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# Inequalities for the Gaussian measure of convex sets

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**Abstract**

This note presents families of inequalities for the Gaussian measure of convex sets which extend the recently proven Gaussian correlation inequality in various directions.

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## 1 Introduction and statement of results

Let $\gamma$ be the standard Gaussian on $\mathbb{R}^n$, defined by

$$\gamma(K) = \int_K \tfrac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\|x\|^2} dx$$

for Lebesgue measurable $K \subseteq \mathbb{R}^n$.

Recently Royen [8] proved that

$$\gamma(A)\gamma(B) \leq \gamma(A \cap B) \tag{1.1}$$

for all dimensions $n$ and all symmetric convex sets $A, B \subseteq \mathbb{R}^n$. This Gaussian correlation inequality (1.1) was previously known as the Gaussian correlation conjecture and was an open problem for over 50 years. See the paper of Latała & Matlak [3] for a discussion of Royen's proof.

The purpose of this note is to offer evidence in support of the following strengthening of inequality (1.1). We will use the notation

$$A + B = \{a + b : a \in A, b \in B\}$$

for the Minkowski sum of two sets.

**Conjecture 1.1.** *The inequality*

$$\gamma(A)\gamma(B) \leq \gamma(A \cap B)\gamma(A + B) \tag{1.2}$$

*holds for all dimensions $n$ and all symmetric convex sets $A, B \subseteq \mathbb{R}^n$.*

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It is obvious that inequality (1.2) holds in dimension $n = 1$. More generally, the inequality holds whenever $A \subseteq B$, since in this case $A = A \cap B$ and $B \subseteq A + B$. Also note that inequality (1.2) holds with equality for any dimension $n > 1$ whenever there is a dimension $m < n$ and sets $\tilde{A} \subseteq \mathbb{R}^m$ and $\tilde{B} \subseteq \mathbb{R}^{n-m}$ such that $A = \tilde{A} \times \mathbb{R}^{n-m}$ and $B = \mathbb{R}^m \times \tilde{B}$, since in this case $A \cap B = \tilde{A} \times \tilde{B}$ and $A + B = \mathbb{R}^n$.

Dar [1] proved that the similar-looking inequality

$$\mathrm{Leb}(A)\mathrm{Leb}(B) \le \mathrm{Leb}(A \cap B)\mathrm{Leb}(A + B) \tag{1.3}$$

holds for all symmetric convex sets $A, B \subseteq \mathbb{R}^n$, where $\mathrm{Leb}$ is the Lebesgue measure on $\mathbb{R}^n$. Since we have the inequality

$$\tfrac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} r_K^2} \, \mathrm{Leb}(K) \le \gamma(K) \le \tfrac{1}{(2\pi)^{n/2}} \mathrm{Leb}(K)$$

for all bounded measurable $K \subset \mathbb{R}^n$, where $r_K = \sup\{\|x\| : x \in K\}$ is the radius of the smallest ball containing $K$, inequality (1.3) implies

$$\gamma(A)\gamma(B) \le \gamma(A \cap B)\gamma(A + B) e^{\frac{1}{2}(r_A + r_B)^2 + \frac{1}{2}(r_A \wedge r_B)^2}. \tag{1.4}$$

Inequality (1.4) does not prove Conjecture 1.1, but it does indicate that the conjecture is plausible. Furthermore, even if Conjecture 1.1 turns out not to be true, inequality (1.4) shows that the correlation inequality (1.1) can be improved when $A$ and $B$ are contained in a sufficiently small ball. Indeed, the right-hand side of inequality (1.4) is smaller than the right-hand side of the correlation inequality (1.1) when $r_A$ and $r_B$ are sufficiently small.

Schechtman, Schlumprecht & Zinn [9, Proposition 3] proved the related inequality that

$$\gamma(A)\gamma(B) \le \gamma\big(\sqrt{2}(A \cap B)\big)\, \gamma\big(\tfrac{1}{\sqrt{2}}(A + B)\big) \tag{1.5}$$

for symmetric convex $A, B \subseteq \mathbb{R}^n$. Using the fact that the map

$$t \mapsto t^{-n}\gamma(tK) = \int_K \tfrac{1}{(2\pi)^{n/2}} e^{-\frac{t^2}{2}\|x\|^2} dx$$

is decreasing for any measurable $K \subseteq \mathbb{R}^n$, inequality (1.5) implies

$$\gamma(A)\gamma(B) \le 2^{n/2}\gamma(A \cap B)\gamma\big(\tfrac{1}{\sqrt{2}}(A + B)\big) \tag{1.6}$$

as was observed by Schechtman, Schlumprecht & Zinn. Note that since $\frac{1}{\sqrt{2}} < 1$, the right-hand side of inequality (1.6) is larger than the right-hand side of the conjectural inequality (1.2). Also note that replacing $A$ and $B$ with $tA$ and $tB$ and sending $t \downarrow 0$ in either inequality (1.5) or (1.6) recovers inequality (1.3).

