Momentum conserving defects in affine Toda field theory

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A Thesis presented for the degree of
Doctor of Philosophy

Durham University

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United Kingdom

January 2018

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Abstract: This thesis expands previous work on defects appearing in classical integrable field theories, with a generalised form for momentum conserving defects being found. It is shown that the defect equations can always be augmented to give a Bäcklund transformation for the bulk theory, and new momentum conserving defects are found for the B_r and D_r ATFTs. Momentum conservation is shown to be a necessary condition if the system is to have an infinite number of conserved quantities for all defects in ATFTs. The D_4 defect in particular is investigated, with the system shown to have a zero curvature representation and soliton-defect interactions being investigated.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification. This thesis contains work carried out in collaboration with my supervisor and published in [BB17], and work published in [Bri17].

Acknowledgements

I would like to thank my supervisor Dr. Peter Bowcock for introducing me to the wonderful world of defects and for his subsequent supervision and encouragement.

Thanks to all of the Durham Maths department, in particular my office-mates Alex, Matt, Richard, Sam, Vaios and Xin. Outside the department special thanks go to Megumi and everyone in Newcastle for making it an enjoyable four years beyond the world of maths.

I would also like to thank my very supportive parents Garry and Tina, and my moderately supportive sister Alison.

Last but certainly not least, thank you Neil.

This work was supported by an STFC studentship.

All maths is friteful and means 0 but if you are a grate brane you hear a tremendous xplosion at about the fifth lesson in trig. This mean that you are through the sound barier and maths hav become what every keen maths master tell you it can be i.e. a LANGWAGE.

— Molesworth in *How to be Topp* by Geoffrey Willans and Ronald Searle

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Chapter 1

Introduction

In this thesis we will be investigating whether a defect can be incorporated into an integrable system without destroying its integrability. A system being integrable implies that it is completely solvable. Integrability was first conceived of in relation to a system with N degrees of freedom specified by N coordinates and N momenta. We can carry out a transformation to a specific set of coordinates known as the action-angle variables, and once in this coordinate system it can be shown that the existence of N independent conserved quantities which are in involution under a Poisson bracket allows the solution to be found by the method of quadratures. This is a Liouville integrable system. For the classical $1+1$ dimensional field theories we will work with there are an infinite number of degrees of freedom, so in order for the system to be (in principle) solvable there must be an infinite number of independent conserved quantities in Poisson involution [FT86; BBT03]. For many integrable field theories there is a class of solutions known as solitons. These appear as stable, localised field configurations and are of immense physical interest [SCM73]. For overviews of the concept of integrability see [FT86; BBT03].

Several methods have been developed for demonstrating the integrability of any particular field theory. It has been shown that being able to use the method of the inverse scattering transform to find solutions to the system implies the existence of an infinite number of conservation laws [ZS72]. Of more interest to us is the method of zero curvature, Lax pairs [Lax68] and the r-matrix [Skl80]; reviews of this method appear in [Sem83; FT86; BBT03]. For a system with a Lax pair which satisfies the zero curvature condition if and only if the equations of motion are satisfied it is possible to use the Lax matrices to write down the monodromy matrix. Evaluating this monodromy matrix at different times then allows the calculation of an infinite number of conserved quantities. If a related r-matrix can be found which satisfies the classical Yang-Baxter equation then these conserved quantities are also in Poisson involution, and the system is Liouville integrable. Here we will be focussing on the existence of an infinite number of conserved quantities. In fact, we initially only consider the conservation of energy and momentum.

Various integrable field theories exist. Much early investigation of integrability took place for the Kortweg-de Vries equation, which was also where the first solitons were observed. The inverse scattering transform was first developed for the Kortweg-de Vries equation and used to find soliton solutions and prove the existence of an infinite number of conserved quantities [GGKM67; ZS72; ZF71]. The method of the Lax pair to prove integrability was also developed for the Kortweg-de Vries equation [Lax68]. Other scalar field theories which have been investigated using the inverse scattering transform are the sine-Gordon model [AKNS73], the Tzitzéica model [BSS93] and the nonlinear Schrödinger equation [ZS72].

The integrable field theories which we will be considering here are the affine Toda field theories (ATFTs). They began life as an investigation into a chain of N particles with certain nearest neighbour interactions [Tod70]. This was later generalised to a vector field theory whose potential is based on the roots of any semi-simple Lie algebra [Bog76; Mik79; MOP81]. The “affine" refers to the fact that the potential is written in terms of the simple roots and the lowest weight root, as the addition of the lowest weight root to a Dynkin diagram gives an affine Dynkin diagram. The ATFT based on the simple and lowest weight roots of Lie algebra $\mathfrak{g}$ is referred to as the $\mathfrak{g}$ ATFT. These were shown to have a zero curvature representation (and so an infinite number of conserved quantities) [MOP81; Wil81] and later shown to be integrable [OT85; OT86] using the method of the Lax pair and r-matrix.

All of the above systems have soliton solutions which are of physical interest [SCM73]. In addition to being integrable solitons (stable by virtue of a balance of nonlinear and dispersive effects) the ATFT solitons are also topological, with their stability guaranteed by the existence of some topological charge. Soliton solutions may be found using the inverse scattering transform, but for ATFTs the Hirota bilinear method, first introduced to find Kortweg-de Vries solitons [Hir71], was used to give soliton solutions. Classical solitons in the A_r ATFTs were first investigated in [Hol92], and their topological charges were found to be weights of the associated Lie algebra. For ATFTs the potential has multiple vacua only if the field takes complex values, but the solitons were found to still have a real mass and energy. In [MM93] all other static solitons were found, and again the topological charges were found to take values in the weight lattice of the relevant Lie algebra, with the multi-soliton solutions given in [McG94a]. Multi-soliton solutions may be constructed and analysed, giving the result that when solitons scatter they undergo a brief deformation when the solitons are in close proximity, before quickly returning to their original shape and velocity, with the only effect of the interaction being a shift in the position of the solitons. For more information on solitons and integrability see [FT86; For90].

Bäcklund transformations are also closely related to integrability [RS02]. They are a set of first order coupled differential equations whose solutions are also the solutions to two uncoupled sets of higher order differential equations [DB76; Miu76]. All of the scalar field theories (but not all ATFTs) mentioned here have a Bäcklund transformation, with [WE73] giving the Kortweg-de Vries equation Bäcklund transformation, [DB76] the sine-Gordon Bäcklund transformation, [Nim83] the nonlinear Schrödinger equation Bäcklund transformation, [BSS93] the Tzitzéica Bäcklund transformation and [FG80] generalising the sine-Gordon Bäcklund transformation to the A_r ATFT Bäcklund transformation. These Bäcklund transformations are useful for finding soliton solutions. By taking the solution of one set of second order equations to be zero it is possible to solve the first order Bäcklund transformation equations, giving a solution to the other set of second order equations. In [LOT93] the Bäcklund transformations of the A_r ATFT are used to generate soliton solutions.

Quantum integrability also exists, although we will not be considering the quantum case at all here. The existence of scattering matrices satisfying the Yang-Baxter equation ensures quantum integrability, with quantum scattering in an integrable system always factorisable to 2-2 scattering. Information on quantum integrability can be found in [ZZ79; Dor91; Dor92; CDS93], with the quantum S-matrix for the Toda chain given in [AFZ79] and the S-matrices for all quantum ATFTs investigated in [BCDS90].

There are many physical examples of integrable systems and solitons; for just a few examples see [SCM73; For90]. In more detail we have [Lam67] for solitons in optics, [Dav77] for solitons in proteins, [Ust98] for solitons in Josephson junctions, [G L93] for the nonlinear Schrödinger equation in optics and [RV04] for solitons in Bose-Einstein condensates. Topological (but, at least in this example, not integrable) solitons are frequently used to model non-perturbative systems of elementary particles [Sky61].

Since (some of) the interest in integrable systems is due to their ability to model physical phenomena whilst remaining exactly solvable it is important to be able to incorporate common physical occurrences without destroying the integrability of the system. In this thesis we are interested in introducing defects to integrable models. A defect is some discontinuity, either in physical media or fields in a mathematical model, and we are aiming to find ways of incorporating a discontinuity into an integrable model without destroying its integrability. Some examples of physical defects have been considered in the Ising model [MP80], in a chain of driven, damped pendula [ABT00] and in semiconductors [Jun+14].

Some of the earliest studies of defects were in quantum integrable field theories, for example in a free fermion theory [DMS94b; DMS94a] and in sine-Gordon theory [KL99], and here it was shown that integrable defects must be purely reflecting or transmitting. Quantum defects have been investigated further in [MRS02; CFG02; MRS03; CMRS05], with a defect in the quantum nonlinear Schrödinger equation appearing in [CMR04], a defect in the quantum sine-Gordon model appearing in [BCZ05] and defects in the quantum ATFTs being investigated in [CZ07; CZ09b; CZ10; CZ11].

From the fact that quantum defects must be purely transmitting came the idea that momentum conservation may be important in the classical case. If a system with a defect is to have soliton solutions (and so likely be integrable) while being purely transmitting then momentum is conserved, at least between very early and very late times (i.e. while the defect and soliton are not interacting). In [BCZ04b], where the Lagrangian approach to classical defects used in this thesis was pioneered, it was found that for a defect in sine-Gordon theory certain equations of motion at the defect ensured that momentum was conserved. The conservation of energy and some higher spin charges was also checked for these momentum conserving defects. These were generalised to give momentum conserving defects in A_r ATFTs, although it was also proved that momentum conserving defects of the particular form found in [BCZ04b; BCZ04a] could never appear in an ATFT based on a different Lie algebra.

An ATFT with a momentum conserving defect has been shown to have a solution where a soliton is transmitted by the defect between the ATFTs on either side, and the defect may delay the soliton or change its topological charge. For certain cases the delay factor may be infinite, meaning that the soliton is absorbed by the defect. In addition to these soliton solutions this system also has infinite number of conserved quantities, so is likely integrable [BCZ04a; CZ07; CZ09b]. However, the integrability of these particular defects has not been proven as they are given in a Lagrangian rather than a Hamiltonian form, meaning that the Poisson brackets and r-matrix satisfying the Yang-Baxter equation required to prove that the charges are in involution are difficult to write down. A Hamiltonian set-up in which the Lax and r-matrix equations are immediately assumed to be satisfied by some matrix associated with the defect are investigated in [AD12a; AD12b; Doi15] for defects in the nonlinear Schrödinger equation, sine-Gordon theory and ATFTs. While these defects are integrable they do not necessarily describe the same systems as the momentum conserving defects found in the Lagrangian set-up. Some attempt to reconcile this Hamiltonian approach and the Lagrangian approach to defects is made in [Cau15; CK15], and a method of moving from the Lagrangian to the Hamiltonian picture was suggested in [CZ09a].

Another interesting observation made for these momentum conserving defects in A_r ATFTs is that the defect equations of motion, when taken to hold everywhere rather than just at the position of the defect, give a Bäcklund transformation between the theories on either side of the defect. This is not completely surprising, as the defect equations couple the two bulk theories on either side of the defect.

In [CZ09a] the momentum conserving defects first found in [BCZ04b] were modified by the addition of a degree of freedom at the defect, allowing a momentum conserving defect in the Tzitzéica model (previously excluded due to not being based on the roots of A_r) to be found. This idea of extra degrees of freedom at the defect, and the fact that one ATFT can be folded to a different ATFT using certain symmetries of the Dynkin diagram [OT83a; OT83b; PS96], was used in [Rob14b] to fold existing A_r ATFT defects to new C_r ATFT defects. The defects appearing in [BCZ04b; BCZ04a; CZ07; CZ09b], which have no additional degrees of freedom at the defect, are referred to as type I defects and the defects with additional degrees of freedom appearing in [CZ09a; Rob14b; Rob14a; Rob15] are referred to as type II defects.

Some investigation into defects in non-relativistic theories such as the nonlinear Schrödinger equation and the Kortweg-de Vries equation have also been made [CZ06; CP16] (also [Doi12; AD12a]).

In this thesis we advance the classical Lagrangian defect story by generalising the type II defects such that we can find momentum conserving defects in the B_r and D_r ATFTs.

In chapter 2 we first provide a little more background on momentum conserving defects in section 2.1 and then present entirely original work, with sections 2.2 and 2.4 giving results which appear in [BB17] and section 2.3 unpublished. In section

2.2 a generalised type II defect is written down, with any number of bulk fields and any number of extra degrees of freedom at the defect. The general form any momentum conserving defect must take and the restrictions which must be satisfied are found. In section 2.3 we consider a moving defect, and show that it is also momentum conserving. Finally in section 2.4 the defect equations are modified to give a Bäcklund transformation. This Bäcklund transformation exists for any bulk theory for which a momentum conserving defect satisfying the constraints detailed in this section can be found.

In chapter 3 the introductory section 3.1 gives some background on ATFTs in general and section 3.2 runs over the proof that momentum conserving type I defects can only appear in A_r ATFTs, the working used to find the type II Tzitzéica defect, and the method for folding two type I A_3 ATFT defects to a single type II C_2 ATFT defect. Section 3.3 is then entirely original work. The general form of a momentum conserving defect in an ATFT in section 3.3 and the specific momentum conserving defects in A_r, B_r, C_r and D_r ATFTs given in sections 3.3.2-3.3.5 can be found in [BB17]. The more detailed working for defects in the D_4 ATFT given in section 3.3.1 appears in [Bri17], and the considerations as to why defects for the ATFTs based on the exceptional Lie algebras have not yet been found in sections 3.3.6-3.3.8 are original and unpublished.

In chapter 4 we focus on the Tzitzéica and D_4 defects. Section 4.1 gives a general introduction to integrability and the condition which must be satisfied for the system with a defect to have infinitely many conserved quantities is found. Section 4.2 uses the form of a generalised momentum conserving defect in an ATFT from section 3.3 to show that momentum conservation is a necessary condition for the defect to be integrable. Taking the specific Tzitzéica and D_4 cases in sections 4.2.1 and 4.2.2 we are able to satisfy the defect zero curvature condition. The results in sections 4.2 and 4.2.2 may be found in [Bri17]. The defect matrix in the Tzitzéica model has been found previously in [AAGZ11].

In chapter 5 the interactions of solitons and defects are considered. Some background on solitons in the D_4 ATFT is given in section 5.1. Section 5.3 contains the soliton delays found for a D_4 ATFT defect as given in section 3.3.1, which appear in [BB17]. This chapter also contains unpublished work on the possible topological charges of the defect and the behaviour of the extra degrees of freedom at the defect as the soliton interacts with it.

Chapter 2

Momentum conserving defects

2.1 Introduction

It is possible for some two-dimensional integrable field theories to accommodate discontinuities in the fields and yet still have several conserved quantities. These include energy and, more remarkably given the breaking of translation invariance by introducing the discontinuity at a specific point in space, momentum. These systems still admit soliton solutions, in the form of solitons in the bulk which are transmitted through the discontinuity with some delay, and have an infinite number of conserved charges, and so likely remain integrable [BCZ04a; CZ07; CZ09b; CZ09a]. This discontinuity is referred to as a defect in the theory, and the fields on either side of the discontinuity are related by some set of defect conditions. There may be a potential and extra degrees of freedom which exist only at the defect and influence the defect conditions.

We take the defect to lie at $x = 0$ (although in section 2.3 we will consider a defect with a time dependent position). The bulk vector field in the region $x \leq 0$ will be called $u(x,t)$, the bulk vector field in the region $x \geq 0$ will be called $v(x,t)$ and any degrees of freedom living on the defect at $x = 0$ are labelled $\lambda(t)$. We shall refer to the $\lambda(t)$ as *auxiliary fields*. The term field may seem a peculiar choice as λ has no spatial dependence; however when we come to consider Bäcklund transformations we will see that it is natural to extend the definition of λ to take values in the bulk. We denote the components of u, v and λ as $u_1, u_2, \dots,\ v_1, v_2, \dots,\ \lambda_1, \lambda_2, \dots$. Additionally we will assume that u and v describe two copies of the same bulk theory but on different sides of the defect, so that the number of components of u and v are equal. There may be any number of components of the auxiliary vector field λ.

The Lagrangian description of the theory in the presence of a defect at $x = 0$ is given in terms of a density

$$\mathcal{L} = \Theta(-x)\mathcal{L}^{(u)} + \Theta(x)\mathcal{L}^{(v)} + \delta(x)\mathcal{L}^{D}, \tag{2.1.1}$$

where the bulk Lagrangian densities

$$\mathcal{L}^{(u)} = \frac{1}{2}(u_{i,t}u_{i,t} - u_{i,x}u_{i,x}) - U(u) \tag{2.1.2}$$

$$\mathcal{L}^{(v)} = \frac{1}{2}(v_{i,t}v_{i,t} - v_{i,x}v_{i,x}) - V(v) \tag{2.1.3}$$

govern the behaviour of the bulk fields u and v. Subscripts of t and x denote partial differentiation with respect to that variable and are separated from subscripts of indices by a comma. Einstein sum notation is used throughout. The two bulk theories are coupled at $x = 0$ via the defect Lagrangian $\mathcal{L}^D$ which depends on u, v and λ. This Lagrangian set-up was pioneered in [BCZ04b].

The form of $\mathcal{L}^D$ we will consider in this chapter is motivated by combining features from existing examples of defects. For the type I defects investigated in [BCZ04b; BCZ04a; CZ04; CZ07; CZ09b] the bulk fields couple to each other at the defect and there are no auxiliary fields. An example of a type I defect coupling multicomponent fields u and v is the defect for A_r ATFT considered in [CZ09b]; its Lagrangian is of the form

$$\mathcal{L}^D = \frac{1}{2}u_i A_{ij} u_{j,t} + \frac{1}{2}v_i A_{ij} v_{j,t} + u_i(\mathbb{1} - A)_{ij} v_{j,t} - F(u, v) \tag{2.1.4}$$

where A is a constant, antisymmetric matrix. However these momentum conserving defects are only compatible with an A_r ATFT in the bulk [BCZ04a]. These type I defects have soliton solutions, where a soliton is transmitted through the defect

[CZ09b]. Constructing the Lax pair showed that the restrictions on the defect which ensured energy and momentum conservation were necessary and sufficient to ensure the existence of an infinite number of conserved charges [BCZ04a; CZ09b]. In [CZ09a] an additional degree of freedom was introduced at the defect, and this modification allowed a momentum conserving defect to appear within the Tzitzéica model (excluded from the integrable type I defects due to being based on the simple roots of folded A_2 rather than purely on A_r). This defect is of the form

$$\mathcal{L}^D = uv_t + 2\lambda\left(u_t - v_t\right) - F(u, v, \lambda) \tag{2.1.5}$$

where u, v and λ are scalar fields. There is a strong body of evidence to suggest that this defect is integrable, namely that momentum and energy are conserved, solitons were able to pass through it with no change other than a delay (determined by the rapidity of the soliton and the defect parameters) [CZ09a], and that the existence of an infinite number of conserved charges has been shown for the Tzitzéica defect [AAGZ11]. Liouville integrability of defects with additional degrees of freedom has been investigated in [AD12a; AD12b], although these defects are not presented in the Lagrangian framework used here.

In both of these examples, the defect Lagrangian consists of two parts: a defect potential $F = F(u, v, \lambda)$ and 'kinetic terms' coupling the time derivatives of the fields to the fields themselves via constant matrices. In this chapter we shall consider the most general defect of this form, combining the vector field aspect of the type I defect (which allowed it to encompass all A_r ATFTs) with the auxiliary field appearing in the type II defect (which allowed a momentum conserving defect to be constructed for an ATFT not based on A_r). The work in [Rob14b] went some way toward combining the two approaches, but required the number of auxiliary fields to be equal to or a multiple of the number of bulk fields.

In section 2.2 we shall derive conditions for a general class of type II defects, where there are any number of bulk fields and any number of extra degrees of freedom confined to the defect, to be momentum conserving. Considering the results in the type I case we are hopeful that the constraints from energy-momentum conservation will be sufficient to ensure integrability. In chapter 4 we will see that this is likely the case for certain Tzitzéica and D_4 ATFT defects.

The initial calculations here are for stationary defects, but in section 2.3 we will also consider momentum conservation for a system with a moving defect.

In [BCZ04b; BCZ04a] it was noted that the defect conditions of any momentum conserving type I defect in an A_r ATFT were a Bäcklund transformation if the defect conditions were taken to hold everywhere, and in [CZ09a] a new Bäcklund transformation for the Tzitzéica model was found by modifying the type II defect conditions. In section 2.4 we show that the defect conditions of the momentum conserving defects investigated can always be augmented to provide a set of equations which are a Bäcklund transformation for the bulk theory. If the defect equations linking the theories on either side are a Bäcklund transformation then we would expect the system to have soliton solutions which pass through the defect, a feature of integrable systems.

2.2 A momentum conserving generalised type II defect

The defect Lagrangian density we consider is

$$\begin{aligned}\mathcal{L}^D =& \frac{1}{2}u_i A_{ij} u_{j,t} + \frac{1}{2}v_i B_{ij} v_{j,t} + u_i C_{ij} v_{j,t} \\ &+ \frac{1}{2}\lambda_i W_{ij} \lambda_{j,t} + \lambda_i X_{ij} u_{j,t} + \lambda_i Y_{ij} v_{j,t} - F(u, v, \lambda),\end{aligned} \tag{2.2.1}$$

where A, B, C, W, X and Y are arbitrary, constant, real coupling matrices. This general form of defect Lagrangian depends on a plethora of unknown couplings contained in the matrices A, B, C, W, X and Y. The main purpose of this section will be to pin down the form of this Lagrangian much more precisely by using our freedom to make field redefinitions and by applying the constraints arising when we require that the system with a defect conserves momentum.

We can immediately see that some of the couplings in the defect Lagrangian (2.2.1) are redundant. The matrices A, B and W can be taken to be antisymmetric as any symmetric part simply adds a total derivative to the Lagrangian which is physically irrelevant, at least in the classical case. Further simplifications can be made by using field redefinitions to put the Lagrangian in a canonical form. Because the auxiliary vector field λ does not appear in the bulk Lagrangians the behaviour of the system is not altered under the redefinition of the auxiliary fields $\lambda_i \to \alpha_{ij}u_j + \beta_{ij}v_j + \gamma_{ij}\lambda_j$. α and β are any matrices and γ is an invertible matrix to ensure the degrees of freedom associated to the auxiliary fields are not removed. The bulk fields can also be transformed as $u_i \to Q_{ij}u_j$, $v_i \to Q'_{ij}v_j$ without changing the general form of the bulk and defect Lagrangians provided Q and Q' are both orthogonal. We intend to use these field redefinitions to simplify the Lagrangian in eq.(2.2.1) as far as possible, 'absorbing' the freedom in the arbitrary coupling matrices into the auxiliary fields. We will find that any momentum conserving defect of the form given above is equivalent, up to some field redefinitions, to a defect in which each component of the fields may couple in either the type I or the type II manner seen in eqs.(2.1.4),(2.1.5).

We begin by further simplifying W, the antisymmetric matrix containing the couplings between auxiliary fields. The spectral theorem states there exists a change of basis $\lambda_i \to \gamma_{ij}\lambda_j$ where the matrix γ is orthogonal, in which the antisymmetric matrix W takes the block-diagonal form

$$W \to \gamma^T W \gamma = \begin{pmatrix} 0 & l_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -l_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & l_k & 0 & \dots & 0 \\ 0 & 0 & \dots & -l_k & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad (2.2.2)$$

where the matrix has $2k$ non-zero eigenvalues, $\pm i l_j$. We can also scale the auxiliary fields $\lambda_i \to c_i \lambda_i$, where c_i are some scalars, to take all entries in this block-diagonal matrix to ± 1. These field redefinitions can be carried out without loss of generality, and so we can always use them to set

$$W = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \qquad (2.2.3)$$

The field redefinition on λ will also affect the matrices X and Y but these changes can be ignored as they amount to redefinitions of what are already arbitrary matrices. With W as above, the components of the auxiliary field, $\{\lambda_i\}$, naturally divide into those for $i = 1 \dots 2k$ which couple to other auxiliary fields, and the remaining components in the zero eigenspace of W which have no coupling to other auxiliary fields in the 'kinetic' part of the defect Lagrangian. The components of λ which couple to other auxiliary fields are relabelled as $\xi_1, \xi_2, \dots$, components of the vector field ξ, and the components of λ which couple to no other auxiliary fields are relabelled as $\mu_1, \mu_2, \ldots,$ components of the vector field μ. In terms of ξ and μ the defect Lagrangian density can now be rewritten as

$$\begin{aligned}\mathcal{L}^D =&\frac{1}{2}u_i A_{ij} u_{j,t} + \frac{1}{2}v_i B_{ij} v_{j,t} + u_i C_{ij} v_{j,t} + \frac{1}{2}\xi_i W_{ij}\xi_{j,t} \\ &+ \mu_i X_{ij} u_{j,t} + \xi_i \hat{X}_{ij} u_{j,t} + \mu_i Y_{ij} v_{j,t} + \xi_i \hat{Y}_{ij} v_{j,t} - F \end{aligned} \qquad (2.2.4)$$

where matrices X and Y have been split into the smaller matrices X, $\hat{X}$, Y and $\hat{Y}$ in order to separate the couplings of the bulk fields to $\{\mu_i\}$ and $\{\xi_i\}$. The matrix W is from now on taken to be

$$W = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & -1 & 0 \end{pmatrix}. \qquad (2.2.5)$$

Having simplified W as far as we can we now turn to the couplings of ξ to the bulk fields. The redefinitions $\xi_i \to W_{ij}\hat{X}_{jk}u_k + W_{ij}\hat{Y}_{jk}v_k + \xi_i$ give

$$\frac{1}{2}\xi_i W_{ij}\xi_{j,t} \to \frac{1}{2}\left(W_{ik}\hat{X}_{kl}u_l + W_{ik}\hat{Y}_{kl}v_l + \xi_i\right) W_{ij}\left(W_{jk}\hat{X}_{kl}u_{l,t} + W_{jk}\hat{Y}_{kl}v_{l,t} + \xi_{j,t}\right). \qquad (2.2.6)$$

Using $W^2 = -\mathbb{1}$ it is then straightforward to show that this provides cancellations which leave the Lagrangian density as

$$\mathcal{L}^D =\frac{1}{2}u_i A_{ij} u_{j,t} + \frac{1}{2}v_i B_{ij} v_{j,t} + u_i C_{ij} v_{j,t} + \frac{1}{2}\xi_i W_{ij}\xi_{j,t} + \mu_i X_{ij} u_{j,t} + \mu_i Y_{ij} v_{j,t} - F. \qquad (2.2.7)$$

As before the effect of these field redefinitions on the arbitrary matrices A, B and C has been negated by an appropriate redefinition of these matrices.

