpha \text{ is primitive}\}/\pm 1$$

be the set of slopes on $\partial Y$. $Sl(Y)$ is a one-dimensional projective space defined over the rational numbers. If we fix a basis $\langle \mu, \lambda \rangle$ for $H_1(\partial Y)$, we can identify $Sl(Y)$ with $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ via the map $a\mu + b\lambda \mapsto a/b$. We denote by $Y(\alpha)$ the closed manifold obtained by Dehn filling $Y$ with slope $\alpha$, and let $K_\alpha \subset Y(\alpha)$ be the core of the filling solid torus.

**Definition 1.1.** If $Y$ is a compact connected oriented three-manifold with torus boundary,

$$\mathcal{L}(Y) = \{\alpha \in Sl(Y) \,|\, Y(\alpha) \text{ is an L-space}\}$$

is the set of *L-space filling slopes* of $Y$.

For the set $\mathcal{L}(Y)$ to be nonempty, we must have $b_1(Y) = 1$, which implies that $Y$ is a rational homology $S^1 \times D^2$. In this paper, we will restrict our attention to manifolds with multiple L-space fillings: that is, for which $|\mathcal{L}(Y)| > 1$. Such manifolds can be easily characterized in terms of their Floer homology. Recall that a knot $K$ in a rational homology sphere $\overline{Y}$ is *Floer simple* [21] if the knot Floer homology $\widehat{HFK}(K) \simeq \mathbb{Z}^{|H_1(\overline{Y})|}$. Equivalently, $K$ is Floer simple if $\overline{Y}$ is an L-space and the spectral sequence from $\widehat{HFK}(K)$ to $\widehat{HF}(\overline{Y})$ degenerates.

**Definition 1.2.** A compact oriented three-manifold $Y$ with torus boundary is *Floer simple* if it has some Dehn filling $Y(\alpha)$ whose core $K_\alpha$ is a Floer simple knot in $Y(\alpha)$.

Then we have

**Proposition 1.3.** $|\mathcal{L}(Y)| > 1$ *if and only if* $Y$ *is Floer simple.*

If $K_\alpha \subset Y(\alpha)$ is Floer simple, then the Floer homology of any surgery on $K_\alpha$ can be determined from $\widehat{HFK}(K_\alpha)$ using the Ozsváth–Szabó mapping cone. The knot Floer homology, in turn, is determined by the Turaev torsion $\tau(Y)$ via the relation

$$\chi(\widehat{HFK}(K_\alpha)) \sim (1 - [\alpha])\tau(Y)$$

established in Proposition 2.1. It follows that if $Y$ is Floer simple, then the Floer homology of any Dehn filling of $Y$ can be determined from the Turaev torsion together with a single $\alpha \in \mathcal{L}(Y)$. In particular, we can determine $\mathcal{L}(Y)$ from this data, as described below.

Write $H_1(Y) = \mathbb{Z} \oplus T$, where $T$ is a torsion group, and let $\phi : H_1(Y) \to \mathbb{Z}$ be the projection. Properly normalized, $\tau(Y)$ can be written as a sum

$$\tau(Y) = \sum_{\substack{h \in H_1(Y) \\ \phi(h) \geq 0}} a_h[h],$$

where $a_h = 1$ for all but finitely many $h \in H_1(Y)$ with $\phi(h) > 0$, and $a_0 \neq 0$. For example, if $H_1(Y) = \mathbb{Z}$, then

$$\tau(Y) = \frac{\Delta(Y)}{1-t} \in \mathbb{Z}[[t]],$$

where the Alexander polynomial $\Delta(Y)$ is normalized to be an element of $\mathbb{Z}[t]$ and we expand the denominator as a Laurent series in positive powers of $t$.

**Proposition 1.4.** *When $Y$ is Floer simple, every coefficient $a_h$ of $\tau(Y)$ is either* 0 *or* 1*.*

Let $S[\tau(Y)] = \{h \in H_1(Y) \,|\, a_h \neq 0\}$ denote the *support* of $\tau(Y)$, and let $\iota : H_1(\partial Y) \to H_1(Y)$ be the map induced by inclusion.

**Definition 1.5.** If $Y$ is a Floer simple manifold, we define

$$\mathcal{D}^\tau(Y) = \{x - y \,|\, x \notin S[\tau(Y)], y \in S[\tau(Y)], \phi(x) \geq \phi(y)\} \cap \operatorname{im} \iota \subset H_1(Y),$$

and write $\mathcal{D}^\tau_{>0}(Y)$ for the subset of $\mathcal{D}^\tau(Y)$ consisting of those elements with $\phi(h) > 0$.

Let $[l] \in Sl(Y)$ be the homological longitude (*i.e.* $l$ is a primitive element of $H_1(\partial Y)$ such that $\iota(l)$ is torsion.) The set $\iota^{-1}(\mathcal{D}^\tau_{>0}(Y))$ is a discrete subset of $Sl(Y)$ whose only limit point is $[l]$. We can now state our first main theorem:

**Theorem 1.6.** *If $Y$ is Floer simple, then either $\mathcal{D}^\tau_{>0}(Y) = \emptyset$ and $\mathcal{L}(Y) = Sl(Y) \setminus [l]$, or $\mathcal{D}^\tau_{>0}(Y) \neq \emptyset$ and $\mathcal{L}(Y)$ is a closed interval whose endpoints are elements of $\iota^{-1}(\mathcal{D}^\tau_{>0}(Y))$ and which contains no element of $\iota^{-1}(\mathcal{D}^\tau_{>0}(Y))$ in its interior.*

Given $\tau(Y)$ and a Floer simple filling slope $\alpha$ for $Y$, it is thus straightforward to determine $\mathcal{L}(Y)$: the torsion determines the set $\mathcal{D}^\tau(Y)$, and $\mathcal{L}(Y)$ is the smallest interval with endpoints in $\iota^{-1}(\mathcal{D}^\tau_{>0}(Y))$ which contains $\alpha$ in its interior.

### *1.1. Splicing*

Theorem 1.6 can be used to address a problem raised by Boyer and Clay in [6]. Suppose that $Y_1$ and $Y_2$ are rational homology solid tori, and that $\varphi : \partial Y_1 \to \partial Y_2$ is an orientation reversing diffeomorphism. The manifold $Y_\varphi = Y_1 \cup_\varphi Y_2$ is said to obtained by splicing $Y_1$ and $Y_2$ together by $\varphi$.

In [6], Boyer and Clay studied how the presence of structure ($*$) on Dehn fillings of the pieces $Y_1$ and $Y_2$ relates to the presence of structure ($*$) on the splice $Y_\varphi$, where structure ($*$) could be one of three things: 1) a coorientable taut foliation; 2) a left-ordering on $\pi_1(Y_\varphi)$; or 3) a nontrivial class in $HF^{red}(Y_\varphi)$ (as $HF^{red}$ vanishes on, and only on, L-spaces). When $Y_1$ and $Y_2$ are graph manifolds, they obtained very strong results in cases 1) and 2), in addition to less complete results in the third case. The analogy with the first two cases suggests the following conjecture, which is implicit in the work of Boyer and Clay and stated explicitly in certain cases by Hanselman [17].

**Conjecture 1.7.** *Suppose that $Y_1$ and $Y_2$ as above are boundary incompressible, and let $\mathcal{L}_i^\circ$ be the interior of $\mathcal{L}(Y_i) \subset Sl(Y_i)$. Then $Y_\varphi$ is an L-space if and only if $\varphi_*(\mathcal{L}_1^\circ) \cup \mathcal{L}_2^\circ = Sl(Y_2)$.*

In particular, the conjecture says that in order for $Y_\varphi$ to be an L-space, both $Y_1$ and $Y_2$ must be Floer simple. Our second main result is

**Theorem 1.8.** *Suppose that $Y_1$ and $Y_2$ as above are Floer simple and have $\mathcal{D}^\tau \neq \emptyset$, and that $\varphi_*(\mathcal{L}_1^\circ) \cap \mathcal{L}_2^\circ \neq \emptyset$. Then $Y_\varphi$ is an L-space if and only if $\varphi_*(\mathcal{L}_1^\circ) \cup \mathcal{L}_2^\circ = Sl(Y_2)$.*

Hanselman and Watson [20] have proved a similar theorem using bordered Floer homology. (Since bordered Floer homology is only defined over $\mathbb{F}_2$, their theorem is about $\mathbb{F}_2$ L-spaces.) The restriction that $\varphi_*(\mathcal{L}_1^\circ) \cap \mathcal{L}_2^\circ \neq \emptyset$ represents a limitation of our approach, rather than anything intrinsic to the problem. To be specific, Theorem 1.8 is proved by writing $Y_\varphi$ as surgery on a connected sum of Floer simple knots. When $\varphi_*(\mathcal{L}_1^\circ) \cap \mathcal{L}_2^\circ = \emptyset$, we have no convenient way of representing the splice as surgery on a knot in an L-space. In contrast, Hanselman and Watson's approach does not require this hypothesis, but does need a condition on the bordered Floer homology, which they call *simple loop type*. In a subsequent joint paper [18], it is shown Floer simple manifolds are all of simple loop type, thus enabling us to remove the hypothesis that $\varphi_*(\mathcal{L}_1^\circ) \cap \mathcal{L}_2^\circ \neq \emptyset$ at the cost of working over $\mathbb{F}_2$ rather than $\mathbb{Z}$. (In fact, the properties of being Floer simple over $\mathbb{F}_2$ and being simple loop type are equivalent.) The proof of this fact relies on Proposition 3.10 of the current paper, where we explicitly compute the bordered Floer homology $\widehat{CFD}(Y, \mu, \lambda)$ of a Floer simple manifold $Y$ for an appropriate choice of $\mu, \lambda \in H_1(\partial Y)$ parametrizing $\partial Y$.

We briefly discuss those aspects of Conjecture 1.7 which are not covered by Theorem 1.8 and its generalizations. As stated, the conjecture implies that a Floer simple manifold $Y$ with $\mathcal{D}^\tau(Y) = \emptyset$ is boundary compressible. This is easily seen to be the case when $H_1(Y) \simeq \mathbb{Z}$, or more generally, when $Y$ is semi-primitive (*cf.* Proposition 1.9 below), but in general we have very little idea how to address this question. (Indeed, this seems like the weakest point of the conjecture.) The other situation which is not addressed by Theorem 1.8 is the case where one or both of $Y_1$ and $Y_2$ is not Floer simple.

It seems plausible that bordered Floer homology could be used to prove the conjecture when $|\mathcal{L}(Y_1)| = 1$ and $|\mathcal{L}(Y_2)| > 1$, or when $|\mathcal{L}(Y_1)| = |\mathcal{L}(Y_2)| = 1$. In contrast, the case where one or both of the $Y_i$ has *no* L-space fillings seems considerably more difficult to address with current technology.

### 1.2. Floer homology solid tori

The class of Floer simple manifolds with $\mathcal{D}^{\tau}_{>0} = \emptyset$ is of special interest. If $Y$ is a rational homology $S^1 \times D^2$, we say that $Y$ is *semi-primitive* if the torsion subgroup of $Y$ is contained in the image of $\iota$, and that $Y$ has genus 0 if $H_2(Y, \partial Y)$ is generated by a surface of genus 0.

**Proposition 1.9.** *If $Y$ is semi-primitive, the following conditions are equivalent:*

(1) *$Y$ is Floer simple and $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$.*
(2) *$Y$ is Floer simple and has genus 0.*
(3) *$Y$ has genus 0 and has an L-space filling.*

For example, if $K \subset S^1 \times S^2$ has a lens space surgery, then the complement of $K$ satisfies the conditions of the proposition. Such knots have been studied by Berge [3], Gabai [16], Cebanu [11], and Buck, Baker and Leucona [2]. Other examples of such manifolds are discussed in section 7.3.

The conditions of Proposition 1.9 are closely related to Watson's notion of a *Floer homology solid torus*. Suppose that $Y$ is a rational homology $S^1 \times D^2$ with homological longitude $l$, and that $m \in H_1(\partial Y)$ satisfies $m \cdot l = 1$.

**Definition 1.10.** [19] $Y$ is a *Floer homology solid torus* if $\widehat{CFD}(Y, m, l) \simeq \widehat{CFD}(Y, m+l, l)$.

**Proposition 1.11.** *If $Y$ satisfies the conditions of Proposition 1.9, then it is a Floer homology solid torus.*

Manifolds with $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$ play an important role in the notion of NLS detection introduced by Boyer and Clay in [6]. If $Y$ is a rational homology $S^1 \times D^2$ and $\alpha \in Sl(Y)$, $\alpha$ is said to be *strongly NLS detected* if $Y(\alpha)$ is not an L-space; $\alpha$ is *NLS detected* if certain splicings of $Y$ with a family of Floer homology solid tori are not L-spaces. (For the precise definition, see section 7.2.) By Theorem 1.6, the set of strongly NLS detected slopes is either a single point, an open interval in $Sl(Y)$, or all of $Sl(Y)$. By combining Theorem 1.8 with some direct geometric computation, we can show

**Corollary 1.12.** *If $Y$ is a rational homology $S^1 \times D^2$, the set of NLS detected slopes in $Sl(Y)$ is the closure of the set of strongly NLS detected slopes.*

### 1.3. Seifert fibered spaces

One of the key motivating examples for the conjecture of [7] is the class of Seifert-fibered spaces. Indeed, building on work of Ozsváth, Szabó, Matić, Naimi, Jankins, Neumann, Eisenbud, and Hirsch [39,32,34,25,12], Lisca and Stipsicz proved

**Theorem 1.13.** *[33] A Seifert fibered space over $S^2$ is an L-space if and only it does not admit a coorientable taut foliation.*

In combination with a result of Boyer, Rolfsen, and Wiest [8], this also implies that a Seifert-fibered space over $S^2$ has non left-orderable $\pi_1$ if and only if it is an L-space. The set of Seifert fibered spaces over $S^2$ which admit a coorientable taut foliation was explicitly described by Jankins and Neumann [25] and Naimi [34], building on a result of Eisenbud, Hirsch, and Neumann [12].

Any Seifert-fibered space over $S^2$ can be obtained by Dehn filling a Seifert fibered space over $D^2$. It follows easily from work of Ozsváth and Szabó [37] that any Seifert fibered space over $D^2$ is Floer simple, so we can compute the set of L-space filling slopes using Theorem 1.6. The resulting description of the set of Seifert fibered spaces which are not L-spaces agrees with the Jankins–Neumann set, thus giving a new direct proof of Theorem 1.13.

### 1.4. Discussion

We conclude with some questions about Floer simple manifolds and their relation to the conjecture of Boyer, Gordon, and Watson. First, we recall the statement of the conjecture.

**Conjecture 1.14.** *[7] If $Y$ is a oriented, closed, prime three-manifold, then $Y$ is an L-space if and only if $\pi_1(Y)$ is non left-orderable.*

A potentially more tractable subset of this problem, raised by Boyer and Clay [6] is:

**Question 1.** Suppose $Y$ is a Floer simple rational homology solid torus. Is $\pi_1(Y(\alpha))$ non left-orderable equivalent to $\alpha$ being an element of $\mathcal{L}(Y)$?

The characterization of $\mathcal{L}(Y)$ given in Theorem 1.6 should make it possible to conduct more detailed tests of Conjecture 1.14. Since there is already considerable experimental evidence in support of the conjecture, we should also consider what circumstances might explain a positive answer to Question 1. One possible explanation is that the condition of being Floer simple is correlated with some strong geometrical property, which in turn can be related to orderings of $\pi_1$.

**Question 2.** Is there a geometric characterization of Floer simple manifolds which can be stated without reference to Floer homology?

More generally, we think that Floer simple manifolds are a natural class of manifolds whose geometrical properties should be investigated for their own sake. Some evidence in support of this idea is provided by the frequency of Floer simple manifolds among geometrically simple 3-manifolds (as measured by the SnapPea census). Proposition 1.3 may lead readers familiar with the example of L-space knots in $S^3$ to suspect that the class of Floer simple manifolds is relatively small, but this is not the case. Of the 59,068 rational homology $S^1 \times D^2$'s in the SnapPy census of manifolds triangulated by at most 9 ideal tetrahedra, nearly 20% have multiple finite fillings, and are thus certifiably Floer simple. Moreover, more than two-thirds of the remaining manifolds have Turaev torsion compatible with their being Floer simple. It seems likely that many of these manifolds are Floer simple as well. (The authors thank Tom Brown for sharing these statistics with them.) For those who like other geometries, we note that every Seifert fibered rational homology $S^1 \times D^2$ is Floer simple.

It would be interesting to know what happens to the density of Floer simple manifolds as the complexity increases. Perhaps the most basic question we could ask along these lines is

**Question 3.** Are there infinitely many irreducible Floer simple manifolds with the same Turaev torsion?

### 1.5. Organization

The remainder of the paper is organized as follows. In section 2, we review some facts about knot Floer homology and the Ozsváth–Szabó mapping cone. These are used in section 3 to prove Proposition 1.3 and to give a characterization of when a given surgery on a Floer simple knot produces an L-space. In this section, we also explain how to compute the bordered Floer homology of a Floer simple manifold. Theorem 1.6 is proved in Section 4. In Section 5 we apply Theorem 1.6 to Seifert fibered spaces, thus giving a new proof of Theorem 1.13. The proof of Theorem 1.8 is given in Section 6. Finally, in Section 7, we discuss manifolds with $\mathcal{D}^{\tau}_{>0} = \emptyset$.

## 2. Knot Floer homology and the Ozsváth–Szabó mapping cone

In this section, we briefly recall some facts about knot Floer homology [38,44,41] which will be used in what follows. First, let us fix some notation. Throughout this section, we assume that $K \subset \overline{Y}$ is an oriented knot in a rational homology sphere. We let $Y = \overline{Y} \setminus \nu(K)$ be its complement, and denote by $\mu \in H_1(\partial Y)$ the class of its meridian. Furthermore, we let $T \subset H_1(Y)$ be the torsion subgroup, and denote by $\phi : H_1(Y) \to \mathbb{Z}$

the projection from $H_1(Y)$ to $H_1(Y)/T \simeq \mathbb{Z}$, where the isomorphism is chosen so that $\phi(\mu) > 0$.

### 2.1. Knot Floer homology

The knot Floer homology $\widehat{HFK}(K)$ is a finitely generated abelian group with an absolute $\mathbb{Z}/2$ grading. It decomposes as a direct sum $\widehat{HFK}(K) = \bigoplus \widehat{HFK}(K, \mathfrak{s})$, where $\mathfrak{s}$ runs over the set $\mathrm{Spin}^c(Y, \partial Y)$ of relative $\mathrm{Spin}^c$ structures on $(Y, \partial Y)$. $\mathrm{Spin}^c(Y, \partial Y)$ is an affine copy of $H_1(Y)$ (*aka* $H_1(Y)$ torsor); it has a free transitive action of $H_1(Y)$. The group $\widehat{HFK}(K, \mathfrak{s})$ is trivial for all but finitely many $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y)$.

Given $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y)$, we consider the formal sum

$$\chi_{\mathfrak{s}}(\widehat{HFK}(K)) := \sum_{h \in H_1(Y)} \chi(\widehat{HFK}(K, \mathfrak{s} + h))[h],$$

where $\chi(\widehat{HFK}(K, \mathfrak{s}))$ is defined using the absolute $\mathbb{Z}/2$ grading. We view $\chi_{\mathfrak{s}}(\widehat{HFK}(K))$ as an element of the group ring $\mathbb{Z}[H_1(Y)]$; it is known as the *graded Euler characteristic of* $\widehat{HFK}(K)$. Clearly

$$\chi_{\mathfrak{s}'}(\widehat{HFK}(K)) = [\mathfrak{s} - \mathfrak{s}']\chi_{\mathfrak{s}}(\widehat{HFK}(K)).$$

From now on, we will drop $\mathfrak{s}$ from the notation and view $\chi(\widehat{HFK}(K))$ as an element of $\mathbb{Z}[H_1(Y)]$, well defined up to global multiplication by elements of $H_1(Y)$. We write $x \sim y$ if $x, y \in \mathbb{Z}[H_1(Y)]$ satisfy $x = [h]y$ for some $h \in H_1(Y)$.

For knots in $S^3$, it is well-known that $\chi(\widehat{HFK}(K))$ is the Alexander polynomial of $K$. More generally, we have

**Proposition 2.1.** $\chi(\widehat{HFK}(K)) \sim (1 - [\mu])\tau(Y)$, *where* $\tau(Y)$ *is the Turaev torsion of* $Y$.

**Proof.** $\widehat{HFK}(K)$ can be identified with the sutured Floer homology $SFH(Y, \gamma_\mu)$ [26], where the suture $\gamma_\mu$ consists of two parallel copies of $\mu$. The Euler characteristic of the sutured Floer homology can be described as an appropriately formulated torsion [14]. When $\partial Y$ is toroidal, this torsion can be expressed in terms of the Turaev torsion, as in Lemma 6.3 of [14]. (This lemma was stated for links in $S^3$, but the proof carries through unchanged.) $\square$

*A priori*, $\tau(Y)$ is an element of the field $Q(H_1(Y))$ obtained by inverting all elements of $\mathbb{Z}[H_1(Y)]$ which are not zero divisors. Note that if $\nu \in H_1(Y)$ satisfies $\phi(\nu) \neq 0$, then $1 - [\nu]$ is not a zero divisor in $\mathbb{Z}[H_1(Y)]$. By hypothesis, $\phi([\mu]) > 0$, so it follows from the proposition that $\tau(Y) \in \mathbb{Z}[H_1(Y)][(1 - [\mu])^{-1}] \subset Q(H_1(Y))$.

Writing $(1 - [\mu])^{-1} = \sum_{i=0}^{\infty} [\mu]^i$ allows us to embed $\mathbb{Z}[H_1(Y)][(1 - [\mu])^{-1}]$ in the Novikov ring

$$\Lambda_\phi[H_1(Y)] = \left\{ \sum_{h \in H_1(Y)} a_h[h] \;\middle|\; \#\{h \,|\, a_h \neq 0, \phi(h) < k\} < \infty \text{ for all } k \right\}.$$

We will view $\tau(Y)$ as an element of $\Lambda_\phi[H_1(Y)]$. By choosing a splitting $H_1(Y) \simeq \mathbb{Z} \oplus T$, we can identify $\Lambda_\phi[H_1(Y)]$ with the Laurent series ring $\mathbb{Z}[t^{-1}, t]] \otimes \mathbb{Z}[T]$, which we shall later sometimes call the "Laurent series group ring."

As an element of the Novikov ring, $\tau(Y)$ is well-defined up to multiplication by elements of $H_1(Y)$. We shall always normalize so that $\tau(Y)$ has the form $\tau(Y) = \sum_h a_h[h]$, where $a_h = 0$ for all $h$ with $\phi(h) < 0$, and $a_0 \neq 0$.

If $H_1(Y) = \mathbb{Z}$, it is well-known that $\tau(Y) \sim \Delta(Y)/(1-t)$, where $\Delta(Y)$ is the Alexander polynomial of $Y$. More generally, if $\Phi : \Lambda_\phi[H_1(Y)] \to \mathbb{Z}[t^{-1}, t]]$ is the map induced by the projection $\phi : H_1(Y) \to \mathbb{Z}$, we define

$$\overline{\tau}(Y) = \Phi(\tau(Y))$$

then $\Delta(Y) \sim (1-t)\overline{\tau}(Y)$ [47, Section 5.2].

If $K$ is a knot in $S^3$, it is well known that $\deg \Delta(t) \leq 2g(K)$, and $\Delta(K)|_{t=1} = 1$. The following result is a simultaneous generalization of these two facts.

**Proposition 2.2** *([47] Lemma II.4.5.1 and Theorem II.4.2.1). If $\|Y\|$ is the Thurston norm of a generator of $H_2(Y, \partial Y)$ and $\tau(Y)$ is normalized as above, then $a_h = 1$ for all $h \in H_1(Y)$ with $\phi(h) > \|Y\|$.*

More generally, it is known that $\widehat{HFK}(K)$ determines both the Thurston norm of $Y$ and whether it is fibered [36,35,27]. Since the knot Floer homology of a Floer simple knot is determined by its Euler characteristic, we have

**Corollary 2.3.** *If $Y$ is boundary incompressible and Floer simple, $\|Y\| = \deg \Delta(Y) - 1$. If $Y$ is also irreducible, then $Y$ fibers over $S^1$ if and only if $\Delta(Y)$ is monic.*

### *2.2. Differentials*

The knot Floer homology of $K$ can be used to compute the Floer homology of surgeries on $K$. Before we explain how to do this, we must understand the relation between $\widehat{HFK}(K)$ and $\widehat{HF}(\overline{Y})$.

We begin by discussing $\text{Spin}^c$ structures. There are maps $i_v, i_h : \text{Spin}^c(Y, \partial Y) \to \text{Spin}^c(\overline{Y})$ which respect the action of $H_1(Y)$, in the sense that $i_v(\mathfrak{s} + a) = i_v(\mathfrak{s}) + i_*(a)$ and $i_h(\mathfrak{s}+a) = i_h(\mathfrak{s})+i_*(a)$ where $i_* : H_1(Y) \to H_1(\overline{Y})$ is the map induced by inclusion. Moreover, $i_v(\mathfrak{s}) - i_h(\mathfrak{s}) = i_*(\lambda)$, where $\lambda$ is a longitude of $K$. We define an equivalence relation on $\text{Spin}^c(Y, \partial Y)$ by declaring $\mathfrak{s}_1 \sim \mathfrak{s}_2$ if $i_v(\mathfrak{s}_1) = i_v(\mathfrak{s}_2)$. It is easy to see that this is the same as requiring that $i_h(\mathfrak{s}_1) = i_h(\mathfrak{s}_2)$, and that the equivalence classes are orbits of $\text{Spin}^c(Y, \partial Y)$ under the action of $\mu$.

Let $\widetilde{\mathfrak{s}}$ be an equivalence class in $\mathrm{Spin}^c(Y, \partial Y)$. After we choose some auxiliary data (a doubly pointed Heegaard diagram for $K$), Heegaard Floer homology constructs for us a graded group

$$\widehat{CFK}(K, \widetilde{\mathfrak{s}}) = \bigoplus_{\mathfrak{s} \in \widetilde{\mathfrak{s}}} \widehat{CFK}(K, \mathfrak{s})$$

together with maps $d_0, d_v, d_h : \widehat{CFK}(K, \widetilde{\mathfrak{s}}) \to \widehat{CFK}(K, \widetilde{\mathfrak{s}})$, which are filtered with respect to the $\mathrm{Spin}^c$ grading in the following sense: if $x \in \widehat{CFK}(Y, \mathfrak{s})$, then $d_0 x \in \widehat{CFK}(Y, \mathfrak{s})$, $d_v x \in \oplus_{k<0} \widehat{CFK}(Y, \mathfrak{s} + k\mu)$ and $d_h x \in \oplus_{k>0} \widehat{CFK}(Y, \mathfrak{s} + k\mu)$. These differentials satisfy the relations $d_0^2 = (d_0 + d_v)^2 = (d_0 + d_h)^2 = 0$. Furthermore, we have

$$\begin{aligned} H(\widehat{CFK}(K, \mathfrak{s}), d_0) &= \widehat{HFK}(K, \mathfrak{s}), \\ H(\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_v) &= \widehat{HFK}(\overline{Y}, i_v(\mathfrak{s})), \\ H(\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_h) &= \widehat{HFK}(\overline{Y}, i_h(\mathfrak{s})). \end{aligned}$$

The $\mathrm{Spin}^c$ grading provides a natural filtration on the latter two complexes, in the sense that $\oplus_{k<n} \widehat{CFK}(K, \mathfrak{s} + k\mu)$ is a subcomplex of $(\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_v)$ and $\oplus_{k>n} \widehat{CFK}(K, \mathfrak{s} + k\mu)$ is a subcomplex of $(\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_h)$. These filtrations give rise to spectral sequences whose $E_1$ term is $\widehat{HFK}(K, \widetilde{\mathfrak{s}})$. We denote by $\tilde{d}_v, \tilde{d}_h$ the induced differentials on the $E_1$ term, so that *e.g.* $\widehat{CFK}(K, d_0 + d_v)$ is homotopy equivalent to $\widehat{HFK}(K, \tilde{d}_v)$. (Note that these are not the same as the $d_1$ differentials in the spectral sequence.)

**Definition 2.4.** For each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, the *bent complex* is $A_{K,\mathfrak{s}} = (\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_{\mathfrak{s}})$, where for $x \in \widehat{CFK}(K, \mathfrak{s} + k\mu)$,

$$d_{\mathfrak{s}}(x) = \begin{cases} d_0(x) + d_v(x) & k < 0 \\ d_0(x) + d_v(x) + d_h(x) & k = 0 \\ d_0(x) + d_h(x) & k > 0 \end{cases}.$$

The bent complexes measure the Heegaard Floer homology of large integer surgery on $K$: $H(A_{K,\mathfrak{s}}) \simeq \widehat{HF}(Y(N\mu + \lambda), i_n(\mathfrak{s}))$ for sufficiently large $N$ and an appropriately chosen $\mathrm{Spin}^c$ structure $i_N(\mathfrak{s})$ on the filling.

The existence of the $\mathrm{Spin}^c$ filtration means there are chain maps

$$\begin{aligned} \pi_v &: A_{K,\mathfrak{s}} \to (\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_v) \\ \pi_h &: A_{K,\mathfrak{s}} \to (\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_h) \end{aligned}$$

given by

$$\pi_v(x) = \begin{cases} 0 & k > 0 \\ x & k \leq 0 \end{cases} \qquad \pi_h(x) = \begin{cases} x & k \geq 0 \\ 0 & k < 0 \end{cases}$$

for $x \in \widehat{CFK}(\mathfrak{s} + k\mu)$.

### *2.3. The Ozsváth–Szabó mapping cone*

Let $\lambda$ be a longitude for $K$, so that $\mu \cdot \lambda = 1$. The mapping cone of Ozsváth and Szabó [41] relates the Heegaard Floer homology of the filling $Y(\lambda)$ to the knot Floer homology of $K$. We recall its construction here.

Since $i_h(\mathfrak{s} - \lambda) = i_v(\mathfrak{s})$, we have

$$H(\widehat{CFK}(K, \mathfrak{s} - \lambda), d_0 + d_h) \simeq \widehat{HF}(Y, i_v(\mathfrak{s})) \simeq H(\widehat{CFK}(K, \mathfrak{s}), d_0 + d_v).$$

This isomorphism is realized by a chain homotopy equivalence

$$j : (\widehat{CFK}(K, \mathfrak{s} - \lambda), d_0 + d_h) \to (\widehat{CFK}(K, \mathfrak{s}), d_0 + d_v).$$

(The map on homology induced by $j$ is the canonical isomorphism of [28], although we will not use this fact here.)

For $\mathfrak{s} \in \mathrm{Spin}^c(Y, \partial Y)$, let $B_{K,\mathfrak{s}} = (\widehat{CFK}(K, \widetilde{\mathfrak{s}}), d_0 + d_v)$. We form two chain complexes

$$\mathbb{A}(K) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} A_{K,\mathfrak{s}} \quad \text{and} \quad \mathbb{B}(K) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} B_{K,\mathfrak{s}}.$$

There is a chain map $f_\lambda : \mathbb{A}(K) \to \mathbb{B}(K)$ given by $f = \pi_v + j \circ \pi_h$. (So if $x \in A_{K,\mathfrak{s}}$, $f_\lambda(x)$ is a sum of terms in $B_{K,\mathfrak{s}}$ and $B_{K,\mathfrak{s}+\lambda}$.) Let $\mathbb{X}_\lambda(K)$ be the mapping cone of $f_\lambda$. In [41], Ozsváth and Szabó prove

**Theorem 2.5.** *[41]* $\widehat{HF}(Y(\lambda)) \simeq H_*(\mathbb{X}_\lambda(K))$.

We make some remarks on the construction. First, it is easy to see that the complex $\mathbb{X}_\lambda(K)$ decomposes as a direct sum of complexes whose underlying groups are of the form

$$\mathbb{X}_\lambda(K, \mathfrak{s}) = \bigoplus_{n \in \mathbb{Z}} A_{K,\mathfrak{s}+n\lambda} \oplus \bigoplus_{n \in \mathbb{Z}} B_{K,\mathfrak{s}+n\lambda}.$$

The summands are on one to one correspondence with elements of the quotient $H_1(Y)/\langle\lambda\rangle \simeq H_1(Y(\lambda))$. The resulting decomposition on homology corresponds to the decomposition of $\widehat{HF}(Y(\lambda))$ by $\mathrm{Spin}^c$ structures.

Second, if $\mathbb{F}_p$ is the field of order $p$, where $p$ is a prime, then we can form the complex $\mathbb{X}_\lambda(K; \mathbb{F}_p) = \mathbb{X}_\lambda(K) \otimes \mathbb{F}_p$. It follows from the universal coefficient theorem that $\widehat{HF}(Y(\lambda); \mathbb{F}_p) \simeq H_*(\mathbb{X}_\lambda(K; \mathbb{F}_p))$.

Finally, it is often convenient to work with the homology of the complexes $A_{K,\mathfrak{s}}$ and $B_{K,\mathfrak{s}}$, rather than the complexes themselves. We can do this if we use field coefficients. Specifically, fix a field $\mathbb{F}_p$, and let $\mathbf{A}_{K,\mathfrak{s}} = H(A_{K,\mathfrak{s}} \otimes \mathbb{F}_p)$, $\mathbf{A}(K) = \oplus \mathbf{A}_{K,\mathfrak{s}}$, $\mathbf{B}_{K,\mathfrak{s}} = H(B_{K,\mathfrak{s}} \otimes \mathbb{F}_p)$, $\mathbf{B}(K) = \oplus \mathbf{B}_{K,\mathfrak{s}}$. Similarly, let $v : \mathbf{A}_{K,\mathfrak{s}} \to \mathbf{B}_{K,\mathfrak{s}}$ be the map induced by $\pi_v$, and $h : \mathbf{A}_{K,\mathfrak{s}} \to \mathbf{B}_{K,\mathfrak{s}+\lambda}$ be the map induced by $j \circ \pi_h$. Finally, let $C_\lambda(K;\mathbb{F}_p)$ be the chain complex whose underlying group is $\mathbf{A}(K) \oplus \mathbf{B}(K)$, with differential given by $dx = v(x) + h(x)$ for $x \in \mathbf{A}(K)$, $dy = 0$ for $y \in \mathbf{B}(K)$.