The new result of this paper is the following:

**Theorem 1.2.** *The inequality*

$$\gamma(A)\gamma(B) \le (1-s)^{-n/2}\, \gamma\left(\sqrt{\tfrac{2(1-s)}{1+t}}(A \cap B)\right) \gamma\left(\sqrt{\tfrac{1-s}{2(1-t)}}(A + B)\right) \tag{1.7}$$

*holds for all dimensions $n$ and all symmetric convex sets $A, B \subseteq \mathbb{R}^n$ and all $\sqrt{s} \le t < 1$.*

The proof of Theorem 1.2 uses a stronger form of the Gaussian correlation inequality (1.1) which already appears in Royen's paper, as well as ideas appearing in the papers of Shao [10] and Schechtman, Schlumprecht & Zinn. We present the proof in the next section.

Note that setting $s = 0$ in inequality (1.7) yields the dimension-independent family of inequalities

$$\gamma(A)\gamma(B) \le \gamma\left(\sqrt{\tfrac{2}{1+t}}(A \cap B)\right)\gamma\left(\tfrac{1}{\sqrt{2(1-t)}}(A + B)\right)$$

which holds for all $0 \le t < 1$. This family interpolates between Schechtman, Schlumprecht & Zinn's inequality (1.5) corresponding to $t = 0$ and Royen's inequality (1.1) corresponding to the limit $t \uparrow 1$. Setting $t = 1/2$ yields

$$\gamma(A)\gamma(B) \le \gamma\left(\tfrac{2}{\sqrt{3}}(A \cap B)\right)\gamma(A + B) \tag{1.8}$$

Note that since $\frac{2}{\sqrt{3}} > 1$, the right-hand side of inequality (1.8) is larger than the right-hand side of the conjectural inequality (1.2).

Note that by setting $s = \frac{1}{2}(1-t)$ for $1/2 \le t < 1$ in inequality (1.7), we have the family of inequalities

$$\gamma(A)\gamma(B) \le \left(\tfrac{2}{1+t}\right)^{n/2}\gamma(A \cap B)\,\gamma\left(\sqrt{\tfrac{1+t}{4(1-t)}}(A + B)\right).$$

Again, the limit $t \uparrow 1$ recovers inequality (1.1). Setting $t = 1/2$ yields

$$\gamma(A)\gamma(B) \le \left(\tfrac{4}{3}\right)^{n/2}\gamma(A \cap B)\,\gamma\left(\tfrac{\sqrt{3}}{2}(A + B)\right). \tag{1.9}$$

Note that since $\frac{1}{\sqrt{2}} < \frac{\sqrt{3}}{2} < 1$ the right-hand side of inequality (1.9) is larger than the right-hand side of the conjectural inequality (1.2), but it is smaller than the right-hand side of inequality (1.6), and therefore improving on the result of Schechtman, Schlumprecht & Zinn. Finally, setting $t = 3/5$ yields

$$\gamma(A)\gamma(B) \le \left(\tfrac{5}{4}\right)^{n/2}\gamma(A \cap B)\,\gamma(A + B)$$

which improves upon inequality (1.4) when either $A$ or $B$ is unbounded.

Finally, note that by setting $s = 2t - 1$ for $1/2 \le t < 1$ in inequality (1.7), we have the family of inequalities

$$\gamma(A)\gamma(B) \le [2(1-t)]^{-n/2}\gamma\left(\sqrt{\tfrac{4(1-t)}{1+t}}(A \cap B)\right)\gamma(A + B).$$

Sending $t \uparrow 1$ yields

$$\begin{aligned}\gamma(A)\gamma(B) &\le \tfrac{1}{(2\pi)^{n/2}}\mathrm{Leb}(A \cap B)\,\gamma(A + B)\\ &\le \gamma(A \cap B)\,\gamma(A + B)\,e^{\frac{1}{2}(r_A \wedge r_B)^2}\end{aligned}$$

which again improves upon inequality (1.4) when either $A$ or $B$ is bounded.