We shall now look for the conditions on the matrices A, B, C, W, X and Y and potential F which arise from demanding that the system described by the Lagrangian in eq.(2.2.7) has a conserved momentum and energy. We expect that demanding momentum conservation will be sufficient to ensure the integrability of the defect.

The Euler-Lagrange equations arising from the Lagrangian density in eq.(2.1.1) with the defect Lagrangian in eq.(2.2.7) give the equations of motion

$$x \leq 0: \qquad 0 = u_{i,tt} - u_{i,xx} + U_{u_i} \tag{2.2.8}$$

$$x \geq 0: \qquad 0 = v_{i,tt} - v_{i,xx} + V_{v_i} \tag{2.2.9}$$

$$x = 0: \qquad u_{i,x} = A_{ij}u_{j,t} + C_{ij}v_{j,t} - X_{ji}\mu_{j,t} - F_{u_i} \tag{2.2.10}$$

$$v_{i,x} = C_{ji}u_{j,t} - B_{ij}v_{j,t} + Y_{ji}\mu_{j,t} + F_{v_i} \tag{2.2.11}$$

$$0 = X_{ij}u_{j,t} + Y_{ij}v_{j,t} - F_{\mu_i} \tag{2.2.12}$$

$$0 = W_{ij}\xi_{j,t} - F_{\xi_i} \tag{2.2.13}$$

where a subscript containing a field denotes partial differentiation with respect to that field.

The total energy of the fields in the bulk is

$$E = \int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{2}\left(u_{i,t}u_{i,t} + u_{i,x}u_{i,x}\right) + U \right) + \int_{0}^{\infty} \mathrm{d}x \left(\frac{1}{2}\left(v_{i,t}v_{i,t} + v_{i,x}v_{i,x}\right) + V \right) \tag{2.2.14}$$

and we expect the conserved total energy to be the sum of this bulk energy plus some contribution from the defect. Differentiating eq.(2.2.14) with respect to t and then using the bulk equations of motion in eqs.(2.2.8), (2.2.9) to rewrite the integrand as a total x derivative allows us to carry out the integration (with $\{u_i\}, \{v_i\} \to$ constant as $x \to \pm\infty$ and U and V having no local minima, only global minima), giving

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(u_{i,x}u_{i,t} - v_{i,x}v_{i,t}\right)\big|_{x=0}. \tag{2.2.15}$$

In order for this term to be conserved we must be able to write the right hand side of this equation as a total time derivative. Using the defect conditions in eqs.(2.2.10), (2.2.11) to remove the x derivatives we find that eq.(2.2.15) may be rewritten as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mathrm{d}F}{\mathrm{d}t}. \tag{2.2.16}$$

Therefore $E + F$ is the conserved energy-like quantity, where E is the bulk energy and F is the defect potential. The introduction of a defect at $x = 0$ does not break the time translation symmetry of the system, so perhaps it is not surprising that it is always possible to construct a conserved energy without placing any further constraints on the couplings in the defect Lagrangian.

Since the defect breaks manifest translation invariance, the system is no longer obviously momentum conserving, and we expect requiring conservation of momentum to be far more restrictive than requiring conservation of energy. Total momentum of the fields in the bulk is given by

$$P = \int_{-\infty}^{0} \mathrm{d}x \, (u_{i,x} u_{i,t}) + \int_{0}^{\infty} \mathrm{d}x \, (v_{i,x} v_{i,t}) \tag{2.2.17}$$

and again we will require that this plus some defect contribution is conserved. Differentiating eq.(2.2.17) with respect to t, using the bulk equations of motion in eqs.(2.2.8), (2.2.9) to rewrite the integrand as a total x derivative and carrying out the integration gives

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \left(\frac{1}{2} \left(u_{i,t} u_{i,t} + u_{i,x} u_{i,x} - v_{i,t} v_{i,t} - v_{i,x} v_{i,x} \right) - U + V \right) \Big|_{x=0} . \tag{2.2.18}$$

In order for the system to be momentum conserving we must be able to rewrite eq.(2.2.18) as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = - \frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{2.2.19}$$

where Ω is the defect contribution to the total momentum of the system.

Using the defect conditions in eqs.(2.2.10)-(2.2.12) we now aim to find the restrictions on the couplings at the defect and the defect potential which are necessary to ensure the system is momentum conserving and so (hopefully) integrable. All fields can be assumed to be evaluated at $x = 0$ from now on. In order for eq.(2.2.18) to be written as a total t derivative the x derivatives must be removed, which can only be done by substituting in eqs.(2.2.10),(2.2.11). We also have the freedom to add any multiple of eqs.(2.2.13),(2.2.12), as such terms are equal to zero. This gives

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} =& \frac{1}{2}u_{i,t}\left(\mathbb{1}-A^2-CC^T\right)_{ij}u_{j,t}-\frac{1}{2}v_{i,t}\left(\mathbb{1}-B^2-C^TC\right)_{ij}v_{j,t} \\ &-u_{i,t}\left(AC-CB\right)_{ij}v_{j,t}+u_{i,t}\left(AX^T-CY^T\right)_{ij}\mu_{j,t} \\ &-v_{i,t}\left(C^TX^T+BY^T\right)_{ij}\mu_{j,t}+\frac{1}{2}\mu_{i,t}\left(XX^T-YY^T\right)_{ij}\mu_{j,t} \\ &-\left(F_{u_i}A_{ij}+F_{v_i}C^T_{ij}\right)u_{j,t}-\left(F_{u_i}C_{ij}-F_{v_i}B_{ij}\right)v_{j,t}+\left(F_{u_i}X^T_{ij}-F_{v_i}Y^T_{ij}\right)\mu_{j,t} \\ &+\frac{1}{2}\left(F_{u_i}F_{u_i}-F_{v_i}F_{v_i}\right)-U+V \\ &+\left(-\xi_{k,t}W_{ki}-F_{\xi_i}\right)\left(\rho_i+\tau_{ij}u_{j,t}+\phi_{ij}v_{j,t}\right) \\ &+\left(u_{k,t}X^T_{ki}+v_{k,t}Y^T_{ki}-F_{\mu_i}\right)\left(\sigma_i+\pi_{ij}u_{j,t}+\chi_{ij}v_{j,t}+\psi_{ij}\mu_{j,t}+\omega_{ij}\xi_{j,t}\right). \quad (2.2.20)\end{aligned}$$

For the right hand side of this equation to be a total t derivative we must remove all terms which are not linear in t derivatives of the fields. The last two lines of this equation contain multiples of the expressions in eqs.(2.2.13),(2.2.12). We have not added multiples of the expressions in eqs.(2.2.10),(2.2.11), as these would reintroduce derivatives of the fields with respect to x which cannot be expressed as t derivatives. Equally the multiplicative factors of the expressions in eqs.(2.2.13),(2.2.12) have been chosen to introduce no higher than quadratic terms of t derivatives of fields into eq.(2.2.20), with any quadratic terms introduced also appearing elsewhere in the expression. This ensures that no terms which would not cancel with any other terms in the expression and which cannot be made into a total t derivative are introduced, as such terms would be immediately set to zero.

Let us begin by considering the term $\mu_{i,t}\left(XX^T-YY^T\right)_{ij}\mu_{j,t}$. For this to be a total t derivative it must identically vanish, and as the quantity XX^T-YY^T is explicitly symmetric, we have that $XX^T=YY^T$. Now consider the case in which a particular auxiliary field decouples from u but not from v. It is always possible to permute the labels on the fields $\{\mu_i\}$ by a field redefinition so that the field μ_1 is the one decoupling from u but not from v, so $X_{1j}=0\ \forall\, j$. The condition $X_{ij}X^T_{jk}=Y_{ij}Y^T_{jk}$ then requires $Y_{1j}Y^T_{jk}=0\ \forall\, k$. One of the conditions from this is $Y_{1j}Y_{1j}=0$ and since all coupling matrices are assumed to be real this is only satisfied if $Y_{1j}=0\ \forall\, j$.

Therefore if an auxiliary field decouples from u it must also decouple from v and vice versa. From eq.(2.2.12) we then have that the equation of motion of the field μ_1 is $F_{\mu_1} = 0$, and so if an auxiliary field decouples completely from all other auxiliary fields and from one of the bulk vector fields it can be made to disappear entirely from the defect Lagrangian.

Now consider the $\mu_{i,t}X_{ij}u_{j,t} + \mu_{i,t}Y_{ij}v_{j,t}$ terms. We take vectors u and v to have r components and vector μ to have m components. The matrix X^T has a kernel which will be some subspace of the vector space μ is living in. By a transformation of μ we can take the basis of the kernel of X^T to be the final k elements of μ. After this transformation the final k columns of X^T will be zero. The final k components of μ completely decouple from u, and so by the argument in the above paragraph they also completely decouple from v, and so Y^T also has the final k columns as zero. The final k components of μ are now auxiliary fields which completely decouple from u and v, and so can be removed from the Lagrangian. The vector μ is now length $m-k$ and the matrices X^T and Y^T must have a kernel of 0, otherwise further μ components should have decoupled. A matrix can only have a zero kernel if the number of rows is greater than or equal to the number of columns, so X and Y are both $(m-k)\times r$ matrices with $m-k \leq r$. The matrix X also has a kernel, and we can take this to have a basis consisting of the first n components of u by an orthogonal transformation of u. These components of u completely decouple from the auxiliary fields, and so we choose to denote the vector containing only these components of u as $u^{(1)}$, where the superscript indicates that these fields couple like a type I defect. We will call the vector containing the remaining components of u $u^{(2)}$. The first n columns of X are then zero, and by rewriting the term $\mu_{i,t}X_{ij}u_{j,t}$ as $\mu_{i,t}\left(0\,X\right)_{ij}u_{j,t} = \mu_{i,t}X_{ij}u^{(2)}_{j,t}$ we have that X is a $(m-k)\times(r-n)$ matrix with zero kernel and so $r-n \leq m-k$. But if $r-n < m-k$ then X^T now has more columns than rows and can no longer have a kernel of zero, therefore X is a square matrix coupling μ and $u^{(2)}$. By the same argument Y is also a $(r-n)\times(r-n)$ matrix, with the first n elements of v now contained in the vector $v^{(1)}$ thanks to an orthogonal transformation of v.

The single bulk vector fields u and v have each been split into two vectors, with u and v arranged so that

$$u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} \qquad\qquad v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix}. \tag{2.2.21}$$

The length n vectors $u^{(1)}$ and $v^{(1)}$ do not couple to any of the auxiliary fields and the length $r-n$ vectors $u^{(2)}$ and $v^{(2)}$ couple to the $r-n$ auxiliary fields which have not been removed by field redefinitions and do not couple to any other auxiliary fields. We relabel the vector field μ as $\mu^{(2)}$ to emphasise that it is coupling to the bulk fields in vectors $u^{(2)}$ and $v^{(2)}$ only. After these field redefinitions the term $\mu_{i,t}X_{ij}u_{j,t} + \mu_{i,t}Y_{ij}v_{j,t}$ has become $\mu^{(2)}_{i,t}X_{ij}u^{(2)}_{j,t} + \mu^{(2)}_{i,t}Y_{ij}v^{(2)}_{j,t}$ with X and Y square with zero kernel. Because they are square with zero kernel both X and Y are invertible, and we can use the field redefinition $\mu^{(2)} \to \left(X^{-1}\right)^T \mu^{(2)}$ to set $X = \mathbb{1}$. The condition $XX^T = YY^T$ becomes $YY^T = \mathbb{1}$, and so Y must be orthogonal. We no longer have complete freedom to carry out orthogonal transformations on bulk vector fields u and v, but orthogonal transformations which do not mix the components of $u^{(1)}$, $v^{(1)}$ with $u^{(2)}$, $v^{(2)}$ are still allowed. So we can use the orthogonal field redefinition $v^{(2)}_i \to -Y^T_{ij}v^{(2)}_j$ to set $Y = -\mathbb{1}$. Finally to keep the type II couplings in the form seen in eq.(2.1.5) we make the field redefinition $\mu^{(2)} \to 2\mu^{(2)}$, setting $X = 2\mathbb{1}$ and $Y = -2\mathbb{1}$.

This splitting of the vector fields u and v into $u^{(1)}$ and $u^{(2)}$ and $v^{(1)}$ and $v^{(2)}$ respectively will also require the coupling matrices A, B and C to be split up. We take

$$A = \begin{pmatrix} A^{(11)} & A^{(12)} \\ -A^{(12)T} & A^{(22)} \end{pmatrix} \quad B = \begin{pmatrix} B^{(11)} & B^{(12)} \\ -B^{(12)T} & B^{(22)} \end{pmatrix} \quad C = \begin{pmatrix} C^{(11)} & C^{(12)} \\ C^{(21)} & C^{(22)} \end{pmatrix} \tag{2.2.22}$$

where $A^{(11)}$, $A^{(22)}$, $B^{(11)}$ and $B^{(22)}$ are antisymmetric to ensure A and B are antisymmetric matrices. The matrices τ, ϕ, π and χ introduced in eq.(2.2.20) split into

$$\tau = \begin{pmatrix} \tau^{(1)} & \tau^{(2)} \end{pmatrix} \quad \phi = \begin{pmatrix} \phi^{(1)} & \phi^{(2)} \end{pmatrix} \quad \pi = \begin{pmatrix} \pi^{(1)} & \pi^{(2)} \end{pmatrix} \quad \chi = \begin{pmatrix} \chi^{(1)} & \chi^{(2)} \end{pmatrix}. \quad (2.2.23)$$

The field redefinition $\mu_i^{(2)} \rightarrow \frac{1}{2}C_{ij}^{(12)T}u_j^{(1)} + \frac{1}{4}A_{ij}^{(22)}u_j^{(2)} + \frac{1}{2}C_{ij}^{(21)}v_j^{(1)} - \frac{1}{4}B_{ij}^{(22)}v_j^{(2)} + \mu_i^{(2)}$ can be used to set $C^{(12)} = A^{(22)} = B^{(22)} = 0$. With this simplification the defect Lagrangian can now be written

$$\begin{aligned}\mathcal{L}^D =& \frac{1}{2}u_i^{(1)}A_{ij}^{(11)}u_{j,t}^{(1)} + u_i^{(1)}A_{ij}^{(12)}u_{j,t}^{(2)} + \frac{1}{2}v_i^{(1)}B_{ij}^{(11)}v_{j,t}^{(1)} + v_i^{(1)}B_{ij}^{(12)}v_{j,t}^{(2)} \\ &+ u_i^{(1)}C_{ij}^{(11)}v_{j,t}^{(1)} + u_i^{(2)}C_{ij}^{(22)}v_{j,t}^{(2)} + 2\mu_i^{(2)}\left(u_{i,t}^{(2)} - v_{i,t}^{(2)}\right) + \frac{1}{2}\xi_i W_{ij}\xi_{j,t} - F. \quad (2.2.24)\end{aligned}$$

Having set the term $\mu_{i,t}\left(XX^T - YY^T\right)_{ij}\mu_{j,t}$ to zero, let us return to the other terms on the right hand side of eq.(2.2.20) which must be a total t derivative for the defect to conserve momentum. The eq.(2.2.20) can now be rewritten as

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} =& \frac{1}{2}u_{i,t}^{(1)}\left(\mathbb{1} - A^{(11)2} - C^{(11)}C^{(11)T} + A^{(12)}A^{(12)T}\right)_{ij}u_{j,t}^{(1)} \\ &+ \frac{1}{2}u_{i,t}^{(2)}\left(\mathbb{1} - C^{(22)}C^{(22)T} + A^{(12)T}A^{(12)} + 4\pi^{(2)}\right)_{ij}u_{j,t}^{(2)} \\ &- \frac{1}{2}v_{i,t}^{(1)}\left(\mathbb{1} - B^{(11)2} - C^{(11)T}C^{(11)} + B^{(12)}B^{(12)T}\right)_{ij}v_{j,t}^{(1)} \\ &- \frac{1}{2}v_{i,t}^{(2)}\left(\mathbb{1} - C^{(22)T}C^{(22)} + B^{(12)^T}B^{(12)} + 4\chi^{(2)}\right)_{ij}v_{j,t}^{(2)} \\ &- u_{i,t}^{(1)}\left(A^{(11)}A^{(12)} - 2\pi^{(1)T}\right)_{ij}u_{j,t}^{(2)} + v_{i,t}^{(1)}\left(B^{(11)}A^{(12)} - 2\chi^{(1)T}\right)_{ij}v_{j,t}^{(2)} \\ &- u_{i,t}^{(1)}\left(A^{(11)}C^{(11)} - C^{(11)}B^{(11)}\right)_{ij}v_{j,t}^{(1)} + 2u_{i,t}^{(2)}\left(\chi^{(2)} - \pi^{(2)T}\right)_{ij}v_{j,t}^{(2)} \\ &- u_{i,t}^{(1)}\left(A^{(12)}C^{(22)} - C^{(11)}B^{(12)} + 2\pi^{(1)T}\right)_{ij}v_{j,t}^{(2)} \\ &+ u_{i,t}^{(2)}\left(A^{(12)^T}C^{(11)} - C^{(22)}B^{(12)T} + 2\chi^{(1)}\right)_{ij}v_{j,t}^{(1)} \\ &+ 2u_{i,t}^{(1)}A_{ij}^{(12)}\mu_{j,t}^{(2)} + u_{i,t}^{(1)}\left(\tau^{(1)}W\right)_{ij}\xi_{j,t} + 2v_{i,t}^{(1)}B_{ij}^{(12)}\mu_{j,t}^{(2)} + v_{i,t}^{(1)}\left(\phi^{(1)T}W\right)_{ij}\xi_{j,t} \\ &+ 2u_{i,t}^{(2)}\left(C^{(22)} + \psi\right)_{ij}\mu_{j,t}^{(2)} + u_{i,t}^{(2)}\left(2\omega + \tau^{(2)T}W\right)_{ij}\xi_{j,t} \\ &- 2v_{i,t}^{(2)}\left(C^{(22)T} + \psi\right)_{ij}\mu_{j,t}^{(2)} - v_{i,t}^{(2)}\left(2\omega - \phi^{(2)T}W\right)_{ij}\xi_{j,t} \\ &+ u_{i,t}^{(1)}\left(A_{ij}^{(11)}F_{u_j^{(1)}} + A_{ij}^{(12)}F_{u_j^{(2)}} - C_{ij}^{(11)}F_{v_j^{(1)}} - \pi_{ij}^{(1)T}F_{\mu_j^{(2)}} - \tau_{ij}^{(1)T}F_{\xi_j}\right) \\ &- u_{i,t}^{(2)}\left(A_{ij}^{(12)T}F_{u_j^{(1)}} + C_{ij}^{(22)}F_{v_j^{(2)}} + \pi_{ij}^{(2)T}F_{\mu_j^{(2)}} + \tau_{ij}^{(2)T}F_{\xi_j} - 2\sigma_i\right)\end{aligned}$$

$$
\begin{aligned}
&- v_{i,t}^{(1)}\left(C_{ij}^{(11)T}F_{u_j^{(1)}} + B_{ij}^{(11)}F_{v_j^{(1)}} + B_{ij}^{(12)}F_{v_j^{(2)}} + \chi_{ij}^{(1)T}F_{\mu_j^{(2)}} + \phi_{ij}^{(1)T}F_{\xi_j}\right)\\
&- v_{i,t}^{(2)}\left(C_{ij}^{(22)T}F_{u_j^{(2)}} - B_{ij}^{(12)T}F_{v_j^{(1)}} + \chi_{ij}^{(2)T}F_{\mu_j^{(2)}} + \phi_{ij}^{(2)T}F_{\xi_j} + 2\sigma_i\right)\\
&+ \mu_{i,t}^{(2)}\left(2F_{u_i^{(2)}} + 2F_{v_i^{(2)}} - \psi_{ij}^{T}F_{\mu_j^{(2)}}\right) - \xi_{i,t}\left(\omega_{ij}^{T}F_{\mu_j^{(2)}} + W_{ij}\rho_j\right)\\
&+ \frac{1}{2}\left(F_{u_i^{(1)}}F_{u_i^{(1)}} + F_{u_i^{(2)}}F_{u_i^{(2)}} - F_{v_i^{(1)}}F_{v_i^{(1)}} - F_{v_i^{(2)}}F_{v_i^{(2)}}\right) - F_{\mu_i^{(2)}}\sigma_i - F_{\xi_i}\rho_i\\
&- U + V. \qquad (2.2.25)
\end{aligned}
$$

Terms in eq.(2.2.25) containing two t derivatives must be set to zero, as they cannot be written as a total t derivative. From the coefficients of $u_{i,t}^{(1)}\mu_{j,t}^{(2)}$ and $v_{i,t}^{(1)}\mu_{j,t}^{(2)}$ in eq.(2.2.25) we have $A^{(12)} = 0$ and $B^{(12)} = 0$. The $u_{i,t}^{(1)}\xi_{j,t}$ and $v_{i,t}^{(1)}\xi_{j,t}$ terms set $\tau^{(1)} = 0$ and $\phi^{(1)} = 0$. The coefficients of $u_{i,t}^{(2)}\xi_{j,t}$ and $v_{i,t}^{(2)}\xi_{j,t}$ constrain $\omega = \frac{1}{2}\phi^{(2)T}W$ and $\tau^{(2)} = -\phi^{(2)}$, whilst we can see that $\pi^{(1)} = 0$ and $\chi^{(1)} = 0$ by looking at the coefficients of $u_{i,t}^{(1)}u_{j,t}^{(2)}$, $v_{i,t}^{(1)}v_{j,t}^{(2)}$, $u_{i,t}^{(1)}v_{j,t}^{(2)}$ and $u_{i,t}^{(2)}v_{j,t}^{(1)}$. For the coefficient of $u_{i,t}^{(2)}v_{j,t}^{(2)}$ to vanish we need that $\chi^{(2)} = \pi^{(2)T}$ and from the coefficients of $u_{i,t}^{(2)}\mu_{j,t}^{(2)}$ and $v_{i,t}^{(2)}\mu_{j,t}^{(2)}$ we find that $\psi = -C^{(22)}$ and that $C^{(22)}$ is symmetric. The field redefinition $\mu_i \to S_{ij}u_j^{(2)} + S'_{ij}v_j^{(2)} + \mu_i$, where S and S' are symmetric, can always be used to set the symmetric part of $C^{(22)}$ (the symmetry of S and S' ensure we do not introduce new terms proportional to $u_{i,t}^{(2)}u_{j,t}^{(2)}$ or $v_{i,t}^{(2)}v_{j,t}^{(2)}$ into the Lagrangian in eq.(2.2.24)). Since $C^{(22)}$ is entirely symmetric we may use this field redefinition to set $C^{(22)} = \mathbb{1}$. The vanishing of the coefficients of $u_{i,t}^{(2)}u_{j,t}^{(2)}$ and $v_{i,t}^{(2)}v_{j,t}^{(2)}$ then set $\chi^{(2)}$ and $\pi^{(2)}$ to be antisymmetric. The coefficient of $u_{i,t}^{(1)}u_{j,t}^{(1)}$would be zero if $\mathbb{1} - A^{(11)2} - C^{(11)}C^{(11)T}$ could be made antisymmetric, but as it is explicitly symmetric we must set it to zero. Following the method in [BCZ04a] we set $C^{(11)}C^{(11)T} = \left(\mathbb{1} - A^{(11)}\right)\left(\mathbb{1} - A^{(11)T}\right)$. The matrix $A^{(11)}$ is antisymmetric and so has purely imaginary eigenvalues, therefore the matrix $(\mathbb{1} - A^{(11)})$ has no zero eigenvalues and we can write $\left(\mathbb{1} - A^{(11)}\right)^{-1}C^{(11)}\left(\left(\mathbb{1} - A^{(11)}\right)^{-1}C^{(11)}\right)^{T} = \mathbb{1}$. Therefore $\left(\mathbb{1} - A^{(11)}\right)^{-1}C^{(11)} = Q$, where Q is an orthogonal matrix and we can set $C^{(11)} = \left(\mathbb{1} - A^{(11)}\right)Q$. As previously mentioned we still have the freedom to carry out an orthogonal transformation on $u^{(1)}$ or $v^{(1)}$ without changing the form of the Lagrangian in eq.(2.2.24), and we can use such transformations to set $C^{(11)} = \left(\mathbb{1} - A^{(11)}\right)$. The condition from the coefficient of $u^{(1)}_{i,t}v^{(1)}_{j,t}$ is now $A^{(11)}\left(\mathbb{1} - A^{(11)}\right) = \left(\mathbb{1} - A^{(11)}\right)B^{(11)}$, and as $\left(\mathbb{1} - A^{(11)}\right)$ is both invertible and commutes with $A^{(11)}$ we have $B^{(11)} = A^{(11)}$. This also ensures that the coefficient of $v^{(1)}_{i,t}v^{(1)}_{j,t}$ vanishes. We will set $A^{(11)} = A$ as the superscript is no longer necessary to identify this matrix. All the coupling matrices apart from A have now been set, either to ensure momentum conservation or via field redefinitions.