**Corollary 2.6.** $\widehat{HF}(Y(\lambda);\mathbb{F}_p) \simeq H(C_\lambda(K;\mathbb{F}_p))$.

**Proof.** The short exact sequence

$$0 \to \mathbb{B}(K) \otimes \mathbb{F}_p \to \mathbb{X}_\lambda(K;\mathbb{F}_p) \to \mathbb{A}(K) \otimes \mathbb{F}_p \to 0$$

gives a long exact sequence

$$\to \mathbf{B}(K) \to \widehat{HF}(Y(\lambda);\mathbb{F}_p) \to \mathbf{A}(K) \to \mathbf{B}(K) \to$$

whose boundary map is given by $v + h$. An exact sequence over a field splits, so we get the statement of the corollary. □

## *2.4. Splicing and surgery*

Suppose $Y_1$ and $Y_2$ are rational homology solid tori, and that $\varphi : \partial Y_1 \to \partial Y_2$ is an orientation reversing diffeomorphism. The manifold $Y_\varphi = Y_1 \cup_\varphi Y_2$ is obtained by splicing $Y_1$ and $Y_2$ together along $\varphi$. Choose a slope $\mu_1 \in Sl(\partial Y_1)$, and let $\mu_2 = \varphi_*(\mu)$ be its image in $Sl(\partial Y_2)$. Let $\overline{Y}_i = Y_i(\mu_i)$ be the corresponding Dehn fillings, and let $K_i = K_{\mu_i}$ be their cores.

**Lemma 2.7.** *$Y_\varphi$ can be obtained by integral surgery on $K_1 \# K_2 \subset \overline{Y}_1 \# \overline{Y}_2$.*

This is well-known, but an understanding of the proof will be useful in what follows, so we sketch it here.

**Proof.** Let $Y'$ be the complement of $K_1 \# K_2$. $Y'$ is obtained by identifying an annulus $\nu(\mu_1) \subset \partial Y_1$ with its image $\nu(\mu_2) = \varphi(\nu(\mu_1)) \subset \partial Y_2$. (Throughout the proof, we use the same symbol to denote both a slope on the torus and a simple closed curve representing it.) Equivalently, $Y'$ can be obtained by starting with the disjoint union of $Y_1, Y_2$ and $S^1 \times I \times I$ and identifying $S^1 \times I \times 0$ with $\nu(\mu_1)$ and $S^1 \times I \times 1$ with $\nu(\mu_2)$. In this model, $\partial Y'$ is a union of four annuli: $\partial Y_1 - \nu(\mu_1)$, $S^1 \times 0 \times I$, $\partial Y_2 - \nu(\mu_2)$, and $S^1 \times 1 \times I$. The meridian $\mu$ of $K_1 \# K_2$ is homotopic to both $\mu_1$ and $\mu_2$ (and to the core of each of the four annuli).

Let $\lambda_1$ be a longitude for $\mu_1$, so that $\lambda_2 = -\varphi(\lambda_1)$ is a longitude for $\mu_2$. We may assume that $\lambda_1 \cap \nu(\mu_1) = p \times I \subset S^1 \times I \simeq \nu(\mu_1)$, and similarly for $\lambda_2$. Let $\lambda_1'$ be the arc obtained by intersecting $\lambda_1$ with $\partial Y_1 - \nu(\mu_1)$, and similarly for $\lambda_2'$. The union of the arcs $\lambda_1', p \times 0 \times I, \lambda_2'$, and $p \times 1 \times I$ is a longitude $\lambda$ for $K_1 \# K_2$. Attaching a 2-handle along $\lambda$ is the same as attaching $I \times I \times I$ to $Y'$, where the top and bottom edges $I \times 1/2 \times 1$ and $I \times 1/2 \times 0$ are identified with $\lambda_1'$ and $\lambda_2'$, and the sides $1 \times 1/2 \times I$ and $0 \times 1/2 \times I$ are identified with the other arcs in $\lambda$. The resulting manifold can be obtained by starting with $Y_1, Y_2$ and $\Sigma \times I$, where $\Sigma$ is a regular neighborhood of the 1-skeleton in $T^2$ and identifying $\Sigma \times 0$ with a tubular neighborhood of $\mu_1 \cup \lambda_1 \subset \partial Y_1$ and $\Sigma \times 1$ with its image under $\varphi$. Finally, filling in the spherical boundary component with $B^3$ gives $Y_1 \cup (T^2 \times I) \cup Y_2 = Y_\varphi$. □

From the proof, we see that $H_1(Y') \simeq H_1(Y_1) \oplus H_1(Y_2)/R$, where $R$ is the subgroup generated by $(\mu_1, \mu_2)$, and that under this isomorphism, $\lambda = (\lambda_1, \varphi_*(\lambda_1)) = (\lambda_1, -\lambda_2)$.

We make two remarks on the utility of this construction. First, it is quite flexible, in the sense that the choice of *any* meridian $\mu_1 \in Sl(\partial Y_1)$ gives a different way of realizing the spliced manifold as a surgery. This flexibility will be useful to us in what follows.

Second, rational surgery on a knot $K \subset \overline{Y}$ amounts to splicing $Y$ with $S^1 \times D^2$. Suppose $\langle \mu, \lambda \rangle$ is our usual basis for $H_1(\partial Y)$, and that $\langle m, l \rangle$ is the standard basis for $H_1(\partial S^1 \times D^2)$ (so $l = [\partial D^2]$). If we glue $\partial Y$ to $\partial(S^1 \times D^2)$ in such a way that $[\partial D^2]$ is identified with $\alpha = p\mu + q\lambda \in H_1(\partial Y)$, then it is easy to see that $\mu$ is identified with $-qm + p^* l$, where $pp^* \equiv 1 \bmod q$. Applying the lemma, we see that $Y(\alpha)$ is obtained by integer surgery on a knot $K' = K \# K_{-q/p} \subset \overline{Y} \# L(q, -p^*) = \overline{Y} \# L(q, -p)$.

The knot $K_{-q/p}$ is the unique knot in $L(q, -p)$ whose complement is $S^1 \times D^2$. (In the notation of [45], it is the simple knot $K(q, -p, 1)$.) It is Floer simple, with Euler characteristic

$$\chi(\widehat{HFK}(K(q, -p, 1)) \sim \frac{t^q - 1}{t - 1}.$$

To use Lemma 2.7 to compute the Floer homology of a splice, we need to know how the knot Floer homology behaves under connected sum.

**Lemma 2.8.** *[42]* $\widehat{HFK}(K_1 \# K_2) \simeq \widehat{HFK}(K_1) \otimes \widehat{HFK}(K_2)$.

The isomorphism is well-behaved with respect to Spin$^c$ structures, in the sense that

$$\chi(\widehat{HFK}(K_1 \# K_2)) \sim \chi(\widehat{HFK}(K_1))\chi(\widehat{HFK}(K_2)).$$

It is also respects the differentials, in the sense that $\widehat{CFK}(K_1 \# K_2, d_0 + d_v)$ is homotopy equivalent to $\widehat{CFK}(K_1, d_0 + d_v) \otimes \widehat{CFK}(K_2, d_0 + d_v)$, and similarly for $d_h$.

In [42], Ozsváth and Szabó combined the observations above with their mapping cone for integer surgeries to express the Floer homology of any rational surgery as a mapping cone.

## 3. Floer simple manifolds

In this section we use Ozsváth and Szabó's mapping cone formula to prove Proposition 1.3 and to derive some basic facts about Floer simple manifolds. For the most part, these are straightforward extensions of results in [40], [45], and [4]. We conclude by explaining how to compute the bordered Floer homology of a Floer simple manifold $Y$ in terms of $\tau(Y)$ and a Floer simple filling slope $\alpha$. Our notation and assumptions are the same as in section 2.

### *3.1. Proof of Proposition 1.3*

Suppose that $K \subset \overline{Y}$ is a knot in an $L$-space, and that some nontrivial surgery on $Y$ is also an L-space.

**Definition 3.1.** We say that $\widehat{HFK}(K, \widetilde{\mathfrak{s}})$ is a *positive chain* if it is generated by elements $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ and the induced differentials $\tilde{d}_h$ and $\tilde{d}_v$ satisfy $\tilde{d}_v(y_i) = \pm x_{i+1}$, $\tilde{d}_h(y_i) = \pm x_i$, and $\tilde{d}_v(x_i) = \tilde{d}_h(x_i) = 0$ for all $i$. More generally, we say that $\widehat{HFK}(K)$ *consists of positive chains* if $\widehat{CFK}(K, \widetilde{\mathfrak{s}})$ is a positive chain for each $\mathfrak{s} \in \text{Spin}^c(Y)$, and that $\widehat{HFK}(K)$ consists of *coherent chains* if either $\widehat{HFK}(K)$ or $\widehat{HFK}(-K)$ consists of positive chains, where $-K \subset -\overline{Y}$ is the mirror knot.

Note that all the $x_i$'s in the definition must have the same relative $\mathbb{Z}/2$ grading, which is opposite that of the $y_i$'s. Since there are more $x_i$'s than $y_i$'s, the $x_i$ contribute to $\chi(\widehat{HFK}(K))$ with positive sign, while the $y_i$'s contribute with negative sign.

If $\widehat{HFK}(-K)$ consists of positive chains, then the dual complex $\widehat{HFK}(K)$ consists of *negative chains*. A negative chain has generators $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ and differentials $\tilde{d}_v(x_i) = \pm y_i$ for $1 \leq i < n$ and $\tilde{d}_h(x_i) = \pm y_{i-1}$ for $1 < i \leq n$. A positive chain with more than one generator cannot be isomorphic to a negative chain, so if both $\widehat{HFK}(K)$ and $\widehat{HFK}(-K)$ consist of positive chains, then $K$ is Floer simple.

Ozsváth and Szabó proved in [40] that if $K \subset S^3$ has an L-space surgery with positive slope, then $\widehat{HFK}(K)$ is a positive chain. The following generalization is an easy consequence of a result of Boileau, Boyer, Cebanu, and Walsh:

**Lemma 3.2.** *Suppose that $K \subset \overline{Y}$ is a knot in an L-space, and that some surgery on $K$ is also an L-space. Then $\widehat{HFK}(K)$ consists of coherent chains.*

**Proof.** A surgery on $K$ is *positive* if the corresponding surgery cobordism is positive definite. Suppose that some positive integral surgery on $K$ is an L-space. By Lemma 6.7 of [4], $H_*(A_{K,\mathfrak{s}}) \simeq \mathbb{Z}$ for all $\mathfrak{s} \in \text{Spin}^c(Y, \partial Y)$. The proof of Theorem 1.2 of [40] carries over unchanged to show that $\widehat{HFK}(K, \widetilde{\mathfrak{s}})$ is a positive chain.

Next, suppose that $Y'$ is obtained by negative integral surgery on $K \subset \overline{Y}$, and that $Y'$ is an L-space. By reversing the orientation of the surgery cobordism, we see that $-Y'$

is obtained by positive surgery on $-K \subset -\overline{Y}$. $-Y'$ is also an L-space, so $\widehat{HFK}(-K)$ consists of positive chains, and $\widehat{HFK}(K)$ consists of negative ones.

Finally, suppose that an L-space $Y'$ is obtained by fractional surgery on $K$. Then $Y'$ is obtained by integral surgery on a knot of the form $K\#K_{-q/p} \subset \overline{Y}\#L(q,-p)$, so $\widehat{HFK}(K\#K_{-q/p}) \simeq \widehat{HFK}(K) \otimes \widehat{HFK}(K_{-q/p})$ is composed of coherent chains. Since $K_{-q/p}$ is Floer simple, it is easy to see that this occurs if and only if $\widehat{HFK}(K)$ is composed of coherent chains. □

**Lemma 3.3.** *If $\widehat{HFK}(K)$ consists of coherent chains, then* $\tau(Y) = \sum_{h \in S[\tau(Y)]} [h]$.

**Proof.** We have

$$\tau(Y) \sim \frac{\chi(\widehat{HFK}(K))}{(1-[\mu])} = \left(\sum_{\overline{\mathfrak{s}} \in M} \chi(\widehat{HFK}(K, \widetilde{\mathfrak{s}}))\overline{\mathfrak{s}}\right)\left(\sum_{i=0}^{\infty}[\mu]^i\right)$$

where $M \subset \mathrm{Spin}^c(Y, \partial Y)$ is a set of coset representatives for the action of $\langle \mu \rangle$ and

$$\chi(\widehat{HFK}(K, \widetilde{\mathfrak{s}})) = \sum_{j \in \mathbb{Z}} \chi(\widehat{HFK}(K, \overline{\mathfrak{s}} + j\mu))[\mu]^j.$$

The hypothesis that $\widehat{HFK}(K)$ consists of coherent chains implies that the nonzero coefficients of $\chi(\widehat{HFK}(K, \widetilde{\mathfrak{s}}))$ alternate between $+1$ and $-1$, and that the outermost coefficients are $+1$. It follows that the coefficients of the product $\chi(\widehat{HFK}(K, \widetilde{\mathfrak{s}}))\left(\sum_{i=0}^{\infty}[\mu]^i\right)$ are all either 0 or $+1$, and hence that all the coefficients of $\tau(Y)$ are either 0 or 1 as well. □

**Corollary 3.4.** *Suppose $\widehat{HFK}(K)$ is composed of coherent chains, and that $\phi(\mu) > \|Y\|$. Then $K$ is Floer simple.*

**Proof.** By hypothesis, $\widehat{HFK}(K)$ is composed of coherent chains, so to prove that $K$ is Floer simple, it suffices to show that every monomial in $\chi(\widehat{HFK}(K))$ appears with a positive coefficient. As usual, we normalize $\tau(Y) = \sum_h a_h[h]$ so that $a_h = 0$ whenever $\phi(h) < 0$, and $a_0 \neq 0$. We have $\chi(\widehat{HFK}(K)) \sim (1-[\mu])\tau(Y)$, so the coefficient of $[h]$ in $\chi(\widehat{HFK}(K))$ is $a_h - a_{h-\mu}$. Both terms in this difference are either 0 or 1. If $\phi(\mu) > \phi(h)$, then $a_{h-\mu} = 0$, while if $\phi(h) \geq \phi(\mu) > \|Y\|$, then $a_h = 1$ by Proposition 2.2. In either case, we see that the coefficient of $[h]$ in $\chi(\widehat{HFK}(K))$ is either 0 or 1. □

**Lemma 3.5.** *If $\widehat{HFK}(K)$ is composed of positive chains, then $\mathcal{L}(Y)$ contains an interval of the form $\{[\mu + a\lambda] \,|\, 0 \leq a \leq \varepsilon\}$ for some $\varepsilon > 0$.*

**Proof.** Since $\widehat{HFK}(K)$ is composed of positive chains, the homology of each of its bent complexes is $\mathbb{Z}$. Since the homology of the bent complexes computes $\widehat{HF}(Y(N\mu + \lambda))$

for some $N \gg 0$, we see that $N\mu + \lambda \in \mathcal{L}(Y)$. Since $\mu \cdot (N\mu + \lambda) = 1$, Proposition 17 of [7] shows that $[b\mu + \lambda] \in \mathcal{L}(Y)$ for all rational numbers $b \geq N$. $\square$

By considering mirrors, we see that if $\widehat{HFK}(K)$ is composed of negative chains, then $\mathcal{L}(Y)$ contains an interval of the form $\{[\mu - a\lambda] \,|\, 0 \leq a \leq \varepsilon\}$ for some $\varepsilon > 0$. It follows that if $K$ is Floer simple, then $\mu$ is an interior point of an interval in $\mathcal{L}(Y)$.

**Proof of Proposition 1.3.** If $Y$ is Floer simple, then it has some filling $Y(\alpha)$ for which $K_\alpha$ is Floer simple. As we observed above, $\alpha$ is contained in the interior of an interval in $\mathcal{L}(Y)$, so clearly $|\mathcal{L}(Y)| > 1$. Conversely, if $\mathcal{L}(Y) > 1$, then $\widehat{HFK}(K)$ is composed of coherent chains, so $\mathcal{L}(Y)$ contains an interval. Now any interval in $Sl(Y)$ contains elements $\alpha$ with $\phi(\alpha)$ arbitrarily large. (To see this, identify $Sl(Y)$ with $\overline{\mathbb{Q}}$ using the canonical meridian and longitude. If $\alpha \mapsto a/b$ under this identification, then $\phi(\alpha) = ka$ for some fixed $k > 0$.) By Corollary 3.4, $K_\alpha \subset Y(\alpha)$ is Floer simple, so $Y$ is Floer simple. $\square$

**Corollary 3.6.** *$K$ is Floer simple if and only if $\mu \in \mathcal{L}^\circ(Y)$.*

**Proof.** As observed above, if $K = K_\mu$ is Floer simple, then $\mu \in \mathcal{L}^\circ(Y)$. Conversely, suppose $\mu \in \mathcal{L}^\circ(Y)$. Then $\mathcal{L}(Y)^\circ \neq \emptyset$, so $Y$ is Floer simple. Since $\mu \in \mathcal{L}(Y)$, $\widehat{HFK}(K)$ is composed of coherent chains. Suppose $\widehat{HFK}(K)$ is composed of negative chains but is not Floer simple. Then some bent group of $K$ has homology of rank $> 1$. This implies that $Y(N\mu + \lambda)$ is not an L-space for $N \gg 0$, which contradicts the assumption that $\mu \in \mathcal{L}(Y)^\circ$. A similar argument applies if $\widehat{HFK}(K)$ is composed of positive chains. $\square$

### *3.2. Surgery on Floer simple knots*

We now suppose that $K \subset \overline{Y}$ is Floer simple. We give a graphical criterion for determining whether a given integer surgery on $K$ is an L-space. To do so, we consider the set $S_{\text{BLACK}} = S[\widehat{HFK}(K)] \subset \text{Spin}^c(Y, \partial Y)$. Since $K$ is Floer simple, $S_{\text{BLACK}}$ is a set of coset representatives for the action of the subgroup $\langle\mu\rangle \subset H_1(Y)$. In other words, every $\mathfrak{s} \in \text{Spin}^c(Y, \partial Y)$ can be written in a unique way as $\bar{\mathfrak{s}} + n\mu$, where $\bar{\mathfrak{s}} \in S_{\text{BLACK}}$ and $n \in \mathbb{Z}$. We color $\mathfrak{s}$ black if $n = 0$, red if $n > 0$, and blue if $n < 0$.

Now suppose we do surgery along $K$ with slope $\lambda$, where $\mu \cdot \lambda = 1$ and $\phi(\lambda) \neq 0$ (*i.e.* $\lambda$ is not the homological longitude). We divide $\text{Spin}^c(Y, \partial Y)$ into cosets for the action of $\langle\lambda\rangle$. Since $\phi(\lambda) \neq 0$, each coset $L$ is an affine copy of $\mathbb{Z}$, so it has a natural ordering. Each element of $L$ is colored either black, red, or blue; elements which are sufficiently negative are all colored blue, and elements which are sufficiently positive are all colored red. We say $L$ is *properly colored* if no red element of $L$ appears before a blue element.

**Proposition 3.7.** *$Y(\lambda)$ is an L-space if and only if every coset for the action of $\langle\lambda\rangle$ is properly colored.*

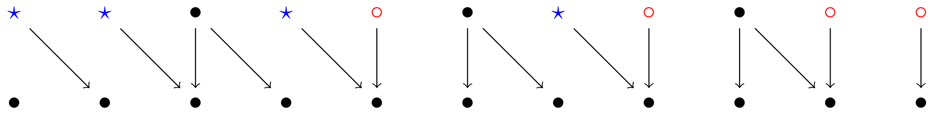

**Fig. 1.** Part of a typical complex $C_L$. Blue dots are shown by stars; red dots by hollow circles. Summands of each of the possible forms are visible. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Proof.** The argument is the same as the proof of Lemma 4.8 in [45]; we sketch it briefly here. We fix a prime $p$ and use the mapping cone to compute $\widehat{HF}(Y(\lambda);\mathbb{F}_p)$. The mapping cone $C_\lambda(K)$ decomposes as a direct sum of chain complexes $C_L$, one for each coset $L$. Since $K$ is Floer simple, the bent groups $\mathbf{A}_{K,\bar{\mathfrak{s}}+n\lambda}$ appearing in one summand are all isomorphic to $\mathbb{F}_p$, as are the groups $\mathbf{B}_{K,\bar{\mathfrak{s}}+n\lambda}$. Let $h_\mathfrak{s}, v_\mathfrak{s}$ be the restriction of the maps $h, v$ to $\mathbf{A}_{K,\mathfrak{s}}$. If $\mathfrak{s}$ is colored red, the map $v_\mathfrak{s}$ is an isomorphism and $h_\mathfrak{s} = 0$; if $\mathfrak{s}$ is colored blue, the map $h_\mathfrak{s}$ is an isomorphism and $v_\mathfrak{s} = 0$; and if $\mathfrak{s}$ is colored black, both $h_\mathfrak{s}$ and $v_\mathfrak{s}$ are isomorphisms.

The complex $C_L$ takes the form shown in Fig. 1, where each colored dot in the top row represents $\mathbf{A}_{K,\bar{\mathfrak{s}}+n\lambda} \simeq \mathbb{F}_p$, each dot in the bottom row represents $\mathbf{B}_{K,\bar{\mathfrak{s}}+n\lambda} \simeq \mathbb{F}_p$, and the arrows represent nonzero differentials. The chain of differentials breaks each time we encounter a red or blue dot, thus decomposing $C_L$ into smaller summands. Summands corresponding to intervals in $L$ whose endpoints are both red or both blue are acyclic; summands whose left endpoint is blue and whose right endpoint is red have homology in even $\mathbb{Z}/2$ homological degree, and summands whose left endpoint is red and whose right endpoint is blue have homology in odd $\mathbb{Z}/2$ homological degree.

It follows that $\widehat{HF}(Y(\lambda), \bar{\mathfrak{s}}) \simeq \mathbb{F}_p$ if and only if $L$ is properly colored, and hence that $Y(\lambda)$ is an $\mathbb{F}_p$ L-space if and only if every coset is properly colored. Finally, the statement of the proposition follows from the fact that $Y(\lambda)$ is an L-space if and only it is an $\mathbb{F}_p$ L-space for every prime $p$. □

### *3.3. Bordered Floer homology of Floer simple manifolds*

In this section, we show that the bordered Floer homology [30] of a Floer simple manifold $Y$ is determined by the Turaev torsion of $Y$ together with a slope in the interior of $\mathcal{L}(Y)$. We very briefly review some facts about bordered Floer homology; for more details see [30,31].

A bordered three-manifold is an oriented three-manifold $Y$ equipped with a *parametrization* (that is, a minimal handle decomposition) of its boundary. We will restrict our attention to the case where $\partial Y = T^2$, in which case a parametrization is specified by a choice of two simple closed curves $\mu, \lambda \in H_1(\partial Y)$ which satisfy $\mu \cdot \lambda = 1$.

The type $D$ bordered Floer homology $\widehat{CFD}(Y, \mu, \lambda)$ is a differential graded module over a certain $\mathbb{F}_2$-algebra $\mathcal{A}(\mathcal{Z})$ associated to the torus. $\mathcal{A}(\mathcal{Z})$ is generated by elements $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}$ and $\rho_{123}$ corresponding to certain arcs on the boundary of the 0-handle in the handle decomposition of $\partial Y$, together with a pair of idempotents $\iota_0, \iota_1$. We let $\mathcal{I} = \langle \iota_0, \iota_1 \rangle \subset \mathcal{A}(\mathcal{Z})$ be the ring of idempotents.

Following Chapter 11 of [30], we can think of the module structure as being specified by a pair of vector spaces $V^0$, $V^1$ over the field of two elements $\mathbb{F}_2$, together with linear maps

$$D_1, D_3, D_{123} : V^0 \to V^1 \qquad D_2 : V^1 \to V^0$$
$$D_{12} : V^0 \to V^0 \qquad D_{23} : V^1 \to V^1$$

where $\widehat{CFD}(Y, \mu, \lambda) = \mathcal{A}(\mathcal{Z}) \otimes_{\mathcal{I}} (V^0 \oplus V^1)$ and for $x \in V^0 \oplus V^1$, the differential is given by $\partial x = \sum \rho_I D_I(x)$.

In writing the above, we have assumed that $\widehat{CFD}(Y, \mu, \lambda)$ has been reduced with respect to all provincial differentials, so that

$$V^0 \simeq SFH(Y, \gamma_\mu) \simeq \widehat{HFK}(K_\mu) \quad V^1 \simeq \widehat{HFK}(Y, \gamma_\lambda) \simeq HFK(K_\lambda)$$

where the suture $\gamma_\mu$ is two parallel copies of $\mu$, and similarly for $\gamma_\lambda$.

Petkova [43] showed that the algebra $\mathcal{A}(\mathcal{Z})$ can be given an absolute $\mathbb{Z}/2$ grading, and that $\widehat{CFD}(Y, \mu, \lambda)$ can be given a $\mathbb{Z}/2$ grading compatible with it. Petkova's grading depends on some auxiliary choices, but we can make some statements which are independent of these choices.

**Lemma 3.8.** *The maps $D_{12}$ and $D_{23}$ preserve the homological $\mathbb{Z}/2$ grading. If $D_1$ has parity $i$ with respect to the $\mathbb{Z}/2$ grading, then $D_2$, $D_3$ and $D_{123}$ have parity $1 + i$, $i$ and $1 + i$, respectively.*

**Proof.** We first consider the absolute grading on $\mathcal{A}(\mathcal{Z})$. By definition, algebra generators corresponding to arcs joining two ends of the same $\alpha$ arc have grading 1. (See definition 11 of [43] and the equations just preceding it.) In our case, this says that $\operatorname{gr} \rho_{12} \equiv \operatorname{gr} \rho_{23} \equiv 1$. From the relations $\rho_1 \cdot \rho_{23} = \rho_{123}$, $\rho_1 \cdot \rho_2 = \rho_{12}$, and $\rho_2 \cdot \rho_3 = \rho_{23}$, we see that $\operatorname{gr} \rho_{123} \equiv \operatorname{gr} \rho_1 + 1$, $\operatorname{gr} \rho_2 \equiv \operatorname{gr} \rho_1 + 1$, and $\operatorname{gr} \rho_3 \equiv \operatorname{gr} \rho_2 + 1 \equiv \operatorname{gr} \rho_1$. The statement now follows from the fact that $\operatorname{gr} \partial \mathbf{x} \equiv \operatorname{gr} \mathbf{x} + 1$. □

We will also need to know how the $D_I$'s behave with respect to the $\text{Spin}^c$ grading. Let us write $V^0_{\mathfrak{s}} := \widehat{HFK}(K_\mu, \mathfrak{s})$, so we have a decomposition $V^0 \simeq \oplus_{\mathfrak{s}} V^0_{\mathfrak{s}}$, and similarly for $V^1$, where the indexing sets in the sums are $\text{Spin}^c(Y, \gamma_\mu)$ and $\text{Spin}^c(Y, \gamma_\lambda)$, as defined in [26].

In what follows, it will be convenient to work with a slightly more general notion of relative $\text{Spin}^c$ structure. Suppose $v \in \Gamma(TY|_{\partial Y})$ is a nonvanishing vector field on $\partial Y$. Elements of $\text{Spin}^c(Y, v)$ are defined to be homology classes of nonvanishing vector fields on $Y$ which restrict to $v$ on $\partial Y$. (Recall that two nonvanishing vector fields are said to be homologous if they are homotopic on the complement of a ball in $Y$.) The set $\text{Spin}^c(Y, \gamma_\mu)$ is defined to be $\text{Spin}^c(Y, v_\mu)$, where $v_\mu$ is a particular nonvanishing vector field defined in section 4 of [26].

If $\mathrm{Spin}^c(Y,v)$ is nonempty, a standard obstruction theory argument shows it is an affine space over $H^2(Y,\partial Y) \simeq H_1(Y)$. Moreover, if $[a] \in H_1(Y)$ is represented by an embedded simple curve $a \subset Y$, a vector field representing $[a]\cdot\mathfrak{s}$ can be obtained from a vector field representing $\mathfrak{s}$ by a local modification in a neighborhood of $a$, as described in section 20.2 of [46]. Finally, if $V$ is a nonvanishing vector field on $T^2\times[0,1]$ which restricts to $v_i$ on $T^2\times\{i\}$ $(i=0,1)$, we obtain a $H_1(Y)$ equivariant bijection $i_V : \mathrm{Spin}^c(Y,v_0) \to \mathrm{Spin}^c(Y,v_1)$ by concatenation with $V$.

**Lemma 3.9.** *There is a bijection $j : \mathrm{Spin}^c(Y,\gamma_\mu) \to \mathrm{Spin}^c(Y,\gamma_\lambda)$ which respects the action of $H_1(Y)$ and for which*

$$D_1 : V^0_{\mathfrak{s}} \to V^1_{j(\mathfrak{s})} \qquad D_2 : V^1_{j(\mathfrak{s})} \to V^0_{\mathfrak{s}-\lambda} \qquad D_3 : V^0_{\mathfrak{s}} \to V^1_{j(\mathfrak{s})+\lambda+\mu}$$
$$D_{12} : V^0_{\mathfrak{s}} \to V^0_{\mathfrak{s}-\lambda} \qquad D_{23} : V^1_{j(\mathfrak{s})} \to V^1_{j(\mathfrak{s})+\mu} \qquad D_{123} : V^0_{\mathfrak{s}} \to V^1_{j(\mathfrak{s})+\mu}$$

This is essentially Lemma 11.42 of [30], but stated so as to clarify the dependence on $\mu$ and $\lambda$.

**Proof.** Huang and Ramos [24] have constructed a grading gr on $\widehat{CFD}(Y,\mu,\lambda)$. This grading lives in a set $S(\mathcal{H})$ of homotopy classes of nonvanishing vector fields on $Y$ which satisfy certain boundary conditions. To be specific, for each elementary idempotent $\iota$ in the algebra $\mathcal{A}(\mathcal{Z})$, there is an associated vector field $v_\iota$ on $\partial Y$, and if $v \in S(\mathcal{H})$, then $v|_{\partial Y}$ should be equal to $v_\iota$ for some elementary idempotent $\iota$. (The vector field $v_{\iota_0}$ does not agree on the nose with the vector field $v_\mu$ defined by Juhász, but there is a natural homotopy which relates them; similarly for $v_{\iota_1}$ and $v_\lambda$.)

Similarly, Huang and Ramos consider the set $G(\mathcal{Z})$ of homotopy classes of nonvanishing vector fields on $\partial Y \times [0,1]$, subject to the constraint that $v|_{\partial Y\times 0} = v_\iota$ and $v|_{\partial Y\times 1} = v_{\iota'}$ for some elementary idempotents $\iota$ and $\iota'$. They show that $G(\mathcal{Z})$ forms a groupoid under concatenation, and that it acts on the grading set $S(\mathcal{H})$, again by concatenation. In section 2.3 of [24], they construct explicit vector fields $v_I$ on $\partial Y\times[0,1]$ associated to each $\rho_I$; the grading of $\rho_I x$ is the vector field $v_I \cdot \mathrm{gr}\, x$, where $\cdot$ denotes the action by concatenation.

The grading of [24] contains the $\mathrm{Spin}^c$ grading, in the sense that if $\mathbf{x}$ is a generator of $\widehat{CFD}(Y,\mu,\lambda)$, then its $\mathrm{Spin}^c$ grading is $\mathfrak{s}(\mathbf{x}) = p(\mathrm{gr}\,\mathbf{x})$, where $p$ is the forgetful map which takes a homotopy class of vector fields to its homology class. By Theorem 1.3 of [24], if $\mathbf{x} \in \widehat{CFD}(Y,\mu,\lambda)$, $\mathrm{gr}\,\partial\mathbf{x} = \sigma^{-1}\cdot\mathrm{gr}\,\mathbf{x}$, where $\sigma$ is a vector field on $\partial Y\times[0,1]$ which is supported in a ball. It follows that $\mathfrak{s}(\partial\mathbf{x}) = \mathfrak{s}(\mathbf{x})$, and hence that $p(v_I)\cdot\mathfrak{s}(D_I\mathbf{x}) = \mathfrak{s}(\mathbf{x})$.

If $\mathfrak{s} \in \mathrm{Spin}^c(Y,\gamma_\mu)$, we define $j(\mathfrak{s}) = p(v_1^{-1})\cdot\mathfrak{s}$. By construction, $D_1 : V^0_{\mathfrak{s}} \to V^1_{j(\mathfrak{s})}$. The fact that $G(\mathcal{Z})$ is a groupoid implies that $j$ is a bijection; $j$ is equivariant with respect to the action of $H_1(Y)$ since we can arrange this action to take place in the interior of $Y$, away from the region in which the concatenation takes place. Similarly, we see that

$$\mathfrak{s}(D_3\mathbf{x}) = p(v_3^{-1})\cdot\mathfrak{s}(\mathbf{x}) = p(v_3^{-1}\cdot v_1)\cdot j(\mathfrak{s}(\mathbf{x})).$$

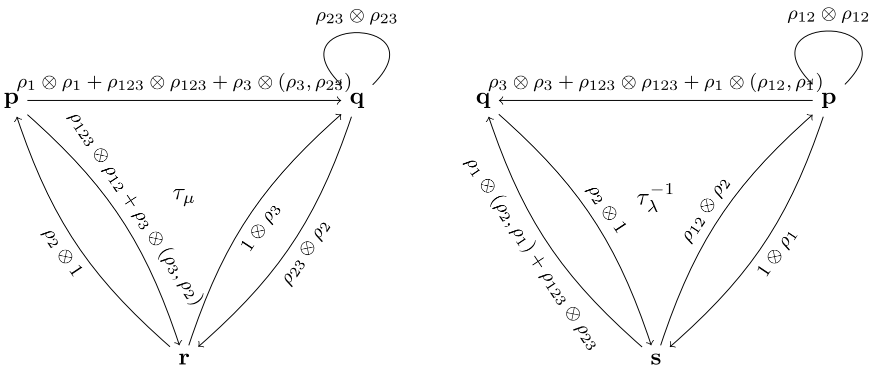


**Fig. 2.** Change of framing bimodules for the torus, taken from figure A.3 of [30].

The set of homology classes of nonvanishing vector fields on $\partial Y \times [0,1]$ which restrict to $v_{\iota_0}$ on one end and $v_{\iota_1}$ on the other is an affine copy of $H_1(\partial Y \times [0,1]) \simeq H_1(\partial Y)$. Thus if $I_1$ is the idempotent of the groupoid $G(\mathcal{Z})$ corresponding to the idempotent $\iota_1$, we must have $p(v_3^{-1} \cdot v_1) = p(I_1) + \alpha$, for some $\alpha \in H_1(\partial Y)$. It follows that $\mathfrak{s}(D_3(\mathbf{x})) = j(\mathfrak{s}(\mathbf{x})) + \alpha$ for some universal element $\alpha \in H_1(\partial Y)$ which does not depend on $Y$ or $\mathbf{x}$. Comparing with Lemma 11.42 of [30], we see that $\alpha = \mu + \lambda$. Thus $D_3 : V^0_{\mathfrak{s}} \to V^1_{j(\mathfrak{s})+\lambda+\mu}$ as desired. The arguments for the other $D_I$'s are very similar. $\square$

**Proposition 3.10.** *Suppose that $Y$ is Floer simple, that $\alpha \in Sl(Y)$ is a Floer simple filling slope, and that $\mu, \lambda \in H_1(\partial Y)$ satisfy $\mu \cdot \lambda = 1$. Then $\widehat{CFD}(Y, \mu, \lambda)$ is determined by $\alpha$ and $\tau(Y)$.*

**Proof.** It suffices to show that $\widehat{CFD}(Y, \mu, \lambda)$ is determined for one particular choice of $\mu$ and $\lambda$, since the invariant of any other choice can then be determined using the change of basis bimodules in [31].