## 2 The proof

Fix the dimension $n$, and for $0 \le t \le 1$ let $\gamma_t$ denote the distribution on $\mathbb{R}^n \times \mathbb{R}^n$ of the jointly normal vector $(X, Y)$ where the distribution of both $X$ and $Y$ is the standard Gaussian measure $\gamma$ and the covariance matrix is $\mathbb{E}(XY^\top) = tI$ where $I$ is the $n \times n$ identity matrix. This family of measures $(\gamma_t)_{0 \le t \le 1}$ interpolates between $\gamma_0(K \times L) = \gamma(K)\gamma(L)$ and $\gamma_1(K \times L) = \gamma(K \cap L)$ and is given explicitly, for $t < 1$, by the formula

$$\gamma_t(H) = \int_H \tfrac{1}{(1-t^2)^{n/2}(2\pi)^n} e^{-\frac{1}{2(1-t^2)}(\|x\|^2 - 2t\langle x,y\rangle + \|y\|^2)}dx\,dy$$

for measurable $H \subseteq \mathbb{R}^n \times \mathbb{R}^n$, where $\langle x, y\rangle = x_1y_1 + \ldots + x_ny_n$ denotes the standard inner product on $\mathbb{R}^n$. We will need a few facts about the measure $\gamma_t$.

**Fact 2.1.** *Fix a measurable set $H \in \mathbb{R}^n \times \mathbb{R}^n$ with the symmetry property that $(x,y) \in H$ implies $(x,-y) \in H$. The map $t \mapsto (1-t^2)^{-n/2}\gamma_t\big(\sqrt{1-t^2}H\big)$ is increasing.*

*Indeed, let*

$$\begin{aligned} f(t) &= (1-t^2)^{-n/2}\gamma_t\big(\sqrt{1-t^2}H\big) \\ &= \int_H \tfrac{1}{(2\pi)^n} e^{-\frac{1}{2}(\|x\|^2 - 2t\langle x,y\rangle + \|y\|^2)} dx\, dy \\ &= \int_H \tfrac{1}{(2\pi)^n} e^{-\frac{1}{2}(\|x\|^2 + \|y\|^2)} \cosh(t\langle x,y\rangle) dx\, dy \end{aligned}$$

*where we have used the symmetry property of $H$ to go from the second to third line. Hence, we have the identity*

$$f'(t) = \int_H \tfrac{1}{(2\pi)^n} e^{-\frac{1}{2}(\|x\|^2+\|y\|^2)} \langle x,y\rangle \sinh(t\langle x,y\rangle)\ dx\ dy.$$

*Since $\theta \sinh\theta \geq 0$ for all real $\theta$, the function $f$ is increasing. A variation of this argument also appears in the paper of Shao [10, Theorem 1.1].*

**Fact 2.2.** *Inspection of Royen's proof [8, equation (2.3)] of the Gaussian correlation inequality* (1.1) *shows that the map*

$$t \mapsto \gamma_t(A \times B)$$

*is increasing on $[0,1]$ for all symmetric convex $A, B \subseteq \mathbb{R}^n$. This monotonicity property was already known for the special case of dimension $n = 2$ by the result of Pitt [5, Theorem 3]. In Appendix A we provide an interesting reformulation of this monotonicity property in terms of the function $\operatorname{sinc} x = \frac{\sin x}{x}$.*

**Fact 2.3.** *Fix $0 \leq t < 1$ and symmetric convex $A, B$. We have the inequality*

$$\gamma_t(A \times B) \leq \gamma\left(\sqrt{\tfrac{2}{1+t}}(A \cap B)\right)\gamma\left(\tfrac{1}{\sqrt{2(1-t)}}(A+B)\right).$$

*To prove the above claim, we will combine a few observations about Gaussian measure. Firstly, we will need the elementary identity that*

$$\gamma_t(K \times L) = \gamma_0\left\{(x,y) : \sqrt{\tfrac{1+t}{2}}x + \sqrt{\tfrac{1-t}{2}}y \in K, \sqrt{\tfrac{1+t}{2}}x - \sqrt{\tfrac{1-t}{2}}y \in L\right\}.$$

*Secondly, fix symmetric convex $A, B \subseteq \mathbb{R}^n$ and real constant $p$ and note that*

$$\gamma_0\{(x,y) : x + py \in A, x - py \in B\} = \int_{\mathbb{R}^n} h(y) d\gamma(y)$$