Putting this all together we have found that in order for a defect to be momentum conserving its Lagrangian must, up to orthogonal transformations of the bulk fields u and v and field redefinitions of the auxiliary fields μ and ξ, be of the form

$$\begin{aligned}\mathcal{L}^D =& \frac{1}{2}u^{(1)}_i A_{ij} u^{(1)}_{j,t} + \frac{1}{2}v^{(1)}_i A_{ij} v^{(1)}_{j,t} + u^{(1)}_i \left(\mathbb{1} - A\right)_{ij} v^{(1)}_{j,t} \\ &+ u^{(2)}_i v^{(2)}_{i,t} + 2\mu^{(2)}_i \left(u^{(2)}_{i,t} - v^{(2)}_{i,t}\right) + \frac{1}{2}\xi_i W_{ij}\xi_{j,t} - F \end{aligned} \tag{2.2.26}$$

where A may be any antisymmetric matrix, W is given in eq.(2.2.5) and the components of the bulk vector fields may be divided in any way between the vector fields $u^{(1)}$, $v^{(1)}$ and $u^{(2)}$, $v^{(2)}$. The Lagrangian appears to have split into a type I defect, a type II defect and some extra degrees of freedom, with these separate systems only interacting through the defect potential. Note that if there are no auxiliary fields, so that $\mu^{(2)}$, ξ, $u^{(2)}$ and $v^{(2)}$ are absent, then this Lagrangian reduces to the form of the A_r ATFT Toda defect in eq.(2.1.4). On the other hand, in the case of a single auxiliary field coupling to single component bulk fields, the fields $u^{(1)}$, $v^{(1)}$ and ξ vanish and the Lagrangian is in the same form as the Lagrangian of the Tzitzéica defect in eq.(2.1.5).

That the defect Lagrangian is in the form eq.(2.2.26) is a necessary but not yet sufficient condition for the defect to be momentum conserving. So far we have eliminated all the terms in eq.(2.2.25) which are quadratic in t derivatives. To ensure that the defect is momentum conserving we must consider the terms which are linear or independent of t derivatives; in this way we shall find additional constraints, in particular on the form of the defect potential F. Applying the constraints on the coupling matrices which we have just found the momentum conservation condition for the defect becomes

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} =&u_{i,t}^{(1)}\left(A_{ij}F_{u_j^{(1)}}-(\mathbb{1}-A)_{ij}\,F_{v_j^{(1)}}\right)-u_{i,t}^{(2)}\left(F_{v_i^{(2)}}-\pi_{ij}^{(2)}F_{\mu_j^{(2)}}-\phi_{ij}^{(2)T}F_{\xi_j}-2\sigma_i\right)\\&-v_{i,t}^{(1)}\left((\mathbb{1}+A)_{ij}\,F_{u_j^{(1)}}+A_{ij}F_{v_j^{(1)}}\right)-v_{i,t}^{(2)}\left(F_{u_i^{(2)}}+\pi_{ij}^{(2)}F_{\mu_j^{(2)}}+\phi_{ij}^{(2)T}F_{\xi_j}+2\sigma_i\right)\\&+\mu_{i,t}^{(2)}\left(2F_{u_i^{(2)}}+2F_{v_i^{(2)}}+F_{\mu_i^{(2)}}\right)+\xi_{i,t}\left(\frac{1}{2}W_{ij}\phi_{jk}^{(2)}F_{\mu_k^{(2)}}-W_{ij}\rho_j\right)\\&+\frac{1}{2}\left(F_{u_i^{(1)}}F_{u_i^{(1)}}+F_{u_i^{(2)}}F_{u_i^{(2)}}-F_{v_i^{(1)}}F_{v_i^{(1)}}-F_{v_i^{(2)}}F_{v_i^{(2)}}-2F_{\mu_i^{(2)}}\sigma_i-2F_{\xi_i}\rho_i\right)\\&-U+V.\end{aligned}\tag{2.2.27}$$

From eq.(2.2.19) we see that the terms involving one t derivative will set the derivatives of the unknown quantity Ω. The terms containing no t derivatives obviously cannot be written as a total t derivative, so must be set to zero. The conditions for momentum conservation are therefore

$$\begin{aligned}\Omega_{u_i^{(1)}} =&-A_{ij}F_{u_j^{(1)}}+(\mathbb{1}-A)_{ij}\,F_{v_j^{(1)}}\\\Omega_{v_i^{(1)}} =&(\mathbb{1}+A)_{ij}\,F_{u_j^{(1)}}+A_{ij}F_{v_j^{(1)}}\\\Omega_{u_i^{(2)}} =&F_{v_i^{(2)}}-\pi_{ij}^{(2)}F_{\mu_j^{(2)}}-\phi_{ij}^{(2)T}F_{\xi_j}-2\sigma_i\\\Omega_{v_i^{(2)}} =&F_{u_i^{(2)}}+\pi_{ij}^{(2)}F_{\mu_j^{(2)}}+\phi_{ij}^{(2)T}F_{\xi_j}+2\sigma_i\\\Omega_{\mu_i^{(2)}} =&-2F_{u_i^{(2)}}-2F_{v_i^{(2)}}-F_{\mu_i^{(2)}}\\\Omega_{\xi_i} =&-\frac{1}{2}W_{ij}\phi_{jk}^{(2)}F_{\mu_k^{(2)}}+W_{ij}\rho_j\end{aligned}\tag{2.2.28}$$

$$\begin{aligned}2(U-V) =&F_{u_i^{(1)}}F_{u_i^{(1)}}+F_{u_i^{(2)}}F_{u_i^{(2)}}-F_{v_i^{(1)}}F_{v_i^{(1)}}-F_{v_i^{(2)}}F_{v_i^{(2)}}\\&-2F_{\mu_i^{(2)}}\sigma_i-2F_{\xi_i}\rho_i\end{aligned}\tag{2.2.29}$$

where $P+\Omega$ is the conserved momentum-like quantity.

At this point we can simplify these momentum conservation conditions significantly by introducing new fields $p_i=\frac{1}{2}(u_i+v_i)$, $q_i=\frac{1}{2}(u_i-v_i)$ and new quantities D, $\bar{D}$ with $F=D+\bar{D}$ and $\Omega=D-\bar{D}$. The vector fields p and q split into $p^{(1)}$, $p^{(2)}$ and $q^{(1)}$, $q^{(2)}$ in exactly the same way as the u and v vector fields split into $u^{(1)}$, $u^{(2)}$ and

$v^{(1)}$, $v^{(2)}$. The momentum conservation conditions in eq.(2.2.28) then simplify to

$$\begin{aligned}
\bar{D}_{p_i^{(1)}} =&0 \\
\bar{D}_{p_i^{(2)}} =&0 \\
D_{q_i^{(1)}} =&-A_{ij}D_{p_j^{(1)}} \\
D_{\mu_i^{(2)}} =&-D_{p_i^{(2)}} \\
2\sigma_i =&-D_{q_i^{(2)}}-\pi_{ij}^{(2)}\left(D_{\mu_j^{(2)}}+\bar{D}_{\mu_j^{(2)}}\right)-\phi_{ij}^{(2)T}\left(D_{\xi_j}+\bar{D}_{\xi_j}\right) \\
2\rho_i =&\phi_{ij}^{(2)}\left(D_{\mu_j^{(2)}}+\bar{D}_{\mu_j^{(2)}}\right)-2W_{ij}\left(D_{\xi_j}-\bar{D}_{\xi_j}\right).
\end{aligned} \tag{2.2.30}$$

The first four of these equations are satisfied if we require the dependencies of D and $\bar{D}$ to be

$$D=D\left(p^{(1)}+Aq^{(1)},p^{(2)}-\mu^{(2)},q^{(2)},\xi\right) \tag{2.2.31}$$

$$\bar{D}=\bar{D}\left(q^{(1)},q^{(2)},\mu^{(2)},\xi\right). \tag{2.2.32}$$

The second two equations simply set the two arbitrary vectors σ and ρ we introduced previously. Rewriting eq.(2.2.29) using eq.(2.2.30) and recalling A and $\pi^{(2)}$ are antisymmetric gives

$$2(U-V)=D_{p_i^{(1)}}\bar{D}_{q_i^{(1)}}+D_{q_i^{(2)}}\bar{D}_{\mu_i^{(2)}}-D_{\mu_i^{(2)}}\bar{D}_{q_i^{(2)}}-4D_{\xi_i}W_{ij}\bar{D}_{\xi_j}. \tag{2.2.33}$$

So a momentum conserving defect has a Lagrangian density which can, using field redefinitions, be written in the form given in eq.(2.2.26) and a defect potential given by $F=D+\bar{D}$ where quantities $D\left(p^{(1)}+Aq^{(1)},p^{(2)}-\mu^{(2)},q^{(2)},\xi\right)$, $\bar{D}\left(q^{(1)},q^{(2)},\mu^{(2)},\xi\right)$ satisfy the momentum conservation condition in eq.(2.2.33). The total conserved energy and momentum of the system are $E+D+\bar{D}$ and $P+D-\bar{D}$, where E and P are the bulk energy and momentum.

From the form of the momentum conservation conditions for D and $\bar{D}$ we can immediately see that multiplying D by a constant, which we will call σ, and multiplying $\bar{D}$ by σ^{-1} does not affect whether they satisfy the momentum conservation conditions. However, it will affect the defect contribution to the total energy and momentum, so the value of this constant is physically important. When we come to write down expressions for D and $\bar{D}$ we will see that D always has an overall multiplier of the arbitrary constant σ and $\bar{D}$ an overall multiplier of σ^{-1}. We call σ the defect parameter.

A redefinition $\mu_i^{(2)} \to \mu_i^{(2)} + f\left(q^{(2)}\right)_{q_i^{(2)}}$ does not alter the defect Lagrangian in eq.(2.2.26) as it only introduces a total t derivative. Redefinitions of the bulk fields which are the orthogonal transformations $u^{(1)} \to Qu^{(1)}$ and $v^{(1)} \to Q^T u^{(1)}$, or the orthogonal transformations $u^{(2)} \to Q'u^{(2)}$, $v^{(2)} \to Q'v^{(2)}$ and $\mu^{(2)} \to Q'^T\mu^{(2)}$, or the shifts $u \to u+c$, $v \to v+d$ (where Q and Q' are any orthogonal matrices and c and d are any constants) alter neither the bulk nor the defect Lagrangian. Therefore none of these redefinitions affect the defect equations or any of the subsequent working to find the momentum conservation condition in eq.(2.2.33). This means that once D and $\bar{D}$ satisfying the momentum conservation condition have been found these field redefinitions can be used to give a family of different defect potentials satisfying the same momentum conservation condition.

The equations of motion at the defect, with the defect Lagrangian given in eq.(2.2.26) with $F = D + \bar{D}$ and written in terms of $p_i = \frac{1}{2}(u_i + v_i)$, $q_i = \frac{1}{2}(u_i - v_i)$, are

$$p_{i,x}^{(1)} = p_{i,t}^{(1)} + 2A_{ij}q_{j,t}^{(1)} - \frac{1}{2}D_{q_i^{(1)}} - \frac{1}{2}\bar{D}_{q_i^{(1)}} \tag{2.2.34}$$

$$q_{i,x}^{(1)} = -q_{i,t}^{(1)} - \frac{1}{2}D_{p_i^{(1)}} \tag{2.2.35}$$

$$p_{i,x}^{(2)} = p_{i,t}^{(2)} - 2\mu_{i,t}^{(2)} - \frac{1}{2}D_{q_i^{(2)}} - \frac{1}{2}\bar{D}_{q_i^{(2)}} \tag{2.2.36}$$

$$q_{i,x}^{(2)} = -q_{i,t}^{(2)} - \frac{1}{2}D_{p_i^{(2)}} \tag{2.2.37}$$

$$0 = q_{i,t}^{(2)} - \frac{1}{4}D_{\mu_i^{(2)}} - \frac{1}{4}\bar{D}_{\mu_i^{(2)}} \tag{2.2.38}$$

$$0 = \xi_{i,t} + W_{ij}D_{\xi_j} + W_{ij}\bar{D}_{\xi_j} \tag{2.2.39}$$

2.3 Moving defects

So far we have only considered a defect at $x = 0$, but it is possible for a defect to have a time dependent position. In [BCZ05] a type I defect with a time dependent position y was considered. Requiring momentum conservation it was found that the defect contribution to momentum was the same as in the stationary case but with a shifted defect parameter σ. The same was true for the defect contribution for the energy. The scattering of a defect off another defect was then considered and the results used to find quantum scattering matrices. We shall carry out the same procedure to construct classical moving defects for the more general class of defects found in section 2.2. We will also consider the effect of Lorentz boosts specifically, based on similar calculations carried out for the type I sine-Gordon defect in [Bow17]. This consideration of moving defects is interesting in its own right and opens up the possibility of further investigations into defect-defect scattering (although we make no consideration of that here). However, more importantly for this thesis, it will provide some motivation for the method of obtaining Bäcklund transformations from the defect equations given in section 2.4. It transpires that in addition to the defect equations of motion for a stationary defect the defect equations for a defect moving with infinite velocity (a space-like defect) are necessary to find a Bäcklund transformation between the bulk theories on either side of the defect.

For a system with a defect at $x = y(t)$, the vector field u to the left of the defect and the vector field v to the right of the defect the Lagrangian density we choose is

$$\mathcal{L} = \Theta(y - x)\mathcal{L}^{(u)} + \Theta(x - y)\mathcal{L}^{(v)} + \delta(x - y)\mathcal{L}^{D}, \tag{2.3.1}$$

which gives the action

$$S = \int_{-\infty}^{\infty} \mathrm{d}t \left(\int_{-\infty}^{y} \mathrm{d}x \mathcal{L}^{(u)} + \int_{y}^{\infty} \mathrm{d}x \mathcal{L}^{(v)} + \mathcal{L}^{D}\Big|_{x=y} \right). \tag{2.3.2}$$

When varying this action and rewriting terms using total t derivatives we must use the Leibniz integral rule, as the limits on the x integrations are time dependent, and recall that the total t derivative of a function evaluated at a time dependent position is given by

$$\frac{\mathrm{d}f|_{x=y}}{\mathrm{d}t} = \left.(f_t + y_t f_x)\right|_{x=y}. \tag{2.3.3}$$

This then gives the Euler-Lagrange equations

$$x \leq y: \quad 0 = \mathcal{L}^{(u)}_{\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}^{(u)}_{\phi_t}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{L}^{(u)}_{\phi_x}\right) \tag{2.3.4}$$

$$x \geq y: \quad 0 = \mathcal{L}^{(v)}_{\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}^{(v)}_{\phi_t}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{L}^{(v)}_{\phi_x}\right) \tag{2.3.5}$$

$$x = y: \quad 0 = \mathcal{L}^{(u)}_{\phi_x} - y_t\mathcal{L}^{(u)}_{\phi_t} - \mathcal{L}^{(v)}_{\phi_x} + y_t\mathcal{L}^{(v)}_{\phi_t} + \mathcal{L}^{D}_{\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}^{D}_{\phi_t}\right) - y_t\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{L}^{D}_{\phi_t}\right) \tag{2.3.6}$$

$$0 = y_t\mathcal{L}^{D}_{\phi_t} - \mathcal{L}^{D}_{\phi_x} \tag{2.3.7}$$

for $\phi = u_i^1, u_i^2, v_i^1, v_i^2, \mu_i^2, \xi_i$. Evidently some of these equations will automatically be satisfied due to $\mathcal{L}^{(u,v)}$ not depending on all fields. For eq.(2.3.7) to be satisfied all derivatives in $\mathcal{L}^D$ must appear as $\partial_t + y_t\partial_x$, that is, they must be along the tangent to the path of the defect. For any field evaluated at the defect its total t derivative is given by eq.(2.3.3).

We take the bulk Lagrangians to be the same as those used in section 2.3, given in eqs.(2.1.2),(2.1.3). Rather than working through momentum conservation completely for a moving defect we instead assume that the defect Lagrangian will be of a similar form to that given in eq.(2.2.26). Replacing all instances of ∂_t appearing there with $\partial_t + y_t\partial_x$ we have the defect Lagrangian

$$\begin{aligned}\mathcal{L}^D =& \frac{1}{2}u_i^{(1)}A_{ij}\left(u_{j,t}^{(1)} + y_t u_{j,x}^{(1)}\right) + \frac{1}{2}v_i^{(1)}A_{ij}\left(v_{j,t}^{(1)} + y_t v_{j,x}^{(1)}\right)\\ &+ u_i^{(1)}\left(\mathbb{1} - A\right)_{ij}\left(v_{j,t}^{(1)} + y_t v_{j,x}^{(1)}\right) + u_i^{(2)}\left(v_{i,t}^{(2)} + y_t v_{i,x}^{(2)}\right)\\ &+ 2\mu_i^{(2)}\left(u_{i,t}^{(2)} + y_t u_{i,x}^{(2)} - v_{i,t}^{(2)} - y_t v_{i,x}^{(2)}\right) + \frac{1}{2}\xi_i W_{ij}\left(\xi_{j,t} + y_t\xi_{j,x}\right) - F.\end{aligned} \tag{2.3.8}$$

Now we can check the energy and momentum conservation of this system. Taking the bulk energy and momentum in eqs.(2.2.14),(2.2.17) we replace 0 with y as the position of the defect and differentiate with respect to t to get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(u_{i,x}u_{i,t} - v_{i,x}v_{i,t} + y_t\left(\frac{1}{2}\left(u_{i,t}u_{i,t} + u_{i,x}u_{i,x} - v_{i,t}v_{i,t} - v_{i,x}v_{i,x}\right) + U - V\right)\right)\Big|_{x=y} \tag{2.3.9}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \left(y_t\left(u_{i,x}u_{i,t} - v_{i,x}v_{i,t}\right) + \frac{1}{2}\left(u_{i,t}u_{i,t} + u_{i,x}u_{i,x} - v_{i,t}v_{i,t} - v_{i,x}v_{i,x}\right) - U + V\right)\Big|_{x=y}. \tag{2.3.10}$$

Taking the defect contributions to the total energy and momentum to be Ψ and Ω respectively we require

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mathrm{d}\Psi|_{x=y}}{\mathrm{d}t} = -\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t} + y_t\frac{\mathrm{d}\Psi}{\mathrm{d}x}\right)\Big|_{x=y} = -\sum_{\phi}\Psi_{\phi}\left(\phi_t + y_t\phi_x\right)|_{x=y} \tag{2.3.11}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{\mathrm{d}\Omega|_{x=y}}{\mathrm{d}t} = -\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t} + y_t\frac{\mathrm{d}\Omega}{\mathrm{d}x}\right)\Big|_{x=y} = -\sum_{\phi}\Omega_{\phi}\left(\phi_t + y_t\phi_x\right)|_{x=y} \tag{2.3.12}$$

where ϕ again runs over all fields.

We will work with fields $p_i = \frac{1}{2}(u_i + v_i)$ and $q_i = \frac{1}{2}(u_i - v_i)$ rather than u_i and v_i. The defect Euler-Lagrange equations in eq.(2.3.6) (along with the Lagrangian in eq.(2.3.8)) then become

$$y_t p^{(1)}_{i,t} + p^{(1)}_{i,x} = p^{(1)}_{i,t} + y_t p^{(1)}_{i,x} + 2A_{ij}\left(q^{(1)}_{j,t} + y_t q^{(1)}_{j,x}\right) - \frac{1}{2}F_{q^{(1)}_i} \tag{2.3.13}$$

$$y_t q^{(1)}_{i,t} + q^{(1)}_{i,x} = -q^{(1)}_{i,t} - y_t q^{(1)}_{i,x} - \frac{1}{2}F_{p^{(1)}_i} \tag{2.3.14}$$

$$y_t p^{(2)}_{i,t} + p^{(2)}_{i,x} = p^{(2)}_{i,t} + y_t p^{(2)}_{i,x} - 2\left(\mu^{(2)}_{i,t} + y_t\mu^{(2)}_{i,x}\right) - \frac{1}{2}F_{q^{(2)}_i} \tag{2.3.15}$$

$$y_t q^{(2)}_{i,t} + q^{(2)}_{i,x} = -q^{(2)}_{i,t} - y_t q^{(2)}_{i,x} - \frac{1}{2}F_{p^{(2)}_i} \tag{2.3.16}$$

$$0 = q^{(2)}_{i,t} + y_t q^{(2)}_{i,x} - \frac{1}{4}F_{\mu^{(2)}_i} \tag{2.3.17}$$

$$0 = \xi_{i,t} + y_t\xi_{i,x} + W_{ij}F_{\xi_j} \tag{2.3.18}$$

where all equations are evaluated at $x = y$. Writing the energy and momentum conservation conditions in eqs.(2.3.9),(2.3.10) in terms of p and q, rearranging these so that all derivatives appear as $\partial_t + y_t\partial_x$ or $y_t\partial_t + \partial_x$ and then using eqs.(2.3.13)-

(2.3.16) to remove all $y_t\partial_t + \partial_x$ derivatives gives

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} =& -\frac{4}{1-y_t}\left(q_{it}^{(2)} + y_t q_{ix}^{(2)}\right)\left(\mu_{it}^{(2)} + y_t \mu_{ix}^{(2)}\right) \\
& -\frac{1}{1+y_t}\left(\left(p_{it}^{(1)} + y_t p_{ix}^{(1)}\right)F_{p_i^{(1)}} + \left(p_{it}^{(2)} + y_t p_{ix}^{(2)}\right)F_{p_i^{(2)}}\right) \\
& -\frac{1}{1-y_t}\left(\left(q_{it}^{(1)} + y_t q_{ix}^{(1)}\right)F_{q_i^{(1)}} + \left(q_{it}^{(2)} + y_t q_{ix}^{(2)}\right)F_{q_i^{(2)}}\right) \\
& -\frac{2y_t}{1-y_t^2}\left(\left(q_{it}^{(1)} + y_t q_{ix}^{(1)}\right)A_{ij}F_{p_j^{(1)}} + \left(\mu_{it}^{(2)} + y_t \mu_{ix}^{(2)}\right)F_{p_i^{(2)}}\right. \\
& \qquad\qquad \left. + \frac{1}{4}\left(F_{p_i^{(1)}}F_{q_i^{(1)}} + F_{p_i^{(2)}}F_{q_i^{(2)}}\right)\right) \\
& + y_t(U-V)
\end{aligned} \tag{2.3.19}$$

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} =& \frac{4}{1-y_t}\left(q_{it}^{(2)} + y_t q_{ix}^{(2)}\right)\left(\mu_{it}^{(2)} + y_t \mu_{ix}^{(2)}\right) \\
& -\frac{1}{1+y_t}\left(\left(p_{it}^{(1)} + y_t p_{ix}^{(1)}\right)F_{p_i^{(1)}} + \left(p_{it}^{(2)} + y_t p_{ix}^{(2)}\right)F_{p_i^{(2)}}\right) \\
& +\frac{1}{1-y_t}\left(\left(q_{it}^{(1)} + y_t q_{ix}^{(1)}\right)F_{q_i^{(1)}} + \left(q_{it}^{(2)} + y_t q_{ix}^{(2)}\right)F_{q_i^{(2)}}\right) \\
& +\frac{2}{1-y_t^2}\left(\left(q_{it}^{(1)} + y_t q_{ix}^{(1)}\right)A_{ij}F_{p_j^{(1)}} + \left(\mu_{it}^{(2)} + y_t \mu_{ix}^{(2)}\right)F_{p_i^{(2)}}\right. \\
& \qquad\qquad \left. + \frac{1}{4}\left(F_{p_i^{(1)}}F_{q_i^{(1)}} + F_{p_i^{(2)}}F_{q_i^{(2)}}\right)\right) \\
& - U + V.
\end{aligned} \tag{2.3.20}$$

This obviously excludes $y_t = \pm 1$, so we will have to consider these cases separately.