We choose $\mu$ to be a slope in the interior of $\mathcal{L}(Y)$ such that $\phi(\mu) > \|Y\|$, and take $\lambda = \lambda_0 - N\mu$, where $\lambda_0$ is some class with $\mu \cdot \lambda_0 = 1$, and $N \gg 0$. (We will specify below how large $N$ needs to be.)

The knots $K_\mu$, $K_\lambda$ are Floer simple, so all the elements of $V_0$ have the same $\mathbb{Z}/2$ grading. Similarly, all elements of $V_1$ have the same $\mathbb{Z}/2$ grading. By Lemma 3.8, either $D_2 = D_{123} = 0$ or $D_1 = D_3 = 0$. To see which of these two options hold, we consider the effect of a Dehn twist along $\mu$. We have

$$\widehat{CFD}(Y, \mu, \lambda + \mu) = \widehat{CFDA}(\tau_\mu) \boxtimes \widehat{CFD}(Y, \mu, \lambda)$$

where the change of framing bimodule $\widehat{CFDA}(\tau_\mu)$ is shown in Fig. 2.

Writing $\widehat{CFD}(Y, \mu, \lambda + \mu) = W^0 \oplus W^1$, we have $W^1 = \mathbf{r} \boxtimes V^0 \oplus \mathbf{q} \boxtimes V^1$. Denote by $D : W^1 \to W^1$ the contribution to $\partial$ coming from provincial differentials; then we have $H(W^1, D) = \widehat{HFK}(K_{\mu+\lambda})$. By choosing $N$ sufficiently large, we can ensure that $\mu + \lambda = \lambda_0 - (N-1)\mu$ is in the interior of $\mathcal{L}(Y)$. It follows that $\widehat{HFK}(K_{\mu+\lambda})$ is Floer simple

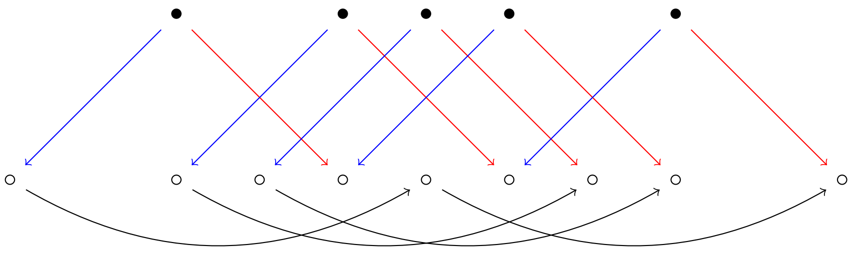

**Fig. 3.** Generators of $\widehat{CFD}(Y, 5m-l, -9m+2l)$, where $Y$ is the complement of the negative trefoil in $S^3$. Dots in the top row represent generators of $V_0$, dots in the bottom row generators of $V_1$. The horizontal position of each generator indicates its Spin$^c$ grading. Potential components of the differential are shown by arrows: red (sloping right) for $D_1$, blue (sloping left) for $D_3$, and black (the arcs) for $D_{23}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and has dimension equal to $|H_1(Y(\mu+\lambda))| = |H_1(Y_\lambda)| - |H_1(Y_\mu)| = \dim V_1 - \dim V_0$. Referring to the figure, we see that the only contribution to the provincial differential $D$ comes from the arrow labeled $1 \otimes \rho_3$. Thus the map $\rho_3 : V^0 \to V^1$ is an injection. Similarly, by considering

$$\widehat{CFD}(Y, \mu+\lambda, \lambda) = \widehat{CFDA}(\tau_\lambda^{-1}) \boxtimes \widehat{CFD}(Y, \mu, \lambda)$$

we deduce that the map $D_1 : V_0 \to V_1$ is injective. Since $D_1$ and $D_3$ are nontrivial, we must have $D_2 = D_{123} = 0$.

Let $\mathfrak{s}_{max} \in S[\widehat{HFK}(K_\mu)]$ be maximal, in the sense that if $\mathfrak{s}_{\max} + \alpha \in S[\widehat{HFK}(K_\mu)]$ (where $\alpha \in H_1(Y)$), then $\phi(\alpha) \leq 0$.

**Lemma 3.11.** *$j(\mathfrak{s}_{\max})$ is maximal in $S[\widehat{HFK}(K_\lambda)]$.*

**Proof.** It is well known [36] that $\widehat{HFK}$ detects the Thurston norm, in the sense that if $K \subset Y(\alpha)$, then

$$\max\{\phi(\mathfrak{s} - \mathfrak{s}') \,|\, \mathfrak{s}, \mathfrak{s}' \in S[\widehat{HFK}(K)]\} = \|Y\| + |\phi(\alpha))|.$$

Choose nonzero elements $\mathbf{x} \in V^0_{\mathfrak{s}_{max}}$, $\mathbf{y} \in V^0_{\mathfrak{s}_{min}}$, where $\phi(\mathfrak{s}_{max} - \mathfrak{s}_{min}) = \|Y\| + \phi(\mu)$. Since $D_1$ and $D_3$ are injective, $j(\mathfrak{s}_{max})$ and $j(\mathfrak{s}_{min}) + \lambda + \mu$ are both in $S[\widehat{HFK}(K_\lambda)]$. We compute

$$\begin{aligned} \phi(j(\mathfrak{s}_{max}) - (j(\mathfrak{s}_{min}) + \mu + \lambda)) &= \|Y\| + |\phi(\lambda)| \\ &= \max\{\phi(\mathfrak{s} - \mathfrak{s}') \,|\, \mathfrak{s}, \mathfrak{s}' \in S[\widehat{HFK}(K_\lambda)]\}. \end{aligned}$$

It follows that $j(\mathfrak{s}_{max})$ must be maximal and $j(\mathfrak{s}_{min} + \mu + \lambda)$ must be minimal. □

We represent $\widehat{CFD}(Y, \mu, \lambda)$ by a directed graph like that shown in Fig. 3, with a vertex for each generator and an edge for each potential component of the differential;

that is, for each pair of generators $\mathbf{x}, \mathbf{y}$ whose $\mathbb{Z}/2$ and $\mathrm{Spin}^c$ gradings are compatible with having $D_I \mathbf{x} = \mathbf{y}$ for some $D_I$, we draw an edge from $\mathbf{x}$ to $\mathbf{y}$ and label it with $D_I$.

**Lemma 3.12.** *Each vertex of the graph associated to $\widehat{CFD}(Y, \mu, \lambda)$ has valence two.*

**Proof.** First suppose that $\mathbf{x}$ is a generator of $V^0$. We have already seen that $D_1$ and $D_3$ are both injective, so $\mathbf{x}$ is the starting point of one arrow labeled with $D_1$ and one arrow labeled with $D_3$. $D_2 = D_{123} = 0$, so the only other possible arrows adjacent to $\mathbf{x}$ are labeled by $D_{12}$. Now $D_{12}$ shifts the $\mathrm{Spin}^c$ grading by $-\lambda$, and $\phi(-\lambda) = N\phi(\mu) - \phi(\lambda_0)$. We choose $N$ sufficiently large that $|\phi(\lambda)| > \max_{\mathfrak{s} \in S} \phi(\mathfrak{s}) - \min_{\mathfrak{s} \in S} \phi(\mathfrak{s})$; then $D_{12}$ vanishes for grading reasons.

Next, if $\mathbf{x}$ is a generator of $V_1$, it can be a terminal point of an arrow labeled $D_1$ or $D_3$, and either an initial or a terminal point of an arrow labeled $D_{23}$. We claim that $\mathbf{x}$ is a terminal point of an arrow of type $D_1$ if and only if it is not an initial point of an arrow of type $D_{23}$. To see this, consider $\mathfrak{s} \in \mathrm{Spin}^c(Y, \gamma_\mu)$. We say $\mathfrak{s}$ is *occupied* if $\mathfrak{s} \in S[\widehat{HFK}(K_\mu)]$, and *unoccupied* otherwise; similarly for $j(\mathfrak{s}) \in \mathrm{Spin}^c(Y, \gamma_\lambda)$, but with $K_\lambda$ in place of $K_\mu$. The claim is equivalent to saying that if $j(\mathfrak{s})$ is occupied, then exactly one of $\mathfrak{s}$ and $j(\mathfrak{s}) + \mu$ is occupied. To prove this, we first record some basic facts about the support.

**Lemma 3.13.** *Let $S = S[\widehat{HFK}(K_\alpha)]$, where $\phi(\alpha) > \|Y\|$. Write $|\mathfrak{s}| = \phi(\mathfrak{s}_{max} - \mathfrak{s})$, and define $\widetilde{\tau}(Y) = \sum_{\phi(h) \leq 0} \widetilde{a}_n[h]$ to be the Turaev torsion of $Y$ expanded as a sum in negative elements of $H_1(Y)$. (This is the opposite of our usual convention.) Then we have*

1. *If $|\mathfrak{s}| < 0$, $\mathfrak{s} \notin S$.*
2. *If $0 \leq |\mathfrak{s}| < \phi(\alpha)$, $\mathfrak{s} \in S$ if and only if the coefficient of $\mathfrak{s} - \mathfrak{s}_{max}$ in $\widetilde{\tau}(Y)$ is* 1.
3. *If $\|Y\| < |\mathfrak{s}| < \phi(\alpha)$, $\mathfrak{s} \in S$.*
4. *If $\phi(\alpha) \leq |\mathfrak{s}| \leq \phi(\alpha) + \|Y\|$, then $\mathfrak{s} \in S$ if and only if $\mathfrak{s} + \alpha \notin S$.*
5. *If $|\mathfrak{s}| > \phi(\alpha) + \|Y\|$, then $\mathfrak{s} \notin S$.*

Our assumption that $\phi(\alpha) > \|Y\|$ implies that the ranges specified in items (2) and (3) overlap.

**Proof.** Item (1) follows from the definition of $\mathfrak{s}_{max}$, while item (5) is the fact that $\widehat{HFK}$ determines the Thurston norm. To prove item (2), we use the relation that $\chi(\widehat{HFK}(K_\alpha)) = (1 - [\alpha])\tau(Y)$. Item (3) follows from the fact that $S$ contains exactly one element in each congruence class modulo $\alpha$ together with the fact that $\phi(\alpha) > \|Y\|$, which implies that if $k \neq 0$, $\mathfrak{s} + k\alpha$ is either in region (1) or in region (5). Finally, in region (4), the only representatives of $\mathfrak{s}$ modulo $\alpha$ which are not in regions (1) or (5) are $\mathfrak{s}$ and $\mathfrak{s} + \alpha$. □

We now check case-by-case (depending on the value of $|\mathfrak{s}|$) that if $j(\mathfrak{s})$ is occupied, then exactly one of $\mathfrak{s}$ and $j(\mathfrak{s}) + \mu$ is occupied.

(i) $|\mathfrak{s}| < 0$. In this case $j(\mathfrak{s})$ is unoccupied, and there is nothing to check.
(ii) $0 \leq |\mathfrak{s}| < \phi(\mu)$. This is in region (2) for both $K_\mu$ and $K_\lambda$, so $\mathfrak{s}$ is occupied if and only if $j(\mathfrak{s})$ is occupied. $|\mathfrak{s}+\mu| > 0$, so $j(\mathfrak{s}) + \mu$ is unoccupied.
(iii) $\phi(\mu) \leq |\mathfrak{s}| \leq \phi(\mu) + \|Y\|$. $\mathfrak{s}$ is in region (4), so $\mathfrak{s}$ is occupied if and only if $\mathfrak{s}+\mu$ is not occupied. $j(\mathfrak{s}) + \mu$ is in region 2), so by (b) it is occupied if and only if $\mathfrak{s}+\mu$ is occupied. Thus $\mathfrak{s}$ is occupied if and only if $j(\mathfrak{s}) + \mu$ is not occupied.
(iv) $\phi(\mu)+\|Y\| < |\mathfrak{s}| \leq |\phi(\lambda)|+\|Y\|$. $\mathfrak{s}$ is in region (5), so it is unoccupied, while $j(\mathfrak{s})+\mu$ is in region (2), since $\phi(\mu) > \|Y\|$. Since $|\mathfrak{s}+\mu| > \|Y\|$, $j(\mathfrak{s}) + \mu$ is occupied.
(v) $|\phi(\lambda)| + \|Y\| < |\mathfrak{s}|$. In this region, $j(\mathfrak{s})$ is unoccupied.

This proves the claim. A very similar argument shows that $\mathbf{x}$ is a terminal point of an arrow of type $D_3$ if and only if it is not the terminal point of an arrow of type $D_{23}$. The statement of the lemma follows. □

Since $K_\mu$ and $K_\lambda$ are Floer simple, each arrow in the diagram corresponds to a map $\mathbb{F}_2 \to \mathbb{F}_2$. To determine the corresponding component of the differential, it suffices to know whether or not this map is 0. We will show that every map corresponding to an arrow in the diagram is nonzero, thus completing the proof of Proposition 3.10. The maps $D_1$ and $D_3$ are injective, so any arrow labeled by $D_1$ or $D_3$ is nonzero. For the arrows labeled by $D_{23}$, we argue as in the proof of Theorem 11.36 in [30]. By Proposition 11.30 of [30], there are maps $D_{012}, D_{01}, D_0, D_{230}$, and $D_{301}$ satisfying

$$D_3 \circ D_{012} + D_{23} \circ D_{01} + D_{123} \circ D_0 = 1_{V_1}$$
$$D_1 \circ D_{230} + D_{01} \circ D_{23} + D_{301} \circ D_2 = 1_{V_1}$$

Since $D_2 = D_{123} = 0$, it follows that if $\mathbf{x}$ is not in the image of $D_3$, it must be in the image of $D_{23}$, and if $\mathbf{x}$ is not in the image of $D_1$, $D_{23}(\mathbf{x}) \neq 0$. Comparing with the proof of Lemma 3.12, we see that every arrow in the diagram must correspond to a nonzero map. □

## 4. Intervals of L-space filling slopes

Now that the "proper coloring" condition of Proposition 3.7 is in place, we are equipped to tackle the problem of describing L-space intervals in terms of $\mathcal{D}^\tau(Y)$ and a slope from the interior of the L-space interval. We begin by establishing some conventions.

### *4.1. Conventions for slopes and homology*

If $Y$ is a compact oriented three-manifold with torus boundary, then a *slope* of $Y$ is a nonseparating, oriented, simple closed curve in $\partial Y$. Such objects correspond bijectively to primitive elements of $H_1(\partial Y)/\{\pm 1\}$, or equivalently, to elements of $\mathbb{P}(H_1(\partial Y))$. Any choice of basis $(m, l)$ for $H_1(\partial Y)$ specifies homogeneous coordinates $nm + n'l \mapsto [n : n']$ on $\mathbb{P}(H_1(\partial Y))$, to which we usually refer in terms of the affinization

$$\begin{aligned} H_1(\partial Y) \setminus \{0\} &\to \mathbb{Q} \cup \{\infty\}, \\ nm + n'l &\mapsto n/n'. \end{aligned} \tag{1}$$

Let $\iota : H_1(\partial Y) \to H_1(Y)$ be the map induced by inclusion. We fix a basis $(m, l)$ for $H_1(\partial Y)$ such that $l$ is a generator of ker $\iota$ and $m \cdot l = 1$. The generator $l$ is the *homological longitude* of $Y$; it is well defined up to sign. In contrast, the choice of $m$ is only well defined up to the addition of a multiple of $l$. Consequently, the numerator of $\pi(nm+n'l) = n/n'$ is canonical (up to sign), but the denominator depends on the choice of $m$.

To Dehn fill $Y$ along a slope $\mu = nm + n'l \in H_1(\partial Y)$, one attaches a 2-handle along the simple closed curve associated to $\mu$, and then fills in the remaining $S^2$ boundary with a 3-ball. The resulting manifold, which we denote by $Y(\mu)$ or $Y(n/n')$, has homology $H_1(Y(\mu)) = H_1(Y)/(\iota(\mu))$, which has order $|n|$ if $H_1(Y)$ is torsion free.

Any non-zero Dehn filling $Y(\mu_{\text{L}})$ produces a knot $K_{\mu_{\text{L}}} := \text{core}(Y(\mu_{\text{L}}) \setminus Y) \subset Y(\mu_{\text{L}})$, on which one can now perform Dehn surgery. Our conventional choice of basis for Dehn *filling* slopes involves a canonical (up to sign) *rational longitude* $l$, with $m$ (satisfying $m \cdot l = 1$) only determined up to addition of copies of $l$. On the other hand, the conventional basis for Dehn *surgery* involves a canonical *meridian*, namely $\mu_{\text{L}}$, for the knot $K_{\mu_{\text{L}}}$, with a longitude $\lambda_{\text{L}} \in H_1(\partial Y)$ (satisfying $\mu_{\text{L}} \cdot \lambda_{\text{L}} = 1$) only determined up to addition of copies of $\mu_{\text{L}}$.

Thus, for an arbitrary slope, say

$$\mu = nm + n'l = \alpha\mu_{\text{L}} + \beta\lambda_{\text{L}} \in H_1(\partial Y), \tag{2}$$

we could describe the Dehn filling $Y(\mu)$ as the $n/n'$-filling of $Y$ (with respect to the basis $(m, l)$), or as the $\alpha/\beta$-surgery along the knot $K_{\mu_{\text{L}}}$ (with respect to the basis $(\mu_{\text{L}}, \lambda_{\text{L}})$). Note that each of these conventional descriptions involves either a denominator or a numerator which is non-canonical. To dodge this problem, we can instead divide the canonical numerator of $n/n'$ by the canonical denominator of $\alpha/\beta$ to obtain $n/\beta$, with

$$n := \mu \cdot l, \qquad \beta := \mu_{\text{L}} \cdot \mu = pn' - qn \text{ (where } \mu_{\text{L}} = pm + ql), \tag{3}$$

and with $|n| = |H_1(Y(\mu))|$ when $H_1(Y)$ is torsion free. Note that $n/\beta$ is not a slope in the conventional sense, since $\mu = n(\mu_{\text{L}}/p) + \beta(l/p)$, with $\mu_{\text{L}}/p, l/p \notin H_1(\partial Y; \mathbb{Z})$, and the projective linear map $\mathbb{P}(H_1(\partial Y)) \to \mathbb{P}(\mathbb{Z}^2)$, $[n : n'] \mapsto [n : \beta]$ is not surjective, having determinant $p$. Still, since this map is injective, it is sufficient for cataloguing slopes. In fact, the reciprocal $\beta/n$ is more convenient for this purpose. Given an initial filling $Y(\mu_{\text{L}})$ on which we wish to perform surgery, we call $(\mu_{\text{L}} \cdot \mu)/(\mu \cdot l) = \beta/n$ the *surgery* $\mu_{\text{L}}$*-label* (or just *surgery label*) of $\mu$. Since

$$\frac{n}{n'} = \frac{p}{q + \beta/n}, \tag{4}$$

the surgery $\mu_{\mathrm{L}}$-label of $\mu$ quantifies the deviation of the Dehn filling slope of $\mu$ from that of $\mu_{\mathrm{L}}$, with a surgery label of $\beta/n = 0$ labeling the original slope $\mu_{\mathrm{L}}$.

We also need conventions for $H_1(Y)$, relative to the map $\iota : H_1(\partial Y) \to H_1(Y)$, restricting to the case of $b_1(Y) = 1$. The Universal Coefficients Theorem implies coker $\iota \cong H^2(Y) \cong \mathrm{Tors}(H_1(Y))$. Thus, setting $T := \mathrm{Tors}(H_1(Y))$ and $T^\partial := \langle \iota(l) \rangle = T \cap \iota(H_1(\partial Y))$, we have coker $\iota = H_1(Y)/\left(\langle \iota(m)\rangle \oplus T^\partial\right) \cong T$, which implies

$$\left(H_1(Y)/T\right)/\iota(m) \cong T^\partial \cong \mathbb{Z}/g, \tag{5}$$

where $g := |T^\partial|$. In other words, any generator $\bar{m}$ for $H_1(Y)/T$ will satisfy $\iota(m) \in \pm g\bar{m} + T$. We shall always choose $\bar{m}$ so that $\iota(m) \in +g\bar{m} + T$.

## 4.2. Conventions for Turaev torsion and $\mathcal{D}^\tau(Y)$

Recall our definition for $\mathcal{D}^\tau(Y) \subset H_1(Y)$ as the finite set

$$\mathcal{D}^\tau(Y) := \{x - y |\, x \notin S[\tau(Y)], y \in S[\tau(Y)]\} \cap \iota(m\mathbb{Z}_{\geq 0} + l\mathbb{Z}), \tag{6}$$

where $\tau(Y)$ is the Turaev torsion of $Y$, which we always normalize so that

$$0 \in S[\tau(Y)], \quad \tau(Y) \in \mathbb{Z}[[t]][T], \tag{7}$$

with $t := [\bar{m}]$ for any generator $\bar{m}$ of $H_1(Y)/T \cong \mathbb{Z}$ satisfying $\iota(m) \in \bar{m}\mathbb{Z}_{>0} + T$.

When $Y$ is Floer simple, we can also define the *torsion complement*,

$$\tau^{\mathrm{c}}(Y) \;:=\; \frac{1}{1-t}\sum_{h \in T}[h] \;-\; \tau(Y), \tag{8}$$

with the Floer simplicity of $Y$ guaranteeing that

$$S[\tau^{\mathrm{c}}(Y)] \;=\; \bar{m}\mathbb{Z}_{\geq 0} \oplus T \;\setminus\; S[\tau(Y)], \tag{9}$$

so that $\mathcal{D}^\tau(Y)$ admits the alternative definition

$$\mathcal{D}^\tau(Y) := (S[\tau^{\mathrm{c}}(Y)] - S[\tau(Y)]) \cap \iota(m\mathbb{Z}_{\geq 0} + l\mathbb{Z}). \tag{10}$$

We shall often want to restrict our attention to the non-torsion elements of $\mathcal{D}^\tau(Y)$,

$$\mathcal{D}^\tau_{>0}(Y) := (S[\tau^{\mathrm{c}}(Y)] - S[\tau(Y)]) \cap \iota(m\mathbb{Z}_{>0} + l\mathbb{Z}) \;=\; \mathcal{D}^\tau(Y) \setminus T. \tag{11}$$

When we wish to emphasize our inclusion of the torsion elements of $\mathcal{D}^\tau(Y)$, we shall write $\mathcal{D}^\tau_{\geq 0}(Y)$ for $\mathcal{D}^\tau(Y)$.

Although we shall not need the following fact until the proof of Theorem 6.2 in Section 6, we lastly remark that the complement of $\mathcal{D}^\tau(Y)$ is a semigroup.

**Proposition 4.1.** *If $Y$ is Floer-simple, then the complement $\Gamma(Y) := \iota(m\mathbb{Z}_{\geq 0}+l\mathbb{Z})\setminus\mathcal{D}^\tau(Y)$ is closed under addition.*

**Proof.** Suppose there exist $x, y \in \Gamma(Y)$ with $x + y \in \mathcal{D}^\tau(Y)$. Since $x + y \in \mathcal{D}^\tau(Y)$, we know there exists $z \in S[\tau(Y)]$ for which $x + y + z \in S[\tau^c(Y)]$. If $z + x \in S[\tau^c(Y)]$ then $x = (x + z) - z \in \mathcal{D}^\tau(Y)$, a contradiction. On the other hand, if $s + x \in S[\tau(Y)]$, then $y = (x + y + z) - (x + z) \in \mathcal{D}^\tau(Y)$, another contradiction. Thus $x + y \notin \mathcal{D}^\tau(Y)$. □

In the case that $Y$ is the complement of an algebraic knot $K \hookrightarrow S^3$ linking the germ of some complex planar curve singularity $(C, \circ) \hookrightarrow (\mathbb{C}^2, \circ)$, $\Gamma(Y)$ coincides with the singularity semigroup (see [10]) associated to the Newton–Puiseux expansion of the singularity.

### *4.3. Notation: truncation and remainders*

Lastly, we need some basic arithmetic notation. Henceforth in this paper, we use the conventional truncations $\lfloor\cdot\rfloor, \lceil\cdot\rceil : \mathbb{Q} \to \mathbb{Z}$,

$$\lfloor r\rfloor := \max\{z \in \mathbb{Z} \,|\, z \leq r\}, \quad \lceil r\rceil := \min\{z \in \mathbb{Z} \,|\, z \geq r\}, \tag{12}$$

and the less conventional notation $[\cdot]_p : \mathbb{Z} \to \{0, \ldots, |p| - 1\}$ to select a representative modulo $p$, by projecting an integer to $\mathbb{Z}/|p|\mathbb{Z}$ and then selecting its preimage in $\{0, \ldots, |p| - 1\} \subset \mathbb{Z}$. In terms of our truncation notation,

$$[a]_b = a - \left\lfloor \frac{a}{b} \right\rfloor b, \quad [-a]_b = -a + \left\lceil \frac{a}{b} \right\rceil b, \ \text{ when } b > 0. \tag{13}$$

### *4.4. Restating Theorem 1.6 as Theorem 4.2*

We are now equipped to re-express Theorem 1.6 in a more practical form, describing the L-space slope interval $\mathcal{L}(Y)$ in terms of any given slope from the interior $\mathcal{L}^\circ(Y)$ of that interval, using the "surgery label" description of slopes. Since the interval of L-space surgery labels always excludes $\infty$—its being the surgery label of the rational longitude $l$—we can always describe a closed interval of L-space surgery labels in terms of its minimum and maximum in $\mathbb{Q}$.

That is, given an L-space slope $\mu_{\text{L}} = pm + ql \in H_1(\partial Y)$ with $\langle\mu_{\text{L}}\rangle \in \mathcal{L}^\circ(Y)$, Theorem 1.6 tells us that when $\mathcal{D}^\tau_{>0} \neq \emptyset$, a Dehn filling $Y(\mu)$ is an L-space if and only if

$$\pi_{\mu_{\text{L}}}(\tilde{\boldsymbol{\delta}}_-) \leq \pi_{\mu_{\text{L}}}(\mu) \leq \pi_{\mu_{\text{L}}}(\tilde{\boldsymbol{\delta}}_+) \quad \text{for all } \boldsymbol{\delta} \in \mathcal{D}^\tau_{>0}(Y), \tag{14}$$

where $\pi_{\mu_{\text{L}}}$ denotes the surgery $\mu_{\text{L}}$-label,

$$\pi_{\mu_{\mathrm{L}}} : H_1(\partial Y) \setminus \{0\} \to \mathbb{Q} \cup \{\infty\}, \quad \mu \mapsto \pi_{\mu_{\mathrm{L}}}(\mu) := (\mu_{\mathrm{L}} \cdot \mu)/(\mu \cdot l), \tag{15}$$

and where, for each $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$, the lifts $\tilde{\boldsymbol{\delta}}_-, \tilde{\boldsymbol{\delta}}_+ \in \iota^{-1}(\boldsymbol{\delta})$, with $\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}_-) < \pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}_+)$, are the two lifts of $\boldsymbol{\delta}$ closest to $\mu_{\mathrm{L}}$ with respect to $\pi_{\mu_{\mathrm{L}}}$, again assuming $\mathcal{D}^{\tau}_{>0}(Y)$ nonempty.

Since $\mathcal{D}^{\tau}_{>0}(Y) \subset \iota(H_1(\partial Y))$, we can express any $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$ as $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l)$. Any lift $\tilde{\boldsymbol{\delta}} \in \iota^{-1}(\boldsymbol{\delta})$ of $\boldsymbol{\delta}$ then takes the form $\tilde{\boldsymbol{\delta}} = \delta m + \tilde{\gamma} l$, satisfying $\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}) = (\mu_{\mathrm{L}} \cdot \tilde{\boldsymbol{\delta}})/\delta = (p\tilde{\gamma} - q\delta)/\delta$, for some $\tilde{\gamma} \equiv \gamma \,(\mathrm{mod}\; g)$. In other words, we have

$$\left\{\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}) \middle| \, \iota(\tilde{\boldsymbol{\delta}}) = \boldsymbol{\delta}\right\} = \frac{[p\gamma - q\delta]_{pg} + pg\mathbb{Z}}{\delta}. \tag{16}$$

Since $\pi_{\mu_{\mathrm{L}}}(\mu_{\mathrm{L}}) = 0$ and $\langle \mu_{\mathrm{L}} \rangle \in \mathcal{L}^{\circ}(Y)$, Theorem 1.6 implies $\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}_-)$ and $\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}_+)$ are respectively the largest negative and smallest positive elements of $\left\{\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}) \middle| \, \iota(\tilde{\boldsymbol{\delta}}) = \boldsymbol{\delta}\right\}$, so that

$$\pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}_-) = \frac{([p\gamma - q\delta]_{pg} - pg)}{\delta} \quad \text{and} \quad \pi_{\mu_{\mathrm{L}}}(\tilde{\boldsymbol{\delta}}_+) = \frac{[p\gamma - q\delta]_{pg}}{\delta} \tag{17}$$

for each $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}$. Thus, proving Theorem 1.6 is equivalent to proving the following.

**Theorem 4.2.** *Suppose $Y$ is Floer simple. If $\mathcal{D}^{\tau}_{>0}(Y) \neq \emptyset$ and $\mu_{\mathrm{L}} = pm + ql \in H_1(\partial Y)$ satisfies $\langle \mu_{\mathrm{L}} \rangle \in \mathcal{L}^{\circ}(Y)$, then the Dehn filling $Y(\mu)$ is an L-space if and only if*

$$\frac{b^{\boldsymbol{\delta}}_-}{\delta} \leq \frac{\mu_{\mathrm{L}} \cdot \mu}{\mu \cdot l} \leq \frac{b^{\boldsymbol{\delta}}_+}{\delta} \quad \textit{for all } \boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{>0}(Y), \tag{18}$$

*where $b^{\boldsymbol{\delta}}_+ := [p\gamma - q\delta]_{pg}$ and $b^{\boldsymbol{\delta}}_- := b^{\boldsymbol{\delta}}_+ - pg$. If $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$, then $Y(\mu)$ is an L-space if and only if $\langle \mu \rangle \neq \langle l \rangle$, or equivalently when $(\mu_{\mathrm{L}} \cdot \mu)/(\mu \cdot l)$ is finite.*

**Remark.** When $\mathcal{D}^{\tau}_{>0}(Y) \neq \emptyset$, the above finite collection of inequalities cuts out $\mathcal{L}(Y)$ as the largest closed interval in $Sl(Y)$ containing $\mu_{\mathrm{L}}$ without containing any elements of $\mathbb{P}(\iota^{-1}(\mathcal{D}^{\tau}_{>0}(Y)))$ in its interior, for $\mathbb{P}(\iota^{-1}(\mathcal{D}^{\tau}_{>0}(Y)))$ the image of $\iota^{-1}(\mathcal{D}^{\tau}_{>0}(Y))$ under the projectivization map $\mathbb{P} : H_1(\partial Y) \setminus \{0\} \to Sl(Y)$. Note that (18) is just a restatement of (14).

Since one could argue that it is difficult to know whether an L-space slope $\mu_{\mathrm{L}}$ satisfies $\langle \mu_{\mathrm{L}} \rangle \in \mathcal{L}^{\circ}(Y)$ without first having computed $\mathcal{L}^{\circ}(Y)$, we point out the following.

**Corollary 4.3.** *If $Y$ is Floer simple and $Y(\mu_{\mathrm{L}})$ is an L-space, for $\mu_{\mathrm{L}} = pm + ql \in H_1(\partial Y)$, then the following are equivalent:*

*(i)* $K_{\mu_{\mathrm{L}}} \subset Y(\mu_{\mathrm{L}})$ *is a Floer simple knot,*
*(ii)* $\langle \mu_{\mathrm{L}} \rangle \in \mathcal{L}^{\circ}(Y)$,
*(iii)* $\langle \mu_{\mathrm{L}} \rangle \notin \mathbb{P}(\iota^{-1}(\mathcal{D}^{\tau}_{>0}(Y)))$,
*(iv)* $b^{\boldsymbol{\delta}}_+ := [p\gamma - q\delta]_{pg} \neq 0$ *for all* $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{>0}(Y)$.

(The equivalence of $(i)$ and $(ii)$ is from Corollary 3.6. For $(iv)$, see the paragraph preceding the theorem statement.) Whereas verifying condition $(iii)$ requires determining membership in an infinite set, testing condition $(iv)$ only requires $|\mathcal{D}^{\tau}_{>0}(Y)|$ comparisons, using numbers we already needed to compute. If it does turn out that some $b^{\boldsymbol{\delta}}_{+} = 0$, then $\langle \mu_{\mathrm{L}} \rangle$ is an endpoint of the interval $\mathcal{L}(Y)$, and the other endpoint of $\mathcal{L}(Y)$ has surgery $\mu_{\mathrm{L}}$-label either $\max(b^{\boldsymbol{\delta}}_{-}/\delta)$ or $\min(b^{\boldsymbol{\delta}}_{+}/\delta)$ over $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$, but determining which of these two endpoints is correct requires more data, such as the knowledge of a second L-space slope.