*where*

$$h(y) = \gamma\left\{(A - py) \cap (B + py)\right\}.$$

*The function $h$ is log-concave by the log-concavity of the Gaussian density, the assumed convexity of $A$ and $B$ and Prékopa's theorem [6, Theorem 6]. For completeness a statement of this important result is included in Appendix B. Since $h$ is even by the assumed symmetry of $A$ and $B$ we have*

$$h(y) \leq h(0) = \gamma(A \cap B)$$

*for all $y \in \mathbb{R}^n$. Furthermore, $h(y) > 0$ only when*

$$(A - py) \cap (B + py) \neq \emptyset,$$

*that is, when there exist points $a \in A$ and $b \in B$ such that*

$$a - py = b + py$$

*and hence*

$$y = \tfrac{1}{2p}(a-b) \in \tfrac{1}{2p}(A+B),$$

*again by the symmetry of $B$. Therefore*

$$\begin{aligned}\int_{\mathbb{R}^n} h(y)d\gamma(y) &\le \int_{y:h(y)>0} h(0)d\gamma(y)\\ &\le \gamma(A\cap B)\,\gamma\big(\tfrac{1}{2p}(A+B)\big).\end{aligned}$$

*Combining these two observations yields*

$$\begin{aligned}\gamma_t(A\times B) &= \gamma_0\left\{(x,y) : x + \sqrt{\tfrac{1-t}{1+t}}y \in \sqrt{\tfrac{2}{1+t}}A,\, x - \sqrt{\tfrac{1-t}{1+t}}y \in \sqrt{\tfrac{2}{1+t}}B\right\}\\ &\le \gamma\left(\sqrt{\tfrac{2}{1+t}}(A\cap B)\right)\gamma\left(\tfrac{1}{\sqrt{2(1-t)}}(A+B)\right)\end{aligned}$$

*The idea establishing the above bound on the Gaussian measure of an intersection was taken from Schechtman, Schlumprecht & Zinn [9, Proposition 3]. In fact, Dar [1, Observation (4)] also employed a similar idea to bound the Lebesgue measure of an intersection, and indeed this type of argument seems to have originated in the paper of Rogers & Shephard [7].*

To prove Theorem 1.2, fix $\sqrt{s} \le t < 1$ and convex symmetric sets $A, B$. We have the following series of inequalities:

$$\begin{aligned}\gamma(A)\gamma(B) &= \gamma_0(A\times B)\\ &\le (1-s)^{-n/2}\gamma_{\sqrt{s}}\big(\sqrt{1-s}(A\times B)\big) && \text{(by Fact 2.1)}\\ &\le (1-s)^{-n/2}\gamma_t\big(\sqrt{1-s}(A\times B)\big) && \text{(by Fact 2.2)}\\ &\le (1-s)^{-n/2}\gamma\left(\sqrt{\tfrac{2(1-s)}{1+t}}(A\cap B)\right)\gamma\left(\sqrt{\tfrac{1-s}{2(1-t)}}(A+B)\right) && \text{(by Fact 2.3)}\end{aligned}$$

as desired. □

**Remark 2.4.** As noted above, Facts 2.1 and 2.2 yield

$$\gamma(A)\gamma(B) \le (1-s)^{-n/2}\gamma_t\big(\sqrt{1-s}(A\times B)\big)$$

for $\sqrt{s} \le t < 1$. Of course, one could also reverse the order by first applying Fact 2.2 and then Fact 2.1 to yield the inequality

$$\begin{aligned}\gamma(A)\gamma(B) &\le \gamma_{\sqrt{s}}(A\times B)\\ &\le \left(\frac{1-s}{1-t^2}\right)^{n/2}\gamma_t\left(\sqrt{\frac{1-t^2}{1-s}}(A\times B)\right).\end{aligned}$$

However, nothing is gained by this reversal as can be seen by setting

$$S = \frac{t^2-s}{1-s}.$$

Note $0 \le S \le t^2$ and that the reversed inequality becomes

$$\gamma(A)\gamma(B) \le (1-S)^{-n/2}\gamma_t\big(\sqrt{1-S}(A\times B)\big).$$

## A A sinc reformulation

In this appendix, we provide an interesting equivalent reformulation of Royen's result that the map $t \mapsto \gamma_t(A \times B)$ is increasing for symmetric convex set $A$ and $B$, where the interpolation measure $\gamma_t$ is defined in section 2.