To the energy conservation equation (2.3.19) we will add eq.(2.3.17) multiplied by by $4(1-y_t)^{-1}\left(\mu_{it}^{(2)} + y_t\mu_{ix}^{(2)}\right) + \sigma_i^E$ and eq.(2.3.18) multiplied by ρ_i^E, and to the momentum conservation equation (2.3.20) we will add eq.(2.3.17) multiplied by $-4(1-y_t)^{-1}\left(\mu_{it}^{(2)} + y_t\mu_{ix}^{(2)}\right) + \sigma_i^P$ and eq.(2.3.18) multiplied by by ρ_i^P. The vectors $\sigma^{E,P}$ and $\rho^{E,P}$ may contain any functions of the fields. This gives

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} =& -\frac{1}{1+y_t}\left(p_{it}^{(1)} + y_t p_{ix}^{(1)}\right)F_{p_i^{(1)}} - \frac{1}{1+y_t}\left(p_{it}^{(2)} + y_t p_{ix}^{(2)}\right)F_{p_i^{(2)}} \\
& -\left(q_{it}^{(1)} + y_t q_{ix}^{(1)}\right)\left(\frac{2y_t}{1-y_t^2}A_{ij}F_{p_j^{(1)}} + \frac{1}{1-y_t}F_{q_i^{(1)}}\right) \\
& -\left(q_{it}^{(2)} + y_t q_{ix}^{(2)}\right)\left(\frac{1}{1-y_t}F_{q_i^{(2)}} - \sigma_i^E\right) \\
& -\left(\mu_{it}^{(2)} + y_t \mu_{ix}^{(2)}\right)\left(\frac{2y_t}{1-y_t^2}F_{p_i^{(2)}} + \frac{1}{1-y_t}F_{\mu_i^{(2)}}\right) + \left(\xi_{it} + y_t\xi_{ix}\right)\rho_i^E
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\frac{y_t}{1-y_t^2}\left(F_{p_i^{(1)}}F_{q_i^{(1)}} + F_{p_i^{(2)}}F_{q_i^{(2)}}\right) - \frac{1}{4}F_{\mu_i^{(2)}}\sigma_i^E - F_{\xi_i}W_{ij}\rho_j^E \\
&+ y_t(U-V) \qquad (2.3.21)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} =& - \frac{1}{1+y_t}\left(p_{it}^{(1)} + y_t p_{ix}^{(1)}\right)F_{p_i^{(1)}} - \frac{1}{1+y_t}\left(p_{it}^{(2)} + y_t p_{ix}^{(2)}\right)F_{p_i^{(2)}} \\
&+ \left(q_{it}^{(1)} + y_t q_{ix}^{(1)}\right)\left(\frac{2}{1-y_t^2}A_{ij}F_{p_j^{(1)}} + \frac{1}{1-y_t}F_{q_i^{(1)}}\right) \\
&+ \left(q_{it}^{(2)} + y_t q_{ix}^{(2)}\right)\left(\frac{1}{1-y_t}F_{q_i^{(2)}} + \sigma_i^P\right) \\
&+ \left(\mu_{it}^{(2)} + y_t \mu_{ix}^{(2)}\right)\left(\frac{2}{1-y_t^2}F_{p_i^{(2)}} + \frac{1}{1-y_t}F_{\mu_i^{(2)}}\right) + \left(\xi_{it} + y_t\xi_{ix}\right)\rho_i^P \\
&+ \frac{1}{2}\frac{1}{1-y_t^2}\left(F_{p_i^{(1)}}F_{q_i^{(1)}} + F_{p_i^{(2)}}F_{q_i^{(2)}}\right) - \frac{1}{4}F_{\mu_i^{(2)}}\sigma_i^P - F_{\xi_i}W_{ij}\rho_j^P \\
&- U + V. \qquad (2.3.22)
\end{aligned}
$$

In the stationary case we immediately recovered $\Psi = F$ and then set $F = D + \bar{D}$ and $\Omega = D - \bar{D}$. We no longer have $\Psi = F$ but can still define functions D and $\bar{D}$ such that $\Psi = D + \bar{D}$ and $\Omega = D - \bar{D}$. Substituting these into eqs.(2.3.11),(2.3.12) and comparing these equations to eqs.(2.3.21),(2.3.22) we find that the defect contributions to the energy and momentum and the defect potential must satisfy the following constraints:

$$
\begin{aligned}
D_{p_i^{(1)}} =& \frac{1}{1+y_t}F_{p_i^{(1)}} \\
D_{p_i^{(2)}} =& \frac{1}{1+y_t}F_{p_i^{(2)}} \\
D_{q_i^{(1)}} =& - \frac{1}{1+y_t}A_{ij}F_{p_j^{(1)}} \\
D_{q_i^{(2)}} =& - \frac{1}{2}\left(\sigma_i^E + \sigma_i^P\right) \\
D_{\mu_i^{(2)}} =& - \frac{1}{1+y_t}F_{p_i^{(2)}} \\
D_{\xi_i} =& - \frac{1}{2}\left(\rho_i^E + \rho_i^P\right) \\
\bar{D}_{p_i^{(1)}} =& 0 \\
\bar{D}_{p_i^{(2)}} =& 0 \\
\bar{D}_{q_i^{(1)}} =& \frac{1}{1-y_t}A_{ij}F_{p_j^{(1)}} + \frac{1}{1-y_t}F_{q_i^{(1)}}
\end{aligned}
$$

$$\begin{aligned}
\bar{D}_{q_i^{(2)}} =& \frac{1}{1-y_t} F_{q_i^{(2)}} - \frac{1}{2}\left(\sigma_i^E - \sigma_i^P\right) \\
\bar{D}_{\mu_i^{(2)}} =& \frac{1}{1-y_t} F_{p_i^{(2)}} + \frac{1}{1-y_t} F_{\mu_i^{(2)}} \\
\bar{D}_{\xi_i} =& -\frac{1}{2}\left(\rho_i^E - \rho_i^P\right).
\end{aligned} \tag{2.3.23}$$

Four of these constraints simply set the arbitrary vectors $\sigma^{E,P}$ and $\rho^{E,P}$, with the remaining constraints setting the dependencies of D and $\bar{D}$ to

$$D = D(p^{(1)} + Aq^{(1)}, p^{(2)} - \mu^{(2)}, q^{(2)}, \xi) \tag{2.3.24}$$

$$\bar{D} = \bar{D}(q^{(1)}, q^{(2)}, \mu^{(2)}, \xi) \tag{2.3.25}$$

$$\tag{2.3.26}$$

and the defect potential to

$$F = (1 + y_t)D + (1 - y_t)\bar{D}. \tag{2.3.27}$$

The final step to ensure energy and momentum conservation is to require the terms containing no derivatives in eqs.(2.3.21),(2.3.22) to vanish. Using the above constraints both the energy and momentum conservation conditions are

$$2(U - V) = D_{p_i^{(1)}} \bar{D}_{q_i^{(1)}} + D_{q_i^{(2)}} \bar{D}_{\mu_i^{(2)}} - D_{\mu_i^{(2)}} \bar{D}_{q_i^{(2)}} - 4 D_{\xi_i} W_{ij} \bar{D}_{\xi_j}. \tag{2.3.28}$$

This is identical to the condition on D and $\bar{D}$ found in the stationary case and given in eq.(2.2.29).

We have not yet shown that a defect moving along any trajectory may be momentum conserving, as eqs.(2.3.9),(2.3.10) were not valid for $y_t = \pm 1$. However, taking the defect equations in eqs.(2.3.13)-(2.3.18) with $y_t = \pm 1$ and carrying out the full energy-momentum conservation calculation for these two specific cases we find that the defect potential is still given by eq.(2.3.27) and D and $\bar{D}$ must still obey the constraints in eqs.(2.3.24),(2.3.25),(2.3.28).

So, if we have found D and $\bar{D}$ which satisfy eqs.(2.3.24),(2.3.25),(2.3.28) for a stationary defect in a certain bulk theory then we can construct a defect moving along an arbitrary path by taking the defect Lagrangian given in eq.(2.3.8) and the defect potential given in eq.(2.3.27). The equations of motion for such a defect are

$$(y_t-1)(p^{(1)}_{i,t}-p^{(1)}_{i,x}) = 2A_{ij}\left(q^{(1)}_{j,t}+y_tq^{(1)}_{j,x}\right)-\frac{1+y_t}{2}D_{q^{(1)}_i}-\frac{1-y_t}{2}\bar{D}_{q^{(1)}_i} \tag{2.3.29}$$

$$(y_t+1)(q^{(1)}_{i,t}+q^{(1)}_{i,x}) = -\frac{1+y_t}{2}D_{p^{(1)}_i} \tag{2.3.30}$$

$$(y_t-1)(p^{(2)}_{i,t}-p^{(2)}_{i,x}) = -2\left(\mu^{(2)}_{i,t}+y_t\mu^{(2)}_{i,x}\right)-\frac{1+y_t}{2}D_{q^{(2)}_i}-\frac{1-y_t}{2}\bar{D}_{q^{(2)}_i} \tag{2.3.31}$$

$$(y_t+1)(q^{(2)}_{i,t}+q^{(2)}_{i,x}) = -\frac{1+y_t}{2}D_{p^{(2)}_i} \tag{2.3.32}$$

$$0 = q^{(2)}_{i,t}+y_tq^{(2)}_{i,x}-\frac{1+y_t}{4}D_{\mu^{(2)}_i}-\frac{1-y_t}{4}\bar{D}_{\mu^{(2)}_i} \tag{2.3.33}$$

$$0 = \xi_{i,t}+y_t\xi_{i,x}+(1+y_t)W_{ij}D_{\xi_j}+(1-y_t)W_{ij}\bar{D}_{\xi_j}. \tag{2.3.34}$$

We will now consider the particular case where we take a stationary defect and apply a Lorentz boost. The bulk theory is relativistic for all defects considered in this thesis. The coordinate transformation

$$t = \cosh(\eta)t'+\sinh(\eta)x' \tag{2.3.35}$$

$$x = \sinh(\eta)t'+\cosh(\eta)x' \tag{2.3.36}$$

is a Lorentz boost with rapidity η. Applying this boost to the momentum conserving defect whose Lagrangian is given in eq.(2.2.26), which is stationary in the (t',x') frame and whose potential we denote as $F'=D'+\bar{D}'$, gives the defect Lagrangian in eq.(2.3.8) with $y_t=\tanh(\eta)$ and defect potential $F=(\cosh(\eta))^{-1}F'$. We have made use of the relations $\Theta(\cosh(\eta)x-\sinh(\eta)t)=\Theta(x-\tanh(\eta)t)$ and $\delta(\cosh(\eta)x-\sinh(\eta)t)=(\cosh(\eta))^{-1}\delta(x-\tanh(\eta)t)$. As the moving defect potential must be of the form given in eq.(2.3.27) for the defect to be momentum conserving we have

$$D = \frac{D'}{\cosh(\eta)(1+\tanh(\eta))}=e^{-\eta}D' \tag{2.3.37}$$

$$\bar{D} = \frac{\bar{D}'}{\cosh(\eta)(1-\tanh(\eta))}=e^{\eta}\bar{D}'. \tag{2.3.38}$$

Recalling that D contains the defect parameter σ (and $\bar{D}$ contains $\frac{1}{\sigma}$) we see that, having taken a stationary defect with defect parameter σ and applied a Lorentz boost with rapidity η, we must also scale the defect parameter by $e^{-\eta}$ if the boosted defect is to be momentum conserving.

Finally we consider $y_t \to -\infty$, or a space-like defect at $t = 0$. This has the Lagrangian density

$$\mathcal{L} = \Theta(-t)\mathcal{L}^{(u)} + \Theta(t)\mathcal{L}^{(v)} + \delta(t)\mathcal{L}^{D} \tag{2.3.39}$$

where $\mathcal{L}^D$ is not dependent on the t derivatives of any field. From eq.(2.3.8) we take the defect Lagrangian to be

$$\begin{aligned}\mathcal{L}^D =& \frac{1}{2}u_i^{(1)}A_{ij}u_{j,x}^{(1)} + \frac{1}{2}v_i^{(1)}A_{ij}v_{j,x}^{(1)} + u_i^{(1)}\left(\mathbb{1} - A\right)_{ij}v_{j,x}^{(1)} \\ &+ u_i^{(2)}v_{i,x}^{(2)} + 2\mu_i^{(2)}\left(u_{i,x}^{(2)} - v_{i,x}^{(2)}\right) + \frac{1}{2}\xi_i W_{ij}\xi_{j,x} - D + \bar{D},\end{aligned} \tag{2.3.40}$$

where D and $\bar{D}$ have the dependencies given in eqs.(2.2.31),(2.2.32) and satisfy the condition in eq.(2.2.33). Moving to fields $p_i = \frac{1}{2}(u_i + v_i)$, $q_i = \frac{1}{2}(u_i - v_i)$ this defect has the equations of motion

$$p_{i,t}^{(1)} = p_{i,x}^{(1)} + 2A_{ij}q_{j,x}^{(1)} - \frac{1}{2}D_{q_i^{(1)}} + \frac{1}{2}\bar{D}_{q_i^{(1)}} \tag{2.3.41}$$

$$q_{i,t}^{(1)} = -q_{i,x}^{(1)} - \frac{1}{2}D_{p_i^{(1)}} \tag{2.3.42}$$

$$p_{i,t}^{(2)} = p_{i,x}^{(2)} - 2\mu_{i,x}^{(2)} - \frac{1}{2}D_{q_i^{(2)}} + \frac{1}{2}\bar{D}_{q_i^{(2)}} \tag{2.3.43}$$

$$q_{i,t}^{(2)} = -q_{i,x}^{(2)} - \frac{1}{2}D_{p_i^{(2)}} \tag{2.3.44}$$

$$0 = q_{i,x}^{(2)} - \frac{1}{4}D_{\mu_i^{(2)}} + \frac{1}{4}\bar{D}_{\mu_i^{(2)}} \tag{2.3.45}$$

$$0 = \xi_{i,x} + W_{ij}D_{\xi_j} - W_{ij}\bar{D}_{\xi_j} \tag{2.3.46}$$

evaluated at $t = 0$. Note that these are the stationary defect Lagrangian in eq.(2.2.26) and stationary defect equations in eqs.(2.2.26),(2.2.34)-(2.2.39) with $t \leftrightarrow x$ and $\bar{D} \to -\bar{D}$.

For this system the energy and momentum before and after the defect occurs are given by

$$E^{(u)} = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{1}{2}\left(u_{i,t}u_{i,t} + u_{i,x}u_{i,x}\right) + U\right) \qquad t \leq 0$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x \Big(\frac{1}{2}(p_{i,t}p_{i,t} + 2p_{i,t}q_{i,t} + q_{i,t}q_{i,t}$$
$$+ p_{i,x}p_{i,x} + 2p_{i,x}q_{i,x} + q_{i,x}q_{i,x}) + U \Big) \tag{2.3.47}$$

$$E^{(v)} = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{1}{2} (v_{i,t}v_{i,t} + v_{i,x}v_{i,x}) + V \right) \qquad t \geq 0$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x \Big(\frac{1}{2}(p_{i,t}p_{i,t} - 2p_{i,t}q_{i,t} + q_{i,t}q_{i,t}$$
$$+ p_{i,x}p_{i,x} - 2p_{i,x}q_{i,x} + q_{i,x}q_{i,x}) + V \Big) \tag{2.3.48}$$

$$P^{(u)} = \int_{-\infty}^{\infty} \mathrm{d}x \, (u_{i,t}u_{i,x}) \qquad t \leq 0$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x \, (p_{i,t}p_{i,x} + p_{i,t}q_{i,x} + q_{i,t}p_{i,x} + q_{i,t}q_{i,x}) \tag{2.3.49}$$

$$P^{(v)} = \int_{-\infty}^{\infty} \mathrm{d}x \, (v_{i,t}v_{i,x}) \qquad t \geq 0$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x \, (p_{i,t}p_{i,x} - p_{i,t}q_{i,x} - q_{i,t}p_{i,x} + q_{i,t}q_{i,x}) \quad . \tag{2.3.50}$$

For $t \neq 0$ we can easily check that these quantities are conserved by differentiating with respect to t, then using the bulk equations of motion to rewrite the integrands as total x derivatives. Using $u, v \to$ constant as $x \to \pm\infty$ and U and V having no local minima, only global minima we evaluate the integrals to find that they are zero. In order to ensure energy and momentum conservation across the defect we simply need to ensure that these quantities match at $t = 0$. To evaluate the quantities

$$E^{(u)} - E^{(v)} = \int_{-\infty}^{\infty} \mathrm{d}x \, (2p_{i,t}q_{i,t} + 2p_{i,x}q_{i,x} + U - V) \tag{2.3.51}$$
$$P^{(u)} - P^{(v)} = \int_{-\infty}^{\infty} \mathrm{d}x \, (2p_{i,t}q_{i,x} + 2q_{i,t}p_{i,x}) \tag{2.3.52}$$

we use eqs.(2.3.41)-(2.3.44) to remove the t derivatives. For eq.(2.3.51) we also use the condition on D and $\bar{D}$ in eq.(2.2.33) to remove U and V. Adding eq.(2.3.45) multiplied by $-2D_{q_i^{(2)}}$ and eq.(2.3.46) multiplied by $-(D_{\xi_i} + \bar{D}_{\xi_i})$ to eq.(2.3.51), and eq.(2.3.45) multiplied by $4q_{i,x}^{(2)}$ and eq.(2.3.46) multiplied by $-(D_{\xi_i} - \bar{D}_{\xi_i})$ to eq.(2.3.52), leaves us with

$$E^{(u)} - E^{(v)} = \int_{-\infty}^{\infty} \mathrm{d}x \Big(- p_{i,x}^{(1)} D_{p_i^{(1)}} - p_{i,x}^{(2)} D_{p_i^{(2)}} - q_{i,x}^{(1)} D_{q_i^{(1)}} - q_{i,x}^{(2)} D_{q_i^{(2)}} - \mu_{i,x}^{(2)} D_{\mu_i^{(2)}} - \xi_{i,x} D_{\xi_i}$$
$$- q_{i,x}^{(1)} \bar{D}_{q_i^{(1)}} - q_{i,x}^{(2)} \bar{D}_{q_i^{(2)}} - \mu_{i,x}^{(2)} \bar{D}_{\mu_i^{(2)}} - \xi_{i,x} \bar{D}_{\xi_i} \Big) \tag{2.3.53}$$

$$P^{(u)} - P^{(v)} = \int_{-\infty}^{\infty} \mathrm{d}x \Big(- p_{i,x}^{(1)} D_{p_i^{(1)}} - p_{i,x}^{(2)} D_{p_i^{(2)}} - q_{i,x}^{(1)} D_{q_i^{(1)}} - q_{i,x}^{(2)} D_{q_i^{(2)}} - \mu_{i,x}^{(2)} D_{\mu_i^{(2)}} - \xi_{i,x} D_{\xi_i} + q_{i,x}^{(1)} \bar{D}_{q_i^{(1)}} + q_{i,x}^{(2)} \bar{D}_{q_i^{(2)}} + \mu_{i,x}^{(2)} \bar{D}_{\mu_i^{(2)}} + \xi_{i,x} \bar{D}_{\xi_i} \Big). \tag{2.3.54}$$

The integrands are now total x derivatives, and so can be evaluated as

$$E^{(u)} - E^{(v)} = -(D + \bar{D})\Big|_{x=\infty} + (D + \bar{D})\Big|_{x=-\infty} \tag{2.3.55}$$

$$P^{(u)} - P^{(v)} = -(D - \bar{D})\Big|_{x=\infty} + (D - \bar{D})\Big|_{x=-\infty}. \tag{2.3.56}$$

So in order for the energy and momentum to match across the defect both D and $\bar{D}$ must have no local minima, only global minima.

In these calculations we only needed to make use of the restriction on D and $\bar{D}$ given in eq.(2.2.33) in the energy conservation calculation. This condition originally arose from requiring momentum conservation in a non translationally invariant system. Here the defect causes the system to no longer be time translationally invariant, so it is not surprising that energy conservation is now the more restrictive condition.

2.4 Defects and Bäcklund transformations

A Bäcklund transformation is a set of coupled first order differential equations whose solutions also satisfy two sets of uncoupled higher order differential equations. These transformations are very closely linked with soliton theory and integrable systems; for more information on Bäcklund transformations and integrability see [Miu76; For90; RS02]. Bäcklund transformations can be used to obtain the soliton solutions of some integrable systems [Lam67; LOT93]. Bäcklund transformations are known for the Kortweg-de Vries equation (the original integrable system with solitons) [WE73] and sine-Gordon [DB76], with the sine-Gordon Bäcklund transformations being generalised to the rest of the A_r ATFTs in [FG80]. The Bäcklund transformation for the Tzitzéica model is also known [BSS93].

In [BCZ04b; BCZ04a] it was noticed that the type I defect equations will, if taken to hold everywhere, provide a Bäcklund transformation for the bulk equations of motion. This link between defects and Bäcklund transformations is not so surprising, as the defects constructed here have defect equations which are first order and couple the fields which appear in the bulk theories on either side of the defect. The link between Bäcklund transformations, defects and integrability has been investigated in [Cau08; AAGZ11; CK15]. It is hoped that this observation can be used to find new Bäcklund transformations for the ATFTs not based on the simple roots of A_r.

However, the defect equations for a type II defect do not give a Bäcklund transformation directly. In [CZ09a] a new Bäcklund transformation of the Tzitzéica model was found by considering the Bäcklund transformation arising from the type I A_2 defect and then folding this model to the Tzitzéica model. In doing so the defect equations for a momentum conserving Tzitzéica defect were retrieved and an additional equation also appeared. Taking the set of momentum conserving defect equations and adding to that the set of defect equations with $t \leftrightarrow x$ and $\bar{D} \to -\bar{D}$, whilst assuming these equations hold simultaneously and over all space, gave the same set of equations as were obtained by folding the A_2 Bäcklund transformation. For more information on the Tzitzéica Bäcklund transformations see references within [CZ09a].

As we are attempting to find Bäcklund transformations for a general field theory with the bulk Lagrangians as given in eqs.(2.1.2),(2.1.3), which is obviously not obtained by folding an A_r ATFT, this observation is crucial. The main stumbling block in getting a Bäcklund transformation directly from the type II defect equations is that these equations involve the auxiliary fields, which are only defined at $x = 0$. However the procedure described above will introduce x derivatives of these fields to the equations. Note that this procedure applied to type I defect equations leaves them unchanged. There may also be some link here with our consideration of moving defects in section 2.3. There we suggested that a defect at a fixed time would have defect equations which were given by taking the defect equations of a momentum conserving defect at a fixed point with $t \leftrightarrow x$ and $\bar{D} \to -\bar{D}$. It may be that we can think of the Bäcklund transformation arising from a general type II defect as the defect equations for both a defect stationary in space and a defect stationary in time, taken to hold simultaneously and over all space and time.

The equations of motion for a momentum conserving defect at $x = 0$ are given in eqs.(2.2.34)-(2.2.39) and these equations with $t \leftrightarrow x$ and $\bar{D} \to -\bar{D}$, that is the equations of motion for a momentum conserving defect at $t = 0$, are given in eqs.(2.3.41)-(2.3.46). Taking this set of twelve equations to hold simultaneously and over all time and space we move to light cone coordinates $x_\pm = \frac{1}{2}(t \pm x)$ (denoting $\partial_{x_\pm}$ as $\partial_\pm$). Removing any repeated equations and rearranging the remaining equations to simplify them gives

$$
\begin{aligned}
p^{(1)}_{i,-} + A_{ij} q^{(1)}_{j,-} =& \frac{1}{2} \bar{D}_{q^{(1)}_i} && (2.4.1)\\
p^{(2)}_{i,-} - \mu^{(2)}_{i,-} =& -\frac{1}{2} \bar{D}_{q^{(2)}_i} && (2.4.2)\\
q^{(1)}_{i,+} =& -\frac{1}{2} D_{p^{(1)}_i} && (2.4.3)\\
q^{(2)}_{i,+} =& -\frac{1}{2} D_{p^{(2)}_i} && (2.4.4)\\
q^{(2)}_{i,-} =& \frac{1}{2} \bar{D}_{\mu^{(2)}_i} && (2.4.5)\\
\mu^{(2)}_{i,+} =& \frac{1}{2} D_{q^{(2)}_i} && (2.4.6)\\
\xi_{i,+} =& -2W_{ij} D_{\xi_j} && (2.4.7)\\
\xi_{i,-} =& -2W_{ij} \bar{D}_{\xi_j}. && (2.4.8)
\end{aligned}
$$

Cross-differentiating these equations and using the dependencies of D and $\bar{D}$ given in eqs.(2.2.31),(2.2.32) and the fact that D and $\bar{D}$ must obey the momentum conservation condition in eq.(2.2.33) we can easily see that these give the bulk equations of motion for vector fields p and q, and also some bulk equations of motion for what were the auxiliary vector fields $\mu^{(2)}$ and ξ.