For completeness, we now pause to re-express Theorem 4.2 in terms of our two more conventional bases, starting with conventional surgery coefficients for surgery along the knot core $K_{\mu_{\mathrm{L}}} \subset Y(\mu_{\mathrm{L}}) \setminus Y$ associated to a given interior L-space slope $\mu_{\mathrm{L}} = pm + ql$ with some specified longitude $\lambda_{\mathrm{L}} := q^*m + p^*l$, $\mu_{\mathrm{L}} \cdot \lambda_{\mathrm{L}} = 1$, so that $pp^* - qq^* = 1$. Next, for each $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$, we express the lifts $\tilde{\boldsymbol{\delta}}_{+}, \tilde{\boldsymbol{\delta}}_{-} \in \iota^{-1}(\boldsymbol{\delta})$ flanking $\mu_{\mathrm{L}}$ as

$$\tilde{\boldsymbol{\delta}}_{+} = a^{\boldsymbol{\delta}}_{+}\mu_{\mathrm{L}} + b^{\boldsymbol{\delta}}_{+}\lambda_{\mathrm{L}}, \qquad \tilde{\boldsymbol{\delta}}_{-} = a^{\boldsymbol{\delta}}_{-}\mu_{\mathrm{L}} + b^{\boldsymbol{\delta}}_{-}\lambda_{\mathrm{L}}, \tag{19}$$

with $b^{\boldsymbol{\delta}}_{+}$, $b^{\boldsymbol{\delta}}_{-}$ $a^{\boldsymbol{\delta}}_{+}$, and $a^{\boldsymbol{\delta}}_{-}$ satisfying

$$b^{\boldsymbol{\delta}}_{+} := [p\gamma - q\delta]_{pg}, \quad b^{\boldsymbol{\delta}}_{-} := b^{\boldsymbol{\delta}}_{+} - pg, \quad \delta \ = \ a^{\boldsymbol{\delta}}_{+}p + b^{\boldsymbol{\delta}}_{+}q^* = a^{\boldsymbol{\delta}}_{-}p + b^{\boldsymbol{\delta}}_{-}q^* > 0. \tag{20}$$

When $p > 0$, a straightforward calculation shows that

$$\frac{a^{\boldsymbol{\delta}}_{-}}{b^{\boldsymbol{\delta}}_{-}} \ < \ -\frac{q^*}{p} \ < \ \frac{a^{\boldsymbol{\delta}}_{+}}{b^{\boldsymbol{\delta}}_{+}} \quad \text{for all } \ \boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y), \tag{21}$$

and Theorem 4.2 takes the following form.

**Corollary 4.4.** *Suppose $Y$ is Floer simple. If $\mathcal{D}^{\tau}_{>0}(Y) \neq \emptyset$, and $\mu_{\mathrm{L}} = pm + ql$ with $p > 0$ is an L-space slope for $Y$ satisfying $b^{\boldsymbol{\delta}}_{+} := [p\gamma - q\delta]_{pg} \neq 0$ for all $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{>0}(Y)$, then for any longitude $\lambda_{\mathrm{L}} = q^*m + p^*l$ (with $\mu_{\mathrm{L}} \cdot \lambda_{\mathrm{L}} = 1$), the $\alpha/\beta$ surgery along $K_{\mu_{\mathrm{L}}} \subset Y(\mu_{\mathrm{L}})$—or equivalently, the Dehn filling $Y(\mu)$ with $\mu := \alpha\mu_{\mathrm{L}} + \beta\lambda_{\mathrm{L}}$—is an L-space if and only if*

$$\frac{\alpha}{\beta} \leq \frac{a^{\boldsymbol{\delta}}_{-}}{b^{\boldsymbol{\delta}}_{-}} \quad \text{or} \quad \frac{a^{\boldsymbol{\delta}}_{+}}{b^{\boldsymbol{\delta}}_{+}} \leq \frac{\alpha}{\beta} \quad \text{for all } \ \boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y), \tag{22}$$

*where $\iota(a^{\boldsymbol{\delta}}_{+}\mu_{\mathrm{L}} + b^{\boldsymbol{\delta}}_{+}\lambda_{\mathrm{L}}) = \iota(a^{\boldsymbol{\delta}}_{-}\mu_{\mathrm{L}} + b^{\boldsymbol{\delta}}_{-}\lambda_{\mathrm{L}}) = \boldsymbol{\delta}$, with $b^{\boldsymbol{\delta}}_{-} := b^{\boldsymbol{\delta}}_{+} - pg$, for each $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$. In such case, the left hand inequality obtains when $\beta/n < 0$, the right hand when $\beta/n > 0$, and we regard both inequalities as vacuously true when $\beta/n = 0$, where $n := \mu \cdot l = \alpha p + \beta q^*$. If $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$, then $Y(\mu)$ is an L-space if and only if $n \neq 0$.*

**Remark.** Note that this makes $\langle \tilde{\boldsymbol{\delta}}_{+} \rangle$ and $\langle \tilde{\boldsymbol{\delta}}_{-} \rangle$ the left-hand and right-hand endpoints, respectively, of $\mathcal{L}(Y) \subset Sl(Y)$ with respect to the orientation induced on $Sl(Y)$ by taking surgery coefficients or Dehn filling slopes. That is because we originally constructed $\tilde{\boldsymbol{\delta}}_+$ and $\tilde{\boldsymbol{\delta}}_-$ to yield the right-hand and left-hand endpoints, respectively, of the space of $\mu_{\mathrm{L}}$-surgery labels for $\mathcal{L}(Y)$, which has the opposite orientation.

One could also characterize L-space slopes in terms of the Dehn filling basis, $m, l$. If we take $\mu_{\mathrm{L}} = pm + ql$ to be an interior L-space slope with $p > 0$, then for any $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{>0}(Y)$, it follows from the two identities in (13) that

$$[p\gamma - q\delta]_{pg} = [-q\delta]_p + p\Big[\gamma - \Big\lceil \tfrac{q}{p}\delta \Big\rceil\Big]_g; \quad -[q\delta - p\gamma]_{pg} = -[q\delta]_p - p\Big[\Big\lfloor \tfrac{q}{p}\delta \Big\rfloor - \gamma\Big]_g, \tag{23}$$

from which it follows that the lifts $\tilde{\boldsymbol{\delta}}_+, \tilde{\boldsymbol{\delta}}_- \in \iota^{-1}(\boldsymbol{\delta})$ adjacent to $\mu_{\mathrm{L}}$ take the form

$$\tilde{\boldsymbol{\delta}}_+ = \delta m + \left(\left\lceil \frac{q}{p}\delta \right\rceil + \left[\gamma - \left\lceil \frac{q}{p}\delta \right\rceil\right]_g\right) l, \quad \tilde{\boldsymbol{\delta}}_- = \delta m + \left(\left\lfloor \frac{q}{p}\delta \right\rfloor - \left[\left\lfloor \frac{q}{p}\delta \right\rfloor - \gamma\right]_g\right) l. \tag{24}$$

As expected, these are the lifts of $\boldsymbol{\delta}$ with Dehn filling slope closest to $p/q$ (regardless of whether $p > 0$), and Theorem 4.2 takes the following form.

**Corollary 4.5.** *Suppose $Y$ is Floer simple. If $\mathcal{D}^{\tau}_{>0}(Y) \neq \emptyset$, and $\mu_{\mathrm{L}} = pm + ql$ is an L-space slope for $Y$ satisfying $p\gamma - q\delta \not\equiv 0 \pmod{pg}$ for all $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{>0}(Y)$, then $\mu = nm + n'l$ is an L-space slope for $Y$ if and only if $\frac{n}{n'} \in I^{\boldsymbol{\delta}}$ for all $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$, where, for each $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)$, $I^{\boldsymbol{\delta}}$ is the closed interval in $\mathbb{Q} \cup \{\infty\}$ which excludes 0 and has endpoints*

$$\frac{\delta}{\left\lceil \frac{q}{p}\delta \right\rceil + \left[\gamma - \left\lceil \frac{q}{p}\delta \right\rceil\right]_g}, \quad \frac{\delta}{\left\lfloor \frac{q}{p}\delta \right\rfloor - \left[\left\lfloor \frac{q}{p}\delta \right\rfloor - \gamma\right]_g}. \tag{25}$$

*If $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$, then $Y(\mu)$ is an L-space if and only if $n \neq 0$.*

**Example.** Suppose $K \subset S^3$ is an L-space knot of positive genus $g(K)$, with Alexander polynomial $\Delta(K)$. Then $Y := S^3 \setminus \nu(K)$ is Floer simple, and since $K \subset S^3$ an L-space knot implies $\deg \Delta(K) = 2g(K)$, the hypothesis $g(K) > 0$ implies $\mathcal{D}^{\tau}_{>0}(Y) \neq \emptyset$. Since $H_1(Y)$ is torsion free, the endpoints of $I^{\boldsymbol{\delta}}$ reduce to $\delta / \left\lceil \frac{q}{p} \right\rceil$ and $\delta / \left\lfloor \frac{q}{p} \right\rfloor$ for each $\boldsymbol{\delta} = \delta\iota(m) \in \mathcal{D}^{\tau}_{>0}(Y)$. We already know that the infinity filling $Y(1m + 0l) = S^3$ is an L-space. Thus (if necessary replacing $K$ with its mirror and using $-\frac{n}{n'}$ for $\frac{n}{n'}$ in (26)), we know that $Y(pm + 1l)$ is an L-space for any $p > 0$ sufficiently large. Taking $p > \max_{\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{>0}(Y)} \delta$ then makes the endpoints of each $I^{\boldsymbol{\delta}}$ become $\delta / \left\lceil \frac{1}{p} \right\rceil = \delta$ and $\delta / \left\lfloor \frac{1}{p} \right\rfloor = +\infty$, and we recover the well known result that for $n' \neq 0$, $Y(nm + n'l)$ is an L-space if and only if

$$\frac{n}{n'} \geq \max_{\delta\iota(m) \in \mathcal{D}^{\tau}_{>0}(Y)} \delta = \deg \tau^{\mathrm{c}}(Y) = (\deg \Delta(K)) - 1 = 2g(K) - 1. \tag{26}$$

### 4.5. Set-up for proof of Theorem 4.2

For the remainder of this section, we fix the following data:

- $Y$ a Floer simple 3-manifold,
- $\mu_{\rm L} = pm + ql \in H_1(\partial Y)$ primitive, with $\langle \mu_{\rm L} \rangle \in \mathcal{L}^\circ(Y)$,
- $\mu = nm + n'l \in H_1(\partial Y)$ primitive, with the goal to determine if $\langle \mu \rangle \in \mathcal{L}(Y)$.

We begin by making some simplifying assumptions, without loss of generality.

**Proposition 4.6.** *For purposes of proving Theorem 4.2, we may assume, without loss of generality, that $p, \beta > 0$, $n \neq 0$, $pg > \deg_{[\bar{m}]} \tau^{\rm c}(Y)$, and $\gcd(p, q) = \gcd(pg, \beta) = 1$, where $\beta := \mu_{\rm L} \cdot \mu$ and $g := |\langle \iota(l) \rangle|$, with $\iota : H_1(\partial Y) \to H_1(Y)$ the map induced by inclusion.*

**Proof.** Theorem 4.2 already correctly characterizes the case of $\beta = 0$, for which $\langle \mu \rangle = \langle \mu_{\rm L} \rangle \in \mathcal{L}^\circ(Y)$, and the case of $n = 0$, for which the filling $Y(\mu) = Y(l)$ is not a rational homology sphere, hence not an L-space. Likewise, we know that any L-space slope $\mu_{\rm L} = pm + ql$ must have $p \neq 0$. Since we are free to replace $\mu_{\rm L}$ with $-\mu_{\rm L}$ or $\mu$ with $-\mu$, we may take $p, \beta > 0$ without loss. Lastly, since $\langle \mu_{\rm L} \rangle \in \mathcal{L}^\circ(Y)$, we can approximate $\mu_{\rm L}$ with another primitive L-space slope $\mu'_{\rm L} = p'm + q'l$ with $\langle \mu'_{\rm L} \rangle \in \mathcal{L}^\circ(Y)$, such that $q' \neq 0$, $p'g > \deg_{[\bar{m}]} \tau^{\rm c}(Y)$, and $\gcd(p'g, \beta') = 1$, where $\beta' := \mu'_{\rm L} \cdot \mu$. □

We henceforth consider the assumptions of Proposition 4.6 to hold.

Our primary tool from Heegaard Floer homology to determine whether $Y(\mu)$ is an L-space is Proposition 3.7, but to exploit this proposition, we must first exhibit $Y(\mu)$ as integer surgery on a Floer simple knot in an L-space. Since $\langle \mu_{\rm L} \rangle \in \mathcal{L}^\circ(Y)$, Corollary 3.6 tells us that the knot core $K_{\mu_{\rm L}} \subset Y(\mu_{\rm L})$ is Floer simple. Thus, we already have $Y(\mu)$ as *rational* surgery on a Floer simple knot. We then appeal to a standard construction for re-expressing a rational surgery on a knot as an integer surgery on the connected sum of that knot with a particular Floer simple knot in an appropriate lens space, called $L(\beta, \alpha^*)$ below.

### 4.6. $(\mu)$ as integer surgery on a Floer simple knot

To describe this construction more explicitly, we first let $K_{\rm U} \subset S^3$ denote the unknot, and take $(m_1, l_1)$ and $(m_2, l_2)$ as respective bases for $H_1(\partial Y)$ and $H_1(\partial(S^3 \setminus K_{\rm U}))$, such that $m_1 \cdot l_1 = m_2 \cdot l_2 = 1$, with $l_1$ generating $\iota_1^{-1}(T_1)$, where $T_1 := \mathrm{Tors}(H_1(Y))$, and with $l_2$ generating $\ker \iota_2$, where $\iota_1 : H_1(\partial Y) \to H_1(Y)$ and $\iota_2 : H_1(\partial(S^3 \setminus K_{\rm U})) \to H_1(S^3 \setminus K_{\rm U})$ are the maps induced on homology by inclusion. This allows us to write

$$\mu := nm_1 + n'l_1, \quad \mu_1 := \mu_{\rm L} = pm_1 + ql_1, \quad \mu_2 := \beta m_2 + \alpha^* l_2, \tag{27}$$

where $\mu_2$ is constructed to produce the desired lens space $(S^3 \setminus K_{\text{U}})(\mu_2) = L(\beta, \alpha^*)$, with $\beta := \mu_{\text{L}} \cdot \mu$ and $\alpha^* := [-n^{-1}p]_\beta$. (Note that the condition $\gcd(pg, \beta) = 1$ from Proposition 4.6, together with the primitivity of $\mu$, imply that $\gcd(n, \beta) = 1$.) Setting $q^* := [-q^{-1}]_p$, write $\alpha$, $p^*$, and $\beta^*$ for the (integer) solutions to the respective equations $n = \alpha p + \beta q^*$, $pp^* - qq^* = 1$, and $\beta\beta^* - \alpha^*\alpha = 1$, so that

$$\lambda_1 := q^* m_1 + p^* l_1, \quad \lambda_2 := \alpha m_2 + \beta^* l_2 \tag{28}$$

serve as longitudes, satisfying $\mu_1 \cdot \lambda_1 = \mu_2 \cdot \lambda_2 = 1$. Note that this makes $\mu = \alpha\mu_1 + \beta\lambda_1$.

Let $Y_\#$ denote the connected sum knot complement

$$Y_\# := Y(\mu_1)\#(S^3 \setminus K_{\text{U}})(\mu_2) \ \setminus \ K_{\mu_1}\#K_{\mu_2}, \tag{29}$$

where $K_{\mu_1} \subset Y(\mu_1)$ and $K_{\mu_2} \subset (S^3 \setminus K_{\text{U}})(\mu_2) = L(\beta, \alpha^*)$ are the knot cores associated to the respective fillings by $\mu_1$ and $\mu_2$. If we write $\iota : H_1(\partial Y_\#) \to H_1(Y_\#)$, $f_1 : H_1(Y) \to H_1(Y_\#)$, and $f_2 : H_1(S^3 \setminus K_{\text{U}}) \to H_1(Y_\#)$ for the maps induced on homology by the corresponding inclusions, then $f_1 \oplus f_2$ descends to the isomorphism,

$$\left(H_1(Y) \oplus H_1(S^3 \setminus K_{\text{U}})\right)/(\iota_1(\mu_1) \sim \iota_2(\mu_2)) \to H_1(Y_\#), \tag{30}$$

which, since $H_1(S^3 \setminus K_{\text{U}})$ is torsion free, restricts to the isomorphism,

$$\left(\iota_1(H_1(\partial Y)) \oplus \iota_2(H_1(\partial(S^3 \setminus K_{\text{U}})))\right)/(\iota_1(\mu_1) \sim \iota_2(\mu_2)) \to \iota(H_1(\partial Y_\#)). \tag{31}$$

For the knot $K_{\mu_1}\#K_{\mu_2}$ with meridian $\mu_\#$, we can splice the longitudes $\lambda_1$ and $\lambda_2$ together to form a longitude of class $\lambda_\# \in \iota^{-1}(f_1(\lambda_1) + f_2(\lambda_2)) \subset \iota(H_1(\partial Y_\#))$. The Dehn filling $Y_\#(\lambda_\#)$ then has homology elements satisfying

$$f_1\iota_1(\mu_1) = f_2\iota_2(\mu_2) = \tfrac{\beta}{\alpha} f_2\iota_2(\lambda_2) = -\tfrac{\beta}{\alpha} f_1\iota_1(\lambda_1), \tag{32}$$

implying that $f_1\iota_1(\mu) = f_1\iota_1(\alpha\mu_1 + \beta\lambda_1) = 0$ in $H_1(Y_\#(\lambda_\#))$. Since, in addition, we know that $Y_\#$ is homeomorphic to $Y$, it follows that $Y(\mu) = Y_\#(\lambda_\#)$, realizing $Y(\mu) = Y_\#(\lambda_\#)$ as zero surgery (relative to $\lambda_\#$) on the Floer simple knot $K_{\mu_1}\#K_{\mu_2} \subset Y(\mu_\#) = Y(\mu_1)\#L(\beta, \alpha^*)$.

Since $\gcd(pg, \beta) = 1$ and $H_1(S^3 \setminus U)$ is torsion free, it follows from the isomorphisms (30) and (31) that $f_1$ restricts to isomorphisms $T_1 \xrightarrow{\sim} T$ and $T_1^\partial \xrightarrow{\sim} T^\partial$, where $T_1 := \text{Tors}(H_1(Y))$, $T := \text{Tors}(H_1(Y_\#))$, $T_1^\partial := T_1 \cap \iota_1(H_1(\partial Y))$, and $T^\partial := T \cap \iota(H_1(\partial Y_\#))$. It also follows that we can choose $l \in \iota^{-1}(T^\partial)$ and $m \in H_1(\partial Y_\#)$ with $m \cdot l = 1$ such that $f_1$ and $f_2$ satisfy

$$\begin{aligned} &f_1 : \iota_1(m_1) \mapsto \beta\iota(m), && f_2 : \iota_2(m_2) \mapsto p\iota(m) + q\xi\iota(l), \\ &f_1 : \iota_1(l_1) \mapsto \beta\xi\iota(l), \end{aligned} \tag{33}$$

on the images of $\iota_1$ and $\iota_2$, for some $\xi \in \mathbb{Z}/g$, with $g := \left|T^\partial\right| = \left|T_1^\partial\right|$. We then have

$$\begin{aligned} \iota(\mu_\#) &= f_1\iota_1(\mu_1) = f_2\iota_2(\mu_2) = \beta p\iota(m) + \beta q\xi\iota(l), \\ \iota(\lambda_\#) &= f_1\iota_1(\lambda_1) + f_2\iota_2(\lambda_2) = n\iota(m) + n'\xi\iota(l), \end{aligned} \tag{34}$$

where we used the facts that

$$n = \alpha p + \beta q^*, \quad n' = \alpha q + \beta p^*. \tag{35}$$

The condition that $\mu_\# \cdot \lambda_\# = 1$ determines $\xi$, which we shall not need.

### 4.7. *Applying the "coloring condition" of Proposition 3.7*

Since this section uses the Euler characteristic of knot Floer homology, which we express in terms of the Turaev torsion, regarded as an element of the Laurent series group ring of homology, we briefly introduce generators $\bar{m}$, $\bar{m}_1$, and $\bar{m}_2$ for $H_1(Y_\#)/T$, $H_1(Y)/T_1$, and $H_1(S^3 \setminus K_{\mathrm{U}})$, respectively, with signs chosen so that

$$\iota(m) \in +g\bar{m} + T, \quad \iota(m_1) \in +g\bar{m}_1 + T_1, \quad \iota(m_2) = \bar{m}_2. \tag{36}$$

For notational brevity, we also set

$$t := [\bar{m}], \quad t_1 := [\bar{m}_1], \quad t_2 := [\bar{m}_2], \tag{37}$$

where $[\cdot]$ indicates inclusion into the Laurent series group ring of the relevant homology group.

In order to use Proposition 3.7, we need the support of the Euler characteristic of the knot Floer homology of the knot core $K_\# \subset Y_\#(\lambda_\#)$. Since $\widehat{HFK}$ tensors on connected sums, its Euler characteristic $\chi^{\widehat{\mathrm{HFK}}}$ turns tensor product into multiplication, and the support function $S[\cdot]$ on (Laurent series) group rings converts this multiplication of polynomials into addition of sets, yielding

$$S\left[\chi^{\widehat{\mathrm{HFK}}}(Y_\#(\lambda_\#), K_\#)\right] = f_1 S\left[\chi^{\widehat{\mathrm{HFK}}}(Y(\mu_1), K_{\mu_1})\right] + f_2 S\left[\chi^{\widehat{\mathrm{HFK}}}((S^3 \setminus K_{\mathrm{U}})(\mu_2), K_{\mu_2})\right]. \tag{38}$$

Proposition 2.1 tells us that

$$\begin{aligned} S\left[\chi^{\widehat{\mathrm{HFK}}}(Y(\mu_1), K_{\mu_1})\right] &= S\left[(1 - [\iota_1(\mu_1)]) \cdot \left((1 - t_1)^{-1} \textstyle\sum_{h \in T_1}[h] \;-\; \tau^{\mathrm{c}}(Y)\right)\right] \\ &= S\left[\frac{1 - t_1^{pg}}{1 - t_1} \textstyle\sum_{h \in T_1}[h] - \tau^{\mathrm{c}}(Y) \;+\; [\iota_1(\mu_1)]\tau^{\mathrm{c}}(Y)\right] \\ &= \left(S[\tau(Y)] \cap (\{0, \ldots, pg - 1\}\bar{m}_1 + T_1)\right) \;\amalg\; \left(S[\tau^{\mathrm{c}}(Y)] + \iota_1(\mu_1)\right), \end{aligned} \tag{39}$$

where $\tau(Y) \in \mathbb{Z}[t^{-1}, t]][T] \supset \mathbb{Z}[H_1(Y)]$ is the Turaev torsion, $\tau^{\mathrm{c}}(Y)$ is the torsion complement as defined in (8), and we used our simplifying assumption that $\deg_{t_1} \tau^{\mathrm{c}}(Y) < pg$. Similarly, we have

$$S\left[\chi^{\widehat{\mathrm{HFK}}}((S^3 \setminus K_{\mathrm{U}})(\mu_2), K_{\mu_2})\right] = S\left[(1 - [(\iota_2(\mu_2)]) \cdot \tau(S^3 \setminus K_{\mathrm{U}})\right] \tag{40}$$
$$= S[(1 - t_2^{\beta})/(1 - t_2)]$$
$$= \{0, \ldots, \beta - 1\}\iota_2(m_2).$$

Thus, if we set

$$A_0 := f_1(S[\tau(Y)] \cap (\{0, \ldots, pg-1\}\bar{m}_1 + T_1)) \ + \ \{0, \ldots, \beta - 1\} f_2 \iota_2(m_2), \tag{41}$$
$$A_1 := f_1 S[\tau^{\mathrm{c}}(Y)] + \iota(\mu_\#) \ + \ \{0, \ldots, \beta - 1\} f_2 \iota_2(m_2),$$

then in the language of Proposition 3.7, we have

$$S_{\mathrm{BLACK}} \ := \ S\left[\chi^{\widehat{\mathrm{HFK}}}(Y_\#(\lambda_\#), K_\#)\right] \ = \ A_0 \amalg A_1, \tag{42}$$
$$S_{\mathrm{RED}} := S_{\mathrm{BLACK}} + \iota(\mu_\#)\mathbb{Z}_{>0}, \qquad S_{\mathrm{BLUE}} := S_{\mathrm{BLACK}} - \iota(\mu_\#)\mathbb{Z}_{>0}.$$

Using the fact that $\iota(\mu_\#) = \beta f_2 \iota_2(m_2)$, one can easily verify that

$$(S_{\mathrm{BLUE}} - S_{\mathrm{RED}}) \cap (\bar{m}\mathbb{Z}_{>0} + T) = ((A_1 - \iota(\mu_\#)) - (A_0 + \iota(\mu_\#))) \cap (\bar{m}\mathbb{Z}_{>0} + T) \tag{43}$$
$$= (f_1(S[\tau^{\mathrm{c}}(Y)] - S[\tau(Y)]) - f_2\iota_2(m_2)\mathbb{Z}_{>0}) \cap (\bar{m}\mathbb{Z}_{>0} + T).$$

Proposition 3.7 then implies $Y_\#(\lambda_\#)$ is an L-space if and only if

$$\iota(\lambda_\#)\mathbb{Z} \cap (S_{\mathrm{BLUE}} - S_{\mathrm{RED}}) \cap (\bar{m}\mathbb{Z}_{>0} + T) = \emptyset. \tag{44}$$

Suppose the above set is nonempty, hence contains some element $b\iota(\lambda_\#)$ such that

$$b\iota(\lambda_\#) = f_1(h_{\mathrm{c}} - h) - k f_2 \iota_2(m_2) \ \in \bar{m}\mathbb{Z}_{>0} + T \tag{45}$$

with $b \in \mathbb{Z}_{\neq 0}$, $k \in \mathbb{Z}_{>0}$, $h_{\mathrm{c}} \in S[\tau^{\mathrm{c}}(Y)]$, and $h \in S[\tau(Y)]$. Since $b\iota(\lambda_\#), k f_2 \iota_2(m_2) \in \iota(H_1(Y_\#))$, we know that $f_1(h_{\mathrm{c}} - h) \in \iota(H_1(Y_\#))$, implying $h_{\mathrm{c}} - h \in \iota(H_1(Y))$. Moreover, since $b\iota(\lambda_\#) \in \bar{m}\mathbb{Z}_{>0} + T$, we know that $h_{\mathrm{c}} - h \in \bar{m}_1\mathbb{Z}_{>0} + T_1$. In other words,

$$h_{\mathrm{c}} - h \in (S[\tau^{\mathrm{c}}(Y)] - S[\tau(Y)]) \cap \iota_1(m_1\mathbb{Z}_{>0} + l_1\mathbb{Z}) =: \mathcal{D}^{\tau}_{>0}(Y). \tag{46}$$

Writing $h_{\mathrm{c}} - h = \delta\iota(m_1) + \gamma\iota(l_1) \in \mathcal{D}^{\tau}_{>0}(Y)$ and evaluating $f_1$, $f_2$, and $\iota(\lambda_\#)$ as expressed in (33) and (34), we transform (45) into

$$(bn)\iota(m) + (bn')\xi\iota(l) = (\beta\delta - kp)\iota(m) + (\beta\gamma - kq)\xi\iota(l), \tag{47}$$

which, since $nm_1 + n'l_1 = \alpha\mu_1 + \beta\lambda_1 = (\alpha p + \beta q^*)m_1 + (\alpha q + \beta p^*)l_1$, yields the two equations

$$b(\alpha p + \beta q^*) = \beta\delta - kp > 0, \tag{48}$$

$$b(\alpha q + \beta p^*) \equiv \beta\gamma - kq \pmod g. \tag{49}$$

One can use the identity $pp^* - qq^* = 1$ to solve the above two equations simultaneously for $b$, obtaining $b \equiv p\gamma - q\delta \pmod g$. Moreover, taking the first equation modulo $p$ implies $b \equiv -q\delta \pmod p$. Thus any solution in $b$ to (47) must satisfy $b \equiv p\gamma - q\delta \pmod{pg}$.

### 4.8. Completing the proof of Theorem 4.2

For each $\boldsymbol{\delta} = \delta\iota(m_1) + \gamma\iota(l_1) \in \mathcal{D}^\tau_{>0}(Y)$, set $b^{\boldsymbol{\delta}}_- := [p\gamma - q\delta]_{pg} - pg$ and $b^{\boldsymbol{\delta}}_+ := [p\gamma - q\delta]_{pg}$. Note that our earlier assumption of $pg > \deg_{t_1} \tau^c(Y)$ ensures that $b^{\boldsymbol{\delta}}_+ \neq 0$ and $|b^{\boldsymbol{\delta}}_-| < pg$.

We claim that $Y_\#(\lambda_{\mu_\#})$ is an L-space—or in other words, that (48) and (49) have no solution $(b,k) \in \mathbb{Z} \times \mathbb{Z}_{>0}$ for any $\delta\iota(m_1) + \gamma\iota(l_1) \in \mathcal{D}^\tau_{>0}(Y)$—if and only if

$$\frac{b^{\boldsymbol{\delta}}_-}{\delta} \leq \frac{\beta}{n} \leq \frac{b^{\boldsymbol{\delta}}_+}{\delta} \quad \text{for all } \boldsymbol{\delta} = \delta\iota(m_1) + \gamma\iota(l_1) \in \mathcal{D}^\tau_{>0}(Y). \tag{50}$$

First, consider the case in which $n > 0$. Suppose there exists $\boldsymbol{\delta} = \delta\iota(m_1) + \gamma\iota(l_1) \in \mathcal{D}^\tau_{>0}(Y)$ for which $\beta/n > b^{\boldsymbol{\delta}}_+/\delta$. Since $n, \delta > 0$, this implies $0 < b^{\boldsymbol{\delta}}_+ n < \beta\delta$. Thus, since $b^{\boldsymbol{\delta}}_+ n \equiv (-q\delta)(\beta q^*) \equiv \beta\delta \pmod p$, there exists $k_0 \in \mathbb{Z}_{>0}$ such that $b^{\boldsymbol{\delta}}_+ n = \beta\delta - k_0 p > 0$. Thus $(b^{\boldsymbol{\delta}}_+, k_0)$ provides a solution for $(b,k)$ in (48), which, together with the relation $b^{\boldsymbol{\delta}}_+ \equiv p\gamma - q\delta \pmod g$, implies (49) also holds for $(b,k) = (b^{\boldsymbol{\delta}}_+, k_0)$, and so $Y_\#(\lambda_\#)$ is not an L-space.

Conversely, still for $n > 0$, suppose that $Y_\#(\lambda_\#)$ is not an L-space. Then there exist $\boldsymbol{\delta} = \delta\iota(m_1) + \gamma\iota(l_1) \in \mathcal{D}^\tau_{>0}(Y)$ and $(b,k) \in \mathbb{Z} \times \mathbb{Z}_{>0}$ for which (48) and (49) hold. In particular, (48) implies $bn < \beta\delta$ and $b > 0$, while (48) and (49) together imply $b \equiv b^{\boldsymbol{\delta}}_+ \pmod{pg}$, requiring $b \geq b^{\boldsymbol{\delta}}_+$. Thus $\beta\delta > bn \geq b^{\boldsymbol{\delta}}_+ n$, implying $\beta/n > b^{\boldsymbol{\delta}}_+/\delta$.

The argument for the case of $n < 0$ is nearly identical, but with a few signs and inequalities reversed. This completes the proof of our claim, and the claim implies the theorem, if we additionally note that (50) holds vacuously when $\mathcal{D}^\tau_{>0}(Y) = \emptyset$, and that $\langle\mu\rangle$ was arbitrary in $Sl(Y) \setminus \{l\}$. □

## 5. Seifert fibered L-spaces

To illustrate the utility of our new L-space interval tool $\mathcal{D}^\tau$, in this section we exploit Theorem 4.2 to offer a simple alternative proof of a known result: namely, the classification of Seifert fibered spaces over $S^2$ which are L-spaces. We restrict to the $S^2$ case because it is the most interesting one, as no higher genus Seifert fibered spaces are L-spaces, and all oriented Seifert fibered spaces over $\mathbb{RP}^2$ are L-spaces [7].

### 5.1. Seifert fibered L-spaces, a history

Up until now, the classification of Seifert fibered L-spaces has relied, at least in one direction, on the classification of oriented Seifert fibered spaces $M$ over $S^2$ admitting transverse foliations, a problem which dates back at least to 1981, when Eisenbud, Hirsch, and Neumann [12] re-expressed this foliations problem in terms of a criterion on representations of $\pi_1(M)$ in $\widetilde{\text{Homeo}}_+S^1$, the universal cover of the group of orientation-preserving homeomorphisms of $S^1$.

A few years later, Jankins and Neumann [25] reformulated the criterion of [12] in terms of Poincaré's "rotation number" invariant on $\widetilde{\text{Homeo}}_+S^1$, a development which, along with the correct conjecture that this criterion is met in $\widetilde{\text{Homeo}}_+S^1$ if and only if it is met in a smooth Lie subgroup thereof, allowed them to write down an explicit characterization of Seifert fibered manifolds over $S^2$ admitting transverse foliations. With the exception of one special case, they also showed that this list was complete. It took more than a decade before Naimi [34] resolved this outstanding case using dynamical methods, and more than a decade after that before Calegari and Walker [9] generalized Naimi's methods to provide a proof of the Jankins–Neumann classification that did not appeal to smooth Lie subgroups.

In the late 1990's, Eliashberg and Thurston [13] proved that one can associate a weakly symplectically fillable contact structure to any $C^2$ cooriented taut foliation on a closed three-manifold—a result which Kazez and Roberts [29], and independently Bowden [5], have recently extended to $C^0$ foliations. Since Ozsváth and Szabó have [39] shown that this contact structure gives rise to a nontrivial class in Heegaard Floer homology, this proves that L-spaces do not admit co-oriented taut foliations.

In the converse direction, Lisca and Matić [32] proved that a Seifert fibered manifold $M$ over $S^2$ admits contact structures in each orientation which are transverse to the fibration if and only if $M$ belongs to the explicit set characterized by Jankins and Neumann. Lisca and Stipsicz then showed [33] that if there is an orientation on a Seifert fibered manifold $M$ over $S^2$ for which no positive contact structure is transverse to the fibration, then $M$ is an L-space.

Since our own answer matches that of Jankins and Neumann, one could take the non-L-space/transverse-foliation equivalence for Seifert fibered manifolds over $S^2$ as a corollary of Theorem 5.1 below. As for our L-space classification itself, however, the proof no longer requires foliations, dynamical methods, or even (after the proof of Theorem 4.2) contact or symplectic geometry. It only uses ordinary homology and one computation of Turaev torsion from a homology presentation.

### 5.2. Conventions and bases

To construct a Seifert-fibered space with $n$ exceptional fibers over $S^2$, we start with the trivial circle fibration $S^1 \times S^2$, and remove $n+1$ solid tori, $S^1 \times D^2_i$, $i \in \{0, \dots, n\}$, yielding a trivial circle fibration over the $n+1$-punctured sphere,

$$\hat{Y} := S^1 \times (S^2 \setminus \coprod_{i=0}^n D_i^2), \qquad \partial\hat{Y} = \coprod_{i=0}^n \partial_i \hat{Y}, \tag{51}$$

where $\partial_i \hat{Y}$ denotes the $i$th toroidal boundary component, $\partial_i \hat{Y} := -\partial(S^1 \times D_i^2)$.