We will use the notation $\mathrm{sinc} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathrm{sinc}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{\sin x_i}{x_i}.$$

**Theorem A.1.** *For all $0 \leq t < 1$ and $n \times n$ matrices $P$ and $Q$ we have*

$$\int \mathrm{sinc}(Px)\mathrm{sinc}(Qy)\langle x, y\rangle \sinh(t\langle x, y\rangle)d\gamma(x)d\gamma(y) \geq 0.$$

To prove Theorem A.1, we will need a lemma about Gaussian Fourier transforms. We note that the idea to study Gaussian (and more general) correlation inequalities via Fourier analysis has appeared in the paper of Koldobsky & Montgomery-Smith [2]. We need some notation. For integrable $f : \mathbb{R}^n \to \mathbb{C}$ define its Fourier transforms $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(u) = \tfrac{1}{(2\pi)^{n/2}} \int e^{\mathrm{i}\langle u,x\rangle} f(x)dx$$

as usual.

**Lemma A.2.** *If $f$ and $g$ are integrable then*

$$\int f(x)g(y)d\gamma_t(x, y) = \int \hat{f}(u)\hat{g}(v)e^{-t\langle u,v\rangle}d\gamma(u)d\gamma(v)$$

*for all $0 \leq t < 1$.*

*Proof.* This is essentially an application of Plancherel's identity. The proof amounts to writing $\hat{f}$ and $\hat{g}$ in terms of their respective Fourier integrals, and since $f$ and $g$ are assumed integrable, Fubini's theorem can be applied. The result is a consequence of the well-known formula

$$\int e^{\mathrm{i}(\langle u,x\rangle+\langle v,y\rangle)-t\langle u,v\rangle}d\gamma(u)d\gamma(v) = \tfrac{1}{(1-t^2)^{n/2}} e^{-\frac{1}{2(1-t^2)}(\|x\|^2-2t\langle x,y\rangle+\|y\|^2)}. \quad \square$$

**Lemma A.3.** *Let $C = [-1, 1]^n = \{x \in \mathbb{R}^n, \max_i |x_i| \leq 1\}$ and set $A = P^\top C$ and $B = Q^\top C$ for $n \times n$ matrices $P$ and $Q$. Then*

$$\gamma_t(A \cap B) = |\det(P)\det(Q)| \int \mathrm{sinc}(Px)\mathrm{sinc}(Qy) \cosh(t\ \langle x, y\rangle)d\gamma(x)d\gamma(y).$$

*Proof.* Note that $\widehat{\mathbb{1}_C}(s) = \mathrm{sinc}(s)$ and hence $\widehat{\mathbb{1}_{P^\top C}}(s) = |\det P|\mathrm{sinc}(Ps)$. By Lemma A.2, we have

$$\gamma_t(A \times B) = |\det(P)\det(Q)| \int \mathrm{sinc}(Px)\mathrm{sinc}(Qy)e^{-t\langle x,y\rangle}d\gamma(x)d\gamma(y).$$

The result follows since $C$ is symmetric and sinc is even. $\square$

The proof of Theorem A.1 follows from differentiating the expression in Lemma A.3 and applying Royen's result on the monotonicity of $t \mapsto \gamma_t(A \times B)$. Notice that since convex sets can be approximated by polyhedra, Theorem A.1 is in fact equivalent to Royen's monotonicity result.

## B Log-concave functions

In this appendix we recall some familiar notions involving log-concavity. A non-negative function $g$ on $\mathbb{R}^n$ is called log-concave if

$$g(\theta x + (1-\theta)y) \geq g(x)^\theta g(y)^{1-\theta}$$

for any $0 \leq \theta \leq 1$ and $x, y \in \mathbb{R}^n$. In particular, the indicator function of a convex set is log-concave. The following fundamental result is due to Prékopa [6, Theorem 6].

**Theorem B.1.** *Suppose that the function $g$ on $\mathbb{R}^{m+n}$ is log-concave. Then the function $h$ on $\mathbb{R}^n$ defined by*

$$h(y) = \int_{\mathbb{R}^m} g(x, y) dx$$

*is also log-concave.*

In section 2 we appeal to Prékopa's theorem with the log-concave function

$$g(x, y) = \tfrac{1}{(2\pi)^{n/2}} \mathbb{1}_{\{(x,y): x+py \in A, x-py \in B\}} e^{-\|x\|^2/2}.$$

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