So the systems of equations $u_{i,tt} - u_{i,xx} + U(u) = 0$ and $v_{i,tt} - v_{i,xx} + V(v) = 0$ where $u_i = p_i + q_i$, $v_i = p_i - q_i$ have a Bäcklund transformation given by eqs.(2.4.1)-(2.4.8) if quantities $D\left(p^{(1)} + Aq^{(1)}, p^{(2)} - \mu^{(2)}, q^{(2)}, \xi\right)$ and $\bar{D}\left(q^{(1)}, q^{(2)}, \mu^{(2)}, \xi\right)$ can be found which satisfy eq.(2.2.33). Here A can be any antisymmetric matrix, W is given by eq.(2.2.5), the bulk fields may be divided between $p^{(1)}$, $q^{(1)}$ and $p^{(2)}$, $q^{(2)}$ in any way and the auxiliary fields may be divided between $\mu^{(2)}$ and ξ in any way as long as $p^{(1)}$ and $q^{(1)}$ are the same length, $p^{(2)}$, $q^{(2)}$ and $\mu^{(2)}$ are the same length and ξ contains an even number of fields due to the form of the matrix W.

Chapter 3

Momentum conserving defects in affine Toda field theory

3.1 Introduction

The ATFTs began life as a description of a one-dimensional lattice of particles with nearest-neighbour interactions, which was shown to be integrable with soliton solutions [Tod70]. The potential of this system contained terms of the form $e^{u_{i-1}-u_i}$, where u_i is the position of particle i, and in [Bog76] these potential terms were generalised to depend on the simple roots of any Lie algebra. In [Mik79] the Toda lattice is taken to a two-dimensional field theory for the A_r and Tzitzéica cases. All ATFTs are given in [MOP81] and their conserved quantities are investigated. ATFTs have been proven to be integrable [Wil81; OT83b; OT85]. The quantum ATFTs have also been shown to be integrable [AFZ79; BCDS90; Dor91; BCDS91; Dor92], although we will not discuss quantum ATFTs or defects here.

In appendix A we make a brief run through some properties of Lie algebras, their representations and their roots and weights to establish notation and recap some important properties which will be useful in this chapter and in chapter 4.

An ATFT is described by the Lagrangian density

$$\mathcal{L}^{(u)} = \frac{1}{2}u_{i,t}u_{i,t} - \frac{1}{2}u_{i,x}u_{i,x} - U \qquad U = \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i e^{\beta(\alpha_i)_j u_j} \tag{3.1.1}$$

where α_i $(i = 1, \ldots, r)$ are the simple root vectors of a Lie algebra as given in eqs.(A.0.18)-(A.0.26), n_i $(i = 1, \ldots, r)$ are a set of integers characteristic of each algebra as given in eqs.(A.0.27)-(A.0.35), $n_0 = 1$ and $\alpha_0 = -\sum_{i=1}^{r} n_i\alpha_i$ gives the root which corresponds to the extra node on an affine Dynkin diagram as given in eq.(A.0.36) [MOP81; OT83a; OT83b]. m is the mass constant, β is the coupling constant and in the classical case we can rescale the field u and the variables t and x to set $m = \beta = 1$. The vector $u = (u_1, \ldots, u_r)^T$ lies in the space spanned by the simple root vectors and the fields $\{u_i\}$ are the projections of u onto the basis of this vector space.

If the term containing the lowest weight root α_0 is not included in this potential then we have the Toda field theories. However, these have no soliton solutions as they are conformal and so are not (as) physically interesting. The Dynkin diagrams given in eqs.(A.0.9)-(A.0.17) encode the inner products between the simple roots for all semi-simple Lie algebras and the affine Dynkin diagrams, which include a node corresponding to the lowest weight root, are given in eqs.(A.0.37)-(A.0.45). Often the affine diagram is distinguished from the non-affine diagram by the addition of a tilde, however here we always take the capital letter to refer to the affine diagram. While we are using the roots as encoded by the affine Dynkin diagrams we are still using the non-affine, finite dimensional generators as defined by the commutation relations in eqs.(A.0.1)-(A.0.5).

Because the simple roots are defined only up to their inner products with other simple roots the potential based on the set of roots $\{\alpha_i\}$ and the potential based on the set of roots $\{Q\alpha_i\}$, where Q is some orthogonal transformation, describe the same ATFT. Because the kinetic part of the bulk Lagrangian is invariant under orthogonal transformations of the fields the ATFTs based on the roots $\{\alpha_i\}$ can be obtained by taking $u \to Qu$ in the ATFT based on the roots $\{Q\alpha_i\}$. In a similar manner we can take the ATFT based on $\{c\alpha_i\}$, where c is a constant, and, with $u \to c^{-1}u$ and a rescaling of the coordinates t and x such that $\partial_{t,x} \to c\partial_{t,x}$, return to the ATFT based on the roots $\{\alpha_i\}$. Similar rescalings to these also allowed us to set $m = \beta = 1$.

This potential has multiple vacua occurring at $2\pi i$ multiples of weights of the Lie algebra the potential is based on, so if the field u is complex then we can have soliton solutions to the ATFT equations of motion which interpolate between different vacua as $x \to \pm\infty$. A soliton can be associated with a particular simple root depending on which weights it interpolates between. Such soliton solutions have been found for all ATFTs [Hol92; MM93; McG94a; Hal94] and will be discussed in more detail in chapter 5.

The non-simply-laced Dynkin diagrams can be obtained by certain foldings of the simply-laced diagrams. A folding is the identification of some simple roots with other simple roots, with the roots which are identified with each other being related by some symmetry of the Dynkin diagram. These same foldings can be applied to the extended or affine Dynkin diagrams, with the lowest weight α_0 root being identified with itself. In terms of the simple roots folding can be thought of as a projection of the simple roots onto a subspace of the root space. If one root has been identified with another root in the folding then both roots will have the same projection. The projected simple roots will be the simple roots of a different Lie algebra to the initial one, and this new set of simple roots must contain fewer roots than the original set of simple roots. These foldings can be used on the ATFT potentials based on one set of simple roots to give an ATFT potential based on a different set of simple roots. When folding an ATFT we begin with an ATFT based on the roots $\{\alpha_i\}$ and containing the field u. Folding a Dynkin diagram would consist of identifying the roots in $\{\alpha_i\}$ with the roots in the smaller set, $\{\tilde{\alpha}_i\}$. To fold the ATFT we identify components of the vector field u with components of the (smaller) vector field $\tilde{u}$ in such a way that $u.u \to \tilde{u}.\tilde{u}$ (so that the kinetic part of the bulk Lagrangian remains in the form in eq.(3.1.1)) and $\alpha_i.u \to \tilde{\alpha}_i.\tilde{u}$. This folding of the vector fields can also be applied to the soliton solutions of the original theory to give soliton solutions of the folded theory. For further information on the folding of ATFTs and how this affects their soliton solutions see [OT83a; OT83b; PS96].

When considering a defect eq.(3.1.1) describes the field theory to the left of the defect and eq.(3.1.1) with the vector u replaced by v describes the field theory to the right. Recall that the components of the vectors u and v appear in the vectors $u^{(1)}$ and $v^{(1)}$ respectively if they do not couple to the auxiliary field $\mu^{(2)}$, and in the vectors $u^{(2)}$ and $v^{(2)}$ if they do couple to $\mu^{(2)}$. We choose to call the vector space in which $u^{(1)}$ (and $v^{(1)}$) live the 1-space and the vector space in which $u^{(2)}$ (and $v^{(2)}$ and $\mu^{(2)}$) live the 2-space. The vectors $u^{(1)}$ and $v^{(1)}$ can be thought of as the projections of u and v onto the 1-space and $u^{(2)}$ and $v^{(2)}$ as the projections of u and v onto the 2-space. The 1-space and 2-space are orthogonal and sum to the vector space in which the vectors u and v live, that is, the space spanned by the simple root vectors. Therefore we can have $\alpha_i^{(1)}$ as the projection of a simple root α_i onto the 1-space and $\alpha_i^{(2)}$ as the projection onto the 2-space.

In this chapter we will first provide some background in the form of previous calculations carried out for defects in ATFTs. We show the results from [BCZ04a], [CZ09a] and [Rob14b], which proved that ATFTs can support momentum conserving type I defects if and only if they are based on the simple roots of A_r, that the type II defects are momentum conserving for the previously excluded Tzitzéica model, and that it is possible to "squeeze" together two type I defects and carry out a folding procedure to obtain a momentum conserving type II defect in a theory not based on the simple roots of A_r. Work similar to the momentum conservation calculation given in section 2.2 was carried out in each of these papers for the less general (type I, type II with a scalar field or folded type II) defects investigated therein. Rather than reproducing the calculations as given in these papers we make use of the more general result found here.

At the start of section 3.3 we give the form which a momentum conserving type II defect in any ATFT must take, then give a detailed analysis of the possible momentum conserving defects in the D_4 ATFT in section 3.3.1. By considering the form of this D_4 defect we are then able to provide examples of momentum conserving type II defects in the A_r, B_r, C_r and D_r ATFTs in sections 3.3.2-3.3.5. These examples were published in [BB17]. Finally we discuss possible reasons why we have not yet been able to find momentum conserving defects in the ATFTs based on the exceptional Lie algebras in sections 3.3.6-3.3.8.

3.2 Previous results

3.2.1 Defects in A_r ATFTs

The purely transmitting classical defects investigated here were first introduced in [BCZ04b], where a momentum conserving type I sine-Gordon defect was found. The sinh-Gordon (or sine-Gordon if the field u is taken to be imaginary rather than real) potential is

$$U = e^u + e^{-u} \tag{3.2.1}$$

and similarly for v. From our work in chapter 2 we see that for a type I sine-Gordon defect the 1-space is 1-dimensional, the 2-space does not exist, there is no ξ field and $A = 0$ (as it is an antisymmetric scalar). Taking these restrictions and eqs.(2.2.26),(2.2.33) gives the momentum conserving defect potential

$$\mathcal{L}^D = uv_t - D - \bar{D} \tag{3.2.2}$$

where $D(p)$ and $\bar{D}(q)$ (with $p = \frac{1}{2}(u+v)$ and $q = \frac{1}{2}(u-v)$) must satisfy

$$2\left(e^{p+q} + e^{-p-q} - e^{p-q} - e^{-p+q}\right) = D_p \bar{D}_q. \tag{3.2.3}$$

This is obviously satisfied by

$$D = 2\sigma\left(e^p + e^{-p}\right) \qquad \bar{D} = \frac{1}{\sigma}\left(e^q + e^{-q}\right) \tag{3.2.4}$$

where σ is an arbitrary constant defect parameter.

We will now consider a momentum conserving type I defect with any number of auxiliary fields, following the calculations made in [BCZ04a]. Again from eqs.(2.2.26),(2.2.33) with no 2-space and the bulk potential given in eq.(3.1.1) we have that the defect potential must be

$$\mathcal{L}^D = \frac{1}{2}u_i A_{ij} u_{j,t} + \frac{1}{2}v_i A_{ij} v_{j,t} + u_i \left(\mathbb{1} - A\right)_{ij} v_{j,t} - D - \bar{D} \tag{3.2.5}$$

where A is any antisymmetric matrix and $D\left(p + Aq\right)$ and $\bar{D}\left(q\right)$ must satisfy

$$2\sum_{i=0}^{r} n_i e^{(\alpha_i)_j p_j} \left(e^{(\alpha_i)_j q_j} - e^{-(\alpha_i)_j q_j}\right) = D_{p_i} \bar{D}_{q_i}. \tag{3.2.6}$$

Since only D is dependent on the field p we can immediately see that D and $\bar{D}$ must take the form

$$D = \sum_{i=0}^{r} x_i e^{(\alpha_i)_j \left(p_j + A_{jk} q_k\right)} \tag{3.2.7}$$

$$\bar{D} = \sum_{j=0}^{r} y_j e^{(z_j)_k q_k} \tag{3.2.8}$$

where x_i and y_j are unknown constants and z_j are unknown vectors. All z_j are different, as if two different z_j were equal then the repeated exponentials could simply be absorbed into the definition of y_j. Substituting these into eq.(3.2.6) and then equating coefficients of exponentials of p we have the momentum conservation condition as

$$2n_i \left(e^{(\alpha_i)_k q_k} - e^{-(\alpha_i)_k q_k}\right) = \sum_{j=0}^{r} x_i y_j (\alpha_i)_k (z_j)_k e^{((z_j)_k + (\alpha_i)_l A_{lk}) q_k} \quad i = 0, \ldots, r. \tag{3.2.9}$$

There are no repeated exponentials on either side, therefore every term on the right hand side must have exactly one equal term on the left hand side. To ensure every term on the left hand side has at least one equal term on the right hand side we take the sets $\{z'_k\}$ and $\{\tilde{z}_k\}$, with $z'_j = \alpha_j + A\alpha_j \in \{z'_k\}$ $(j = 0, \ldots, r)$ and $\tilde{z}_j = -\alpha_j + A\alpha_j \in \{\tilde{z}_k\}$ $(j = 0, \ldots, r)$, and take the z_j in eq.(3.2.9) to run over both of these sets. All elements within the same set are different (by the invertibility of $(\mathbb{1} \pm A)$) but there may be overlap between the sets which introduces repeated exponentials.

All z_j are different. Therefore for each value of i there will be, on the right hand side of eq.(3.2.9), a term with z_i', a term with $\tilde{z}_i$ and then all other terms going to zero. So $z_j.\alpha_i = 0$ when $z_j \neq z_i', \tilde{z}_i$, and $z_j.\alpha_i \neq 0$ when $z_j = z_i', \tilde{z}_i$ to ensure each term on the left hand side of eq.(3.2.9) has an equal on the right hand side. If a particular z_j only appears in $\{z_k'\}$ then $z_j.\alpha_i = 0 \quad \forall i \neq j$ and $z_j.\alpha_j \neq 0$. For $j \neq 0$ we therefore require $z_j.\alpha_0 = 0$, but this sets either $z_j.\alpha_j = 0$ or $z_j.\alpha_i \neq 0$ for some $i \neq j$. For $j = 0$ we have $z_0.\alpha_i = 0 \quad \forall i \neq 0$, but then by the definition of α_0 we cannot have $z_0.\alpha_0 \neq 0$. So no z_j can only appear in the set $\{z_k'\}$. Either by the same argument, or by the fact that $\{z_k'\}$ and $\{\tilde{z}_k\}$ contain the same number of elements, we then see that no z_j can only appear in the set $\{\tilde{z}_k\}$. All z_j must appear in both $\{z_k'\}$ and $\{\tilde{z}_k\}$ and the two sets overlap completely.

We have two distinct choices, either to set $z_j = \alpha_j + A\alpha_j$ or $z_j = -\alpha_j + A\alpha_j$. We first tackle the $z_j = \alpha_j + A\alpha_j$ case. The momentum conservation condition in eq.(3.2.9) becomes

$$\begin{aligned} 2n_i\left(e^{(\alpha_i)_k q_k} - e^{-(\alpha_i)_k q_k}\right) = &\sum_{j\neq i} x_i y_j (\alpha_i)_k (\mathbb{1} + A)_{kl} (\alpha_j)_l e^{((\alpha_j)_k + (\alpha_i - \alpha_j)_l A_{lk}) q_k} \\ &+ x_i y_i (\alpha_i)_k (\alpha_i)_k e^{(\alpha_i)_k q_k} \qquad\qquad i = 0, \ldots, r. \end{aligned} \tag{3.2.10}$$

For every α_i there must be exactly one α_j such that the expression $-\alpha_i = \alpha_j - A(\alpha_i - \alpha_j)$ holds, so every simple root and the lowest weight root must be related to one other simple or lowest weight root by the expression $(-\mathbb{1} + A)\alpha_i = (\mathbb{1} + A)\alpha_j$. This ensures that every term on the left hand side of eq.(3.2.10) has one matching term on the right hand side. It is evident that we cannot have $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_i$. Taking dot products with simple roots we have $\alpha_i.(\mathbb{1} + A)\alpha_j = \alpha_i.(-\mathbb{1} + A)\alpha_i = -|\alpha_i|^2$ and $\alpha_j.(-\mathbb{1} + A)\alpha_i = \alpha_i.(-\mathbb{1} - A)\alpha_j = \alpha_j.(\mathbb{1} + A)\alpha_j = |\alpha_j|^2$, and so $|\alpha_i| = |\alpha_j| \quad \forall i, j$. The ATFT supporting the defect must be based on a simply laced Lie algebra.

Now consider the terms in eq.(3.2.10) which must vanish. We must have $\alpha_k.z_j = 0$ except when $k = i, j$, where the i is such that $(-\mathbb{1} + A)\alpha_i = (\mathbb{1} + A)\alpha_j$. So z_j is orthogonal to all but two of the simple roots and the lowest weight root. Recalling that the fundamental weights are defined by $\alpha_i.w_j = \delta_{ij}$ $(i, j = 1, \ldots, r)$ and span the root space we will choose to write z_j in terms of the weights rather than the roots. For $i, j \neq 0$ and $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_j$ taking $z_j = c_i w_i + d_j w_j$, where c_i and d_j are unknown, ensures $\alpha_k.z_j = 0$ for $k \neq i, j$ is satisfied. z_0 must be orthogonal to all simple roots except for α_i such that $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_0$, so we take $z_0 = c_i w_i$. The z_j such that $(-\mathbb{1} + A)\alpha_0 = (\mathbb{1} + A)\alpha_j$ is also orthogonal to all but one simple root, so we take $z_j = d_j w_j$. If we define $w_0 = 0$ then we can set $z_j = c_i w_i + d_j w_j$ where $(-\mathbb{1} + A)\alpha_i = (\mathbb{1} + A)\alpha_j$.

Taking this together with our original definition $z_j = \alpha_j + A\alpha_j$ we consider the inner products of z_j with the roots. Take α_i to be such that $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_j$, then $\alpha_i.z_j = \alpha_i.(\mathbb{1} + A)\alpha_j = \alpha_i.(-\mathbb{1} + A)\alpha_i = -|\alpha_i|^2$ and $\alpha_i.z_j = c_i\alpha_i.w_i + d_j\alpha_i.w_j = c_i$, giving $c_i = -|\alpha_i|^2$. From $\alpha_j.z_j = \alpha_j.(\mathbb{1}+A)\alpha_j = |\alpha_j|^2$ and $\alpha_j.z_j = c_i\alpha_j.w_i + d_j\alpha_j.w_j = d_j$ we have $d_j = |\alpha_j|^2$, giving $z_j = |\alpha_j|^2(w_j - w_i)$.

Finally take $z_j = |\alpha_j|^2(w_j - w_i) = (\mathbb{1}+A)\alpha_j$ and $z_k = |\alpha_k|^2(w_k - w_j) = (\mathbb{1}+A)\alpha_k$ with $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_j$ and $(-\mathbb{1}+A)\alpha_j = (\mathbb{1}+A)\alpha_k$. These give $z_j - z_k = |\alpha_j|^2(-w_i + 2w_j - w_k)$ and $z_j - z_k = (\mathbb{1} + A)\alpha_j - (\mathbb{1} + A)\alpha_k = (\mathbb{1} + A)\alpha_j - (-\mathbb{1} + A)\alpha_j = 2\alpha_j$, so $\alpha_j = \frac{|\alpha_j|^2}{2}(-w_i + 2w_j - w_k)$. Recalling the definition of the fundamental weights this means that every simple root is non-orthogonal with a maximum of two other simple roots, and so each node on the Dynkin diagram is connected to a maximum of two other nodes. Looking at the Dynkin diagrams we immediately see that out of the simple laced algebras this restricts us to A_r. A nearly identical calculation can be carried out for $z_j = -\alpha_j + A\alpha_j$ which reaches the same conclusion. For the A_r simple roots α_j is non-orthogonal to α_{j-1} and α_{j+1}, so we may set either $i = j - 1$ and $k = j + 1$ or $i = j + 1$ and $k = j - 1$, taking all subscripts to be $\mod r$.

With this information we can now write down D and $\bar{D}$ which satisfy eq.(3.2.6). We set $|\alpha_i| = \sqrt{2}$ and have $\alpha_i.\alpha_j = -1$ for $j = i - 1, i + 1$ with $\alpha_i.\alpha_j = 0$ otherwise.

There are two distinct cases, one where the roots are related by $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_{i+1} \quad \forall i$ and another where they are related by $(-\mathbb{1}+A)\alpha_i = (\mathbb{1}+A)\alpha_{i-1} \quad \forall i$. These transformations move round the affine Dynkin diagram in opposite directions. So we just need to find a matrix A which satisfies one of these relations and set the constants x_i, y_i to some suitable value. The calculation for $z_j = -\alpha_j + A\alpha_j$ gives the same two possible cases.

As in [CZ09b] we define the matrix

$$B = \sum_{a=1}^{r} w_a (w_a - w_{a+1})^T \tag{3.2.11}$$

which gives

$$B\alpha_j = \sum_{a=1}^{r} w_a (w_a - w_{a+1})^T \alpha_j = w_j - w_{j-1} \tag{3.2.12}$$

$$B^T\alpha_j = \sum_{a=1}^{r} (w_a - w_{a+1}) w_a^T \alpha_j = w_j - w_{j+1} \tag{3.2.13}$$

for $j = 0, \ldots, r$ and with a taken to be $\mod r$. Then taking $A = \mathbb{1} - 2B$ (which can be checked to be antisymmetric) ensures $(\mathbb{1}+A)\alpha_i = (-\mathbb{1}+A)\alpha_{i+1}$ is satisfied, or alternatively taking $A = \mathbb{1} - 2B^T$ ensures $(\mathbb{1}+A)\alpha_i = (-\mathbb{1}+A)\alpha_{i-1}$ is satisfied.

For $A = \mathbb{1} - 2B$ the momentum conserving defect Lagrangian is

$$\mathcal{L}^D = \frac{1}{2} u_i (\mathbb{1} - 2B)_{ij} u_{j,t} + \frac{1}{2} v_i (\mathbb{1} - 2B)_{ij} v_{j,t} + 2u_i B_{ij} v_{j,t} - D - \bar{D} \tag{3.2.14}$$

with

$$D = \sum_{i=0}^{r} x_i e^{(\alpha_i)_j \left(p_j + q_j - 2B_{jk} q_k\right)} \tag{3.2.15}$$

$$\bar{D} = \sum_{i=0}^{r} y_i e^{2(\alpha_i)_j B_{jk} q_k} \tag{3.2.16}$$

and the momentum conservation condition in eq.(3.2.6) is satisfied if we set $x_i = \sigma$, $y_i = \sigma^{-1}$ where σ is the arbitrary, constant defect parameter.

For $A = \mathbb{1} - 2B^T$ the momentum conserving defect Lagrangian is

$$\mathcal{L}^D = \frac{1}{2} u_i (\mathbb{1} - 2B^T)_{ij} u_{j,t} + \frac{1}{2} v_i (\mathbb{1} - 2B^T)_{ij} v_{j,t} + 2u_i B^T_{ij} v_{j,t} - D - \bar{D} \tag{3.2.17}$$

with

$$D = \sum_{i=0}^{r} x_i e^{(\alpha_i)_j \left(p_j + q_j - 2B^T_{jk} q_k\right)} \tag{3.2.18}$$

$$\bar{D} = \sum_{i=0}^{r} y_i e^{2(\alpha_i)_j B^T_{jk} q_k} \tag{3.2.19}$$

where again $x_i = \sigma$, $y_i = \sigma^{-1}$.

The orthogonal transformation $u \to Qu$, $v \to Qv$ where Q is an orthogonal matrix such that $Q^T BQ = B^T$ (so $Q^T AQ = -A$) moves from the first case for an ATFT based on the roots $\{\alpha_i\}$ to the second case for an ATFT based on the roots $\{Q^T \alpha_i\}$. We have not included here the proof that the set of roots for the two bulk ATFTs must be the same.

The transmission of solitons through these defects has been investigated in [BCZ04a; CZ07; CZ09b] and it was found that the solitons are delayed but otherwise unchanged, with the delay dependent on the defect parameter σ and the rapidity of the incoming soliton. One point to note about these soliton transmissions is that the same soliton will have different delays when travelling through the two different defects, and that the delay of the soliton associated with α_i when travelling through the $A = \mathbb{1} - 2B$ defect is the same as that of the α_{r+1-i} soliton passing through the $A = \mathbb{1} - 2B^T$ defect and vice versa. We do not give the calculations for the transmission of solitons through type I defects in this thesis, but the method used to find the type I defect delays is employed in chapter 5 when investigating the transmission of solitons through a defect in the D_4 ATFT.