Next, we choose presentations for $H_1(\hat{Y})$ and $H_1(\partial_i \hat{Y})$ in terms of the regular fiber class $f \in H_1(\hat{Y})$ and classes horizontal to this fiber. For each $i \in \{0, \ldots, n\}$, we take $(\tilde{f}_i, -\tilde{h}_i)$ as a reverse-oriented basis for $H_1(\partial_i \hat{Y})$. Here, $\tilde{h}_i \in H_1(\partial_i \hat{Y})$ denotes the meridian of the excised solid torus $S^1 \times D_i^2$, and if we write $\hat{\imath}_i : H_1(\partial_i \hat{Y}) \to H_1(\hat{Y})$ for the map induced by inclusion, then $\tilde{f}_i \in \hat{\imath}_i^{-1}(f)$ denotes the lift of $f$ satisfying $(\tilde{f}_i \cdot \tilde{h}_i)|_{\partial_i \hat{Y}} = 1$. Setting each $h_i := \hat{\imath}_i(\tilde{h}_i) \in H_1(\hat{Y})$, we note that there must be a relation among the $h_i$, since the $n+1$-punctured sphere is the same as the $n$-punctured disk, with first Betti number $n$. In fact, since any one of the $h_i$ can be regarded as the class of minus the boundary of this disk, with the remaining $h_i$ summing to a class equal to the boundary of the disk, we have $\sum_{i=0}^n h_i = 0$, so that $H_1(\hat{Y})$ has presentation

$$H_1(\hat{Y}) = \langle f, h_0, \ldots, h_n \,|\, \textstyle\sum_{i=0}^n h_i = 0 \rangle . \tag{52}$$

To specify a Seifert fibered space, one simply lists the Dehn filling slopes, in terms of the basis $(\tilde{f}_i, -\tilde{h}_i)$ for each $H_1(\partial_i \hat{Y})$, of the $n+1$ toroidal boundary components of $\hat{Y}$, conventionally filling $\partial_0 Y$ with an integer slope and the remaining $\partial_i Y$ with noninteger slopes. That is, for any $e_0, r_1, \ldots, r_n \in \mathbb{Z}$ and $s_1, \ldots, s_n \in \mathbb{Z}_{\neq 0}$ with each $\frac{r_i}{s_i} \notin \mathbb{Z}$, the Seifert fibered space $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ denotes the Dehn filling of $\hat{Y}$ along the slopes

$$\begin{aligned} \mu_0 &:= e_0 \tilde{f}_0 - \tilde{h}_0, \\ \mu_i &:= r_i \tilde{f}_i - s_i \tilde{h}_i, \;\; i \in \{1, \ldots, n\}. \end{aligned} \tag{53}$$

The resulting manifold has first homology

$$H_1\Big(M(e_0; \tfrac{r_1}{s_1}, \ldots, \tfrac{r_n}{s_n})\Big) = \langle f, h_0, \ldots, h_n \,|\, \textstyle\sum_{i=0}^n h_i = \hat{\imath}_0(\mu_0) = \ldots = \hat{\imath}_n(\mu_n) = 0 \rangle . \tag{54}$$

Note that for any $(z_0, \ldots, z_n) \in \mathbb{Z}^{n+1}$ satisfying $\sum_{i=0}^n z_i = 0$, the change of basis $h_i \mapsto h_i + z_i f$, $i \in \{0, \ldots, n\}$, yields the reparameterization

$$M(e_0; \tfrac{r_1}{s_1}, \ldots, \tfrac{r_n}{s_n}) \mapsto M(e_0 + z_0; \tfrac{r_1}{s_1} + z_1, \ldots, \tfrac{r_n}{s_n} + z_n). \tag{55}$$

In addition, $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ admits an orientation reversing homeomorphism,

$$-M(e_0; \tfrac{r_1}{s_1}, \ldots, \tfrac{r_n}{s_n}) = M(-e_0; -\tfrac{r_1}{s_1}, \ldots, -\tfrac{r_n}{s_n}). \tag{56}$$

### *5.3. Statement of L-space classification*

We are now able to state our result.

**Theorem 5.1.** *If $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ denotes a Seifert fibered space over $S^2$ with $n > 0$ exceptional fibers, then $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is* not *an L-space if and only if $e_0 + \sum_{i=1}^{n} \frac{r_i}{s_i} = 0$ or*

$$-e_0 + \min_{0<x<s} -\frac{1}{x}\left(-1 + \sum_{i=1}^{n}\left\lceil\frac{r_i x}{s_i}\right\rceil\right) \;<\; 0 \;<\; -e_0 + \max_{0<x<s} -\frac{1}{x}\left(1 + \sum_{i=1}^{n}\left\lfloor\frac{r_i x}{s_i}\right\rfloor\right), \tag{57}$$

*where $s$ denotes the least common positive multiple of $s_1, \ldots, s_n$.*

**Remark.** If we take each $s_i > 0$, then inequality (57) is equivalent to the condition that

$$\min_{0<x<s} \frac{1}{x}\left(1 - \sum_{i=1}^{n}\frac{[-r_i x]_{s_i}}{s_i}\right) \;<\; e_0 + \sum_{i=1}^{n}\frac{r_i}{s_i} \;<\; \max_{0<x<s} \frac{1}{x}\left(-1 + \sum_{i=1}^{n}\frac{[r_i x]_{s_i}}{s_i}\right). \tag{58}$$

The middle expression, $e_0 + \sum_{i=1}^{n} \frac{r_i}{s_i}$, is the orbifold Euler characteristic. If $e_0 + \sum_{i=1}^{n} \frac{r_i}{s_i} = 0$, then (58) fails to hold when $n \leq 2$, in which case all three expressions are equal.

Theorem 5.1 makes it easy to deduce the L-space filling slope interval for any regular-fiber complement in a Seifert fibered space. That is, for any $j \in \{1, \ldots, n\}$, the above theorem implies that $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is an L-space if and only if

$$-e_0 x - \left(-1 + \sum_{i \neq j}\left\lceil\frac{r_i x}{s_i}\right\rceil\right) \geq \left\lceil\frac{r_j x}{s_j}\right\rceil \quad \text{or} \quad -e_0 x - \left(1 + \sum_{i \neq j}\left\lfloor\frac{r_i x}{s_i}\right\rfloor\right) \leq \left\lfloor\frac{r_j x}{s_j}\right\rfloor \tag{59}$$

for all $x \in \{1, \ldots, s-1\}$. Since the above expressions are integers, (59) holds if and only if

$$-e_0 x - \left(-1 + \sum_{i \neq j}\left\lceil\frac{r_i x}{s_i}\right\rceil\right) \geq \frac{r_j x}{s_j} \quad \text{or} \quad -e_0 x - \left(1 + \sum_{i \neq j}\left\lfloor\frac{r_i x}{s_i}\right\rfloor\right) \leq \frac{r_j x}{s_j}. \tag{60}$$

Dividing both sides by $x$ then gives the following result.

**Corollary 5.2.** *If $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$, with each $s_i > 0$, denotes a Seifert fibered space over $S^2$ with $n > 1$ exceptional fibers, then for any $j \in \{1, \ldots, n\}$, $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is an L-space if and only if $e_0 + \sum_{i=1}^{n} \frac{r_i}{s_i} \neq 0$ and*

$$\frac{r_j}{s_j} \leq -e_0 + \min_{0<x<s} -\frac{1}{x}\left(-1 + \sum_{i \neq j}\left\lceil\frac{r_i x}{s_i}\right\rceil\right) \quad \text{or} \quad -e_0 + \max_{0<x<s} -\frac{1}{x}\left(1 + \sum_{i \neq j}\left\lfloor\frac{r_i x}{s_i}\right\rfloor\right) \geq \frac{r_j}{s_j}, \tag{61}$$

*where $s$ is the least common multiple of those $s_i$ with $i \neq j$.*

*5.4. Set-up for proof of Theorem 5.1: Dehn filling a Floer simple manifold*

We begin by expressing $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ as the Dehn filling of a Floer simple manifold $Y$. For now, we demand that $0 < r_i < s_i$ and $\gcd(r_i, s_i) = 1$ for each $i \in \{1, \ldots, n\}$. Let $Y$ denote the regular-fiber complement

$$Y := M(0; \tfrac{r_1}{s_1}, \ldots, \tfrac{r_n}{s_n}) \setminus (S^1 \times D_0^2), \tag{62}$$

so that $Y(\mu_0) = M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$. Regarding $Y$ as a partial Dehn filling of $\hat{Y}$, we have

$$H_1(Y) = \langle f, h_0, \ldots, h_n \,|\, \textstyle\sum_{i=0}^n h_i = \hat{\iota}_1(\mu_1) = \ldots = \hat{\iota}_n(\mu_n) = 0 \rangle\,. \tag{63}$$

Writing $\iota_0 : H_1(\partial Y) \to H_1(Y)$ for the map induced by inclusion, and identifying $\tilde{h}_0$ and $\tilde{f}_0$ with their respective images under the canonical isomorphism $H_1(\partial_0 \hat{Y}) \to H_1(\partial Y)$, we again have $\iota_0(\tilde{h}_0) = h_0$ and $\iota_0(\tilde{f}_0) = f$, but in the sense of the above presentation for $H_1(Y)$.

Define

$$S_{\mathrm{gcd}} := \gcd\left(\frac{\prod_{i=1}^n s_i}{s_1}, \ldots, \frac{\prod_{i=1}^n s_i}{s_n}\right), \quad s := \frac{\prod_{i=1}^n s_i}{S_{\mathrm{gcd}}}, \tag{64}$$

noting that this makes $s$ the least common multiple of $s_1, \ldots, s_n$. Note that if we set

$$l := p\tilde{f}_0 + q^*\tilde{h}_0, \quad \text{with} \quad p := \sum_{i=1}^n \frac{r_i}{s_i}\frac{s}{g}, \quad q^* := \frac{s}{g}, \quad g := \gcd\left(\sum_{i=1}^n \frac{r_i}{s_i} s,\ s\right), \tag{65}$$

then $l$ is primitive in $H_1(\partial Y)$. In addition, since $h_0 = -\sum_{i=1}^n h_i$, we have

$$0 \;=\; \sum_{i=1}^n \frac{s}{s_i} \hat{\iota}_i(\mu_i) \;=\; \sum_{i=1}^n \frac{r_i}{s_i} sf \;+\; sh_0 \;=\; g\iota_0(l). \tag{66}$$

Thus $\iota_0(l) \in H_1(Y)$ is torsion, and so $l$ is also a rational longitude. Moreover, since $g\iota_0(l) = \sum_{i=1}^n \frac{s}{s_i}\hat{\iota}_i(\mu_i) = 0$ is a primitive linear combination of the relations in the presentation of $H_1(Y)$ in (63), we have $g = |\langle \iota_0(l) \rangle|$. Choosing any $m \in H_1(\partial Y)$ satisfying $m \cdot l = 1$, and writing $m = -q\tilde{f}_0 - p^*\tilde{h}_0$, allows one to solve for $\tilde{f}_0$ and $\tilde{h}_0$ in terms of $m$ and $l$.

Now, since all $\frac{r_i}{s_i} > 0$ by assumption, we know from Ozsváth and Szabó in [37] that $Y(-h_0) = M(0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is an L-space, so we may take $\mu_{\mathrm{L}} := -\tilde{h}_0$ as our given L-space filling slope, and choose $\lambda_{\mathrm{L}} = \tilde{f}_0$ for its longitude, with $\mu_{\mathrm{L}} \cdot \lambda_{\mathrm{L}} = -\tilde{h}_0 \cdot \tilde{f}_0 = 1$. We then have

$$\mu_{\mathrm{L}} := -\tilde{h}_0 = pm + ql, \qquad \lambda_{\mathrm{L}} := \tilde{f}_0 = q^*m + p^*l, \tag{67}$$

with $p$ and $q^*$ as in (65), and with $q$ and $p^*$ solving the Diophantine equation $pp^* - qq^* = 1$.

### 5.5. Computation of $\mathcal{D}^\tau(Y)$

To compute $\mathcal{D}^\tau(Y)$, we need the Turaev torsion, $\tau(Y)$. Recall that $Y$ is a union along torus boundaries of trivial circle fibrations,

$$Y = S^1 \times (S^2 \setminus \coprod_{i=0}^n D_i^2) \,\cup\, S_1^1 \times D_1^2 \,\cup\, \ldots \,\cup\, S_n^1 \times D_n^2. \tag{68}$$

The leftmost $S^1$ above, corresponding to the regular fiber in $\hat{Y}$, has class $\hat{\iota}_0(\lambda_{\mathrm{L}}) = f \in H_1(\hat{Y})$. Similarly, for $i \in \{1, \ldots, n\}$, each $S_i^1$ above has class $\hat{\iota}_i(\lambda_i) \in H_1(\hat{Y})$, where $\lambda_i$ is any longitude satisfying $(\mu_i \cdot \lambda_i)|_{\partial_i \hat{Y}} = 1$. Since each $\hat{\iota}_i(\mu_i) = 0$, each class $\hat{\iota}_i(\lambda_i)$ is independent of the choice of $\lambda_i$. The Turaev torsion then obeys a product rule for unions along torus boundaries [47, Thm. VII.1.4], yielding

$$\begin{aligned} \tau(Y) &:= (1 - [\hat{\iota}_0(\lambda_{\mathrm{L}})])^{-\chi(S^2 \setminus \coprod_{i=0}^n D_i^2)} \textstyle\prod_{i=1}^n (1 - [\hat{\iota}_i(\lambda_i)])^{-\chi(D_i^2)} \\ &:= (1 - [\hat{\iota}_0(\lambda_{\mathrm{L}})])^{n-1} \textstyle\prod_{i=1}^n (1 - [\hat{\iota}_i(\lambda_i)])^{-1}, \end{aligned} \tag{69}$$

where $[\cdot]$ denotes inclusion of $H_1(\hat{Y})$ into the Laurent series group ring for $H_1(\hat{Y})$.

These $\hat{\iota}_i(\lambda_i)$ bear simple relationships to $\iota_0(\mu_{\mathrm{L}})$ and $\iota_0(\lambda_{\mathrm{L}})$. That is, we claim that

$$\iota_0(\mu_{\mathrm{L}}) = \textstyle\sum_{i=1}^n r_i \hat{\iota}_i(\lambda_i), \;\text{ and }\; \iota_0(\lambda_{\mathrm{L}}) = s_i \hat{\iota}_i(\lambda_i) \;\text{ for each }\; i \in \{1, \ldots, n\}. \tag{70}$$

To see this, note that since each $\mu_i = r_i \tilde{f}_i - s_i \tilde{h}_i$, with $(\tilde{f}_i \cdot \tilde{h}_i)|_{\partial_i \hat{Y}} = (\mu_i \cdot \lambda_i)|_{\partial_i \hat{Y}} = 1$, we know there exist $r_i^*, s_i^* \in \mathbb{Z}$ such that

$$\hat{\iota}_i(\lambda_i) = s_i^* f + r_i^* h_i, \quad r_i r_i^* + s_i s_i^* = 1, \tag{71}$$

implying that

$$r_i \hat{\iota}_i(\lambda_i) = s_i^*(r_i f) + r_i r_i^* h_i = s_i^*(s_i h_i) + r_i r_i^* h_i = h_i, \tag{72}$$

$$s_i \hat{\iota}_i(\lambda_i) = s_i s_i^* f + r_i^*(s_i h_i) = s_i s_i^* f \,+\, r_i^*(r_i f) = f = \iota_0(\lambda_{\mathrm{L}}). \tag{73}$$

Thus, since $\iota_0(\mu_{\mathrm{L}}) = -h_0 = \sum_{i=1}^n h_i$, (70) holds in $H_1(Y)$.

Since $\iota_0(\lambda_{\mathrm{L}}) = s_i \hat{\iota}_i(\lambda_i)$ for each $i \in \{1, \ldots, n\}$, we may rewrite $\tau(Y)$ as

$$\tau(Y) = \frac{1}{1 - [\iota_0(\lambda_{\mathrm{L}})]} \prod_{i=1}^n \frac{1 - [\hat{\iota}_i(\lambda_i)]^{s_i}}{1 - [\hat{\iota}_i(\lambda_i)]}, \tag{74}$$

which has support

$$S[\tau(Y)] = \{\iota_0(\lambda_{\mathrm{L}})\mathbb{Z}_{\geq 0}\} + \{\textstyle\sum_{i=1}^n y_i \hat{\iota}_i(\lambda_i) \,|\, y_i \in \{0, \ldots, s_i - 1\}\}. \tag{75}$$

Since $Y$ has multiple L-space fillings, it is Floer simple, and so each element of $H_1(Y)$ has coefficient 0 or 1 in $\tau(Y)$, and the torsion complement $\tau^{\mathrm{c}}(Y)$ has support

$$S[\tau^{\mathrm{c}}(Y)] = \{-j\iota_0(\lambda_{\mathrm{L}}) + \textstyle\sum_{i=1}^{n} y_i\hat{\iota}_i(\lambda_i) \,|\, j \in \{1,\dots,n-1\}, y_i \in \{0,\dots,s_i-1\}\} \\ \cap H_1(Y)_{\geq 0}, \tag{76}$$

where $H_1(Y)_{\geq 0} := \{w \in H_1(Y) | \phi(w) \geq 0\}$ for any homomorphism $\phi : H_1(Y) \to \mathbb{Z}$ satisfying $\phi(\iota_0(m)) > 0$.

Since $s_i\hat{\iota}_i(\lambda_i) = \iota_0(\lambda_{\mathrm{L}})$ for each $i \in \{1,\dots,n\}$, it follows from (75) that $S[\tau(Y)]$ is additively closed, which, in turn, implies that

$$(S[\tau^{\mathrm{c}}(Y)] - S[\tau(Y)]) \cap H_1(Y)_{\geq 0} = S[\tau^{\mathrm{c}}(Y)], \tag{77}$$

so that $\mathcal{D}^{\tau}(Y)$ is the intersection $\mathcal{D}^{\tau}(Y) = S[\tau^{\mathrm{c}}(Y)] \cap \iota_0(H_1(\partial Y))$. By (70), we know that

$$\begin{aligned}\iota_0(H_1(\partial Y)) &= \mathrm{Span}\,\{\iota_0(\mu_{\mathrm{L}}), \iota_0(\lambda_{\mathrm{L}})\} \qquad (78)\\ &= \mathrm{Span}\,\{\iota_0(\mu_{\mathrm{L}}) = \textstyle\sum_{i=1}^{n} r_i\hat{\iota}_i(\lambda_i),\ \ \iota_0(\lambda_{\mathrm{L}}) = s_1\hat{\iota}_1(\lambda_1) = \dots = s_n\hat{\iota}_n(\lambda_n)\}\,.\end{aligned}$$

Now, for any $j \in \{1,\dots,n-1\}$ and $(y_1,\dots,y_n) \in \prod_{i=1}^{n}\{0,\dots,s_i-1\}$, we have

$$-j\iota_0(\lambda_{\mathrm{L}}) + \textstyle\sum_{i=1}^{n} y_i\hat{\iota}_i(\lambda_i) = \sum_{i=1}^{n}(y_i + z_is_i)\hat{\iota}_i(\lambda_i) - (j + \sum_{i=1}^{n} z_i)\iota_0(\lambda_{\mathrm{L}}) \tag{79}$$

for any $(z_1,\dots,z_n) \in \mathbb{Z}^n$. Thus, $-j\iota_0(\lambda_{\mathrm{L}}) + \sum_{i=1}^{n} y_i\hat{\iota}_i(\lambda_i) \in \iota_0(H_1(\partial Y))$ if and only if there exist $(z_1,\dots,z_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$ for which

$$(y_1 + z_1s_1,\dots,y_n + z_ns_n) = (r_1x,\dots,r_nx). \tag{80}$$

In such case, we have $y_i = [r_ix]_{s_i}$ and $z_i = \left\lfloor \frac{r_ix}{s_i} \right\rfloor$ for each $i \in \{1,\dots,n\}$.

We can therefore parameterize $\mathcal{D}^{\tau}(Y) = S[\tau^{\mathrm{c}}(Y)] \cap \iota_0(H_1(\partial Y))$ as

$$\begin{aligned}&\mathcal{D}^{\tau}(Y) = \left\{\boldsymbol{\delta}_x^j \,|\, j \in \{1,\dots,n-1\},\ x \in \{1,\dots,s-1\},\ \delta_x^j \geq 0\right\}, \text{ with} \qquad (81)\\ &\boldsymbol{\delta}_x^j := a_x^{j-}\iota_0(\mu_{\mathrm{L}}) + b_x^{j-}\iota_0(\lambda_{\mathrm{L}}), \qquad a_x^{j-} := x, \qquad b_x^{j-} := -j - \sum_{i=1}^{n}\left\lfloor\frac{r_ix}{s_i}\right\rfloor,\\ &\delta_x^j := a_x^{j-}p + b_x^{j-}q^* = \frac{s}{g}\left(-j + \sum_{i=1}^{n}\frac{[r_ix]_{s_i}}{s_i}\right),\end{aligned}$$

where $\delta_x^j := \tilde{\boldsymbol{\delta}}_x^j \cdot l$ for any $\tilde{\boldsymbol{\delta}}_x^j \in \iota_0^{-1}(\boldsymbol{\delta}_x^j)$. Since $\boldsymbol{\delta}_x^j$ is invariant under the action $x \mapsto x + s$, it suffices to choose a fundamental domain of length $s$ for $x \in \mathbb{Z}$. The above expression for $\mathcal{D}^{\tau}(Y)$ uses the fundamental domain $x \in \{0,\dots,s-1\}$, but excludes 0, since $\delta_0^j < 0$ for all $j \in \{1,\dots,n-1\}$.

### 5.6. Application of Theorem 4.2/Corollary 4.4

This particular choice of fundamental domain ensures that for all $\boldsymbol{\delta}_x^j \in \mathcal{D}^\tau_{>0}(Y)$, we have $b_x^{j-} = b_-^{\boldsymbol{\delta}_x^j}$ and $a_x^{j-} = a_-^{\boldsymbol{\delta}_x^j}$ in the sense of Corollary 4.4. That is, for all $j \in \{1, \dots, n-1\}$ and $x \in \{1, \dots, s-1\}$ with $\delta_x^j > 0$, we have

$$0 \; < \; -b_x^{j-} \; = \frac{a_x^{j-}p}{q^*} - \frac{\delta_x^j}{q^*} = \sum_{i=1}^n \frac{r_i x}{s_i} - \frac{g}{s}\delta_x^j < \sum_{i=1}^n \frac{r_i s}{s_i} - 0 = \; pg. \tag{82}$$

This makes $a_x^{j-}\mu_{\text{L}} + b_x^{j-}\lambda_{\text{L}} \in \iota_0^{-1}(\boldsymbol{\delta}_x^j)$ one of the two lifts of $\boldsymbol{\delta}_x^j$ closest to $\mu_{\text{L}}$ in $\mathbb{P}(H_1(\partial Y))$, and the closest lift of $\boldsymbol{\delta}_x^j$ on the other side of $\mu_{\text{L}}$ is $a_x^{j+}\mu_{\text{L}} + b_x^{j+}\lambda_{\text{L}} \in \iota_0^{-1}(\boldsymbol{\delta}_x^j)$, where

$$\begin{aligned} a_x^{j+} &:= a_+^{\boldsymbol{\delta}_x^j} \; = a_x^{j-} - q^* g \; = -(s-x), \\ b_x^{j+} &:= b_+^{\boldsymbol{\delta}_x^j} \; = b_x^{j-} + pg \; = -j + \sum_{i=1}^n \left\lceil \frac{r_i(s-x)}{s_i} \right\rceil . \end{aligned} \tag{83}$$

To use Corollary 4.4 on $M(e_0; \frac{r_1}{s_1}, \dots, \frac{r_n}{s_n}) = Y(\mu_0)$, we shall also want the $(\mu_{\text{L}}, \lambda_{\text{L}})$-surgery coefficients for $\mu_0$, and the value of $\mu_0 \cdot l$. Since $\mu_0 = e_0\tilde{f}_0 - \tilde{h}_0$ and $l = p\tilde{f}_0 + q^*\tilde{h}_0$, with $\mu_{\text{L}} = -\tilde{h}_0$, $\lambda_{\text{L}} = \tilde{f}_0$, $p = \frac{s}{g}\sum_{i=1}^n \frac{r_i}{s_i}$, and $q^* = \frac{s}{g}$, we have

$$\begin{aligned} \mu_0 &= \alpha\mu_{\text{L}} + \beta\lambda_{\text{L}}, \qquad\qquad \alpha := 1, \quad \beta := e_0, \\ \mu_0 \cdot l &= e_0 q^* + p = \frac{s}{g}\left(e_0 + \sum_{i=1}^n \frac{r_i}{s_i}\right). \end{aligned} \tag{84}$$

Since $Y(\mu_0)$ is never an L-space when $\mu_0 \cdot l = 0$, and since the case of $e_0 + \sum_{i=1}^n \frac{r_i x}{s_i} = 0$ is treated separately in the theorem statement, we henceforth restrict to the case of $\mu_0 \cdot l \neq 0$.

Suppose that $\mathcal{D}^\tau_{>0}(Y) \neq \emptyset$. In this case, Corollary 4.4 tells us that $M(e_0; \frac{r_1}{s_1}, \dots, \frac{r_n}{s_n}) = Y(\mu_0)$ is an L-space if and only if

$$\frac{\alpha}{\beta} := \frac{1}{e_0} \;\leq\; \frac{x}{b_x^{j-}} =: \frac{a_x^{j-}}{b_x^{j-}} \qquad \text{or} \qquad \frac{a_x^{j+}}{b_x^{j+}} := \frac{-(s-x)}{b_x^{j+}} \;\leq\; \frac{1}{e_0} =: \frac{\alpha}{\beta} \tag{85}$$

for all $j \in \{1, \dots, n-1\}$ and $x \in \{1, \dots, s-1\}$ with $\delta_x^j > 0$, and moreover the left-hand (respectively right-hand) inequality obtains only if $\beta/(\mu_0 \cdot l) < 0$ (respectively $\beta/(\mu_0 \cdot l) > 0$).

Further suppose that $\beta = e_0 < 0$. Then $M(e_0; \frac{r_1}{s_1}, \dots, \frac{r_n}{s_n})$ is an L-space if and only if

$$\begin{cases} 0 \geq -e_0 + b_x^{j-}/x & \text{for all } j \text{ and } x \text{ with } \delta_x^j > 0 & \text{if } \mu_0 \cdot l > 0 \\ 0 \leq -e_0 - b_x^{j+}/(s-x) & \text{for all } j \text{ and } x \text{ with } \delta_x^j > 0 & \text{if } \mu_0 \cdot l < 0 \, . \\ \text{never (case already excluded)} & & \text{if } \mu_0 \cdot l = 0 \end{cases} \tag{86}$$

Note that for all $j \in \{1, \ldots, n-1\}$ and $x \in \{1, \ldots, s-1\}$, $\delta_x^j = a_x^{j\pm} p + b_x^{j\pm} q^*$ implies

$$b_x^{j-}/x = \delta_x^j/(q^* x) - p/q^*, \qquad -b_x^{j+}/(s-x) = -\delta_x^j/(q^*(s-x)) - p/q^*. \tag{87}$$

Thus $b_x^{j-}/x$ is never maximized and $-b_x^{j+}/(s-x)$ is never minimized when $\delta_x \leq 0$, so we can remove the $\delta_x^j > 0$ conditions from (86). Moreover, $b_x^{j-}/x$ is never maximized and $-b_x^{j+}/(s-x)$ is never minimized when $j > 1$, so it suffices to fix $j = 1$. Reparameterizing the second case of (86) by $s - x \mapsto x$ then transforms (86) into the condition

$$0 \leq -e_0 + \min_{0<x<s} -\frac{1}{x}\left(-1 + \sum_{i=1}^{n} \left\lceil \frac{r_i x}{s_i} \right\rceil \right) \quad \text{or} \quad -e_0 + \max_{0<x<s} -\frac{1}{x}\left(1 + \sum_{i=1}^{n} \left\lfloor \frac{r_i x}{s_i} \right\rfloor \right) \leq 0, \tag{88}$$

which is the negation of the theorem statement's inequality for non-L-spaces.

When $e_0 \geq 0$, $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is always an L-space, since $e_0 = 0$ corresponds to our initial L-space $Y(\mu_{\text{L}})$, and since when $e_0 < 0$, the right-hand inequality in (85) holds for all $j \in \{1, \ldots, n-1\}$ and $x \in \{1, \ldots, s-1\}$. Accordingly, when $e \geq 0$, (88) always holds (via its right-hand inequality).

Lastly, suppose that $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$. Since we have excluded the case of $\mu_0 \cdot l = 0$, this implies that $Y(\mu_0) = M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is an L-space, so we must show that (88) holds. To see this, first note that the negation of (88) is equivalent to the inequality

$$\min_{0<x<s} \frac{1}{x}\left(1 - \sum_{i=1}^{n} \frac{[-r_i x]_{s_i}}{s_i}\right) < e_0 + \sum_{i=1}^{n} \frac{r_i}{s_i} < \max_{0<x<s} \frac{1}{x}\left(-1 + \sum_{i=1}^{n} \frac{[r_i x]_{s_i}}{s_i}\right). \tag{89}$$

Since $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$ implies $\delta_x^j \leq 0$ for all $j \in \{1, \ldots, n-1\}$ and $x \in \{1, \ldots, s-1\}$, we have

$$1 - \sum_{i=1}^{n} \frac{[-r_i x]_{s_i}}{s_i} = -\delta_{s-x}^{j=1} \geq 0, \qquad -1 + \sum_{i=1}^{n} \frac{[r_i x]_{s_i}}{s_i} = \delta_x^{j=1} \leq 0 \tag{90}$$

for all $x \in \{1, \ldots, s-1\}$. Thus (89) fails and (88) holds.

We have finished showing that, when $0 < r_i < s_i$ and $\gcd(r_i, s_i) = 1$ for each $i \in \{1, \ldots, n\}$, $M(e_0; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ is an L-space if and only if $e_0 + \sum_{i=1}^{n} \frac{r_i}{s_i} \neq 0$ and (88) holds. Moreover, since (88) is invariant under any map $\frac{r_i}{s_i} \mapsto \frac{dr_i}{ds_i}$ with $d \in \mathbb{Z}_{\neq 0}$, or under any reparameterization of the type in (55), we can remove our initial restrictions that $0 < r_i < s_i$ and $\gcd(r_i, s_i) = 1$, completing the proof of the theorem.

## 6. Gluings along torus boundaries

The introduction to Section 5 discusses how, for Seifert fibered spaces over $S^2$ (although the same is true for all Seifert fibered spaces [7,15]), the property of admitting a cooriented taut foliation is equivalent to the property of not being an L-space.

### 6.1. Equivalent properties for Seifert fibered spaces

In fact, this pair of equivalent properties belongs to a larger list.

**Theorem 6.1** *([12,39,33,8]). Suppose M is a Seifert fibered space over $S^2$. Then the following are equivalent:*

(1) *$M$ admits a cooriented taut foliation.*
(2.$\rho$) *There exists a homomorphism $\rho : \pi_1(M) \to \mathrm{Homeo}_+\mathbb{R}$ with non-trivial image.*
(2.LO) *The fundamental group $\pi_1(M)$ admits a left ordering.*
(3) *$M$ is not an L-space.*

**Summary of Proof.** Our idiosyncratic numbering owes to a result of Boyer, Rolfsen, and Wiest [8], which implies that (2.$\rho$) = (2.LO) for (a superset of) all closed, prime, oriented three-manifolds. We also have (1) ⇒ (3) for all closed oriented three-manifolds, as shown by Ozsváth and Szabó in the case of $C^2$ foliations [39], a result recently extended to $C^0$ foliations by Kazez and Roberts [29], and independently by Bowden [5].

More is known for Seifert fibered spaces. For Seifert fibrations over $S^2$, we have (1) = (2) as a corollary of a result by Eisenbud, Hirsh, and Neumann [12]. The result that (3) ⇒ (1) is due to Lisca, Matić, and Stipsicz for fibrations over $S^2$ [32,33], Boyer, Gordon, and Watson for fibrations over $\mathbb{RP}^2$ [7], and Gabai for fibrations with positive first Betti number [15]. One could also regard the classification by Jankins, Neumann [25], and Naimi [34] of Seifert fibered spaces over $S^2$ satisfying (1), together with the classification in the present article's Theorem 5.1 of Seifert fibered L-spaces over $S^2$, as an alternative proof that (1) = (3). □

The above result motivated a conjecture of Boyer, Gordon, and Watson [7] that properties (2) and (3) above are equivalent for all closed, prime, oriented three-manifolds.

### 6.2. Gluing results

To further explore the relationship of the above properties, Boyer and Clay [6] studied how each of these properties glue together when one splices together Seifert fibered spaces along the toroidal boundaries of fiber complements to form a graph manifold. In the process, Boyer and Clay observed that properties (1) and (2) obey a similar criterion determining when they admit compatible gluings. The property (3) of being a non-L-space proved less tractable for this exercise, but Boyer and Clay conjectured that property (3) should follow a similar gluing pattern to that of (1) and (2).

We are now able to confirm their conjecture in the case in which two Floer simple manifolds glued along their torus boundaries have the interiors of their L-space intervals overlap via the gluing map. In fact, there is no requirement that these Floer simple manifolds be graph manifolds.

**Theorem 6.2.** *Suppose that $Y_1$ and $Y_2$ are Floer simple manifolds glued together along their boundary tori. Such gluing is specified by a linear map $\varphi : H_1(\partial Y_1) \to H_1(\partial Y_2)$ with $\det\varphi = -1$, descending to a map $\varphi_{\mathbb{P}} : Sl(Y_1) \to Sl(Y_2)$ on Dehn filling slopes, where $Sl(Y_i) = \mathbb{P}(H_1(\partial Y_i))$. Let $\mathcal{L}(Y_i) \subset Sl(Y_i)$ denote the interval (with interior $\mathcal{L}^\circ(Y_i)$) of L-space filling slopes for $Y_i$, for each $i \in \{1, 2\}$, and suppose that $\varphi_{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cap \mathcal{L}^\circ(Y_2)$ is nonempty. Then $Y_1 \cup_\varphi Y_2$ is an L-space if and only if $\varphi_{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = Sl(Y_2)$ if both $\mathcal{D}^\tau_{\geq 0}(Y_i)$ are nonempty, and if and only if $\varphi_{\mathbb{P}}(\mathcal{L}(Y_1)) \cup \mathcal{L}(Y_2) = Sl(Y_2)$ otherwise.*

**Notation.** For brevity, we henceforth write

$$I_i := \mathcal{L}(Y_i) \quad \text{and} \quad \dot{I}_i := \mathcal{L}^\circ(Y_i).$$

*6.3. Set-up for proof: conventions and simplifying assumptions*

We begin by choosing bases $(m_i, l_i)$ for $H_1(\partial Y_i)$ and $\bar{m}_i$ for $H_1(Y_i)/\mathrm{Tors}(Y_i)$, for each $i \in \{1, 2\}$, according to the conventions of Section 4.1. Thus, if we write $\iota_i : H_1(\partial Y_i) \to H_1(Y_i)$ for the map induced on homology by inclusion of the boundary, then $l_i$ generates $\iota^{-1}(T_i)$, where $T_i := \mathrm{Tors}(H_1(Y_i))$, $m_i$ satisfies $m_i \cdot l_i = 1$, and $\bar{m}_i$ satisfies $\iota_i(m_i) \in g_i\bar{m}_i + T_i$, where $g_i := |T_i^\partial|$, with $T_i^\partial := \iota(\langle l_i \rangle) = T_i \cap \iota(H_1(\partial Y_i))$.