3.2.2 Defects in the Tzitzéica model

That these defects only appeared in A_r ATFTs suggests there is something fundamental missing. In [CZ09a] an extra degree of freedom was introduced by writing down a defect Lagrangian for a scalar bulk field and one extra field confined to the defect. The Lagrangian for such a defect will have no 1-space and a scalar 2-space, so from the work in section 2.2 and eq.(2.2.26) it will be

$$\mathcal{L}^D = uv_t + 2\mu\,(u_t - v_t) - D - \bar{D}. \tag{3.2.20}$$

The Tzitzéica potential is eq.(3.1.1) with

$$\alpha_0 = -2 \qquad \alpha_1 = 1 \qquad n_0 = 1 \qquad n_1 = 2 \tag{3.2.21}$$

and is evidently not covered by the previous case as the roots are of different lengths. For a defect in this theory $D(p-\mu, q)$ and $\bar{D}(q,\mu)$ must satisfy the momentum conservation condition

$$2e^{-2(p+q)} + 4e^{p+q} - 2e^{-2(p-q)} - 4e^{p-q} = D_q\bar{D}_\mu - D_\mu\bar{D}_q. \tag{3.2.22}$$

Because only D is dependent on p and the right hand side must be overall independent of μ we can write

$$D = x_0(q)e^{-2p+2\mu} + x_1(q)e^{p-\mu} \tag{3.2.23}$$

$$\bar{D} = y_0(q)e^{-2\mu} + y_1(q)e^{\mu}. \tag{3.2.24}$$

Unfortunately the form of the right hand side of eq.(3.2.22) means that taking the same approach as in the type I case, putting all the fields into exponentials and then identifying terms which must be zero, would be significantly more difficult. This is due to the existence of cancellations between terms on the right hand side which did not appear in the type I case. Instead we will write down a set of differential equations for $x_{0,1}$ and $y_{0,1}$ to be solved.

At the end of section 2.2 we noted that the redefinition $\mu \to \mu + f(q)$ of the auxiliary field, where f is any function, does not change the kinetic part of the defect Lagrangian and so can be used to give a family of D and $\bar{D}$ satisfying the same momentum conservation condition. In order to simplify the differential equations to be solved we will use the field redefinition $\mu \to \mu - \frac{1}{2}\ln x_1$ to set $x_1 = 1$. The other coefficients are currently arbitrary, so can be redefined to include this. Substituting eqs.(3.2.23),(3.2.24) into eq.(3.2.22) and equating exponentials of p and μ gives a set of four differential equations which are solved by

$$
\begin{aligned}
x_0 =& \frac{1}{2c}(e^q + e^{-q})^2 & y_0 =& c \\
x_1 =& 1 & y_1 =& 4(e^q + e^{-q})
\end{aligned}
\tag{3.2.25}
$$

where c is a constant. We now have a specific solution,

$$
D = \frac{1}{2c}(e^q + e^{-q})^2 e^{-2p+2\mu} + e^{p-\mu} \tag{3.2.26}
$$

$$
\bar{D} = ce^{-2\mu} + 4(e^q + e^{-q})e^{\mu}. \tag{3.2.27}
$$

We can choose to take $\mu \to \mu + \frac{1}{3}\ln c$ and multiply D by $c^{\frac{1}{3}}$ and $\bar{D}$ by $c^{-\frac{1}{3}}$. This removes all instances of the constant c. To introduce as much freedom as is possible we then make the field redefinition $\mu \to \mu + f(q)$ and multiply D by the arbitrary constant σ and $\bar{D}$ by σ^{-1}, giving

$$
D = \sigma\left(\frac{1}{2}(e^q + e^{-q})^2 e^{2f} e^{-2p+2\mu} + e^{-f} e^{p-\mu}\right) \tag{3.2.28}
$$

$$
\bar{D} = \frac{1}{\sigma}\left(e^{-2f} e^{-2\mu} + 4(e^q + e^{-q})e^{f} e^{\mu}\right) \tag{3.2.29}
$$

as the solutions to eq.(3.2.22).

There is also some freedom to redefine the external fields. We can shift u or v by an integer multiple $2\pi i$ without affecting the bulk Lagrangians or the kinetic part of the defect Lagrangian. Taking $u \to u + 2\pi i n$, $v \to v + 2\pi i m$ (so $p \to p + \pi i(n+m)$, $q \to q + \pi i(n-m)$) gives the defect potential

$$
D = \sigma\left(\frac{1}{2}e^{2f}(e^{2q} + e^{-2q} + 2)e^{-2p+2\mu} + (-1)^{n+m} e^{-f} e^{p-\mu}\right) \tag{3.2.30}
$$

$$
\bar{D} = \frac{1}{\sigma}\left(e^{-2f} e^{-2\mu} + 4(-1)^{n-m} e^{f}(e^q + e^{-q})e^{\mu}\right). \tag{3.2.31}
$$

But we can also immediately take the redefinition $\mu \to \mu + \pi i(n+m)$ to return to the D and $\bar{D}$ given in eqs.(3.2.28),(3.2.29), and since the freedom to shift the external fields corresponds to a shift in the auxiliary fields the entire family of momentum conserving defects satisfying the momentum conservation condition in eq.(3.2.22) have a potential given by eqs.(3.2.28),(3.2.29).

The interactions of solitons with this defect were investigated in [CZ09a], and a similar situation to the A_r ATFT case was found, with the defect able to delay or absorb solitons and change their topological charge.

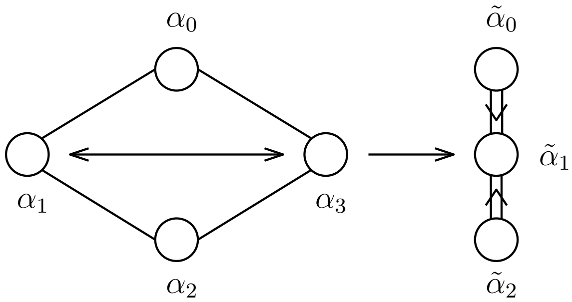
3.2.3 Folding defects

In [Rob14b; Rob14a; Rob15] this idea of additional degrees of freedom at the defect, and the folding of A_2 ATFT Bäcklund transformations to give the Tzitzéica defect equations plus the extra equations required for a Bäcklund transformation, gave rise to some generalisation of the type II defects.

An A_r ATFT can support multiple defects while remaining momentum conserving. For two defects at different positions there will be (in addition to u ad v to the left and right of both defects respectively) some bulk field which is only defined between the two defects. The position of the defects does not affect their momentum conservation, so we can then take the position of both defects to $x = 0$. What was the bulk field between the two defects is now confined to the point $x = 0$, and should play the same role as the extra degree of freedom at the defect introduced in [CZ09a].

To take these defects to some new defects not in an A_r ATFT we then change the bulk theory by folding. This will extend the type II case introduced in [CZ09a] to a defect with more than one component in the bulk vector fields. Of the Dynkin diagrams given in eqs.(A.0.38)-(A.0.45) the only one which can be obtained by a folding of the A_r Dynkin diagram is C_r. The possible foldings of Dynkin diagrams and ATFTs are discussed in [OT83a; PS96]. This means that the only new defect we can find with this method will be a C_r ATFT defect. This folding involves identifying the simple root α_i with α_{r+1-i} and the α_0 root with itself. When A_3 is folded to C_2

the Dynkin diagram undergoes the folding



(3.2.32)

There are other possible foldings of A_r, but we will not discuss the Dynkin diagrams and ATFTs which these give rise to.

We have already given a brief explanation of folding in the bulk. To fold a (type I) defect we take the identifications of components of u and v with components of $\tilde{u}$ and $\tilde{v}$ which folded the bulk ATFTs and apply them to the fields appearing in the defect Lagrangian. When it comes to folding defects there is an additional consideration-the transmission of solitons. When we fold an ATFT the solitons also undergo a folding, with solitons associated to roots which are identified with each other being folded to the soliton associated with the resultant single node on the folded Dynkin diagram. Therefore if during the folding two roots are to be identified with each other then the solitons associated with them must have the same overall delay when passing through the defect(s). This ensures that the resultant folded soliton can actually be transmitted through the folded defect. In section 3.2.1 we saw that, for the two species of defect in eqs.(3.2.14),(3.2.17), the soliton associated with α_i passing through one defect had the same delay as the soliton associated with α_{r+1-i} passing through the other defect. Taking one defect of each species with the same defect parameter will ensure that the overall delays of the α_i and α_{r+1-i} solitons are the same after passing through both defects.

We will, as a small example, follow [Rob14b] and fold defects in an A_3 ATFT to a defect in a C_2 ATFT. We identify α_1 with α_3, as shown in eq.(3.2.32). To fold the ATFT we need some identification $\tilde{u} = u$ such that $\tilde{\alpha}_0.\tilde{u} = \alpha_0.u$, $\tilde{\alpha}_1.\tilde{u} = \tilde{\alpha}_3.\tilde{u} = \alpha_1.u$ and $\tilde{\alpha}_2.\tilde{u} = \alpha_2.u$, where $\{\tilde{\alpha}_i\}$ are the A_3 simple roots and $\{\alpha_i\}$ are the C_2 simple roots. Taking the bulk ATFT given by the Lagrangian in eq.(3.1.1) and the A_3 roots as given in eq.(A.0.18) and making the identifications

$$\tilde{u}_1 = \frac{1}{\sqrt{2}}u_1 \qquad \tilde{u}_2 = \frac{1}{\sqrt{2}}u_2 \qquad \tilde{u}_3 = -\frac{1}{\sqrt{2}}u_2 \qquad \tilde{u}_4 = -\frac{1}{\sqrt{2}}u_1 \tag{3.2.33}$$

gives the folded ATFT Lagrangian

$$\mathcal{L}^{(u)} = \frac{1}{2}\left(u_{1,t}u_{1,t} + u_{2,t}u_{2,t} - u_{1,x}u_{1,x} - u_{2,x}u_{2,x}\right) - e^{-\sqrt{2}u_1} - 2e^{\frac{1}{\sqrt{2}}(u_1-u_2)} - e^{\sqrt{2}u_2}. \tag{3.2.34}$$

This C_2 ATFT potential is based on the simple roots given in eq.(A.0.20) scaled by $\frac{1}{\sqrt{2}}$.

Now that we have the bulk folding we can consider the defect folding. Consider a system containing a defect at $x = x_1$ described by eqs.(3.2.14),(3.2.15),(3.2.16) with field $\tilde{u}$ to the left of the defect and field $\tilde{\mu}$ to the right, then a defect at $x = x_2 > x_1$ described by eqs.(3.2.17),(3.2.18),(3.2.19) with field $\tilde{\mu}$ to the left of the defect and field $\tilde{v}$ to the right. The defect parameter σ is the same for both defects. Fields $\tilde{u}$, $\tilde{\mu}$ and $\tilde{v}$ all obey the bulk equations of motion for the A_3 ATFT. This set-up ensures that the solitons associated with the α_1 and α_3 simple roots in A_3 have the same delay after passing through both defects, as they will need to act as a single C_2 soliton associated with α_1 after folding. We then take $x_{1,2} \to 0$, giving a system with the same form we investigated in chapter 2 (in eq.(2.1.1)) with defect Lagrangian

$$\begin{aligned}\mathcal{L}^D =& \frac{1}{2}\tilde{u}_i\left(\mathbb{1} - 2B\right)_{ij}\tilde{u}_{j,t} + \frac{1}{2}\tilde{v}_i\left(\mathbb{1} - 2B^T\right)_{ij}\tilde{v}_{j,t} - 2\tilde{\mu}_i B^T_{ij}(\tilde{u}_{j,t} - \tilde{v}_{j,t}) \\ & - D_1 - \bar{D}_1 - D_2 - \bar{D}_2.\end{aligned} \tag{3.2.35}$$

The field $\tilde{\mu}$ is now confined to $x = 0$ so has no bulk equations of motion. The defect potential terms D_1 and $\bar{D}_1$ are given by eqs.(3.2.15),(3.2.16) with $u = \tilde{u}$ and $v = \tilde{\mu}$ (so $p = \frac{1}{2}(\tilde{u} + \tilde{\mu})$, $q = \frac{1}{2}(\tilde{u} - \tilde{\mu})$) and D_2 and $\bar{D}_2$ are given by eqs.(3.2.18),(3.2.19) with $u = \tilde{\mu}$ and $v = \tilde{v}$ (so $p = \frac{1}{2}(\tilde{\mu} + \tilde{v})$, $q = \frac{1}{2}(\tilde{\mu} - \tilde{v})$).

The matrix B is given by eq.(3.2.11) and the fundamental weights for A_3 given in eq.(A.0.46) and is

$$B = \frac{1}{8}\begin{pmatrix} 3 & -3 & -1 & 1 \\ 1 & 3 & -3 & -1 \\ -1 & 1 & 3 & -3 \\ -3 & -1 & 1 & 3 \end{pmatrix}. \tag{3.2.36}$$

This would appear to disagree with B being defined as $\frac{1}{2}(\mathbb{1} - A)$ where A is antisymmetric. However, the root space of A_3 is three dimensional, and for the simple roots we are using is the space orthogonal to $(1\,1\,1\,1)^T$. The action of B on this space can be checked to be the action of the identity matrix plus some antisymmetric matrix.

Carrying out the folding of the bulk fields given in eq.(3.2.33) on $\tilde{u}$ and $\tilde{v}$ for the defect Lagrangian in eq.(3.2.35) gives

$$\begin{aligned} \mathcal{L}^D =& \frac{1}{2\sqrt{2}}\tilde{\mu}_1\left(-\left(u_{1,t} - v_{1,t}\right) - 3\left(u_{2,t} - v_{2,t}\right)\right) + \frac{1}{2\sqrt{2}}\tilde{\mu}_2\left(\left(u_{1,t} - v_{1,t}\right) - \left(u_{2,t} - v_{2,t}\right)\right) \\ &+ \frac{1}{2\sqrt{2}}\tilde{\mu}_3\left(\left(u_{1,t} - v_{1,t}\right) + 3\left(u_{2,t} - v_{2,t}\right)\right) + \frac{1}{2\sqrt{2}}\tilde{\mu}_4\left(\left(u_{1,t} - v_{1,t}\right) - \left(u_{2,t} - v_{2,t}\right)\right) \\ &- D_1 - D_2 - \bar{D}_1 - \bar{D}_2. \end{aligned} \tag{3.2.37}$$

To put this into the standard form found in the previous section we set the auxiliary fields as

$$\begin{aligned} \tilde{\mu}_1 =& \sqrt{2}\left(-\mu_1 - \mu_2 + \mu_3\right) + \frac{1}{2\sqrt{2}}\left(u_1 + v_1 + u_2 + v_2\right) \\ \tilde{\mu}_2 =& \sqrt{2}\left(\mu_1 - 3\mu_2 + \mu_4\right) + \frac{1}{2\sqrt{2}}\left(-u_1 - v_1 + 3u_2 + 3v_2\right) \\ \tilde{\mu}_3 =& \sqrt{2}\mu_3 \\ \tilde{\mu}_4 =& \sqrt{2}\left(2\mu_3 - \mu_4\right). \end{aligned} \tag{3.2.38}$$

This also removes the μ_3 and μ_4 auxiliary fields from the kinetic part of the Lagrangian.

The effect of these field redefinitions on the defect potential is to set

$$D_1 = \sigma\Big(e^{\frac{1}{\sqrt{2}}(-2p_1 - q_1 + 2\mu_1)} + e^{\frac{1}{\sqrt{2}}(2p_1 + q_1 - 2\mu_1 - 2p_2 + 4\mu_2 + 2\mu_3 - 2\mu_4)}$$

$$+ e^{\frac{1}{\sqrt{2}}(2p_2+q_2-2\mu_2)} + e^{\frac{1}{\sqrt{2}}(-q_2-2\mu_2-2\mu_3+2\mu_4)}\Big) \tag{3.2.39}$$

$$D_2 = \sigma\Big(e^{\frac{1}{\sqrt{2}}(-2p_1+q_1+2\mu_1)} + e^{\frac{1}{\sqrt{2}}(2p_1-q_1-2\mu_1-2p_2+4\mu_2+2\mu_3-2\mu_4)}$$

$$+ e^{\frac{1}{\sqrt{2}}(2p_2-q_2-2\mu_2)} + e^{\frac{1}{\sqrt{2}}(q_2-2\mu_2-2\mu_3+2\mu_4)}\Big) \tag{3.2.40}$$

$$\bar{D}_1 = \frac{1}{\sigma}\Big(e^{\frac{1}{\sqrt{2}}(-q_1-2\mu_1)} + e^{\frac{1}{\sqrt{2}}(-p_1+2\mu_1+p_2-q_2-4\mu_2-2\mu_3+2\mu_4)}$$

$$+ e^{\frac{1}{\sqrt{2}}(q_2+2\mu_2)} + e^{\frac{1}{\sqrt{2}}(p_1+q_1-p_2+2\mu_2+2\mu_3-2\mu_4)}\Big) \tag{3.2.41}$$

$$\bar{D}_2 = \frac{1}{\sigma}\Big(e^{\frac{1}{\sqrt{2}}(q_1-2\mu_1)} + e^{\frac{1}{\sqrt{2}}(-p_1+2\mu_1+p_2+q_2-4\mu_2-2\mu_3+2\mu_4)}$$

$$+ e^{\frac{1}{\sqrt{2}}(-q_2+2\mu_2)} + e^{\frac{1}{\sqrt{2}}(p_1-q_1-p_2+2\mu_2+2\mu_3-2\mu_4)}\Big). \tag{3.2.42}$$

Because fields $\mu_{3,4}$ no longer appear in the kinetic part of the defect Lagrangian their equations of motion are $F_{\mu_3} = 0$ and $F_{\mu_4} = 0$ where $F = D_1 + D_2 + \bar{D}_1 + \bar{D}_2$. This sets

$$e^{\sqrt{2}(\mu_3-\mu_4)} = e^{\frac{1}{\sqrt{2}}(-p_1+\mu_1+p_2-3\mu_2)}\left(e^{\frac{1}{\sqrt{2}}q_1} + e^{-\frac{1}{\sqrt{2}}q_1}\right)^{-\frac{1}{2}}\left(e^{\frac{1}{\sqrt{2}}q_2} + e^{-\frac{1}{\sqrt{2}}q_2}\right)^{\frac{1}{2}}, \tag{3.2.43}$$

and substituting this back in to the defect potential gives

$$D = \sigma\Bigg(\left(e^{\frac{1}{\sqrt{2}}q_1} + e^{-\frac{1}{\sqrt{2}}q_1}\right)e^{\sqrt{2}(-p_1+\mu_1)}$$

$$+ 2\left(e^{\frac{1}{\sqrt{2}}q_1} + e^{-\frac{1}{\sqrt{2}}q_1}\right)^{\frac{1}{2}}\left(e^{\frac{1}{\sqrt{2}}q_2} + e^{-\frac{1}{\sqrt{2}}q_2}\right)^{\frac{1}{2}} e^{\frac{1}{\sqrt{2}}(p_1-p_2-\mu_1+\mu_2)}$$

$$+ \left(e^{\frac{1}{\sqrt{2}}q_2} + e^{-\frac{1}{\sqrt{2}}q_2}\right)e^{\sqrt{2}(p_2-\mu_2)}\Bigg) \tag{3.2.44}$$

$$\bar{D} = \frac{1}{\sigma}\Bigg(\left(e^{\frac{1}{\sqrt{2}}q_1} + e^{-\frac{1}{\sqrt{2}}q_1}\right)e^{-\sqrt{2}\mu_1}$$

$$+ 2\left(e^{\frac{1}{\sqrt{2}}q_1} + e^{-\frac{1}{\sqrt{2}}q_1}\right)^{\frac{1}{2}}\left(e^{\frac{1}{\sqrt{2}}q_2} + e^{-\frac{1}{\sqrt{2}}q_2}\right)^{\frac{1}{2}} e^{\frac{1}{\sqrt{2}}(\mu_1-\mu_2)}$$

$$+ \left(e^{\frac{1}{\sqrt{2}}q_2} + e^{-\frac{1}{\sqrt{2}}q_2}\right)e^{\sqrt{2}\mu_2}\Bigg). \tag{3.2.45}$$

This can be checked to satisfy the momentum conservation condition in eq.(2.2.33).

To obtain the defect potential for a C_2 ATFT based on the simple roots given in eq.(A.0.20), rather than these roots scaled by $\frac{1}{\sqrt{2}}$, a scaling of the fields is all that is required. Taking the above defect potential and scaling all fields by $\sqrt{2}$ will give a defect potential which satisfies the momentum conservation condition for ATFT potentials based on the simple roots of C_2 as given in eq.(A.0.20).

3.3 Defects in ATFTs

When considering the general defect found in chapter 2 with a particular potential the fact that we carried out rotations on the external fields in order to simplify the defect Lagrangian becomes relevant. Fortunately, rotations of the fields in the bulk do not fundamentally change the bulk Lagrangian and potential given in eq.(3.1.1). For the calculations here we want to be able to take $\{\alpha_i\}$ to be fixed to certain, reasonably simple vectors. Over the course of the calculations in section 2.2 the external fields have undergone the transformations $u \to Qu$ and $v \to Q'v$, where Q and Q' are orthogonal and arbitrary. The sets of simple roots $\{Q^T\alpha_i\}$ and $\{Q'^T\alpha_i\}$ have the same Dynkin diagram as $\{\alpha_i\}$. If we choose to begin with the bulk potentials from eq.(3.1.1) dependent on $\{Q^T\alpha_i\}$ for U and $\{Q'^T\alpha_i\}$ for V then after u and v have undergone their field redefinitions both U and V will be dependent on $\{\alpha_i\}$. Because the root space splits into the 1-space and 2-space we will need to choose an orthonormal basis for the simple roots which also provides orthonormal bases for the 1-space and 2-space. We have not been able to find a systematic way of determining either the splitting of the root space or the choice of basis.

By considering the exponentials of the field p in the momentum conservation condition in eq.(2.2.33) when we use the potentials U and V as given in eq.(3.1.1), and the dependencies of D and $\bar{D}$ in eqs.(2.2.31),(2.2.32), we see that they must take the form

$$D = \sigma \sum_{i=0}^{n} x_i \left(q^{(2)}, \xi\right) e^{(\alpha_i)_j^{(1)}\left(p_j^{(1)} + A_{jk} q_k^{(1)}\right) + (\alpha_i)_j^{(2)}\left(p_j^{(2)} - \mu_j^{(2)}\right)} \tag{3.3.1}$$

$$\bar{D} = \frac{1}{\sigma} \sum_{i=0}^{n} y_i \left(q^{(1)}, q^{(2)}, \xi\right) e^{-(\alpha_i)_j^{(1)} A_{jk} q_k^{(1)} + (\alpha_i)_j^{(2)} \mu_j^{(2)}} \tag{3.3.2}$$

where σ is a constant and x_i and y_i are functions yet to be determined. Because every term contains a simple (or lowest weight) root we can talk about terms being associated with a particular root or Dynkin diagram node. Mainly due to the aforementioned difficulty with determining how the root space should be split there is no obvious systematic way of ensuring that D and $\bar{D}$ satisfy the momentum conservation condition in eq.(2.2.33) for a particular set of simple roots. Instead we have used trial and error to find momentum conserving defects for some ATFTs.

Using eqs.(3.3.1),(3.3.2) in the momentum conservation condition and equating powers of p we have

$$2n_i\left(e^{(\alpha_i)_k q_k}-e^{-(\alpha_i)_k q_k}\right)=\sum_{j=0}^{r}\Big(x_i(\alpha_i)_k y_{j,q_k}+x_i y_j(\alpha_i)^{(1)}_k A_{kl}(\alpha_j)^{(1)}_l+x_{i,q^{(2)}_k}(\alpha_j)^{(2)}_k y_j$$
$$-4x_{i,\xi_k}W_{kl}y_{j,\xi_l}\Big)e^{(\alpha_i-\alpha_j)^{(1)}_k A_{kl}q^{(1)}_l-(\alpha_i-\alpha_j)^{(2)}_k\mu_k} \quad (3.3.3)$$

for $i=0,\dots,r$ as the momentum conservation conditions.

In order to solve the momentum conservation conditions we must now move to specific cases and use the simple roots as given in appendix A. All of the following results require making a particular choice for the 1-space and 2-space splitting. This splitting was simply found by guesswork for the D_4 case, and the defect was then found by explicitly solving the coupled differential equations in eq.(3.3.3) for x_i and y_i. The subsequent cases used the form of the D_4 defect to inform the choices made for the splitting and the functions x_i and y_i.

3.3.1 D_4 defect

For the D_4 ATFT potential we use the simple roots from eq.(A.0.21), the lowest weight root from eq.(A.0.36) and the n_i values from eq.(A.0.30) in the potential given in eq.(3.1.1). In [BB17] it was found that taking the 1-space to have the basis (e_1, e_4) and the 2-space to have the basis (e_2, e_3), giving two auxiliary fields μ_2 and μ_3, and taking $A=0$ and no ξ fields gave a defect which, with the correct choice of potential, was momentum conserving. The full set of momentum conserving defect potentials was found in [Bri17]. With these choices of 1-space and 2-space the defect

Lagrangian in eq.(2.2.26) becomes

$$\mathcal{L}^D = u_1 v_{1,t} + u_2 v_{2,t} + u_3 v_{3,t} + u_4 v_{4,t} + 2\mu_2\left(u_{2,t} - v_{2,t}\right) + 2\mu_3\left(u_{3,t} - v_{3,t}\right) - D - \bar{D} \tag{3.3.4}$$

where $D(p_1, p_2-\mu_2, p_3-\mu_3, p_4, q_2, q_3)$ and $\bar{D}(q_1, q_2, q_3, q_4, \mu_2, \mu_3)$ (with $p_i = \frac{1}{2}(u_i+v_i)$, $q_i = \frac{1}{2}(u_i - v_i)$) must satisfy

$$2(U-V) = D_{p_1}\bar{D}_{q_1} + D_{q_2}\bar{D}_{\mu_2} - D_{\mu_2}\bar{D}_{q_2} + D_{q_3}\bar{D}_{\mu_3} - D_{\mu_3}\bar{D}_{q_3} + D_{p_4}\bar{D}_{q_4}. \tag{3.3.5}$$

From eqs.(3.3.1),(3.3.2) we expect D and $\bar{D}$ to be

$$\begin{aligned} D = \sigma\Big(&x_0(q_2,q_3)e^{-p_1-p_2+\mu_2} + x_1(q_2,q_3)e^{p_1-p_2+\mu_2} + x_2(q_2,q_3)e^{p_2-p_3-\mu_2+\mu_3} \\ &+ x_3(q_2,q_3)e^{p_3-p_4-\mu_3} + x_4(q_2,q_3)e^{p_3+p_4-\mu_3}\Big) \end{aligned} \tag{3.3.6}$$

$$\begin{aligned} \bar{D} = \frac{1}{\sigma}\Big(&y_0(q_1,q_2,q_3,q_4)e^{-\mu_2} + y_1(q_1,q_2,q_3,q_4)e^{-\mu_2} + y_2(q_1,q_2,q_3,q_4)e^{\mu_2-\mu_3} \\ &+ y_3(q_1,q_2,q_3,q_4)e^{\mu_3} + y_4(q_1,q_2,q_3,q_4)e^{\mu_3}\Big) \end{aligned} \tag{3.3.7}$$

where x_i and y_i are unknown functions. As some terms in $\bar{D}$ have the same exponentials of μ we can redefine some of these currently arbitrary functions as $y_1 \to y_1 - y_0$ and $y_3 \to y_3 - y_4$ to set $y_0 = 0$ and $y_4 = 0$. We can also use the field redefinitions $\mu_2 \to \mu_2 - \left(\int^{q_2} \ln x_0(q_2', q_3)\mathrm{d}q_2'\right)_{q_2}$ and $\mu_3 \to \mu_3 - \left(\int^{q_2} \ln x_0(q_2', q_3)\mathrm{d}q_2'\right)_{q_3}$ to set $x_0 = 1$. The rest of the x_i and y_i can simply be redefined to include this extra function.

Using these choices in eq.(3.3.3) and equating powers of $\mu_{2,3}$ we find a set of differential equations which x_i and y_i must satisfy as a momentum conservation condition. There are two distinct solutions,

$$\begin{aligned} x_0 &= 1 & & \\ x_1 &= 1 & y_1 &= \left(e^{q_1} + e^{-q_1}\right)\left(e^{q_2} + e^{-q_2}\right) \\ x_2 &= 2g(q_3)\left(e^{q_2} + e^{-q_2}\right) & y_2 &= g(q_3)^{-1}\left(e^{q_3} + e^{-q_3}\right) \\ x_3 &= \frac{1}{c}g(q_3)^{-1}\left(e^{q_3} + e^{-q_3}\right) & y_3 &= cg(q_3)\left(e^{q_4} + e^{-q_4}\right) \\ x_4 &= \frac{1}{c}g(q_3)^{-1}\left(e^{q_3} + e^{-q_3}\right) & & \end{aligned} \tag{3.3.8}$$

and

$$\begin{aligned}
x_0 =& 1 & & \\
x_1 =& -1 & y_1 =& \left(e^{q_1} - e^{-q_1}\right)\left(e^{q_2} - e^{-q_2}\right) \\
x_2 =& -2g(q_3)\left(e^{q_2} - e^{-q_2}\right) & y_2 =& g(q_3)^{-1}\left(e^{q_3} - e^{-q_3}\right) \\
x_3 =& -\frac{1}{c}g(q_3)^{-1}\left(e^{q_3} - e^{-q_3}\right) & y_3 =& cg(q_3)\left(e^{q_4} - e^{-q_4}\right) \\
x_4 =& \frac{1}{c}g(q_3)^{-1}\left(e^{q_3} - e^{-q_3}\right) & &
\end{aligned} \tag{3.3.9}$$

where the constant c and function $g(q_3)$ are free (and may be different in each case). When used to write down D and $\bar{D}$ from eqs.(3.3.6),(3.3.7) these will give two separate possibilities for the momentum conserving defect potential.

We can use our freedom to carry out field redefinitions to remove the constant c and function g in both cases. For the first solution taking $\mu_2 \to \mu_2 - \frac{1}{3}\ln c$, $\mu_3 \to \mu_3 - \frac{2}{3}\ln c$ and $\sigma \to c^{\frac{1}{3}}\sigma$ removes (or absorbs into the definition of $\mu^{(2)}$ and σ) the constant c and taking $\mu_2 \to \mu_2$, $\mu_3 \to \mu_3 - \ln g(q_3)$ removes the function $g(q_3)$. Reintroducing all possible freedom available from auxiliary field redefinitions by taking $\mu_2 \to \mu_2 + f(q_2,q_3)_{q_2}$, $\mu_3 \to \mu_3 + f(q_2,q_3)_{q_3}$ (where f may be any function) we now have, from the first set of solutions, the defect potential

$$\begin{aligned}
D^+ =& \sigma\Bigg(e^{f_{q_2}}\left(e^{p_1} + e^{-p_1}\right)e^{-p_2+\mu_2} + 2e^{-f_{q_2}+f_{q_3}}\left(e^{q_2} + e^{-q_2}\right)e^{p_2-p_3-\mu_2+\mu_3} \\
&+ e^{-f_{q_3}}\left(e^{q_3} + e^{-q_3}\right)\left(e^{p_4} + e^{-p_4}\right)e^{p_3-\mu_3}\Bigg)
\end{aligned} \tag{3.3.10}$$

$$\begin{aligned}
\bar{D}^+ =& \frac{1}{\sigma}\Bigg(e^{-f_{q_2}}\left(e^{q_1} + e^{-q_1}\right)\left(e^{q_2} + e^{-q_2}\right)e^{-\mu_2} + e^{f_{q_2}-f_{q_3}}\left(e^{q_3} + e^{-q_3}\right)e^{\mu_2-\mu_3} \\
&+ e^{f_{q_3}}\left(e^{q_4} + e^{-q_4}\right)e^{\mu_3}\Bigg).
\end{aligned} \tag{3.3.11}$$

The + superscripts will differentiate this from the defect potential arising from the second set of solutions, and refer to the fact that terms of the form $(e^q + e^{-q})$ appear here.

For the second solution taking $\mu_2 \to \mu_2 - \frac{1}{3}\ln c$, $\mu_3 \to \mu_3 - \frac{2}{3}\ln c$, $\sigma \to c^{\frac{1}{3}}\sigma$ and $\mu_3 \to \mu_3 - \ln g(q_3)$ again removes the constant c and function $g(q_3)$. Reintroducing all possible freedom available from auxiliary field redefinitions by taking $\mu_2 \to \mu_2 + f(q_2,q_3)_{q_2}$, $\mu_3 \to \mu_3 + f(q_2,q_3)_{q_3}$ (where f may be any function) we now have, from the second set of solutions, the defect potential

$$\begin{aligned} D^- =&\sigma\Bigg(e^{f_{q_2}}\left(e^{p_1}-e^{-p_1}\right)e^{-p_2+\mu_2} - 2e^{-f_{q_2}+f_{q_3}}\left(e^{q_2}-e^{-q_2}\right)e^{p_2-p_3-\mu_2+\mu_3} \\ &+ e^{-f_{q_3}}\left(e^{q_3}-e^{-q_3}\right)\left(e^{p_4}-e^{-p_4}\right)e^{p_3-\mu_3}\Bigg) \end{aligned} \tag{3.3.12}$$

$$\begin{aligned} \bar{D}^- =&\frac{1}{\sigma}\Bigg(- e^{-f_{q_2}}\left(e^{q_1}-e^{-q_1}\right)\left(e^{q_2}-e^{-q_2}\right)e^{-\mu_2} + e^{f_{q_2}-f_{q_3}}\left(e^{q_3}-e^{-q_3}\right)e^{\mu_2-\mu_3} \\ &+ e^{f_{q_3}}\left(e^{q_4}-e^{-q_4}\right)e^{\mu_3}\Bigg). \end{aligned} \tag{3.3.13}$$

The $-$ superscripts here refer to the fact that terms of the form $(e^q - e^{-q})$ appear.

There is still the freedom to carry out field redefinitions on the bulk fields. The bulk fields may be shifted by any $2\pi i$ multiple of a weight of D_4 without affecting the bulk Lagrangians. If u and v have the same shift then p is also shifted by a $2\pi i$ multiple of a weight, and as exponentials of p in D all appear in the form $e^{\alpha_i\cdot p}$ they remain unchanged. q would remain completely unchanged. So as in the Tzitzéica case it is the relative shift between u and v which is important. We will consider shifts of v proportional to the fundamental weights given in eqs.(A.0.47).

Acting on the defect potential given by D^+, $\bar{D}^+$ in eqs.(3.3.10),(3.3.11) with $v \to v + 2\pi i w_1$, where w_1 is one of the fundamental weights given in eq.(A.0.47), and also performing the shift $\mu_3 \to \mu_3 + \pi i$ on the auxiliary fields and the redefinition $\sigma \to -\sigma$ gives D^+, $\bar{D}^+$. The freedom from this external field redefinition is equivalent to the freedom we already have to redefine the auxiliary fields and the defect parameter, and does not give a defect potential that is materially different. Carrying out an identical set of redefinitions on D^-, $\bar{D}^-$ returns to D^-, $\bar{D}^-$ also.

Acting on D^+, $\bar{D}^+$ with $v \to v + 2\pi i w_2$ immediately returns D^+, $\bar{D}^+$, and likewise acting on D^-, $\bar{D}^-$ with $v \to v + 2\pi i w_2$ immediately returns D^-, $\bar{D}^-$.

Acting on D^+, $\bar{D}^+$ with $v \to v + 2\pi i w_3$ and $\mu_3 \to \mu_3 - \frac{\pi i}{2}$ gives D^-, $\bar{D}^-$, so the two defect potentials, while not linked by any redefinitions of the auxiliary fields, are linked by a shift of the bulk fields. Using the same shift and set of redefinitions on D^-, $\bar{D}^-$ returns D^+, $\bar{D}^+$.

Finally acting on D^+, $\bar{D}^+$ with $v \to v + 2\pi i w_4$, the shifts $\mu_2 \to \mu_2 + \pi i$, $\mu_3 \to \mu_3 - \frac{\pi i}{2}$ and the redefinition $\sigma \to -\sigma$ gives D^-, $\bar{D}^-$. Unsurprisingly the same set of field redefinitions take D^-, $\bar{D}^-$ to D^+, $\bar{D}^+$.

A shift of a $2\pi i$ multiple of fundamental weights $w_{1,2}$ has no effect on either defect potential beyond utilising the freedom to make auxiliary field redefinitions which is already encapsulated by the presence of the arbitrary function f in the potentials. A shift which is a $2\pi i$ multiple of fundamental weights $w_{3,4}$ links the two distinct defect potentials.

The choice for the 1-space and 2-space splitting and these corresponding momentum conserving defect potentials can be used to give Bäcklund transformations for the D_4 ATFT as discussed in section 2.4.

3.3.2 D_r defect

For this ATFT and the ATFTs in the following subsections we will not attempt to find all possible momentum conserving potentials for a particular defect. Instead we will use the 1-space and 2-space splitting for the D_4 case and the form of the defect potential in eqs.(3.3.10),(3.3.11) to inform us as to the likely splittings and defect potentials for other momentum conserving defects.

For the D_r ATFT potential we use the simple roots from eq.(A.0.21), the lowest weight root from eq.(A.0.36) and the n_i values from eq.(A.0.30) in the potential given in eq.(3.1.1). In the D_4 defect the bulk fields which lived in the 2-space were those which appeared in the term in the bulk potential associated with the central node on the Dynkin diagram. To move from D_4 to D_r we assume that the fields appearing in the terms of the bulk potential associated with the central chain of nodes on the Dynkin diagram will be the fields in the 2-space, so we take the basis of the 1-space to be (e_1, e_r) and the basis of the 2-space to be $(e_2, \ldots, e_{r-1})$. We also have $A = 0$ and no ξ fields [BB17]. With these choices the defect Lagrangian in eq.(2.2.26) becomes

$$\mathcal{L}^D = u_1 v_{1,t} + u_r v_{r,t} + \sum_{i=1}^{r-1} \left(u_i v_{i,t} + 2\mu_i \left(u_{i,t} - v_{i,t} \right) \right) - D - \bar{D}. \tag{3.3.14}$$

We can assume that the terms appearing in D and $\bar{D}$ of the D_4 defect given in eqs.(3.3.10),(3.3.11) which are associated with the central (or outer) nodes of the Dynkin diagram will have the same form as the terms appearing in D and $\bar{D}$ of the D_r defect which are associated with the central chain of nodes (or the outer nodes). This gives a possible choice which ensures $D(p_1, p_2 - \mu_2, \ldots, p_{r-1} - \mu_{r-1}, p_r, q_2, \ldots, q_{r-1})$ and $\bar{D}(q_1, \ldots, q_r, \mu_2, \ldots, \mu_{r-1})$ satisfy the momentum conservation condition in eq.(2.2.33),

$$D = \sigma \Bigg(\left(e^{p_1} + e^{-p_1} \right) e^{-p_2 + \mu_2} + 2 \sum_{i=2}^{r-2} \left(e^{q_i} + e^{-q_i} \right) e^{p_i - p_{i+1} - \mu_i + \mu_{i+1}} \\ + \left(e^{q_{r-1}} + e^{-q_{r-1}} \right) \left(e^{p_r} + e^{-p_r} \right) e^{p_{r-1} - \mu_{r-1}} \Bigg) \tag{3.3.15}$$

$$\bar{D} = \frac{1}{\sigma} \Bigg(\left(e^{q_1} + e^{-q_1} \right) \left(e^{q_2} + e^{-q_2} \right) e^{-\mu_2} + \sum_{i=2}^{r-2} \left(e^{q_{i+1}} + e^{-q_{i+1}} \right) e^{\mu_i - \mu_{i+1}} \\ + \left(e^{q_r} + e^{-q_r} \right) e^{\mu_{r-1}} \Bigg). \tag{3.3.16}$$

As for the D_4 defect it is possible to use redefinitions of the μ_i fields and shifts of the external fields to give different defect potentials satisfying the same momentum conservation condition. It may be that there are also some other defect potentials which are momentum conserving but not linked to this potential by any field redefinitions.

3.3.3 A_r defect

For the A_r ATFT potential we use the simple roots from eq.(A.0.18), the lowest weight root from eq.(A.0.36) and the n_i values from eq.(A.0.27) in the potential given in eq.(3.1.1). The bulk fields in the A_r ATFT have the additional constraint $\sum_{i=1}^{r+1} u_i = 0$, $\sum_{i=1}^{r+1} v_i = 0$. Because the Dynkin diagram of A_r (and so the terms appearing in the bulk potential) looks like the central chain of nodes in the D_r

diagram and ATFT we will take all the fields to be projections onto the 2-space. There will be r auxiliary fields which satisfy the constraint $\sum_{i=1}^{r+1}\mu_i = 0$. We also have $A = 0$ and no ξ fields [BB17]. With these choices the defect Lagrangian in eq.(2.2.26) becomes

$$\mathcal{L}^D = \sum_{i=1}^{r+1}\left(u_i v_{i,t} + 2\mu_i\left(u_{i,t} - v_{i,t}\right)\right) - D - \bar{D}. \tag{3.3.17}$$

All terms in D and $\bar{D}$ are taken to be of a similar form to those associated with the central chain of nodes in eqs.(3.3.15),(3.3.16), giving

$$D = \sigma\left(\sum_{i=1}^{r}\left(e^{q_i} + e^{-q_i}\right)e^{p_i - p_{i+1} - \mu_i + \mu_{i+1}} + \left(e^{q_{r+1}} + e^{-q_{r+1}}\right)e^{p_{r+1} - p_1 - \mu_{r+1} + \mu_1}\right) \tag{3.3.18}$$

$$\bar{D} = \frac{1}{\sigma}\left(\sum_{i=1}^{r}\left(e^{q_{i+1}} + e^{-q_{i+1}}\right)e^{\mu_i - \mu_{i+1}} + \left(e^{q_1} + e^{-q_1}\right)e^{\mu_{r+1} - \mu_1}\right). \tag{3.3.19}$$

This is the same as the defect given by squeezing two A_r defects together [CZ09a; Rob14b]. Once again it is possible to use redefinitions of the μ_i fields and shifts of the external fields to give different defect potentials, and it may be that there are other defect potentials which are momentum conserving but not linked to this potential by any field redefinitions.

3.3.4 B_r defect

For the B_r ATFT potential we use the simple roots from eq.(A.0.19), the lowest weight root from eq.(A.0.36) and the n_i values from eq.(A.0.28) in the potential given in eq.(3.1.1). The Dynkin diagram for B_r has two nodes, α_0 and α_1, which look like the same nodes in D_r, with the rest of the diagram is similar to the central chain of D_r. So we will take the fields in the terms of the bulk potential associated with this central chain part to be the projections onto the 2-space. This gives the basis of the 1-space to be (e_1) and the basis of the 2-space to be $(e_2, \ldots, e_r)$. With $A = 0$ and no ξ fields this gives (from eq.(2.2.26)) the defect Lagrangian

$$\mathcal{L}^D = u_1 v_{1,t} + \sum_{i=2}^{r}\left(u_i v_{i,t} + 2\mu_i\left(u_{i,t} - v_{i,t}\right)\right) - D - \bar{D}. \tag{3.3.20}$$

Once again we look at the form of the terms in eqs.(3.3.15),(3.3.16) to guess the form of D and $\bar{D}$. Taking

$$\begin{aligned} D =&\sigma\Bigg(\left(e^{p_1}+e^{-p_1}\right)e^{-p_2+\mu_2} \\ &+2\sum_{i=1}^{r-1}\left(e^{q_i}+e^{-q_i}\right)e^{p_i-p_{i+1}-\mu_i+\mu_{i+1}}+2\left(e^{q_r}+e^{-q_r}\right)e^{p_r-\mu_r}\Bigg) \end{aligned} \tag{3.3.21}$$

$$\bar{D} =\frac{1}{\sigma}\left(\left(e^{q_1}+e^{-q_1}\right)\left(e^{q_2}+e^{-q_2}\right)e^{-\mu_2}+\sum_{i=1}^{r-1}\left(e^{q_{i+1}}+e^{-q_{i+1}}\right)e^{\mu_i-\mu_{i+1}}+e^{\mu_r}\right) \tag{3.3.22}$$

satisfies the momentum conservation condition in eq.(2.2.33). This is a new momentum conserving defect. Once again it is possible to use field redefinitions to give different defect potentials, and it may be that there are other defect potentials which are momentum conserving but not linked to this potential by any field redefinitions.

3.3.5 C_r defect

For the C_r ATFT potential we use the simple roots from eq.(A.0.20), the lowest weight root from eq.(A.0.36) and the n_i values from eq.(A.0.29) in the potential given in eq.(3.1.1). The Dynkin diagram for C_r had two nodes, α_0 and α_r, which look like the α_r node in B_r and the rest of the diagram is similar to the central chain of D_r or B_r. So we will take there to be no 1-space and the entire root space to be the 2-space, as the field appearing in the α_r bulk potential term in B_r was a projection onto the 2-space. With $A=0$ and no ξ fields this gives (from eq.(2.2.26)) the defect Lagrangian

$$\mathcal{L}^D =\sum_{i=1}^{r}\left(u_i v_{i,t}+2\mu_i\left(u_{i,t}-v_{i,t}\right)\right)-D-\bar{D}. \tag{3.3.23}$$

Considering the form of the terms in eqs.(3.3.21),(3.3.22) we choose to take

$$\begin{aligned} D =&\sigma\Bigg(\frac{1}{2}\left(e^{q_1}+e^{-q_1}\right)^2e^{-2p_1+2\mu_1} \\ &+2\sum_{i=1}^{n-1}\left(e^{q_{i+1}}+e^{-q_{i+1}}\right)e^{p_i-p_{i+1}-\mu_i+\mu_{i+1}}+\frac{1}{2}e^{2p_n-2\mu_n}\Bigg) \end{aligned} \tag{3.3.24}$$

$$\bar{D} = \frac{1}{\sigma}\left(e^{-2\mu_1} + 2\sum_{i=1}^{n-1}\left(e^{q_i} + e^{-q_i}\right)e^{\mu_i - \mu_{i+1}} + \left(e^{q_n} + e^{-q_n}\right)^2 e^{2\mu_n}\right), \tag{3.3.25}$$

which satisfies the momentum conservation condition in eq.(2.2.33). For C_2 this momentum conserving defect is the same as that found in section 3.2 (up to a redefinition of the μ_1 and μ_2 auxiliary fields), following the method in [Rob14b], by squeezing together A_3 type I defects and then carrying out a folding procedure. Once again it is possible to use field redefinitions to give different defect potentials, and it may be that there are other defect potentials which are momentum conserving but not linked to this potential by any field redefinitions.

3.3.6 E_r defect

Some attempts have been made to find a defect for one of the E series ATFTs, with the focus on E_6 as the simplest of the three. However, so far no progress has been made. As we are simply using trial and error there is little useful to say here. In all attempts we have used no ξ fields, as the presence of these significantly complicates the differential equations which x_i and y_i must satisfy.

In section 4.2 an observation about a likely constraint on the splitting of the root space into the 1-space and 2-space is made. Every simple and lowest weight root must either have $\alpha_i^{(1)} = 0$ or some other root α_j must exist such that $(\mathbb{1} + A)\alpha_i^{(1)} = (-\mathbb{1} + A)\alpha_j^{(1)}$ and $\alpha_i^{(2)} = \alpha_j^{(2)}$. While this constraint was not proved to be necessary for a defect to have an infinite number of conserved quantities it does hold for the 1-space and 2-space splittings found so far. Satisfying this may require rotating the simple roots as given in eq.s(A.0.22),(A.0.36) so that $\alpha_i = (\alpha_i^{(1)}, \alpha_i^{(2)})$, that is, the 1-space and 2-space each have a basis given by some of the standard orthonormal basis vectors e_i.

From this we have tried to choose the 1-space and 2-space in such a way that $(\alpha_0)^{(2)} = (\alpha_1)^{(2)} = (\alpha_5)^{(2)}$, $(\alpha_2)^{(2)} = (\alpha_4)^{(2)} = (\alpha_5)^{(2)}$, $(\alpha_3)^{(1)} = 0$, and the matrix A can be found such that $(\alpha_0)^{(1)}$, $(\alpha_1)^{(1)}$, $(\alpha_5)^{(1)}$ and $(\alpha_2)^{(1)}$, $(\alpha_4)^{(1)}$, $(\alpha_6)^{(1)}$ are correctly related. This has not yet been possible.

We have also tried choose the 1-space in such a way that $(\alpha_0)^{(2)} = (\alpha_6)^{(2)}$, $(\alpha_1)^{(2)} = (\alpha_2)^{(2)}$, $(\alpha_4)^{(2)} = (\alpha_5)^{(2)}$, $(\alpha_3)^{(1)}$=0 and then as $A = 0$ $(\alpha_0)^{(1)} = -(\alpha_6)^{(1)}$ and $(\alpha_1)^{(1)} = -(\alpha_2)^{(1)}$ and $(\alpha_4)^{(1)} = -(\alpha_5)^{(1)}$. This has also not yet been possible.

3.3.7 F_4 defect

The F_4 ATFT can be found by folding the E_6 ATFT, so it is unlikely that any progress will be made here before the E_6 defect has been found. Some attempts have been made with $A = 0$ and no ξ fields, but no useful information has been gleaned. If the F_4 defect were found before the E_6 defect it may be that the connection between the two ATFTs would give hints as to the form of the E_6 defect.

3.3.8 G_2 defect

The G_2 ATFT is given by a folding of the D_4 ATFT, and so as the D_4 defects are now known we would expect to be able to apply the folding procedure from section 3.2 and [Rob14b] to obtain a momentum conserving G_2 defect. This has not yet been achieved.

Chapter 4

Conserved quantities of defects in affine Toda field theory

4.1 Introduction

The study of integrability of systems is a large and important area in mathematical physics. Here we work with classical $1 + 1$ dimensional field theories, and define these as Liouville integrable if it can be shown that there are an infinite number of conserved quantities which are independent and in Poisson involution. In the classical case the existence of a Lax pair satisfying the zero curvature condition implies the existence of an infinite number of conserved quantities. The existence of an r-matrix satisfying the classical Yang-Baxter equation ensures that the conserved quantities are independent and in Poisson involution [Lax68; Sem83; FT86].