We shall break the operation of torus boundary gluing into three steps more amenable to Heegaard Floer computation: those of Dehn filling, connected sum, and Dehn surgery. In preparation, assuming $\varphi_{\mathbb{P}}(\dot{I}_1) \cap \dot{I}_2$ nonempty, choose $\mu_1 \in \mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2)) \subset H_1(\partial Y_1)$ and a longitude $\lambda_1 \in H_1(\partial Y_1)$ satisfying $\mu_1 \cdot \lambda_1 = 1$. Set $\mu_2 := \varphi(\mu_1)$ and $\lambda_2 := -\varphi(\lambda_1) \in H_1(\partial Y_2)$, noting that this makes $\lambda_2$ a longitude relative to $\mu_2$, since $\mu_1 \cdot \lambda_1 = 1$ and $\det\varphi = -1$ imply $\mu_2 \cdot \lambda_2 = 1$. Write $\mu_i = p_i m_i + q_i l_i$ and $\lambda_i = q_i^* m_i + p_i^* l_i$, with $p_i p_i^* - q_i q_i^* = 1$, for each $i \in \{1, 2\}$. Note that the invariant $q^* := q_1^* p_2 + q_2^* p_1$ is independent of choices of $\mu_1$ and $\lambda_1$. That is, if we write $(\phi_{ij})$ for the entries of the matrix for $\varphi$ with respect to the bases $(m_1, l_1)$ and $(m_2, l_2)$, then

$$q^* = p_2 q_1^* + q_2^* p_1 = (\phi_{11} p_1 + \phi_{12} q_1) q_1^* - (\phi_{11} q_1^* + \phi_{12} p_1^*) p_1 = -\phi_{12}. \tag{91}$$

Before using $\mu_i$ and $\lambda_i$ to splice together $Y_1$ and $Y_2$, we first pause to make some simplifying assumptions, without loss of generality.

**Proposition 6.3.** *Suppose $\varphi_{\mathbb{P}}(\dot{I}_1) \cap \dot{I}_2 \neq \emptyset$. For purposes of proving Theorem 6.2, it is sufficient to take $q^* > 0$, and we may choose $\mu_1 \in \mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2)) \subset H_1(\partial Y_1)$ to satisfy $\gcd(p_i, q_i) = \gcd(p_1, p_2) = \gcd(p_1, g_2) = \gcd(p_2, g_1) = 1$, $p_1, p_2 > q^* > 0$, and $p_i > (1 + \deg_{[\bar{m}_1]}\tau^{\mathrm{c}}(Y_1))(1 + \deg_{[\bar{m}_2]}\tau^{\mathrm{c}}(Y_2))$ for $i \in \{1, 2\}$, where $p_i m_i + q_i l_i = \mu_i$, $q_i^* m_i + p_i^* l_i = \lambda_i$, $\mu_2 := \varphi(\mu_1)$, $\lambda_2 := -\varphi(\lambda_1)$, and $q^* := q_1^* p_2 + q_2^* p_1$ for $i \in \{1, 2\}$. We call such $\mu_1$* "judiciously chosen."

**Proof.** We summarily dispense with the case in which $q^* = 0$, since then $\varphi_{\mathbb{P}}(\dot{I}_1) \cup \dot{I}_2 \neq \mathbb{P}(H_1(\partial Y_2))$ and $Y_1 \cup_\varphi Y_2$ is not a rational homology sphere, hence not an L-space. If

$q^* < 0$, then we may send $q^*$ to $-q^*$ by making the changes of basis $(m_i, l_i) \mapsto (m_i, -l_i)$ while simultaneously reversing the orientations of both $Y_1$ and $Y_2$. This preserves the positivity of $p_1$ and $p_2$, and leaves invariant the questions of whether $Y_1 \cup_\varphi Y_2$ is an L-space and whether $\varphi_{\mathbb{P}}(\dot{I}_1) \cap \dot{I}_2 = \mathbb{P}(H_1(\partial Y_2))$, or $\varphi_{\mathbb{P}}(I_1) \cap I_2 = \mathbb{P}(H_1(\partial Y_2))$. Thus we henceforth take $q^* > 0$.

We can construct a judicious choice of $\mu_1$ as an approximation of a primitive representative $P_1 m_1 + Q_1 l_1 \in \mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2))$ with $P_1 > 0$. Since $\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2)$ contains an open ball, we can demand that $P_i$ and $Q_i$ are nonzero for $i \in \{1, 2\}$, where $P_2 m_2 + Q_2 l_2 = \varphi(P_1 m_1 + Q_1 l_1)$. If $P_2 < 0$, we repair this sign with the change of basis $(m_2, l_2) \mapsto (-m_2, -l_2)$. Writing $M_\varphi = (\phi_{ij})$ for the matrix for $\varphi$ with respect to the bases $(m_1, l_1)$ and $(m_2, l_2)$, choose $s \in \mathbb{Z}$ such that $x := \phi_{22} + \phi_{12} s$ and $y := -\phi_{21} - \phi_{11} s$ are nonzero, with $\gcd(x, g_2) = 1$, noting that we now have $M_\varphi (x, y)^\top = (-1, s)^\top$. Next, set

$$D := |g_1 g_2 x y (y P_1 - x Q_1)(P_1 + x P_2)|\,, \tag{92}$$

and define $\mu_1 := p_1 m_1 + q_1 l_1$ and $\mu_2 := p_2 m_2 + q_2 l_2 = \varphi(\mu_1)$, with

$$\begin{aligned} p_1 &:= P_1 D N + x, & p_2 &:= P_2 D N - 1, \\ q_1 &:= Q_1 D N + y, & q_2 &:= Q_2 D N + s \end{aligned} \tag{93}$$

for some integer $N > q^*(1 + \deg_{[\bar{m}_1]} \tau^{\mathrm{c}}(Y_1))(1 + \deg_{[\bar{m}_2]} \tau^{\mathrm{c}}(Y_2))$ chosen large enough to make $\mu_1 := p_1 m_1 + q_1 l_1$ lie in $\mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2))$. Then $\gcd(p_1, g_2) = \gcd(p_2, g_1) = 1$, and one can use the facts that $p_1/x - q_1/y = (yP_1 - xQ_1)(D/(xy))N$ is relatively prime to $p_1/x$ and that $p_1/x + p_2 = (P_1 + xP_2)(D/x)N$ is relatively prime to $p_2$ to argue, respectively, that $\gcd(p_1, q_1) = 1$ and $\gcd(p_1, p_2) = 1$, the former of which statements implies $\gcd(p_2, q_2) = 1$. □

### *6.4. Dehn filling a Floer simple manifold*

We are now ready to construct $Y_1 \cup_\varphi Y_2$ as the Dehn filling of a Floer simple manifold $Y$. For each $i \in \{1, 2\}$, perform the (L-space) Dehn filling $Y_i(\mu_i)$, writing $K_{\mu_i}$ for the knot core of $Y_i(\mu_i) \setminus Y_i$. Next, let $Y$ denote the (Floer simple) knot complement

$$Y := Y_1(\mu_1) \# Y_2(\mu_2) \setminus K_{\mu_1} \# K_{\mu_2} \tag{94}$$

of the connected sum $K_{\mu_1} \# K_{\mu_2} \subset Y_1(\mu_1) \# Y_2(\mu_2) = Y(\mu_{\mathrm{L}})$, where $\mu_{\mathrm{L}}$ denotes the meridian of $K_{\mu_1} \# K_{\mu_2}$, and as usual, write $\iota : H_1(\partial Y) \to H_1(Y)$ for the map induced on homology by inclusion of the boundary, and set $T := \mathrm{Tors}(H_1(Y))$ and $T^\partial := \iota(H_1(\partial Y)) \cap T$. The maps $f_i : H_1(Y_i) \longrightarrow H_1(Y)$ induced by inclusion descend to an isomorphism $f_1 \oplus f_2 : (H_1(Y_1) \oplus H_1(Y_2))/(\iota_1(\mu_1) \sim \iota_2(\mu_2)) \xrightarrow{\sim} H_1(Y)$ that identifies meridians, via $f_1 \iota_1(\mu_1) = f_2 \iota_2(\mu_2) = \iota(\mu_{\mathrm{L}})$. In addition, $K_{\mu_1} \# K_{\mu_2}$ has a longitude $\lambda_{\mathrm{L}}$ satisfying $f_1(\iota_1(\lambda_1)) + f_2(\iota_2(\lambda_2)) = \iota(\lambda_{\mathrm{L}})$.

Consider the Dehn filling $Y(\lambda_L)$, which one could regard as 0-surgery with respect to the basis $(\mu_L, \lambda_L)$ along the knot $K_{\mu_1} \# K_{\mu_2} \subset Y(\mu_L) = Y_1(\mu_1) \# Y_2(\mu_2)$, with $Y(\mu_L)$ an L-space. Since $Y$ already identifies $\iota_1(\mu_1)$ with $\iota_2(\varphi(\mu_1))$, and since setting $\iota(\lambda_L) = 0$ identifies $\iota_1(\lambda_1)$ with $\iota_2(\varphi(\lambda_1))$, we have

$$Y(\lambda_L) = Y_1 \cup_\varphi Y_2. \tag{95}$$

To describe $Y(\lambda_L)$ more explicitly, one can deduce that $f_1 \oplus f_2$ restricts to an isomorphism

$$(\iota_1(H_1(\partial Y_1)) \oplus \iota_2(H_1(\partial Y_2)))/(\iota_1(\mu_1) \sim \iota_2(\mu_2)) \xrightarrow{\sim} \iota(H_1(\partial Y)) \oplus \langle \sigma_0 \rangle, \tag{96}$$

for some $\sigma_0 \in T$ with $|\langle \sigma_0 \rangle| = \gcd(g_1, g_2)$. That is, if we define

$$g_0 := \gcd(g_1, g_2), \quad \hat{g}_1 := g_1/g_0, \quad \hat{g}_2 := g_2/g_0, \quad g := g_1 g_2/g_0 = \hat{g}_1 \hat{g}_2 g_0, \tag{97}$$

then for $l \in H_1(\partial Y)$ an appropriately signed generator of $\iota^{-1}(T)$ and any $m \in H_1(\partial Y)$ satisfying $m \cdot l = 1$, there are $\sigma_0 \in T$ of order $g_0$ and $\xi \in \mathbb{Z}/g$ such that

$$\begin{aligned} &f_1 \colon \iota_1(m_1) \mapsto p_2 \iota(m) + q_2 \hat{g}_1 \xi \iota(l) - q_1 \sigma_0, && f_2 \colon \iota_2(m_2) \mapsto p_1 \iota(m) + q_1 \hat{g}_2 \xi \iota(l) + q_2 \sigma_0, \\ &f_1 \colon \iota_1(l_1) \mapsto p_2 \hat{g}_2 \xi \iota(l) + p_1 \sigma_0, && f_2 \colon \iota_2(l_2) \mapsto p_1 \hat{g}_1 \xi \iota(l) - p_2 \sigma_0. \end{aligned} \tag{98}$$

Thus, $g = |T^\partial|$, and if we write

$$\mu_L = pm + ql, \qquad \lambda_L = q^* m + p^* l, \tag{99}$$

then $p$, $q$, $q^*$, and $p^*$ satisfy

$$\begin{aligned} &p = p_1 p_2, && q \equiv (q_1 p_2 g_2 + q_2 p_1 g_1)\xi \pmod{g}, \\ &q^* = q_1^* p_2 + q_2^* p_1, && p^* \equiv ((p_1 p_2^* + q_1^* q_2) g_1 + (p_2 p_1^* + q_2^* q_1) g_2)\xi \pmod{g}. \end{aligned} \tag{100}$$

Again, the condition $\mu_L \cdot \lambda_L = 1$ determines the value of $\xi$, which we shall not need. Of course, it will often be more convenient to express this restriction of $\iota_i(H_1(\partial Y_i))$ to $f_1 \oplus f_2$ in terms of the bases $(\iota_i(\mu_i), \iota_i(\lambda_i))$ for $\iota_i(H_1(\partial Y_i))$ and $(\iota(\mu_L), \iota(\lambda_L))$ for $\iota(H_1(\partial Y))$, as we shall describe explicitly in the proof of Proposition 6.5.

In either case, we see that $q^* = q_1^* p_2 + q_2^* p_1$ makes its appearance as $\lambda_L \cdot l$. Thus, $Y_1 \cup_\varphi Y_2 = Y(\lambda_L)$ can be regarded as surgery with label $(\mu_L \cdot \lambda_L)/(\lambda_L \cdot l) = 1/q^*$ along $K_{\mu_1} \# K_{\mu_2} \subset Y(\mu_L)$.

### *6.5. Computation of $\mathcal{D}^\tau(Y)$*

For the remainder of Section 6, we regard the entire preceding construction, along with the hypotheses of Theorem 6.2, as fixed initial data. We are now ready to compute

$\mathcal{D}^\tau(Y)$, which we shall call $\mathcal{D}^\tau_{\geq 0}(Y)$ to emphasize that in this case we are not excluding torsion elements.

**Proposition 6.4.** *Suppose that $\mu_1$ is "judiciously chosen" from $\mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2))$ nonempty, and that $Y$ is constructed as above. If we set $t_\partial := [(\iota(m))]$, then $\mathcal{D}^\tau_{\geq 0}(Y) = R_0 \amalg (R_1 \cup R_2) \amalg R_3$, with*

$$\begin{aligned} R_0 &:= S\left[\frac{1}{1-t_\partial} - \frac{1-t_\partial^{p_1p_2}}{(1-t_\partial^{p_1})(1-t_\partial^{p_2})}\right] + T^\partial, \qquad (101)\\ R_1 &:= f_1(\mathcal{D}^\tau_{\geq 0}(Y_1)) \ + \ f_2\big(\{0,\dots,p_2-1\}\iota_2(m_2) + T_2^\partial\big)\,,\\ R_2 &:= f_2(\mathcal{D}^\tau_{\geq 0}(Y_2)) \ + \ f_1\big(\{0,\dots,p_1-1\}\iota_1(m_1) + T_1^\partial\big)\,,\\ R_3 &:= \iota(\mu_{\mathrm{L}}) + f_1(\mathcal{D}^\tau_{\geq 0}(Y_1)) + f_2(\mathcal{D}^\tau_{\geq 0}(Y_2)). \end{aligned}$$

**Proof.** To compute $\mathcal{D}^\tau_{\geq 0}(Y)$, we need the Turaev torsion $\tau(Y)$ and torsion complement $\tau^{\mathrm{c}}(Y)$. In order to write these down, we first choose generators $\bar{m}$ for $H_1(Y)/T$ and $\bar{m}_i$ and $H_1(Y_i)/T_i$ satisfying

$$\iota(m) \in g\bar{m} + T, \quad \iota_i(m_i) \in g_i\bar{m}_i + T_i, \quad i \in \{1,2\}. \qquad (102)$$

Recall that the above condition only constrains the signs of $\bar{m}$ and $\bar{m}_i$. We shall write

$$t := [\bar{m}] \in \mathbb{Z}[H_1(Y)], \quad t_i := [\bar{m}_i] \in \mathbb{Z}[H_1(Y_i)], \quad i \in \{1,2\}, \qquad (103)$$

for the inclusions of $\bar{m}$ and $\bar{m}_i$ into their respective group rings.

Invoking the standard gluing rules for Turaev torsion yields

$$\tau(Y) = (1 - [\iota(\mu_{\mathrm{L}})])\, \tilde{f}_1(\tau(Y_1))\, \tilde{f}_2(\tau(Y_2)), \qquad (104)$$

where each $\tilde{f}_i$ denotes the lift of $f_i$ to the Laurent series group ring $\mathbb{Z}[t_i^{-1}, t_i]][T_i] \supset \mathbb{Z}[H_1(Y_i)]$. (One could also obtain this result by using Proposition 2.1 and the fact that Heegaard Floer homology tensors on connected sums.)

For $i \in \{1,2\}$, set $P_T := \sum_{h\in T}[h] \in \mathbb{Z}[H_1(Y)]$ and $P_{T_i} := \sum_{h_i \in T_i}[h_i] \in \mathbb{Z}[H_1(Y_i)]$, and let $P$ and $P_i$ denote the Laurent series $P := P_T/(1-t)$ and $P_i := P_{T_i}/(1-t_i)$, the latter with polynomial truncations

$$\bar{P}_i \ := \ (1 - [\iota_i(\mu_i)])P_i \ = \ \frac{1-t_i^{p_ig_i}}{1-t_i} P_{T_i}. \qquad (105)$$

The torsion complements $\tau^{\mathrm{c}}(Y) := P - \tau(Y)$ and $\tau^{\mathrm{c}}(Y_i) := P_i - \tau(Y_i)$ then satisfy

$$\begin{aligned} \tau^{\mathrm{c}}(Y) &= P - (1 - [\iota(\mu_{\mathrm{L}})])\tilde{f}_1(P_1 - \tau^{\mathrm{c}}(Y_1))\tilde{f}_2(P_2 - \tau^{\mathrm{c}}(Y_2)) \qquad (106)\\ &= r_0^{\mathrm{c}} + r_{12}^{\mathrm{c}} + r_3^{\mathrm{c}}, \end{aligned}$$

$$\text{with } r_0^{\rm c} := P - (1 - [\iota(\mu_{\rm L})])\tilde{f}_1(P_1)\tilde{f}_2(P_2),$$
$$r_{12}^{\rm c} := \tilde{f}_1(\tau^{\rm c}(Y_1))\tilde{f}_2(\bar{P}_2) + \tilde{f}_1(\bar{P}_1)\tilde{f}_2(\tau^{\rm c}(Y_2)) - \tilde{f}_1(\tau^{\rm c}(Y_1))\tilde{f}_2(\tau^{\rm c}(Y_2)),$$
$$r_3^{\rm c} := [\iota(\mu_{\rm L})]\tilde{f}_1(\tau^{\rm c}(Y_1))\tilde{f}_2(\tau^{\rm c}(Y_2)).$$

It is straightforward to show that each of $r_0^{\rm c}$, $r_{12}^{\rm c}$, and $r_3^{\rm c}$ is an element of $\mathbb{Z}[H_1(Y)]$ with coefficients in $\{0,1\}$, and that the three sets $S[r_0^{\rm c}]$, $S[r_{12}^{\rm c}]$, and $S[r_3^{\rm c}]$ are disjoint. In particular, $r_0^{\rm c}$ satisfies the property

$$\big(S[r_0^{\rm c}] - S[\tilde{f}_1(P_1)\tilde{f}_2(P_2)]\big) \cap S[P] = S[r_0^{\rm c}], \tag{107}$$

while $r_{12}^{\rm c}$ satisfies

$$S[r_{12}^{\rm c}] = S[\tilde{f}_1(\tau^{\rm c}(Y_1))\tilde{f}_2(\bar{P}_2) + \tilde{f}_1(\bar{P}_1)\tilde{f}_2(\tau^{\rm c}(Y_2))]. \tag{108}$$

On the other hand, since each $(1 - [\iota_i(\mu_i)])\tau(Y_i)$ has no negative coefficients, it follows from (104) that $\tau(Y)$ has support

$$S[\tau(Y)] = S[\tilde{f}_1(\tau(Y_1))\,\tilde{f}_2(\tau(Y_2))] \subset H_1(Y). \tag{109}$$

Lastly, we compute $\mathcal{D}^{\tau}_{\geq 0}(Y) := (S[\tau^{\rm c}(Y)] - S[\tau(Y)]) \cap \iota(m\mathbb{Z}_{\geq 0} + l\mathbb{Z})$. Using the facts that $0 \in S[\tau(Y_i)]$ for each $i \in \{1,2\}$ (as per the convention stated in (7) in Section 4.2) and that $\iota(H_1(\partial Y)) \subset f_1\iota_1(H_1(\partial Y_1)) \oplus f_2\iota_2(H_1(\partial Y_2))$, we obtain $\mathcal{D}^{\tau}_{\geq 0}(Y) = R_0 \amalg (R_1 \cup R_2) \amalg R_3$, with

$$\begin{aligned} R_0 &= S[r_0^{\rm c}] \cap \iota(m\mathbb{Z}_{\geq 0} + lZ), \\ R_1 &= f_1(\mathcal{D}^{\tau}_{\geq 0}(Y_1)) + f_2(S[\bar{P}_2] \cap \iota_2(H_1(Y_2))), \\ R_2 &= f_2(\mathcal{D}^{\tau}_{\geq 0}(Y_2)) + f_1(S[\bar{P}_1] \cap \iota_1(H_1(Y_1))), \\ R_3 &= \iota(\mu_{\rm L}) + f_1(\mathcal{D}^{\tau}_{\geq 0}(Y_1)) + f_2(\mathcal{D}^{\tau}_{\geq 0}(Y_2)), \end{aligned} \tag{110}$$

where property (107) has made any remaining subsets of $S[\tau^{\rm c}(Y)] - S[\tau(Y)]$—such as, for example, $f_1(S[\tau^{\rm c}(Y_1)] - S[\tau(Y_1)]) \cap (m\mathbb{Z}_{<0} + T)$—land in $S[r_0^{\rm c}]$. It is straightforward to show that the above $R_i$ are equal to those enumerated in the statement of the proposition. □

### *6.6. Computation of L-space interval for Y*

Having determined $\mathcal{D}^{\tau}(Y)$, we can apply Theorem 4.2 to compute the L-space interval for $Y$.

**Proposition 6.5.** *Suppose that $\mu_1$ is "judiciously chosen" from $\mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2))$ nonempty, and that $Y$ is constructed as above. For each $i \in \{1,2\}$, set $\bar{q}_i := [q_i^*]_{p_i}$ and* *let $B_i$ denote the set $B_i := \left\{ [p_i\gamma_i - q_i\delta_i]_{p_ig_i} \middle| \boldsymbol{\delta}_i = \delta_i\iota_i(m_i) + \gamma_i\iota_i(l_i) \in \mathcal{D}^{\tau}_{\geq 0}(Y_i)\right\}$. Then $Y_1 \cup_\varphi Y_2$ is an L-space if and only if condition* (L.*i*) *holds for each $b_1 \in B_1$,* (L.*ii*) *holds for each $b_2 \in B_2$, and* (L.*iii*) *holds for each $(b_1, b_2) \in B_1 \times B_2$ with $b_1 \equiv b_2 \pmod{g_0}$:*

$$\text{(L.}i\text{)}\quad \frac{1}{b}\left\lfloor\frac{b\bar{q}_1}{p_1}\right\rfloor + \frac{1}{b}\left\lfloor\frac{b\bar{q}_2}{p_2}\right\rfloor \geq 1 \ \ \forall\, b \equiv b_1 \pmod{p_1g_1},\ 0 < b < pg,$$

$$\text{(L.}ii\text{)}\quad \frac{1}{b}\left\lfloor\frac{b\bar{q}_1}{p_1}\right\rfloor + \frac{1}{b}\left\lfloor\frac{b\bar{q}_2}{p_2}\right\rfloor \geq 1 \ \ \forall\, b \equiv b_2 \pmod{p_2g_2},\ 0 < b < pg,$$

$$\text{(L.}iii\text{)}\quad \frac{1}{b}\left\lfloor\frac{b\bar{q}_1}{p_1}\right\rfloor + \frac{1}{b}\left\lfloor\frac{b\bar{q}_2}{p_2}\right\rfloor > 1 \ \ \forall\, b \equiv b_1 \pmod{p_1g_1},\ b \equiv b_2 \pmod{p_2g_2},\ 0 < b < pg,$$

*where $p := p_1p_2$ and $g := g_1g_2/g_0$, with $g_0 = \gcd(g_1, g_2)$.*

**Proof.** We begin by ensuring that $\mathcal{D}^{\tau}_{\geq 0}(Y)$ meets the conditions of Theorem 4.2 Since $R_0 \not\subset T$ implies $\mathcal{D}^{\tau}_{>0}(Y) \neq \emptyset$, it remains to verify, for each $\boldsymbol{\delta} = \delta\iota(m)+\gamma\iota(l) \in \mathcal{D}^{\tau}_{\geq 0}(Y)$, that $b_{\boldsymbol{\delta}} := [p\gamma - q\delta]_{pg} (\equiv \mu_{\text{L}} \cdot \iota^{-1}(\boldsymbol{\delta}) \pmod{pg})$ is nonzero, or equivalently, that $\boldsymbol{\delta} \notin \langle \iota(\mu_{\text{L}})\rangle$. Now, the definition of $\mathcal{D}^{\tau}_{\geq 0}$ already implies $0 \notin \mathcal{D}^{\tau}_{\geq 0}(Y)$. Recalling the result of Proposition 6.4, and that $\iota(\mu_{\text{L}}) = p\iota(m) + q\iota(l)$ with $p := p_1p_2$, we know that the inclusions

$$R_0 \subset \{1, \ldots, p_1p_2 - p_1 - p_2\}\iota(m) + T^{\partial}, \tag{111}$$

$$\begin{aligned} R_1 \cup R_2 &\subset f_1\big(\{0, \ldots, p_1 - 1\}\iota_1(m_1) + T^{\partial}_1\big) + f_2\big(\{0, \ldots, p_2 - 1\}\iota_2(m_2) + T^{\partial}_2\big) \\ &= (\{0, \ldots, p_1 - 1\}p_2 + \{0, \ldots p_2 - 1\}p_1)\iota(m) + T^{\partial} \end{aligned} \tag{112}$$

imply that $\langle\iota(\mu_{\text{L}})\rangle \cap (R_0 \cup R_1 \cup R_2) = \emptyset$. Lastly, since our "judiciously chosen" hypothesis makes $\deg_{[\bar{m}_i]} \tau^{c}(Y_i) < p_ig_i = \deg_{[\bar{m}_i]}[\iota_i(\mu_i)]$, and since the kernel of $f_1 \oplus f_2$ is generated by $(\iota_1(\mu_1), -\iota_2(\mu_2))$, we know that $\langle\iota(\mu_{\text{L}})\rangle \cap R_3 = \emptyset$. Thus, Theorem 4.2 applies.

Since we can regard $Y_1 \cup_\varphi Y_2 = Y(\lambda_{\text{L}})$ as surgery with label $1/q^*$ along $K_{\mu_1}\#K_{\mu_2} \subset Y(\mu_{\text{L}})$, Theorem 4.2 tells us that $Y_1 \cup_\varphi Y_2$ is an L-space if and only if

$$\frac{b_{\boldsymbol{\delta}} - p}{\delta} \leq \frac{1}{q^*} \leq \frac{b_{\boldsymbol{\delta}}}{\delta} \tag{113}$$

for all $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{>0}(Y)$ $(= \mathcal{D}^{\tau}_{\geq 0}(Y) \setminus T)$. Now, since $b_{\boldsymbol{\delta}} \equiv \mu_{\text{L}} \cdot \tilde{\boldsymbol{\delta}} \pmod{pg}$ for any lift $\tilde{\boldsymbol{\delta}} \in \iota^{-1}(\boldsymbol{\delta})$, there always exists a unique $a_{\boldsymbol{\delta}} \in \mathbb{Z}$ for which $\boldsymbol{\delta} = \iota(a_{\boldsymbol{\delta}}\mu_{\text{L}} + b_{\boldsymbol{\delta}}\lambda_{\text{L}})$. Such $a_{\boldsymbol{\delta}} \in \mathbb{Z}$ satisfies $\delta = a_{\boldsymbol{\delta}}p + b_{\boldsymbol{\delta}}q^*$. Taking this as a definition for $a_{\boldsymbol{\delta}} \in \mathbb{Z}$, we note that, since $b_{\boldsymbol{\delta}} - p < 0$ and $q^* > 0$, the left-hand inequality in (113) is vacuous, whereas the right-hand inequality is equivalent to the condition $a_{\boldsymbol{\delta}} \leq 0$.

Since $b_{\boldsymbol{\delta}}q^* > 0$ for all $\boldsymbol{\delta} = \delta\iota(m) + \gamma\iota(l) \in \mathcal{D}^{\tau}_{\geq 0}(Y)$, we obtain $a_{\boldsymbol{\delta}} \leq 0$ automatically whenever $\delta < p$. In particular, $a_{\boldsymbol{\delta}} \leq 0$ for all $\boldsymbol{\delta} \in R_0$ and for any $\boldsymbol{\delta} \in \mathcal{D}^{\tau}_{\geq 0}(Y) \cap (0\iota(m) + T^{\partial})$. Now, the latter case is, strictly speaking, irrelevant to the question of whether $Y_1 \cup_\varphi Y_2$ is an L-space, but the fact that the condition $a_{\boldsymbol{\delta}} \leq 0$ is vacuous on torsion elements of $\mathcal{D}^{\tau}_{\geq 0}(Y)$ allows us to apply the condition to all of $\mathcal{D}^{\tau}_{\geq 0}(Y)$, thereby simplifying our bookkeeping.

It remains to apply the condition $a_{\boldsymbol{\delta}} \leq 0$ to each of $R_1$, $R_2$, and $R_3$, from which we shall obtain the respective conditions (L.$i$), (L.$ii$), and (L.$iii$). To do this, we first, for each $i \in \{1,2\}$, consider the bijection,

$$\begin{aligned} \{0,\dots,p_i g_i - 1\} &\longrightarrow \{0,\dots,p_i-1\}\iota_i(m_i) + T_i^{\partial} \ \subset\ \iota_i(H_1(\partial Y_i)), \\ b_i &\longmapsto \iota_i(-\left\lfloor \frac{b_i q_i^*}{p_i} \right\rfloor \mu_i \ +\ b_i\lambda_i) \ \in\ [b_i\bar{q}_i]_{p_i}\iota_i(m_i) + T_i^{\partial}, \end{aligned} \tag{114}$$

recalling that $\mu_i = p_i m_i + q_i l_i$, $\lambda_i = q_i^* m_i + p_i^* l_i$, and $g_i := |T_i^{\partial}|$ with $T_i^{\partial} = \langle \iota_i(l_i)\rangle$. The inverse map sends

$$\mathbf{x}_i := x_i\iota_i(m_i) + y_i\iota_i(l_i) \longmapsto b_i^{\mathbf{x}_i} := [\mu_i \cdot (x_i m_i + y_i l_i)]_{p_i g_i} = [p_i y_i - q_i x_i]_{p_i g_i}. \tag{115}$$

Thus, if we define $a_i^{\mathbf{x}_i} := -(b_i^{\mathbf{x}_i} q_i^* - [b_i^{\mathbf{x}_i} q_i^*]_{p_i})/p_i$, then for any $\mathbf{x}_i := x_i\iota_i(m_i) + y_i\iota_i(l_i)$ with $x_i \in \{0,\dots,p_i-1\}$, and for any $s_i \in \mathbb{Z}$, we have

$$\mathbf{x}_i = \iota_i(a_i^{\mathbf{x}_i}\mu_i + b_i^{\mathbf{x}_i}\lambda_i) = \iota_i((a_i^{\mathbf{x}_i} - q_i^* g_i s_i)\mu_i + (b_i^{\mathbf{x}_i} + p_i g_i s_i)\lambda_i), \tag{116}$$

with $s_i \in \mathbb{Z}$ parametrizing the lifts $\iota_i^{-1}(\mathbf{x}_i)$ of $\mathbf{x}_i$.

Since $f_1\iota_1(\mu_1) = f_2\iota_2(\mu_2) = \iota(\mu_{\text{L}})$ and $f_1\iota_1(\lambda_1) + f_2\iota_2(\lambda_2) = \iota(\lambda_{\text{L}})$, we deduce from (116) that $f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2) \in \iota(H_1(\partial Y))$ if and only if there exist $s_1, s_2 \in \mathbb{Z}$ such that $b_1^{\mathbf{x}_1} + p_1 g_1 s_1 = b_2^{\mathbf{x}_2} + p_2 g_2 s_2$, which, in turn, occurs if and only if $b_1^{\mathbf{x}_1} \equiv b_2^{\mathbf{x}_2} (\text{mod } g_0)$, since $g_0 = \gcd(p_1 g_1, p_2 g_2) = \gcd(g_1, g_2)$. In such case, if we write $f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2) = \iota(a\mu_{\text{L}} + b\lambda_{\text{L}})$ with $b \in \{0,\dots,pg-1\}$, then $b$ is the unique solution in $\{0,\dots,pg-1\}$ to the equivalences $b \equiv b_1^{\mathbf{x}_1}$ (mod $p_1 g_1$), $b \equiv b_2^{\mathbf{x}_2}$ (mod $p_2 g_2$). Setting $b = b_i^{\mathbf{x}_i} + p_i g_i s_i$ makes $g_i s_i = (b - b_i^{\mathbf{x}_i})/p_i$ for each $i \in \{1,2\}$, so that we obtain

$$\begin{aligned} a &= \textstyle\sum_{i\in\{1,2\}} (a_i^{\mathbf{x}_i} - q_i^* g_i s_i) \\ &= \textstyle\sum_{i\in\{1,2\}} \Big( -(b_i^{\mathbf{x}_i} q_i^* - [b_i^{\mathbf{x}_i} q_i^*]_{p_i})/p_i \ - \ q_i^*(b - b_i^{\mathbf{x}_i})/p_i \Big) \\ &= -b(\bar{q}_1 p_2 + \bar{q}_2 p_1 - p_1 p_2)/p_1 p_2 \ + \ [b_1^{\mathbf{x}_1} q_1^*]_{p_1}/p_1 \ + \ [b_2^{\mathbf{x}_2} q_2^*]_{p_2}/p_2 \\ &= -b - \left\lfloor \frac{b\bar{q}_1}{p_1} \right\rfloor - \left\lfloor \frac{b\bar{q}_2}{p_2} \right\rfloor, \end{aligned} \tag{117}$$

where the third line uses the identity

$$q^* := q_1^* p_2 + q_2^* p_1 = \bar{q}_1 p_2 + \bar{q}_2 p_1 - p_1 p_2.$$

(Here, the lefthand side is just the definition of $q^*$. For the righthand side, the "judiciously chosen" hypotheses $0 < q^* < p_i$ imply that $0 < q^* < \bar{q}_1 p_2 + \bar{q}_2 p_1$ and $q^* =$

$[\bar{q}_1 p_2 + \bar{q}_2 p_1]_{p_1 p_2}$. Thus, since $0 \le \bar{q}_i < p_i$, we have $p_1p_2 < q^* + p_1p_2 \le \bar{q}_1 p_2 + \bar{q}_2 p_1 < 2p_1p_2$, forcing the middle two terms of this inequality to be equal.)

Since we may write any $\boldsymbol{\delta} \in R_1$ as

$$\boldsymbol{\delta} \;=\; f_1(\boldsymbol{\delta}_1) + f_2(\mathbf{x}_2) \;=\; \iota(a\mu_{\mathrm{L}} + b\lambda_{\mathrm{L}}) \tag{118}$$

with $\boldsymbol{\delta}_1 \in \mathcal{D}^{\tau}_{\ge 0}(Y_1)$, $\mathbf{x}_2 \in \{0,\dots,p_2-1\}\iota_2(m_2) + \{0,\dots,g_2-1\}\iota_2(l_2)$ satisfying $b_2^{\mathbf{x}_2} \equiv b_1^{\boldsymbol{\delta}_1} \pmod{g_0}$, $0 < b < pg$, and $a$ as determined in (117), we have $a_{\boldsymbol{\delta}} = a$, and demanding $a_{\boldsymbol{\delta}} \le 0$ yields condition $(i)_{\mathrm{L}}$. Likewise, applying $a_{\boldsymbol{\delta}} \le 0$ for all $\boldsymbol{\delta} \in R_2$ yields condition $(ii)_{\mathrm{L}}$. The case of $R_3$ is similar, except that since $\boldsymbol{\delta} = \iota(\mu_{\mathrm{L}}) + f_1(\boldsymbol{\delta}_1) + f_2(\boldsymbol{\delta}_2)$, we need $a_{\boldsymbol{\delta}} = 1 + a \le 0$ and $b_1^{\boldsymbol{\delta}_1} \equiv b_2^{\boldsymbol{\delta}_2} \pmod{g_0}$, yielding condition $(\mathrm{iii}_{\mathrm{L}})$. □

### 6.7. *Determining when gluing hypothesis is met*

We next turn our attention to the L-space filling slope intervals $I_i \subset \mathbb{P}(H_1(\partial Y_i))$, to determine when they combine according to the hypotheses of the theorem.

**Proposition 6.6.** *Suppose that $\mu_1$ is "judiciously chosen" from $\mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2))$ nonempty, and that $Y$ is constructed as above. For each $i \in \{1,2\}$, set $\bar{q}_i := [q_i^*]_{p_i}$ and let $B_i$ denote the set $B_i := \left\{ [p_i\gamma_i - q_i\delta_i]_{p_i g_i} \middle| \, \boldsymbol{\delta}_i = \delta_i \iota_i(m_i) + \gamma_i \iota_i(l_i) \in \mathcal{D}^{\tau}_{\ge 0}(Y_i) \right\}$. Then $\varphi_{\mathbb{P}}(\dot{I}_1) \cup \dot{I}_2 = \mathbb{P}(H_1(\partial Y_2))$ when both $\mathcal{D}^{\tau}_{\ge 0}(Y_i)$ are nonempty—and $\varphi_{\mathbb{P}}(I_1) \cup I_2 = \mathbb{P}(H_1(\partial Y_2))$ when one or both $\mathcal{D}^{\tau}_{\ge 0}(Y_i)$ are empty—if and only if the following three conditions hold:*

$$(\mathrm{I}.i) \quad \frac{1}{b_1}\left\lfloor \frac{b_1 \bar{q}_1}{p_1} \right\rfloor + \frac{1}{b_1}\left\lfloor \frac{b_1 \bar{q}_2}{p_2} \right\rfloor \ge 1 \;\; \textit{for all} \;\; b_1 \in B_1,$$

$$(\mathrm{I}.ii) \quad \frac{1}{b_2}\left\lfloor \frac{b_2 \bar{q}_1}{p_1} \right\rfloor + \frac{1}{b_2}\left\lfloor \frac{b_2 \bar{q}_2}{p_2} \right\rfloor \ge 1 \;\; \textit{for all} \, b_2 \in B_2,$$

$$(\mathrm{I}.iii) \quad \frac{1}{b_1}\left\lfloor \frac{b_1 \bar{q}_1}{p_1} \right\rfloor + \frac{1}{b_2}\left\lfloor \frac{b_2 \bar{q}_2}{p_2} \right\rfloor > 1 \;\; \textit{for all} \, (b_1, b_2) \in B_1 \times B_2.$$

**Proof.** For $i \in \{1,2\}$, let $\pi_i$ denote the "surgery label" map, $\pi_i : H_1(\partial Y_i) \backslash \{0\} \longrightarrow \mathbb{Q} \cup \infty$,

$$\pi_i \;:\; \alpha_i\mu_i + \beta_i\lambda_i \longmapsto \frac{\mu_i \cdot (\alpha_i\mu_i + \beta_i\lambda_i)}{(\alpha_i\mu_i + \beta_i\lambda_i) \cdot l_i} = \frac{\beta_i}{\alpha_i p_i + \beta_i q_i^*}, \tag{119}$$

and for each $\boldsymbol{\delta}_i = \delta_i\iota_i(m_i) + \gamma_i\iota_i(l_i) \in \mathcal{D}^{\tau}_{\ge 0}(Y_i)$, let $\tilde{\boldsymbol{\delta}}_{i+}, \tilde{\boldsymbol{\delta}}_{i-} \in \iota_i^{-1}(\boldsymbol{\delta}_i)$ denote the two lifts of $\boldsymbol{\delta}_i$ closest to $\mu_i$ with respect to surgery label, *i.e.*,

$$\tilde{\boldsymbol{\delta}}_{i+} = a_{i+}^{\boldsymbol{\delta}_i}\mu_i + b_{i+}^{\boldsymbol{\delta}_i}\lambda_i, \quad b_{i+}^{\boldsymbol{\delta}_i} := [p_i\gamma_i - q_i\delta_i]_{p_i g_i}, \tag{120}$$

$$\tilde{\boldsymbol{\delta}}_{i-} = a_{i-}^{\boldsymbol{\delta}_i}\mu_i + b_{i-}^{\boldsymbol{\delta}_i}\lambda_i := \left(a_{i+}^{\boldsymbol{\delta}_i} + q_i^* g_i\right)\mu_i + \left(b_{i+}^{\boldsymbol{\delta}_i} - p_i g_i\right)\lambda_i. \tag{121}$$

Note that since $p_i > \deg_{[\bar{m}_i]}\tau^{\mathrm{c}}(Y_i)$ implies $\delta_i < p_i$, we have

$$\delta_i = a_{i+}^{\boldsymbol{\delta}_i} p_i + b_{i+}^{\boldsymbol{\delta}_i} q_i^* = [b_{i+}^{\boldsymbol{\delta}_i} q_i^*]_{p_i} = a_{i-}^{\boldsymbol{\delta}_i} p_i + b_{i-}^{\boldsymbol{\delta}_i} q_i^* = [b_{i-}^{\boldsymbol{\delta}_i} q_i^*]_{p_i} \geq 0. \tag{122}$$

Note also that $\pi_i(\tilde{\boldsymbol{\delta}}_{i-}) < 0 < \pi_i(\tilde{\boldsymbol{\delta}}_{i+})$ unless $\delta_i = 0$, in which case $\pi_i(\tilde{\boldsymbol{\delta}}_{i-}) = \pi_i(\tilde{\boldsymbol{\delta}}_{i+}) = \infty$.

Corollary 4.5 then implies that, for $\mathcal{D}^{\tau}_{\geq 0}(Y_i)$ nonempty, $\tilde{I}_i := \mathbb{P}^{-1}(I_i) \subset H_1(\partial Y_i)$ takes the form $\tilde{I}_i = \bigcap_{\boldsymbol{\delta}_i \in \mathcal{D}^{\tau}_{\geq 0}(Y_i)} \tilde{I}_i^{\boldsymbol{\delta}_i}$, where

$$\tilde{I}_i^{\boldsymbol{\delta}_i} := \left\{ \mu \in H_1(\partial Y_i) \setminus \{0\} \,\middle|\, \begin{array}{ll} \pi_i(\tilde{\boldsymbol{\delta}}_{i-}) \leq \pi_i(\mu) \leq \pi_i(\tilde{\boldsymbol{\delta}}_{i+}) & \text{if } \delta_i > 0, \\ \pi_i(\mu) \neq \infty \; (= \pi_i(\tilde{\boldsymbol{\delta}}_{i-}) = \pi_i(\tilde{\boldsymbol{\delta}}_{i+})) & \text{if } \delta_i = 0 \end{array} \right\}. \tag{123}$$

If $\mathcal{D}^{\tau}_{\geq 0}(Y_i) = \emptyset$, then, similarly to the case in which $\delta_i = 0$, $\tilde{I}_i$ is the complement of $\pi_i^{-1}(\infty)$.

Note that we always have $\infty \notin \pi_i(\tilde{I}_i)$. Thus, a necessary condition to achieve $\varphi_{\mathbb{P}}(I_1) \cup I_2 = \mathbb{P}(H_1(\partial Y_2))$ or $\varphi_{\mathbb{P}}(\dot{I}_1) \cup \dot{I}_2 = \mathbb{P}(H_1(\partial Y_2))$ is to have

$$(\infty.i) \quad \infty \in \pi_2 \circ \varphi(\tilde{I}_1), \qquad (\infty.ii) \quad \infty \in \pi_1 \circ \varphi^{-1}(\tilde{I}_2). \tag{124}$$

We claim that conditions $(\infty.i)$ and $(\infty.ii)$ are respectively equivalent to (I.$i$) and (I.$ii$). First note that it is sufficient to prove the equivalence of $(\infty.i)$ and (I.$i$), since the maps

$$\varphi : \alpha\mu_1 + \beta\lambda_1 \mapsto \alpha\mu_2 - \beta\lambda_2, \quad \varphi^{-1} : \alpha\mu_2 + \beta\lambda_2 \mapsto \alpha\mu_1 - \beta\lambda_1 \tag{125}$$

are exchanged by swapping $i = 1$ with $i = 2$. Also, when $\mathcal{D}^{\tau}_{\geq 0}(Y_1) = \emptyset$, in which case (I.$i$) holds vacuously, our hypothesis that $q^* \neq 0$, ensuring that $\pi_2\varphi\pi_1^{-1}(\infty) \neq \infty$, implies $(\infty.i)$ holds automatically. Thus, we henceforth assume that $\mathcal{D}^{\tau}_{\geq 0}(Y_1)$ is nonempty.

For any $a_1\mu_1 + b_1\lambda_1 \in H_1(\partial Y_1) \setminus \{0\}$, it is straightforward to show that the map

$$\pi_2 \circ \varphi : a_1\mu_1 + b_1\lambda_1 \mapsto \frac{-b_1}{a_1 p_2 - b_1 q_2^*} \tag{126}$$

has denominator satisfying

$$a_1 p_2 - b_1 q_2^* = \frac{p_2}{p_1}(a_1 p_1 + b_1 q_1^*) - b_1 \frac{q^*}{p_1}. \tag{127}$$

In particular, since $q^* > 0$, and since $\delta_1 = a_{1-}^{\boldsymbol{\delta}_1} p_1 + b_{1-}^{\boldsymbol{\delta}_1} q_1^* \geq 0$ and $b_{1-}^{\boldsymbol{\delta}_1} < 0$ for any $\boldsymbol{\delta}_1 \in \mathcal{D}^{\tau}_{\geq 0}(Y_1)$, we have

$$a_{1-}^{\boldsymbol{\delta}_1} p_2 - b_{1-}^{\boldsymbol{\delta}_1} q_2^* > 0, \;\; \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) > 0 \quad \text{for all } \boldsymbol{\delta}_1 \in \mathcal{D}^{\tau}_{\geq 0}(Y_1). \tag{128}$$

Now, there are two ways in which $\pi_2 \circ \varphi(\tilde{I}_1^{\boldsymbol{\delta}_1})$ could contain $\infty$. One is if $\infty$ is contained as an endpoint of $\pi_2 \circ \varphi(\tilde{I}_1^{\boldsymbol{\delta}_1})$, in which case, since $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) \neq \infty$, we must have

$\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}) = \infty$, or equivalently, $a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2 = 0$. Conveniently, the condition $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) \neq \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+})$ also implies that $\pi_2 \circ \varphi(\tilde{I}^{\boldsymbol{\delta}_1}_1)$ is closed in this case. The other possibility is that $\infty$ lies in the interior of $\pi_2 \circ \varphi(\tilde{I}^{\boldsymbol{\delta}_1}_1)$. Since $\pi_1^{-1}$, $\varphi$, and $\pi_2$ are each orientation reversing, this is equivalent to the condition that $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) \leq \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+})$, which, since $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) > 0$, implies $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}) > 0$ and hence $a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2 < 0$. In fact, the converse is also true: using the substitutions $a^{\boldsymbol{\delta}_1}_{1-} = a^{\boldsymbol{\delta}_1}_{1+} + q^*_1 g_1$ and $b^{\boldsymbol{\delta}_1}_{1-} = b^{\boldsymbol{\delta}_1}_{1+} - p_1 g_1$, and the fact that $a^{\boldsymbol{\delta}_1}_{1+}p_1 + b^{\boldsymbol{\delta}_1}_{1+}q^*_1 \geq 0$, it is straightforward to show that the inequalities $a^{\boldsymbol{\delta}_1}_{1-}p_2 - b^{\boldsymbol{\delta}_1}_{1-}q^*_2 > 0$ (from (128)) and $a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2 < 0$ imply that $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) \leq \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+})$. Thus, in summary, ($\infty$.$i$) holds if and only if $a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2 \leq 0$ for all $\boldsymbol{\delta}_1 \in \mathcal{D}^{\tau}_{\geq 0}(Y_1)$, or equivalently, if and only if

$$a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2 \leq -[b^{\boldsymbol{\delta}_1}_{1+}q^*_2]_{p_2} \quad \text{for all } \boldsymbol{\delta}_1 \in \mathcal{D}^{\tau}_{\geq 0}(Y_1), \tag{129}$$

which, after substituting $a^{\boldsymbol{\delta}_1}_{1+} = ([b^{\boldsymbol{\delta}_1}_{1+}q^*_1]_{p_1} - b^{\boldsymbol{\delta}_1}_{1+}q^*_1)/p_1$ and $q^*_1 p_2 + q^*_2 p_1 = \bar{q}_1 p_2 + \bar{q}_2 p_1 - p_1 p_2$, becomes condition (I.$i$).

Thus, conditions ($\infty$.$i$) and ($\infty$.$ii$) are respectively equivalent to conditions (I.$i$) and (I.$ii$). When one or both of $\mathcal{D}^{\tau}_{\geq 0}(Y_i)$ are empty, (I.$iii$) holds vacuously, and ($\infty$.$i$) and ($\infty$.$ii$) are jointly equivalent to the condition that $\varphi_{\mathbb{P}}(I_1) \cup I_2 = \mathbb{P}(H_1(\partial Y_2))$. We henceforth assume that each $\mathcal{D}^{\tau}_{\geq 0}(Y_i) \neq \emptyset$, and that conditions (I.$i$) and (I.$ii$), hence ($\infty$.$i$) and ($\infty$.$ii$), hold.

For each $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_1) \in \mathcal{D}^{\tau}_{\geq 0}(Y_1) \times \mathcal{D}^{\tau}_{\geq 0}(Y_2)$, the substitutions $a^{\boldsymbol{\delta}_1}_{1+} = ([b^{\boldsymbol{\delta}_1}_{1+}q^*_1]_{p_1} - b^{\boldsymbol{\delta}_1}_{1+}q^*_1)/p_1$, $a^{\boldsymbol{\delta}_2}_{2+} = ([b^{\boldsymbol{\delta}_2}_{2+}q^*_2]_{p_2} - b^{\boldsymbol{\delta}_2}_{2+}q^*_2)/p_2$, and $q^*_1 p_2 + q^*_2 p_1 = \bar{q}_1 p_2 + \bar{q}_2 p_1 - p_1 p_2$ make the condition

$$\frac{1}{b^{\boldsymbol{\delta}_1}_{1+}}\left\lfloor \frac{b^{\boldsymbol{\delta}_1}_{1+}\bar{q}_1}{p_1} \right\rfloor + \frac{1}{b^{\boldsymbol{\delta}_2}_{2+}}\left\lfloor \frac{b^{\boldsymbol{\delta}_2}_{2+}\bar{q}_2}{p_2} \right\rfloor > 1 \tag{130}$$

equivalent to the inequality

$$a^{\boldsymbol{\delta}_1}_{1+}b^{\boldsymbol{\delta}_2}_{2+} + a^{\boldsymbol{\delta}_2}_{2+}b^{\boldsymbol{\delta}_1}_{1+} < 0, \tag{131}$$

which, after we multiply by $-p_2$ and add $b^{\boldsymbol{\delta}_2}_{2+}(a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2)$ to both sides, becomes

$$-b^{\boldsymbol{\delta}_1}_{1+}(a^{\boldsymbol{\delta}_2}_{2+}p_2 + b^{\boldsymbol{\delta}_2}_{2+}q^*_2) > b^{\boldsymbol{\delta}_2}_{2+}(a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2), \tag{132}$$

which, since $-b^{\boldsymbol{\delta}_1}_{1+}(a^{\boldsymbol{\delta}_2}_{2+}p_2 + b^{\boldsymbol{\delta}_2}_{2+}q^*_2) \leq 0$, implies $a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2 \neq 0$, and hence $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}) \neq \infty$. Note that when $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}) \neq \infty$, condition ($\infty$.$i$) is equivalent to the condition

$$(0 <)\ \ \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) \leq \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}). \tag{133}$$

If $\delta_2 = 0$, then $\tilde{I}^{\boldsymbol{\delta}_2}_2$ is the complement of $\pi_2^{-1}(\infty)$, and so (133) is equivalent to the condition that $\varphi_{\mathbb{P}}(\mathring{I}^{\boldsymbol{\delta}_1}_1) \cup \mathring{I}^{\boldsymbol{\delta}_2}_2 = \mathbb{P}(H_1(\partial Y_2))$, where $\mathring{I}^{\boldsymbol{\delta}_i}_i$ denotes the interior of $\mathbb{P}(\tilde{I}^{\boldsymbol{\delta}_i}_i)$ for each $i \in \{1, 2\}$. If $\delta_2 > 0$, so that $\pi_2(\tilde{\boldsymbol{\delta}}_{2-}) < 0 < \pi_2(\tilde{\boldsymbol{\delta}}_{2+})$, then dividing (132) by

$\delta_2(a^{\boldsymbol{\delta}_1}_{1+}p_2 - b^{\boldsymbol{\delta}_1}_{1+}q^*_2)$ makes (132) equivalent to the inequality $\pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}) < \pi_2(\tilde{\boldsymbol{\delta}}_{2+})$, which, combined with (133), becomes

$$\pi_2(\tilde{\boldsymbol{\delta}}_{2-}) < 0 < \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1-}) \leq \pi_2 \circ \varphi(\tilde{\boldsymbol{\delta}}_{1+}) < \pi_2(\tilde{\boldsymbol{\delta}}_{2+}), \tag{134}$$

which again is equivalent to the condition that $\varphi_{\mathbb{P}}(\dot{I}^{\boldsymbol{\delta}_1}_1) \cup \dot{I}^{\boldsymbol{\delta}_2}_2 = \mathbb{P}(H_1(\partial Y_2))$. Thus condition (I.*iii*), which takes (130) over all $(\boldsymbol{\delta}_1, \boldsymbol{\delta}_1) \in \mathcal{D}^{\tau}_{\geq 0}(Y_1) \times \mathcal{D}^{\tau}_{\geq 0}(Y_2)$, is equivalent to the condition that $\varphi_{\mathbb{P}}(\dot{I}_1) \cup \dot{I}_2 = \mathbb{P}(H_1(\partial Y_2))$. □

### 6.8. Comparison of L-space classification with gluing hypothesis

Now that we have both classified when $Y_1 \cup_\varphi Y_2$ is an L-space, and classified when it satisfies the gluing hypothesis in terms of the union of the L-space intervals of $Y_1$ and $Y_2$, it remains to show that these two classifications are equivalent.

**Proposition 6.7.** *Suppose that $\mu_1$ is "judiciously chosen" from $\mathbb{P}^{-1}(\dot{I}_1 \cap \varphi^{-1}_{\mathbb{P}}(\dot{I}_2))$ nonempty, and that $Y$ is constructed as above. For each $i \in \{1,2\}$, set $\bar{q}_i := [q^*_i]_{p_i}$ and let $B_i$ denote the set $B_i := \Big\{ [p_i\gamma_i - q_i\delta_i]_{p_i g_i} \big| \, \boldsymbol{\delta}_i = \delta_i\iota_i(m_i) + \gamma_i\iota_i(l_i) \in \mathcal{D}^{\tau}_{\geq 0}(Y_i) \Big\}$. Then condition (I.$i$) (respectively (I.$ii$)) from Proposition 6.6 holds if and only if condition (L.$i$) (respectively (L.$ii$)) from Proposition 6.5 holds for all $b_1 \in B_1$ (respectively $b_2 \in B_2$).*

**Proof.** If $B_1 = \emptyset$, then conditions (I.$i$) and (L.$i$) hold vacuously, hence are equivalent. We therefore assume $B_1$ is nonempty and fix some $b_1 \in B_1$. Clearly (L.$i$) implies the statement of (I.$i$) for that particular $b_1$, since $b_1 \in \{b \in \mathbb{Z} \,|\, b \equiv b_1 \,(\mathrm{mod}\, p_1), 0 < b < p_1g_1p_2g_2/g_0\}$.

Conversely, suppose (I.$i$) holds for that $b_1$. Substituting $q^* = \bar{q}_1p_2 + \bar{q}_2p_1 - p_1p_2$ gives

$$\frac{b_1q^*}{p_1p_2} \geq \frac{[b_1\bar{q}_1]_{p_1}}{p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{p_2}. \tag{135}$$

Thus, for any $b := b_1 + yp_1g_1$ with $y \in \{0, \dots, p_2g_2/g_0 - 1\}$, we have

$$\begin{aligned} \frac{bq^*}{p_1p_2} &\geq \left( \frac{[b_1\bar{q}_1]_{p_1}}{p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{p_2} \right) + \frac{yp_1g_1(p_1\bar{q}_2 - (p_1 - \bar{q}_1)p_2)}{p_1p_2} \\ &\geq \frac{[b\bar{q}_1]_{p_1}}{p_1} + \left( \frac{[b_1\bar{q}_2]_{p_2}}{p_2} + \frac{yg_1[p_1\bar{q}_2]_{p_2}}{p_2} \right) \\ &\geq \frac{[b\bar{q}_1]_{p_1}}{p_1} + \frac{[b\bar{q}_2]_{p_2}}{p_2}, \end{aligned} \tag{136}$$

which is equivalent to the inequality in condition (L.$i$). An analogous argument proves the equivalence of conditions (L.$ii$) and (I.$ii$) for any $b_2 \in B_2$. □

**Proposition 6.8.** *Suppose that $\mu_1$ is "judiciously chosen" from $\mathbb{P}^{-1}(\dot{I}_1 \cap \varphi_{\mathbb{P}}^{-1}(\dot{I}_2))$ nonempty, and that $Y$ is constructed as above. For each $i \in \{1,2\}$, set $\bar{q}_i := [q_i^*]_{p_i}$ and let $B_i$ denote the set $B_i := \left\{ [p_i\gamma_i - q_i\delta_i]_{p_i g_i} \middle| \boldsymbol{\delta}_i = \delta_i \iota_i(m_i) + \gamma_i \iota_i(l_i) \in \mathcal{D}^{\tau}_{\geq 0}(Y_i) \right\}$. Suppose conditions* (I.$i$) *and* (I.$ii$) *from Proposition 6.6 hold. Then condition* (I.$iii$) *from Proposition 6.6 holds if and only if condition* (L.$iii$) *from Proposition 6.5 holds for all $(b_1, b_2) \in B_1 \times B_2$ with $b_1 \equiv b_2 \pmod{g_0}$.*

**Proof.** We henceforth assume that $\mathcal{D}^{\tau}_{\geq 0}(Y_1)$ and $\mathcal{D}^{\tau}_{\geq 0}(Y_2)$ are nonempty, since otherwise conditions (I.$iii$) and (L.$iii$) hold vacuously in all cases.

If condition (I.$iii$) holds, then it holds for any $(b_1, b_2) \in B_1 \times B_2$ with $b_1 \equiv b_2 \pmod{g_0}$. In this case, the unique $b \in \{0, \dots, p_1 p_2 g - 1\}$ satisfying $b \equiv b_1 \pmod{p_1 g_1}$ and $b \equiv b_2 \pmod{p_2 g_2}$ also satisfies $[b\bar{q}_1]_{p_1} = [b_1\bar{q}_1]_{p_1}$ and $[b\bar{q}_2]_{p_2} = [b_2\bar{q}_2]_{p_2}$, so that we have

$$\frac{1}{b}\left\lfloor \frac{b\bar{q}_1}{p_1} \right\rfloor + \frac{1}{b}\left\lfloor \frac{b\bar{q}_2}{p_2} \right\rfloor = \frac{1}{b_1}\left\lfloor \frac{b_1\bar{q}_1}{p_1} \right\rfloor + \frac{1}{b_2}\left\lfloor \frac{b_2\bar{q}_2}{p_2} \right\rfloor + \left(\frac{1}{b_1} - \frac{1}{b}\right)\frac{[b_1\bar{q}_1]_{p_1}}{p_1} + \left(\frac{1}{b_2} - \frac{1}{b}\right)\frac{[b_2\bar{q}_2]_{p_2}}{p_2}$$
$$> 1. \tag{137}$$

Thus (I.$iii$) implies (L.$iii$) for all $(b_1, b_2) \in B_1 \times B_2$ with $b_1 \equiv b_2 \pmod{g_0}$, and it remains to prove the converse.

**Claim.** *Suppose that conditions* (I.$i$) *and* (I.$ii$) *hold, and that there exists some $(b_1, b_2) \in B_1 \times B_2$ for which the statement of* (I.$iii$) *fails, or equivalently (using the substitution $q^* = \bar{q}_1 p_2 + \bar{q}_2 p_1 - p_1 p_2$), for which*

$$\frac{q^*}{p_1 p_2} \leq \frac{[b_1\bar{q}_1]_{p_1}}{b_1 p_1} + \frac{[b_2\bar{q}_2]_{p_2}}{b_2 p_2}. \tag{138}$$

*Then we have the inequalities*

$$(i) \quad \frac{[b_1\bar{q}_1]_{p_1}}{b_1} \geq \frac{[b_2\bar{q}_1]_{p_1}}{b_2}, \qquad (ii) \quad \frac{[b_2\bar{q}_2]_{p_2}}{b_2} \geq \frac{[b_1\bar{q}_2]_{p_2}}{b_1}, \tag{139}$$

*and conditions* (I.$i$) *and* (I.$ii$) *for this particular $(b_1, b_2) \in B_1 \times B_2$ become the equalities*

$$(i) \quad \frac{q^*}{p_1 p_2} = \frac{[b_1\bar{q}_1]_{p_1}}{b_1 p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{b_1 p_2}, \qquad (ii) \quad \frac{q^*}{p_1 p_2} = \frac{[b_2\bar{q}_1]_{p_1}}{b_2 p_1} + \frac{[b_2\bar{q}_2]_{p_2}}{b_2 p_2}. \tag{140}$$

**Proof of Claim.** Using the substitution $q^* = \bar{q}_1 p_2 + \bar{q}_2 p_1 - p_1 p_2$, we can re-express conditions (I.$i$) and (I.$ii$) as

$$(i) \quad \frac{q^*}{p_1 p_2} \geq \frac{[b_1\bar{q}_1]_{p_1}}{b_1 p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{b_1 p_2}, \qquad (ii) \quad \frac{q^*}{p_1 p_2} \geq \frac{[b_2\bar{q}_1]_{p_1}}{b_2 p_1} + \frac{[b_2\bar{q}_2]_{p_2}}{b_2 p_2}. \tag{141}$$

Concatenating (138) with (141.$i$) (respectively, (141.$ii$)) then yields inequality (139.$ii$) (respectively, (139.$i$)). Setting $\delta_1 := [b_1\bar{q}_1]_{p_1} \in \mathcal{D}^{\tau}_{\geq 0}(Y_1)$ and $\delta_2 := [b_2\bar{q}_2]_{p_2} \in \mathcal{D}^{\tau}_{\geq 0}(Y_2)$, we note that (139.$i$) implies

$$\frac{\delta_2}{b_2 p_2} < \frac{1}{b_1}, \tag{142}$$

since otherwise, applying (139.$i$) and $1/b_1 \leq \delta_2/(b_2p_2)$ in succession would yield

$$\begin{aligned}\frac{[b_2\bar{q}_1]_{p_1}}{b_2p_1} &\leq \frac{\delta_1}{p_1}\cdot\frac{1}{b_1} \leq \frac{\delta_1}{p_1}\cdot\frac{\delta_2}{b_2p_2} = \frac{\delta_1\delta_2/p_2}{b_2p_1} \\ &< \frac{(1+\deg_{t_1}\tau^{\mathrm{c}}(Y_1))(1+\deg_{t_2}\tau^{\mathrm{c}}(Y_2)/p_2}{b_2p_1} < \frac{1}{b_2p_1},\end{aligned}$$

making $[b_2\bar{q}_1]_{p_1} < 1$, a contradiction. Thus (142) must hold.

Applying (141.$i$), (138), and (142) in succession, we obtain

$$\begin{aligned}\frac{[b_1\bar{q}_1]_{p_1}}{b_1p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{b_1p_2} &\leq \frac{q^*}{p_1p_2} \qquad (143)\\ &\leq \frac{[b_1\bar{q}_1]_{p_1}}{b_1p_1} + \frac{[b_2\bar{q}_2]_{p_2}}{b_2p_2} \\ &< \frac{[b_1\bar{q}_1]_{p_1}}{b_1p_1} + \frac{1}{b_1}. \qquad (144)\end{aligned}$$

Subtracting $\frac{[b_1\bar{q}_1]_{p_1}}{b_1p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{b_1p_2}$ from lines (143) and (144) then yields

$$0 \leq \frac{q^*}{p_1p_2} - \left(\frac{[b_1\bar{q}_1]_{p_1}}{b_1p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{b_1p_2}\right) < \frac{1}{b_1} - \frac{[b_1\bar{q}_2]_{p_2}}{b_1p_2}, \tag{145}$$

but we also know that

$$\frac{q^*}{p_1p_2} - \left(\frac{[b_1\bar{q}_1]_{p_1}}{b_1p_1} + \frac{[b_1\bar{q}_2]_{p_2}}{b_1p_2}\right) \in \frac{1}{b_1}\mathbb{Z}. \tag{146}$$

Thus, (140.$i$) must hold, and (140.$ii$) follows from symmetry, proving our Claim. $\square$

Having proven our Claim, we pause to introduce the notation $b_i \mapsto \boldsymbol{\delta}_i^{b_i}$ for the bijection

$$\begin{aligned}\{0,\ldots,p_ig_i-1\} &\to \{0,\ldots,p_i-1\}\iota_i(m_i)+T_i^{\partial}, \qquad (147)\\ b_i &\mapsto \boldsymbol{\delta}_i^{b_i} := \iota_i(-\left\lfloor\frac{b_iq_i^*}{p_i}\right\rfloor\mu_i + b_i\lambda_i) \in [b_i\bar{q}_i]_{p_i}\iota_i(m_i)+T_i^{\partial},\end{aligned}$$

whose inverse we used to define each $B_i$ as a set of integers indexing the elements of $\mathcal{D}^{\tau}_{\geq 0}(Y_i)$.

We now proceed with an inductive argument. Suppose that (L.$iii$) holds for all $(b_1, b_2) \in B_1 \times B_2$ satisfying $b_1 \equiv b_2 \pmod{g_0}$, and that (I.$i$) and (I.$ii$) hold, but that there exist $b_i \in B_i$ and $b_I \in B_I$, with $\{i, I\} = \{1, 2\}$ and $b_i \leq b_I$, for which (I.*iii*) fails, *i.e.*, for which

$$\frac{q^*}{p_1 p_2} \leq \frac{[b_i \bar{q}_i]_{p_i}}{b_i p_i} + \frac{[b_I \bar{q}_I]_{p_I}}{b_I p_I}. \tag{148}$$

Equation (140) from our Claim then tells us that

$$\frac{1}{b_i}\left\lfloor \frac{b_i \bar{q}_i}{p_i} \right\rfloor + \frac{1}{b_i}\left\lfloor \frac{b_i \bar{q}_I}{p_I} \right\rfloor = 1. \tag{149}$$

This means that $b_i \notin B_I$, since otherwise, setting $b := b_i \in B_i \cap B_I = B_1 \cap B_2$ would make (149) contradict condition (L.*iii*). Thus, $\boldsymbol{\delta}_I^{b_i} \notin \mathcal{D}^{\tau}_{\geq 0}(Y_I)$ and $b_i < b_I$.

We next apply (139) from our Claim, to obtain

$$[b_i \bar{q}_I]_{p_I} \leq \frac{b_i}{b_I}[b_I \bar{q}_I]_{p_I} < [b_I \bar{q}_I]_{p_I}. \tag{150}$$

Since $\boldsymbol{\delta}_I^{b_I} - \boldsymbol{\delta}_I^{b_i} \in ([b_I \bar{q}_I]_{p_I} - [b_i \bar{q}_I]_{p_I})\, \iota_I(m_I) + T_I^{\partial}$, the above inequality implies $\boldsymbol{\delta}_I^{b_I} - \boldsymbol{\delta}_I^{b_i} \in \iota_I(m_I \mathbb{Z}_{\geq 0} + l_I \mathbb{Z})$. Thus, since $\boldsymbol{\delta}_I^{b_i} \notin \mathcal{D}^{\tau}_{\geq 0}(Y_I)$ and $\boldsymbol{\delta}_I^{b_I} \in \mathcal{D}^{\tau}_{\geq 0}(Y_I)$, the additive closure of $\iota_I(m_I \mathbb{Z}_{\geq 0} + l_I \mathbb{Z}) \setminus \mathcal{D}^{\tau}_{\geq 0}(Y_I)$ from Proposition 4.1 tells us that $\boldsymbol{\delta}_I^{b_I} - \boldsymbol{\delta}_I^{b_i} \in \mathcal{D}^{\tau}_{\geq 0}(Y_I)$. Since (150) implies $([b_I \bar{q}_I]_{p_I} - [b_i \bar{q}_I]_{p_I}) = [(b_I - b_i)\bar{q}_I]_{p_I}$, we actually have $\boldsymbol{\delta}_I^{b_I} - \boldsymbol{\delta}_I^{b_i} = \boldsymbol{\delta}_I^{b_I - b_i} \in \mathcal{D}^{\tau}_{\geq 0}(Y_I)$, implying $b_I - b_i \in B_i$. We furthermore have

$$\frac{[b_i \bar{q}_I]_{p_I}}{b_i} \leq \frac{[b_I \bar{q}_I]_{p_I}}{b_I} \quad \Longrightarrow \quad \frac{[b_I \bar{q}_I]_{p_I}}{b_I} \leq \frac{[(b_I - b_i)\bar{q}_I]_{p_I}}{b_I - b_i}, \tag{151}$$

so that (148) implies

$$\frac{q^*}{p_1 p_2} \leq \frac{[b_i \bar{q}_i]_{p_i}}{b_i p_i} + \frac{[(b_I - b_i)\bar{q}_I]_{p_I}}{(b_I - b_i) p_I}, \tag{152}$$

with $b_i \in B_i$ and $b_I - b_i \in B_I$, mimicking our initial conditions.

We then iterate the process, at each iteration redefining $i, I \in \{1, 2\}$, $b_i$, and $b_I$ so that

$$b_i^{\text{new}} := \min\{b_i^{\text{old}}, b_I^{\text{old}} - b_i^{\text{old}}\}, \qquad b_I^{\text{new}} := \max\{b_i^{\text{old}}, b_I^{\text{old}} - b_i^{\text{old}}\}. \tag{153}$$

Like any Euclidean Algorithm, this strictly decreasing sequence bounded by zero must terminate at zero, with its last two nonzero entries equal to

$$b_i^{\text{final}} = b_I^{\text{final}} = \gcd(b_i^{\text{original}}, b_I^{\text{original}}). \tag{154}$$

Setting $b := b_i^{\text{final}} = b_I^{\text{final}} \in B_1 \cap B_2$ then makes (149) contradict condition (L.*iii*).

This completes the proof of the proposition, thereby completing the proof of Theorem 6.2 □

## 7. Generalized solid tori and NLS detection

In this section, we study manifolds with $\mathcal{D}^{\tau}_{>0} = \emptyset$. Unless otherwise specified, we assume that $Y$ is a rational homology $S^1 \times D^2$ with $H_1(Y) = \mathbb{Z} \oplus T$, and that $\phi : H_1(Y) \to H_1(Y)/T \simeq \mathbb{Z}$ is the projection. We define $g_Y > 0$ by the relation im $\phi = g_Y\mathbb{Z} \subset \mathbb{Z}$. The number $g_Y$ is the minimal intersection number of a curve on $\partial Y$ with a surface generating $H_2(Y, \partial Y)$. Equivalently, it is the minimal number of boundary components of such a surface, or the order of the homological longitude $l$ in $H_1(Y)$. Finally, we define $k_Y$ to be the order of the group $T/(T \cap \text{im } \iota)$, so that $|T| = k_Y g_Y$.

### *7.1. Generalized solid tori*

The Seifert fibered spaces $N_g = M(\emptyset; 1/g, -1/g)$ provide a motivating example of a class of manifolds with $\mathcal{L}(N_g) = Sl(N_g) \setminus [l]$. They were studied in [7] (for $g = 2$) and subsequently by Watson [19] for arbitrary values of $g$. We briefly describe them here. First, we have

$$H_1(N_g) = \langle f, h_1, h_2 \mid f + gh_1 = f - gh_2 = 0\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}/g.$$

The $\mathbb{Z}$ summand is generated by $h_1$, and the $\mathbb{Z}/g$ summand is generated by $\sigma = h_1 + h_2$. $H_1(\partial N_g) = \langle f, \sigma\rangle$, so $\iota(H_1(\partial N_g)) = g\mathbb{Z} \oplus \mathbb{Z}/g \subset H_1(N_g)$. The Turaev torsion is

$$\tau(N_g) \sim \frac{1 - [f]}{(1 - [h_1])(1 - [h_2])} = \frac{1 - t^g}{(1 - t)(1 - t\sigma)}$$

so the Milnor torsion is

$$\overline{\tau}(N_g) = \tau(N_g)|_{\sigma=1} = \frac{1 - t^g}{(1 - t)^2} = 1 + 2t + 3t^2 + \ldots + (g - 1)t^{g-1} + gt^g + gt^{g+1} + \ldots$$

It is easy to see that if $x \notin S[\tau(N_g)]$, $y \in S[\tau(N_g)]$ with $\phi(x) > \phi(y)$, then $\phi(x - y) < g$. If $x - y \in \text{im } \iota$, we must have $\phi(x - y) = 0$, so $\mathcal{D}^{\tau}_{>0}(N_g) = \emptyset$. More generally, the same argument shows that

**Proposition 7.1.** *If $Y$ is Floer simple and* $\deg \Delta(Y) < g_Y$*, then* $\mathcal{D}^{\tau}_{>0}(Y) = \emptyset$.

Motivated by this, we make the following

**Definition 7.2.** A *generalized solid torus* is a Floer simple manifold $Y$ with $\deg \Delta(Y) < g_Y$.

If $Y$ is such a manifold, Corollary 2.3 implies that $\|Y\| \leq g_Y - 2$. On the other hand, an embedded surface which generates $H_2(Y, \partial Y)$ has at least $g_Y$ boundary components, so a norm-minimizing surface must have genus 0.

The Milnor torsion of a generalized solid torus is determined by $g_Y$ and $k_Y$.

**Lemma 7.3.** *Suppose that $Y$ is a rational homology $S^1 \times D^2$. If $p : \mathbb{Z}[t] \to \mathbb{Z}[t]/(t^{g_Y} - 1)$ is the projection, then $p(\Delta(Y)) = k_Y(1 - t^{g_Y})/(1 - t)$.*

**Proof.** The usual product formula for the torsion implies that

$$\tau(Y(l)) = j_{1*}(\tau(S^1 \times D^2)j_{2*}(\tau(Y))$$

where $j_1 : S^1 \times D^2 \to Y(l)$ and $j_2 : Y \to Y(l)$ are the inclusions. It follows that

$$\overline{\tau}(Y(l)) = \frac{\overline{\tau}(Y)}{1 - t^{g_Y}}.$$

By [47], Lemma 3.2, we have

$$\overline{\tau}(Y(l)) = \frac{t^c|H_1(Y(l)|}{(1-t)^2} + P(t)$$

where $c \in \mathbb{Z}$ and $p(t) \in \mathbb{Z}[t^{\pm 1}]$. $|H_1(Y(l))| = k_Y$. Combining the two formulas, we see that

$$\Delta(Y) = \frac{k_Y t^c(1 - t^{g_Y})}{1 - t} + (1 - t^{g_Y})(1 - t)P(t). \quad \square$$

Combining the lemma with the requirement that $\deg \Delta(Y) < g_Y$ gives

**Corollary 7.4.** *If $Y$ is a generalized solid torus, $\Delta(Y) \sim k_Y(1 - t^{g_Y})/(1 - t)$.*

In contrast, $\tau(Y)$ is not determined by the fact that $Y$ is a generalized solid torus, as can be seen by considering the Seifert-fibered spaces $M(\emptyset; a/g, -a/g)$.

**Proposition 7.5.** *A generalized solid torus is a Floer homology solid torus in the sense of Watson [19].*

**Proof.** Let $g = g_Y$. Recall that $Y$ is a Floer homology solid torus if $\widehat{CFD}(Y, m, l) \simeq \widehat{CFD}(Y, m + l, l)$, where $l$ is the rational longitude and $m \cdot l = 1$. By composing with an appropriate change of basis bimodule, we see that this is equivalent to saying that for some $\mu, \lambda$ with $\mu \cdot \lambda = 1$, we have $\widehat{CFD}(Y, \mu, \lambda) \simeq \widehat{CFD}(Y, \tau_l(\mu), \tau_l(\lambda))$, where $\tau_l$ is the Dehn twist along $l$.

Suppose that $Y$ is a generalized solid torus. By Proposition 3.10, we can explicitly compute $\widehat{CFD}(Y, \mu, \lambda)$ for an appropriate choice of $\mu$ and $\lambda$. In fact, $\widehat{CFD}(\mu, \lambda)$ is determined by the polynomials $\chi(\widehat{HFK}(K_\mu))$ and $\chi(\widehat{HFK}(K_\lambda))$, which are in turn determined by $\Delta(Y)$, $\iota(\mu)$, and $\iota(\lambda)$. Since $\|Y\| = g - 2$, the criteria of Proposition 3.10 will be satisfied if we take $\mu = m$ and $\lambda = l - Nm$, where $N \gg 0$.

Let $S_\mu \subset H_1(Y)$ be the support of $\widehat{HFK}(K_\mu)$, normalized so that if $x \in S_\mu$, then $0 \leq \phi(x) \leq 2g - 2$. $S_\mu$ is determined by the conditions that for $0 \leq \phi(x) \leq g - 1$,

$x \in S_\mu$ if and only if $x \in S[\tau(Y)]$, and for $g-1 \le \phi(\mu) \le 2g-2$, $x \in S[\mu]$ if and only if $x - \mu \notin S[\tau(Y)]$.

Similarly, let $S_\lambda \subset H_1(Y)$ be the support of $\widehat{HFK}(K_\lambda)$, normalized so that if $x \in S_\lambda$, then $0 \le \phi(x) \le (N+1)g-2$. $S_\lambda$ is determined by the conditions that for $0 \le \phi(x) \le g-1$, $x \in S_\lambda$ if and only if $x \in S[\tau(Y)]$, and for $g-1 \le \phi(x) \le (N+1)g-2$, $x \in S_\lambda$ if and only if $x + \lambda \notin S[\tau(Y)]$. (Note that $\phi(\lambda) < 0$, so we need $x + \lambda$ here rather than $x - \lambda$).

Now let $\mu' = \tau_l(m) = \mu + l$ and $\lambda' = \tau_l(\lambda) = \lambda - Nl$. The supports $S_{\mu'}$ and $S_{\lambda'}$ can be described similarly.

We define an isomorphism $f : \widehat{CFD}(Y, \mu, \lambda) \to \widehat{CFD}(Y, \mu', \lambda')$. The map $f : \widehat{HFK}(K_\mu) \to \widehat{HFK}(K_{\mu'})$ is given as follows. If $x \in S_\mu$, then $f$ takes the unique nonzero element of $\widehat{HFK}(K_\mu)$ supported at $x$ to the unique nonzero element of $\widehat{HFK}(K_{\mu'})$ supported at $x + \lfloor \phi(x)/g \rfloor l$. Using the description of the sets $S_\mu$ and $S'_\mu$ given above, together with the fact that $\phi(\mu) = g$, it is easy to see that $f$ is a bijection. Similarly, if $x \in S_\lambda$, we define $f$ to take the unique nonzero element supported at $x$ to the unique nonzero element of $\widehat{HFK}(K_{\lambda'})$ supported at $x + \lfloor \phi(x)/g \rfloor l$.

It remains to check that $f$ carries the arrows in the diagram for $C = \widehat{CFD}(Y, \mu, \lambda)$ to the arrows in the diagram for $C' = \widehat{CFD}(Y, \mu', \lambda')$. Suppose $x$ and $y$ are the initial and terminal ends of an arrow of type $D_{23}$ in $C$, so that $y - x = \mu$. Then $\phi(y) - \phi(x) = g$, so $f(y) - f(x) = \mu + l = \mu'$, so $f(y)$ and $f(x)$ are the endpoints of an arrow of type $D_{23}$ in $C'$. A very similar argument shows that arrows of types $D_1$ and $D_3$ are preserved as well. $\square$

We can prove a partial converse to Proposition 7.1. Recall that $Y$ is said to be semi-primitive if $T \subset \text{im}\ \iota$. Equivalently, $Y$ is semi-primitive if $k_Y = 1$.

**Proposition 7.6.** *Suppose that $Y$ is semi-primitive and Floer simple. If $\mathcal{D}^\tau_{>0}(Y) = \emptyset$, then $Y$ is a generalized solid torus.*

**Proof.** Let $g = g_Y$. Since $Y$ is semiprimitive, we have $H_1(Y) = \mathbb{Z} \oplus (\mathbb{Z}/g)$ and also $\text{im}\ \iota = g\mathbb{Z} \oplus \mathbb{Z}/g \subset H_1(Y)$. Let $t, \sigma$ be generators of the $\mathbb{Z}$ and $\mathbb{Z}/g$ summands respectively, so that $\tau(Y) = \sum_{i=0}^\infty q_i(\sigma) t^i$, where $q_i(\sigma)$ is a sum of powers of $\sigma$. Suppose that for some value of $i$, $q_i(1) < g$ and $q_{i-g}(1) > 0$. Then we can find $x \notin S[\tau(Y)]$ with $\phi(x) = i$ and $y \in S[\tau(Y)]$ with $\phi(y) = i - g$. It follows that $x - y \in \text{im}\ \iota$, which contradicts $\mathcal{D}^\tau_{>0}(Y) = 0$. We conclude that for a fixed value of $k$ there is at most one value of $n$ for which $q_{k+ng}(1) \neq 0, g$.

The Milnor torsion of $Y$ is $\overline{\tau}(Y) = \Delta(Y)/(1-t) = \displaystyle\sum_{i=0}^{\infty} a_i t^i$, where $a_i = q_i(1)$.

**Lemma 7.7.** *There is a constant $c$ so that for all $k \in \mathbb{Z}/g$, $\displaystyle\sum_{i \equiv k\ (g)} a_i \equiv k + c\ (g)$.*

Note that all but finitely many of the $a_i$ are equal to either 0 or $g$, so the sum is a well-defined element of $\mathbb{Z}/g$.

**Proof.** We say that $f(t) \in \mathbb{Z}[t]$ has property (*) if the statement of the corollary holds for $a_i$ given by $f(t)/(1-t) = \sum_{i=0}^{\infty} a_i t^i$. It is easy to see that $f(t) = 1 + t + \ldots + t^{g-1}$ has property (*), and that if $f(t)$ has property (*), then so do $f(t) + t^i - t^{g+i}$ and $t^c f(t)$. Lemma 7.3 implies that $\Delta(Y)$ can be obtained from $1 + t + \ldots + t^{g-1}$ by a sequence of operations of the first type plus a single operation of the second type, so $\Delta(Y)$ has property (*). □

The lemma implies that after an appropriate shift in the indexing of the $a_i$'s (so that $\overline{\tau}(Y)$ is no longer constrained to have $t^0$ as its lowest order term) the subsequence $(a_{k+ng})$ has the form $\ldots, 0, 0, 0, k, g, g, g \ldots$, where $0 \leq k \leq g$. In other words, each subsequence is determined up to a global shift, and it remains to see how these shifts fit together.

We claim that the sequence $(a_i)$ has the form $\ldots 0, 0, 0, 1, 2, \ldots, g-1, g, g, g \ldots$. Equivalently,

$$\overline{\tau}(Y) \sim \overline{\tau}_0 = t + 2t^2 + \ldots + (g-1)t^{g-1} + gt^g + gt^{g+1} + \ldots = \frac{t(1-t^g)}{(1-t)^2}$$

To see this, let us say that $Q(t) \in \mathbb{Z}[t^{-1}, t]]$ is obtained from $P(t)$ by an elementary shift if $Q(t) - P(t) = at^i + (g-a)t^{i+g}$ for some $a, i \in \mathbb{Z}$. We have shown above that $\overline{\tau}(Y)$ is obtained from $\overline{\tau}_0$ by a sequence of elementary shifts.

Next, we consider the effect of an elementary shift on the Alexander polynomial. If $Q(t) \in \mathbb{Z}[t^{-1}, t]]$, let $F(Q(t)) = p((1-t)Q(t))$, where $p : \mathbb{Z}[t] \to \mathbb{Z}[t]/(t^g - 1)$ is the projection, so that $F(\overline{\tau}_0) = 1 + \ldots + t^{g-1}$. An easy calculation shows that if $Q(t) - P(t) = at^i + (g-a)t^{i+g}$, then $F(Q(t)) - F(P(t)) = gt^i - gt^{i+1}$. It follows that if $Q(t)$ is obtained from $\overline{\tau}_0$ by a sequence of elementary shifts and $F(Q(t)) = F(\overline{\tau}_0)$, then $Q(t)$ is obtained from $\overline{\tau}_0$ by a global shift; that is, each residue class is shifted by the same number of elementary shifts. To sum up, we have proved that $\overline{\tau}(Y) \sim \overline{\tau}_0$, so $Y$ is a generalized solid torus. □

As we observed above, if $Y$ is a generalized solid torus, $H_2(Y, \partial Y)$ is generated by a surface of genus 0. It follows that $Y(l) = Z\#(S^1 \times S^2)$, where $Z$ is a rational homology sphere. Conversely, we have

**Proposition 7.8.** *Suppose that $K \subset Z\#(S^1 \times S^2)$ has an L-space surgery. Then the complement of $K$ is a generalized solid torus.*

**Proof.** We use the exact triangle with twisted coefficients, as formulated by Ai and Peters in [1]. We briefly recall their statement. Given a class $\eta \in H_1(Y)$ and $\mu \in Sl(Y)$, we can form $\omega_\mu = PD(j_*(\eta)) \in H^2(Y(\mu))$, where $j : Y \to Y(\mu)$ is the inclusion. The twisted Floer homology $\widehat{HF}(Y(\mu); \Lambda_{\omega_\mu})$ is a module over the universal Novikov ring

$$\Lambda = \left\{ \sum a_r t^r \mid r \in \mathbb{R}, a_r \in \mathbb{Z}, \#\{r < C \mid a_r \neq 0\} < \infty \text{ for all } C \in \mathbb{R} \right\}.$$

If the image of $\omega_\mu$ in $H^2(Y(\mu),\mathbb{R})$ is 0, then $\widehat{HF}(Y(\mu);\Lambda_{\omega_\mu}) = \widehat{HF}(Y(\mu)) \otimes \Lambda$. Ai and Peters show that if $\mu \cdot \lambda = 1$, there is a long exact sequence

$$\to \widehat{HF}(Y(\mu);\Lambda_{\omega_\mu}) \to \widehat{HF}(Y(\lambda);\Lambda_{\omega_\lambda}) \to \widehat{HF}(Y(\mu+\lambda);\Lambda_{\omega_{\mu+\lambda}}) \\ \to \widehat{HF}(Y(\mu);\Lambda_{\omega_\mu}) \to .$$

Let $Y$ be the complement of $K$, so $Y(l) = Z\#(S^1 \times S^2)$. Choose $\eta \in H_1(Y)$ with $\phi(\eta) = 1$, so that $\omega_l$ generates $H_2(Y(l)) = \mathbb{Z}$. By [1] Proposition 2.2, $\widehat{HF}(Y(l);\Lambda_{\omega_\lambda}) = 0$.

Now suppose there is some $m$ with $m \cdot l = 1$ and $m \in \mathcal{L}(Y)$. In this case $H^2(Y(m);\mathbb{R}) \simeq H^2(Y(m+l);\mathbb{R}) = 0$. The exact triangle shows that $\widehat{HF}(Y(m)) \otimes \Lambda \simeq \widehat{HF}(Y(m+l)) \otimes \Lambda$, which implies that $\widehat{HF}(Y(m)) \simeq \widehat{HF}(Y(m+l))$. Since $H_1(Y(m)) \simeq H_1(Y(m+l))$, it follows that $m + l \in \mathcal{L}(Y)$. Repeating, we find that $m + nl \in \mathcal{L}(Y)$ for all $n > 0$, and thus that $l$ is a limit point of $\mathcal{L}(Y)$. It follows that $Y$ is Floer simple and $\mathcal{D}^\tau_{>0}(Y) = \emptyset$.

For the general case, suppose that $\mu \in \mathcal{L}(Y)$. Then $Y(l)$ is obtained by integer surgery on $K_\mu \# K_{-q/p} \subset Y(\mu)\#L(q,-p)$ for an appropriate choice of $p$ and $q$. Let $Y'$ be the complement of this knot. The argument above shows that every non-longitudinal filling of $Y'$ is an L-space. An infinite family of these fillings are also obtained by Dehn filling on $Y$, so $Y$ is Floer simple.

To conclude the argument, we compute $\overline{\tau}(Y)$. Let $j_1 : Y \to Y(l)$ and $j_2 : S^1 \times D^2 \to Y(l)$ be the inclusions. The usual product formula for the torsion says that

$$\overline{\tau}(Y(l)) \sim j_{1*}(\overline{\tau}(Y))j_{2*}(\overline{\tau}(S^1 \times D^2)).$$

Here

$$\overline{\tau}(Y(l)) = \overline{\tau}(Z\#(S^1 \times S^2)) \sim \frac{|H_1(Z)|}{(1-t)^2}.$$

It is easy to see that the map $j_{1*} : H_1(Y)/Tors \to H_1(Y(l))/Tors$ is an isomorphism, while the map $j_{2*} : H_1(S^1 \times D^2) \to H^1(Y(l))/Tors$ is multiplication by $g$, so

$$\frac{|H_1(Z)|}{(1-t)^2} \sim \frac{\overline{\tau}(Y)}{1-t^g}.$$

Equivalently

$$\overline{\tau}(Y) \sim |H_1(Z)| \frac{1-t^g}{(1-t)^2}.$$

It follows that $Y$ is a generalized solid torus. □

Proposition 1.9 from the introduction is an immediate consequence of Propositions 7.6 and 7.8, and Proposition 1.11 follows from Proposition 7.5.

*7.2. NLS detection*

Next, we study the notion of NLS detection introduced by Boyer and Clay in [6]. Suppose that $Y_1$ is a rational homology solid torus and that $Y_2$ is a semi-primitive generalized solid torus. Given a primitive class $\alpha \in H_1(Y_1)$, choose an orientation reversing homeomorphism $\varphi : \partial Y_1 \to \partial Y_2$ with $\varphi_*(\alpha) = l$, where $l \in H_1(\partial Y_2)$ is the homological longitude. Since $Y_2$ is a Floer homology solid torus, $\widehat{HF}(Y_\varphi)$ is well defined, in the sense that any $\phi$ satisfying $\varphi_*(\alpha) = l$ will give the same result. We say that $\alpha$ is NLS detected by $Y_2$ if $Y_\varphi$ is not an L-space.

If $Y_1$ is Floer simple, it follows from Theorem 1.8 that $\alpha$ is NLS detected by $Y_2$ if and only if $\alpha$ is not in the interior of $\mathcal{L}(Y)$. In fact, there is a direct proof of this fact for any $Y_1$.

**Proposition 7.9.** *The slope $\alpha$ is NLS detected by $Y_2$ if and only if $\alpha$ is not in the interior of $\mathcal{L}(Y_1)$.*

**Proof.** Suppose that $\alpha$ is not NLS detected by $Y_2$. Then $Y_{\varphi_i}$ is an L-space for every $\varphi_i$ with $\varphi_{i*}(\alpha) = l$. The manifolds $Y_{\varphi_i}$ are all obtained by Dehn filling a manifold $Y'$ which is constructed by identifying $\nu(\alpha) \subset \partial Y_1$ with $\nu(l) \subset \partial Y_2$, as in the proof of Lemma 2.7. It follows that $Y'$ is Floer simple.

Let $\mu \in H_1(\partial Y')$ be the class which represents the common image of $\alpha \in H_1(\partial Y_1)$ and $l \in H_1(\partial Y_2)$. The sutured manifold $(Y', \gamma_\mu)$ contains an essential annulus $A$ which separates $Y_1$ from $Y_2$. The boundary of $A$ is a pair of curves parallel to $\mu$. We choose the position of the sutures so that one component of $\partial A$ lies in $R_+(\gamma_\mu)$ and the other component is in $R_-(\gamma_\mu)$. Decomposing $(Y', \gamma_\mu)$ along $A$ gives a new sutured manifold which is the disjoint union of $(Y_1, \gamma_\alpha)$ and $(Y_2, \gamma_l)$. $A$ is a product annulus, so it follows from Lemma 8.9 of [27] that

$$SFH(Y', \gamma_\mu) = SFH(Y_1, \gamma_\alpha) \otimes SFH(Y_2, \gamma_l).$$

Since $\partial Y' = T^2$, there is a natural injection $c : \text{Spin}^c(Y', \gamma_\mu) \to H_1(Y')$ given by the formula $j(\mathfrak{s}) = PD(c_1(\mathfrak{s}))$, and similarly for $Y_1$ and $Y_2$. The tensor product respects the decomposition into $\text{Spin}^c$ structures in the sense that $c(x \otimes y) = j_{1*}(c(x)) + j_{2*}(c(y))$, where $j_i : Y_i \to Y'$ is the inclusion.

In the case at hand, $H_1(Y') = H_1(Y_1) \oplus H_1(Y_2)/\langle \alpha = l\rangle$, and $H_1(Y_2) \simeq \mathbb{Z} \oplus \mathbb{Z}/g_{Y_2}$, where the $\mathbb{Z}/g_{Y_2}$ summand is generated by $l$. Thus $H_1(Y') \simeq \mathbb{Z} \oplus (H_1(Y_1)/\langle g_{Y_2}\alpha\rangle)$. Now $\alpha$ is a nontorsion element of $H_1(Y_1)$ (otherwise $Y_\varphi$ is not a rational homology sphere), so the image of $j_{1*}$ is contained in the torsion subgroup of $H_1(Y')$.

If $Y$ is a rational homology $S^1 \times D^2$ and $\beta \in Sl(Y)$, then $SFH(Y, \gamma_\beta, \mathfrak{s}) = 0$ whenever $\phi(c(\mathfrak{s})) > \|Y\| + |\phi(\beta)|$. The set $O_{(Y,\gamma_\beta)} = \{\mathfrak{s} \in \text{Spin}^c(Y, \gamma_\beta) \,|\, \phi(c(\mathfrak{s})) = \|Y\| + |\phi(\beta)|\}$ is the set of *outer* $\text{Spin}^c$ structures for $(Y, \gamma_\beta)$ [27]. We write

$$SFH(Y,\gamma_\beta,O) = \bigoplus_{\mathfrak{s}\in O_{(Y,\gamma_\beta)}} SFH(Y,\gamma_\beta,\mathfrak{s}).$$

Since the image of $j_{1*}$ is contained in the torsion subgroup, we have

$$SFH(Y',\gamma_\mu,O) \simeq SFH(Y_1,\gamma_\alpha)\otimes SFH(Y_2,\gamma_l,O). \tag{155}$$

In particular, $\|Y'\| = \|Y_2\| = g_{Y_2} - 2 = g_{Y'} - 2$, so $Y'$ is a generalized solid torus.

To conclude the proof we use the following two lemmas. The first is probably well-known, but we give a proof just in case.

**Lemma 7.10.** *Suppose $Y$ is an incompressible rational homology $S^1\times D^2$, that $l\in H_1(\partial Y)$ is the homological longitude, and that $m\cdot l = 1$. Then*

$$SFH(Y,\gamma_l,O)\simeq SFH(Y,\gamma_m,O)\otimes H_*(S^1).$$

**Proof.** Let $S\subset Y$ be a properly embedded surface generating $H_2(Y,\partial Y)$. If we decompose $(Y,\gamma_m)$ along $S$, we get a sutured manifold $(Z,\gamma_Z)$, where $\partial Z$ is a union of two copies of $S$ glued together their boundaries, and there is one suture for each component of $\partial S$. Decomposing $(Y,\gamma_l)$ along $S$ gives $(Z,\gamma'_Z)$, where the suture $\gamma'_Z$ is the same as $\gamma_Z$ except that there are three parallel sutures along one component of $\partial S$ instead of one. By Proposition 9.2 of [27], $SFH(Z,\gamma'_Z)\simeq SFH(Z,\gamma_Z)\otimes H_*(S^1)$. □

**Lemma 7.11.** *If $Y$ is a generalized solid torus and $m\in H_1(\partial Y)$ satisfies $\phi(m) = g_Y$, then $SFH(Y,\gamma_m,O)\simeq \mathbb{Z}^{k_Y}$.*

**Proof.** $SFH(Y,\gamma_m,O) = \widehat{HFK}(K_m,O)$, where $K_m\subset Y(m)$ is the dual knot. Since $Y$ is a generalized solid torus, the latter group is Floer simple, hence determined by its Euler characteristic. By Lemma 7.3,

$$\phi(\chi(\widehat{HFK}(K_m))) = \frac{k_Y(1-t^{g_Y})^2}{(1-t)^2}.$$

It follows that $\widehat{HFK}(K_m,O)\simeq\mathbb{Z}^{k_Y}$. □

Applying the lemmas to $Y_2$, which has $k_{Y_2} = 1$, we see that $SFH(Y_2,\gamma_l,O)\simeq H_*(S^1)$. For $Y'$, suppose that $H_1(Y_1(\alpha)) = H_1(Y_1)/\langle\alpha\rangle$ has order $d$. The torsion subgroup of $H_1(Y')$ is $H_1(Y_1)/\langle g_{Y_2}\alpha\rangle$, so it has order $g_{Y_2}d$. Since $g_{Y'} = g_{Y_2}$, we see that $k_{Y'} = d$. Since $\mu$ is the homological longitude of $Y'$, $SFH(Y',\gamma_\mu,O)\simeq H_*(S^1)\otimes\mathbb{Z}^d$. Comparing with equation (155), we see that $SFH(Y_1,\gamma_\alpha)\simeq\mathbb{Z}^d$. Now if $K_\alpha\subset Y_1(\alpha)$ is the dual knot, then $\widehat{HFK}(K_\alpha) = SFH(Y_1,\gamma_\alpha)\simeq\mathbb{Z}^d$, where $d = |H_1(Y_1(\alpha))|$. So $K_\alpha$ is Floer simple, which implies that $Y_1$ is Floer simple and that $\alpha$ is in the interior of $\mathcal{L}(Y_1)$.

Conversely, if $\alpha$ is in the interior of $\mathcal{L}(Y)$, Theorem 1.8 implies that $Y_\varphi$ is an L-space, so $\alpha$ is not NLS detected by $Y_2$. □

Boyer and Clay define $\alpha$ to be NLS detected if it is NLS detected by some $N_g$, where $N_g = M(1/g, -1/g)$ is the original family of Floer homology solid tori discussed above. The proposition shows that $\alpha$ is NLS detected by one $N_g$ if and only if it is NLS detected by all $N_g$ if and only if $\alpha$ is not the interior of $\mathcal{L}(Y)$. This proves Corollary 1.12.

### *7.3. Examples*

We conclude by constructing some examples of generalized solid tori. Some of these were previously known to Hanselman and Watson [19] and Vafaee [49]. We start with the following observation.

**Corollary 7.12.** *If $Y$ is an irreducible, semi-primitive generalized solid torus, then $Y$ is the complement of a closed $g_Y$-strand braid in $S^1 \times S^2$.*

**Proof.** The hypotheses imply that $\Delta(Y) \sim (1-t^g)/(1-t)$ and that $H_2(Y, \partial Y)$ is generated by a $g_Y$-times punctured sphere. By Corollary 2.3, it follows that $Y$ fibers over $S^1$ with fiber of genus 0. □

By Proposition 7.8, the complement of any knot in $S^1 \times S^2$ with a lens space surgery is a generalized solid torus. Cebanu [11] showed that a knot of this form is a closed braid in $S^1 \times S^2$. Examples of such knots were studied by Buck, Baker, and Leucona in [2]. Many (but not all) of them are derived from knots in the solid torus which have solid torus surgeries. These knots were completely classified by Gabai [16] and Berge [3].

To find other examples, we look for braids in $S^1 \times S^2$ which have L-space surgeries. One criterion for finding such examples is given here. Suppose $\sigma$ is an ordinary $g$ strand braid in $D^2 \times I$. We can close $\sigma$ to get a closed braid in $S^1 \times D^2$. Dehn filling $S^1 \times D^2$ along $S^1 \times p$ gives the ordinary braid closure $\overline{\sigma} \subset S^3$. We can also fill $S^1 \times D^2$ along $\partial D^2$ to get a closed braid in $S^1 \times S^2$, which we denote by $\widetilde{\sigma}$. Let $\Delta \in Br_g$ be the full twist on $g$-strands.

**Proposition 7.13.** *Suppose that $\sigma$ is a braid with the property that $K_n = \overline{\Delta^n \sigma}$ is an L-space knot in $S^3$ for all $n \geq 0$. Then the complement of $\widetilde{\sigma}$ is a semi-primitive generalized solid torus.*

**Proof.** Let $L \subset S^3$ be the link which is the union of $K = \overline{\sigma}$ and the braid axis $B$. The braid $\tilde{\sigma}$ is the image of $K$ in the $S^1 \times S^2$ obtained by doing 0-surgery on $B$.

Let $L(a, c)$ be the manifold obtained by doing $a$ surgery on $K$ and $c$ surgery on $B$, where $a \in \mathbb{Z}$ and $c \in \mathbb{Q}$. Then $L(a, -1/n)$ is the result of $a + ng^2$ surgery on $K_n$. Using Seifert's algorithm, it is easy to see that there is a constant $C(\sigma)$ with the property that $g(K_n) \leq C(\sigma) + ng(g-1)/2$. Thus if $a > 2C(\sigma)$, then $a + ng^2 \geq 2g(K_n) - 1$ for all $n \geq 0$. By hypothesis, $K_n$ is a positive L-space knot, so $L(a, -1/n)$ is an L-space for all $n > 0$.

Now let $\overline{Y}$ be the manifold obtained by doing $a$ surgery on $K$, and let $Y = \overline{Y} - \nu(B)$. There is a slope $\alpha_0 \in Sl(Y)$ so that $Y(\alpha_0) = L(a, 0)$, and a sequence of slopes $\alpha_{-1/n} \in$

$Sl(Y)$ which converge to $\alpha_0$ such that $Y(\alpha_{-1/n}) = L(a, -1/n)$. It follows that $Y$ is Floer simple and that $\alpha_0$ is in the closure of $\mathcal{L}(Y)$. Since $\alpha_0$ is not the homological longitude of $Y$, $\alpha_0 \in \mathcal{L}(Y)$, so $L(a, 0)$ is an L-space. By Proposition 7.8, $Y$ is a generalized solid torus. □

We call a closed braid in the solid torus which satisfies the criterion a *L-space braid.* Examples include:

- Knots in the solid torus with solid torus surgeries (*aka* Berge–Gabai knots)
- The twisted torus knots $T(p, kp \pm 1; 2, 1)$ studied by Vafaee [48]
- Cables of L-space braids [22]
- Satellites where the pattern knot is a Berge–Gabai knot and the companion is an L-space braid [23]

We conclude with two remarks. First, we conjecture that every positive one-bridge braid (not just the Berge–Gabai knots) is an L-space braid. Since the knot obtained by applying a full twist to a one-bridge braid is again a one-bridge braid, this is equivalent to showing that the closure of any positive one-bridge braid is an L-space knot in $S^3$. Second, in light of the last two items, it would be interesting to know if a satellite where both the pattern and the companion are L-space braids is also an L-space braid.

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