For the systems containing defects investigated in the previous chapters there is some evidence that they are likely integrable (namely that they give a Bäcklund transformation and admit soliton solutions) [BCZ04b; BCZ04a; CZ09b; CZ09a]. An infinite number of conserved quantities have been generated for the type I defects [BCZ04b; BCZ04a; CZ07; Cau08; CZ09b] and a type II defect in the Tzitzéica model is shown to have an infinite number of conserved quantities in [AAGZ11].

However, the discontinuity at the defect makes it difficult to move from the Lagrangian description we have used so far to the Hamiltonian description required to find an r-matrix satisfying the classical Yang-Baxter equation, and so prove that the charges are in involution. A Hamiltonian set-up in which the Lax and r-matrix equations are immediately assumed to be satisfied by some matrix associated with the defect is investigated in [AD12a; AD12b; Doi15; Doi16] for defects in the nonlinear Schrödinger and sine-Gordon equations. While these defects are integrable they do not necessarily describe the same systems as the momentum conserving defects found in the Lagrangian set-up. Some attempt to reconcile this Hamiltonian approach and the Lagrangian approach to defects is made in [Cau15; CK15] for the nonlinear Schrödinger equation and sine-Gordon cases. The type I and type II Lagrangians are rewritten as Hamiltonians with second class constraints in [CZ09a]. Some information on quantum integrability in field theories is given in [ZZ79; Dor91; Dor92] and the quantum integrability of defects is investigated in [CZ11; CZ10].

Here we follow [BCZ04a] by considering whether the system with a defect has a zero curvature representation, and so an infinite number of conserved quantities, but not whether these conserved quantities are in Poisson involution. We will first give a brief introduction to the Lax pair, the zero curvature condition and how this is used to generate an infinite number of conserved quantities.

The Lax pair is a pair of matrices $a_0(t,x,\lambda)$ and $a_1(t,x,\lambda)$ such that for a vector field $\Psi(t,x)$

$$\frac{\mathrm{d}\Psi(t,x)}{\mathrm{d}t} = -\, a_0(t,x,\lambda)\Psi(t,x) \tag{4.1.1}$$
$$\frac{\mathrm{d}\Psi(t,x)}{\mathrm{d}x} = -\, a_1(t,x,\lambda)\Psi(t,x) \tag{4.1.2}$$

where λ is the spectral parameter. Using $f(q+\delta q) = f(q)+\delta q f(q)_q$ and $e^{\delta q f} = \mathbb{1}+\delta q f$ we have

$$\Psi(t+\delta t,x) =(\mathbb{1} - \delta t a_0(t,x,\lambda))\Psi(t,x) = e^{-\delta t a_0(t,x,\lambda)}\Psi(t,x) \tag{4.1.3}$$

$$\Psi(t,x+\delta x) =(\mathbb{1} - \delta x a_1(t,x,\lambda))\Psi(t,x) = e^{-\delta x a_1(t,x,\lambda)}\Psi(t,x). \tag{4.1.4}$$

These infinitesimal translations of Ψ may then be used to build any path, giving the transport matrices

$$\Psi(t_2, x, \lambda) = Pe^{-\int_{t_1}^{t_2} \mathrm{d}t' a_0(t',x,\lambda)} \Psi(t_1, x, \lambda) \tag{4.1.5}$$

$$\Psi(t, x_2, \lambda) = Pe^{-\int_{x_1}^{x_2} \mathrm{d}x' a_1(t,x',\lambda)} \Psi(t, x_1, \lambda) \tag{4.1.6}$$

where P denotes path ordering. The transport matrices themselves are also solutions to eqs.(4.1.1),(4.1.2) respectively.

If we can show that the transport of Ψ along two infinitesimally different paths with the same endpoint is path independent then we can use this to build up any paths between the same two endpoints and have the transport of Ψ be path independent, so have a system with a zero curvature representation.

$$\begin{tikzcd}
\Psi(t+\delta t, x) \arrow[r, "e^{-\int a_1}"] & \Psi(t+\delta t, x+\delta x) \\
\Psi(t, x) \arrow[u, "e^{-\int a_0}"] \arrow[r, "e^{-\int a_1}"'] & \Psi(t, x+\delta x) \arrow[u, "e^{-\int a_0}"']
\end{tikzcd} \tag{4.1.7}$$

From this picture we see that for path independence we require

$$\begin{aligned}
&Pe^{-\int_t^{t+\delta t} \mathrm{d}t' a_0(t',x+\delta x,\lambda)} Pe^{-\int_x^{x+\delta x} \mathrm{d}x' a_1(t,x',\lambda)} \\
&\qquad = Pe^{-\int_x^{x+\delta x} \mathrm{d}x' a_1(t+\delta t,x',\lambda)} Pe^{-\int_t^{t+\delta t} \mathrm{d}t' a_0(t',x,\lambda)} \\
&(\mathbb{1} - \delta t a_0(t, x+\delta x, \lambda))(\mathbb{1} - \delta x a_1(t, x, \lambda)) = (\mathbb{1} - \delta x a_1(t+\delta t, x, \lambda))(\mathbb{1} - \delta t a_0(t, x, \lambda)),
\end{aligned} \tag{4.1.8}$$

and expanding this equation gives the zero curvature condition

$$a_{1,t} - a_{0,x} + [a_0, a_1] = 0. \tag{4.1.9}$$

This must be satisfied by the Lax pair if we are to generate an infinite number of conserved quantities. It is the same as the condition found if the overdetermined system of equations in eqs.(4.1.1),(4.1.2) is required to be consistent. The gauge transformation

$$a_0 \to \tilde{a}_0 = -G_t G^{-1} + G a_0 G^{-1} \tag{4.1.10}$$

$$a_1 \to \tilde{a}_1 = -G_x G^{-1} + G a_1 G^{-1} \tag{4.1.11}$$

leaves the zero curvature condition unchanged.

The system in the bulk is some field u which is governed by an equation of motion. If a pair of matrices which are dependent on u and the spectral parameter λ satisfy eq.(4.1.9) if and only if u satisfies the equations of motion of the system then we have a Lax pair of the system. To generate the infinite number of conserved quantities we need the monodromy matrix

$$T(t,\lambda) = Pe^{-\int_{-\infty}^{\infty} \mathrm{d}x' a_1(t,x',\lambda)}. \tag{4.1.12}$$

The conserved quantities generated by the monodromy matrix will then be dependent on u, and so will be conserved quantities of the specific system. The time translation matrices at $x \to \pm\infty$ are

$$S_\pm(t_2,t_1,\lambda) = Pe^{-\int_{t_1}^{t_2} \mathrm{d}t' a_0(t',\pm\infty,\lambda)} \tag{4.1.13}$$

and provided that a_0 satisfies

$$a_0(t,\infty,\lambda) = Q a_0(t,-\infty,\lambda) Q^{-1}, \tag{4.1.14}$$

where Q is some constant matrix we can use $e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$ to show that the time translation matrices at $x \to \pm\infty$ satisfy

$$\begin{aligned}
S_+(t_2,t_1,\lambda) =& Pe^{-\int_{t_1}^{t_2} \mathrm{d}t' a_0(t',\infty,\lambda)} \\
=& Pe^{Q(a)\left(-\int_{t_1}^{t_2} \mathrm{d}t' a_0(t',-\infty,\lambda)\right) Q(a)^{-1}} \\
=& \sum_{n=0}^{\infty} \frac{1}{n!} Q(a) \left(-\int_{t_1}^{t_2} \mathrm{d}t' a_0(t',-\infty,\lambda)\right)^n Q(a)^{-1} \\
=& Q(a) Pe^{-\int_{t_1}^{t_2} \mathrm{d}t' a_0(t',-\infty,\lambda)} Q(a)^{-1} \\
=& Q(a) S_-(t_2,t_1,\lambda) Q(a)^{-1}.
\end{aligned} \tag{4.1.15}$$

We can now consider transport from $(t_1, -\infty)$ to (t_2, ∞) along two different paths.

$$\begin{tikzcd}
\Psi(t_2,-\infty) \arrow[r, "T"] & \Psi(t_2,\infty) \\
\Psi(t_1,-\infty) \arrow[u, "S_-"] \arrow[r, "T"'] & \Psi(t_1,\infty) \arrow[u, "S_+"']
\end{tikzcd} \tag{4.1.16}$$

Because we have zero curvature this gives the relation

$$\begin{aligned}
T(t_2,\lambda)S_-(t_2,t_1,\lambda) =& S_+(t_2,t_1,\lambda)T(t_1,\lambda)\\
T(t_2,\lambda)S_-(t_2,t_1,\lambda) =& QS_-(t_2,t_1,\lambda)Q^{-1}T(t_1,\lambda)\\
Q^{-1}T(t_2,\lambda) =& S_-(t_2,t_1,\lambda)Q^{-1}T(t_1,\lambda)S_-(t_2,t_1,\lambda)^{-1}
\end{aligned} \tag{4.1.17}$$

and by taking the trace of this and using the cyclicity of the trace we have

$$\mathrm{tr}\left(Q^{-1}T(t_2,\lambda)\right) = \mathrm{tr}\left(Q^{-1}T(t_1,\lambda)\right). \tag{4.1.18}$$

Therefore the trace of the matrix $Q^{-1}T(t,\lambda)$ is time independent. T is an exponential of a matrix containing λ, so can be expanded infinitely in λ, and as zero curvature must hold for any value of λ all of the coefficients of λ must be independently conserved. This gives an infinite number of conserved quantities associated with the system whose Lax pair is a_0, a_1.

For an ATFT the Lax pair is

$$a_0 = \frac{1}{2}\left(u_x.H + \frac{1}{\sqrt{2}}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|e^{\frac{1}{2}\alpha_i.u}\left(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i}\right)\right) \tag{4.1.19}$$

$$a_1 = \frac{1}{2}\left(u_t.H + \frac{1}{\sqrt{2}}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|e^{\frac{1}{2}\alpha_i.u}\left(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i}\right)\right) \tag{4.1.20}$$

[MOP81] where H are the Cartan generators and E_{α_i} is the generator associated with the root α_i. These matrices obey the relations given in eqs.(A.0.1)-(A.0.5). The α_i are the simple and lowest weight roots given in eqs.(A.0.18)-(A.0.26),(A.0.36) and the marks n_i are given in eqs.(A.0.27)-(A.0.35). Using these matrices in eq.(4.1.9) we can check that it is satisfied provided that the equations of motion of the ATFT (given by the Lagrangian density in eq.(3.1.1)) are satisfied. This matrix a_0 can be checked to satisfy eq.(4.1.14) for $Q = e^{\pi i(w_- - w_+).H}$, where the field u takes values of $2\pi i w_\pm$ as $x \to \pm\infty$ and $w_\pm$ are weights of the Lie algebra.

When investigating defects we follow [BCZ04b; BCZ04a; CZ09b], only considering whether the system posesses a zero curvature representation (and so the existence of an infinite number of conserved quantities), not the existence of an r-matrix. The defects are always taken to appear in integrable theories, so the bulk fields theories have a zero curvature representation and we only need to consider curvature across the defect. We take there to be a time dependent matrix which acts to move from the left of the defect to the right of the defect without changing position. This calculation of the defect zero curvature condition is not specific to defects in ATFTs, but can be applied to a defect in any integrable theory.

Consider an integrable theory in the region $x \le 0$ with the Lax pair $a_0^<(t,x)$, $a_1^<(t,x)$ satisfying the zero curvature condition in eq.(4.1.9), an integrable theory in the region $x \ge 0$ with the Lax pair $a_0^>(t,x)$, $a_1^>(t,x)$ also satisfying eq.(4.1.9) and with a defect at $x = 0$. The matrices $a_0^<(t,x)$, $a_1^<(t,x)$ depend on the field u and $a_0^>(t,x)$, $a_1^>(t,x)$ depend on the field v. We consider the transport of the vector Ψ in the region of the defect.

$$
\begin{tikzcd}
u & v \\
\Psi(t+\delta t,0) \arrow[r, "K"] & \Psi(t+\delta t,0) \\
\Psi(t,0) \arrow[u, "e^{-\int a_0^<}"] \arrow[r, "K"'] & \Psi(t,0) \arrow[u, "e^{-\int a_0^>}"']
\end{tikzcd}
\tag{4.1.21}
$$

The defect transport matrix K introduced here depends on both the u and v fields evaluated at $x = 0$ and on any auxiliary fields which are confined to the defect. For zero curvature this gives

$$K(t+\delta t) P e^{-\int_t^{t+\delta t} \mathrm{d}t' a_0^<(t',0)} = P e^{-\int_t^{t+\delta t} \mathrm{d}t' a_0^>(t',0)} K(t). \tag{4.1.22}$$

Expanding this in δt we have

$$K_t = K a_0^< - a_0^> K \tag{4.1.23}$$

evaluated at $x = 0$. An extremely similar calculation which gave the same zero curvature condition for the defect was carried out in [BCZ04a]. The bulk zero curvature condition in eq.(4.1.9) is satisfied if and only if the bulk equations of motion are satisfied, and this extra defect zero curvature condition must be satisfied if and only if the defect equations are satisfied. Note that eq.(4.1.23) is equivalent to K being a gauge transformation between the operators $\partial_t + a_0^<$ and $\partial_t + a_0^>$, with $\partial_t + a_0^< = K^{-1}(\partial_t + a_0^>)K$. Carrying out a gauge transform of G on $a_0^<$ and G' on $a_0^>$ (as given in eq.(4.1.10)) along with the gauge transformation $K \to K' = G'KG^{-1}$ leaves this defect zero curvature condition unchanged.

In this chapter we first use the general defect in an ATFT given by the Lagrangian in eq.(2.2.26) and the defect potential in eqs.(3.3.1),(3.3.2) to make some comments on how the defect zero curvature condition in eq.(4.1.23) may be satisfied. We then explicitly calculate K for the Tzitzéica and D_4 defects, proving that these two systems have an infinite number of conserved quantities. The defect matrix for the Tzitzéica model has been found previously in [AAGZ11].

4.2 Zero curvature for momentum conserving defects in ATFTs

Using the ATFT a_0 matrix given in eq.(4.1.19) and taking it to depend on $u = p + q$ to give $a_0^<$ and $v = p - q$ to give $a_0^>$ the zero curvature condition on the defect becomes

$$\begin{aligned}2K_t =& p_{j,x}\,[K, H_j] + q_{j,x}\{K, H_j\} \\ &+ \frac{1}{\sqrt{2}}\sum_{i=0}^{r} \sqrt{n_i}|\alpha_i| e^{\frac{1}{2}(\alpha_i)_j p_j} \left(\lambda \left(e^{\frac{1}{2}(\alpha_i)_j q_j} K E_{\alpha_i} - e^{-\frac{1}{2}(\alpha_i)_j q_j} E_{\alpha_i} K\right)\right.\end{aligned}$$

$$-\frac{1}{\lambda}\left(e^{\frac{1}{2}(\alpha_i)_j q_j}KE_{-\alpha_i}-e^{-\frac{1}{2}(\alpha_i)_j q_j}E_{-\alpha_i}K\right)\Bigg) \quad (4.2.1)$$

where square brackets indicate a commutator and curly brackets an anticommutator (not a Poisson bracket).

We will begin by taking the defect to be of the general form given in eq.(2.2.26), which has defect equations given in eqs.(2.2.34)-(2.2.39), where D and $\bar{D}$ must have the dependencies given in eqs.(2.2.31),(2.2.32) and satisfy the additional momentum conservation condition in eq.(2.2.33). Using eqs.(2.2.34)-(2.2.37) to remove all x derivatives from eq.(4.2.1) gives

$$\begin{aligned}2K_t=&\left(p^{(1)}_{j,t}-2q^{(1)}_{k,t}A_{kj}-\frac{1}{2}D_{q^{(1)}_j}-\frac{1}{2}\bar{D}_{q^{(1)}_j}\right)\left[K,H^{(1)}_j\right]\\&+\left(p^{(2)}_{j,t}-2\mu^{(2)}_{j,t}-\frac{1}{2}D_{q^{(2)}_j}-\frac{1}{2}\bar{D}_{q^{(2)}_j}\right)\left[K,H^{(2)}_j\right]\\&+\left(-q^{(1)}_{j,t}-\frac{1}{2}D_{p^{(1)}_j}\right)\{K,H^{(1)}_j\}+\left(-q^{(2)}_{j,t}-\frac{1}{2}D_{p^{(2)}_j}\right)\{K,H^{(2)}_j\}\\&+\frac{1}{\sqrt{2}}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|e^{\frac{1}{2}(\alpha_i)_j p_j}\Bigg(\lambda\left(e^{\frac{1}{2}(\alpha_i)_j q_j}KE_{\alpha_i}-e^{-\frac{1}{2}(\alpha_i)_j q_j}E_{\alpha_i}K\right)\\&\qquad-\frac{1}{\lambda}\left(e^{\frac{1}{2}(\alpha_i)_j q_j}KE_{-\alpha_i}-e^{-\frac{1}{2}(\alpha_i)_j q_j}E_{-\alpha_i}K\right)\Bigg).\end{aligned} \quad (4.2.2)$$

Every Cartan generator is associated with one of the orthonormal basis vectors of the root space, so $H^{(1)}$ denotes the Cartan generators which are associated with the orthonormal basis vectors which form a basis of the 1-space and $H^{(2)}$ denotes the Cartan generators associated with the orthonormal basis vectors of the 2-space. The t derivatives on the right hand side can be removed by applying the transformation

$$K=e^{-\frac{1}{2}(p_j+q_j)H_j+q^{(1)}jA_{jk}H^{(1)}_k+\mu^{(2)}_j H^{(2)}_j}\hat{K}e^{\frac{1}{2}(p_j-q_j)H_j-q^{(1)}_j A_{jk}H^{(1)}_k-\mu^{(2)}_j H^{(2)}_j} \quad (4.2.3)$$

to give

$$\begin{aligned}&4\hat{K}_t+D_{p_j}\{\hat{K},H_j\}+(D_{q_j}+\bar{D}_{q_j})\left[\hat{K},H_j\right]\\&=\sqrt{2}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|\Bigg(\lambda e^{(\alpha_i)_j p_j+(\alpha_i)^{(1)}_j A_{jk}q^{(1)}_k-(\alpha_i)^{(2)}_j\mu^{(2)}_j}\left[\hat{K},E_{\alpha_i}\right]\\&\qquad-\frac{1}{\lambda}e^{-(\alpha_i)^{(1)}_j A_{jk}q^{(1)}_k+(\alpha_i)^{(2)}_j\mu^{(2)}_j}\left(e^{(\alpha_i)_j q_j}\hat{K}E_{-\alpha_i}-e^{-(\alpha_i)_j q_j}E_{-\alpha_i}\hat{K}\right)\Bigg).\end{aligned} \quad (4.2.4)$$

If $\hat{K}$ is dependent on a field then the term $\hat{K}_t$ introduces a t derivative of that field, which will not appear anywhere else in eq.(4.2.4). For the fields $q^{(2)}$ and ξ we can remove the t derivative using eq.(2.2.38) and eq.(2.2.39) respectively. For the fields $p^{(1)}$, $q^{(1)}$, $p^{(2)}$ and $\mu^{(2)}$ the t derivative cannot be removed (except by the introduction of an x derivative, which returns us to the previous step in our calculation) so $\hat{K}$ cannot be dependent on these fields. The same argument can be used to show that $\hat{K}$ cannot depend on the derivatives of fields as well. With $\hat{K}$ only dependent on $q^{(2)}$ and ξ we have $\hat{K}_t = \hat{K}_{q_i^{(2)}} q_{i,t}^{(2)} + \hat{K}_{\xi_i}\xi_{i,t}$, and using this and eqs.(2.2.38),(2.2.39) the zero curvature condition becomes

$$\begin{aligned}
&\hat{K}_{q_i^{(2)}}(D_{\mu_i^{(2)}} + \bar{D}_{\mu_i^{(2)}}) - 4\hat{K}_{\xi_i}W_{ij}(D_{\xi_j} + \bar{D}_{\xi_j}) + D_{p_j}\{\hat{K}, H_j\} + (D_{q_j} + \bar{D}_{q_j})\left[\hat{K}, H_j\right] \\
&= \sqrt{2}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|\Bigg(\lambda e^{(\alpha_i)_j p_j + (\alpha_i)_j^{(1)}A_{jk}q_k^{(1)} - (\alpha_i)_j^{(2)}\mu_j^{(2)}}\left[\hat{K}, E_{\alpha_i}\right] \\
&\qquad - \frac{1}{\lambda}e^{-(\alpha_i)_j^{(1)}A_{jk}q_k^{(1)} + (\alpha_i)_j^{(2)}\mu_j^{(2)}}\left(e^{(\alpha_i)_j q_j}\hat{K}E_{-\alpha_i} - e^{-(\alpha_i)_j q_j}E_{-\alpha_i}\hat{K}\right)\Bigg). \qquad (4.2.5)
\end{aligned}$$

To progress further we now need a specific form for the defect potential. In section 3.3 we stated that for a defect in an ATFT to be momentum conserving D and $\bar{D}$ must be of the form given in eqs.(3.3.1),(3.3.2). Using this in the zero curvature condition we have

$$\begin{aligned}
&\sigma\sum_{i=0}^{r} e^{(\alpha_i)_j p_j + (\alpha_i)_j^{(1)}A_{jk}q_k^{(1)} - (\alpha_i)_j^{(2)}\mu_j^{(2)}}\Big(- x_i(\alpha_i)_j^{(2)}\hat{K}_{q_j^{(2)}} + 4x_{i,\xi_j}W_{jk}\hat{K}_{\xi_k} \\
&\qquad + x_i(\alpha_i)_j^{(1)}A_{jk}\left[\hat{K}, H_k^{(1)}\right] + x_{i,q_j^{(2)}}\left[\hat{K}, H_j^{(2)}\right] + x_i(\alpha_i)_j\{\hat{K}, H_j\}\Big) \\
&+ \frac{1}{\sigma}\sum_{i=0}^{r} e^{-(\alpha_i)_j^{(1)}A_{jk}q_k^{(1)} + (\alpha_i)_j^{(2)}\mu_j^{(2)}}\Big(y_i(\alpha_i)_j^{(2)}\hat{K}_{q_j^{(2)}} + 4y_{i,\xi_j}W_{jk}\hat{K}_{\xi_k} \\
&\qquad - y_i(\alpha_i)_j^{(1)}A_{jk}\left[\hat{K}, H_k^{(1)}\right] + y_{i,q_j}\left[\hat{K}, H_j\right]\Big) \\
&= \sqrt{2}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|\Bigg(\lambda e^{(\alpha_i)_j p_j + (\alpha_i)^{(1)})_j A_{jk}q_k^{(1)} - (\alpha_i)_j^{(2)}\mu_j^{(2)}}\left[\hat{K}, E_{\alpha_i}\right] \\
&\qquad - \frac{1}{\lambda}e^{-(\alpha_i)_j^{(1)}A_{jk}q_k^{(1)} + (\alpha_i)_j^{(2)}\mu_j^{(2)}}\left(e^{(\alpha_i)_j q_j}\hat{K}E_{-\alpha_i} - e^{-(\alpha_i)_j q_j}E_{-\alpha_i}\hat{K}\right)\Bigg). \qquad (4.2.6)
\end{aligned}$$

Equating exponents of p splits this into $r+2$ equations,

$$\sqrt{2}\sqrt{n_i}|\alpha_i|\rho\left[\hat{K}, E_{\alpha_i}\right] = - x_i(\alpha_i)_j^{(2)}\hat{K}_{q_j^{(2)}} + 4x_{i,\xi_j}W_{jk}\hat{K}_{\xi_k}$$

$$+ x_i(\alpha_i)^{(1)}_j A_{jk}\left[\hat{K}, H^{(1)}_k\right] + x_{i,q^{(2)}_j}\left[\hat{K}, H^{(2)}_j\right]$$
$$+ x_i(\alpha_i)_j\{\hat{K}, H_j\} \tag{4.2.7}$$

for $i = 0, \ldots, r$ and

$$\begin{aligned}
&-\sqrt{2}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|e^{-(\alpha_i)^{(1)}_j A_{jk} q^{(1)}_k + (\alpha_i)^{(2)}_j \mu^{(2)}_j}\left(e^{(\alpha_i)_j q_j}\hat{K}E_{-\alpha_i} - e^{-(\alpha_i)_j q_j}E_{-\alpha_i}\hat{K}\right)\\
&\quad = \rho\sum_{i=0}^{r} e^{-(\alpha_i)^{(1)}_j A_{jk} q^{(1)}_k + (\alpha_i)^{(2)}_j \mu^{(2)}_j}\Big(- y_i(\alpha_i)^{(1)}_j A_{jk}\left[\hat{K}, H^{(1)}_k\right] + y_{i,q_j}\left[\hat{K}, H_j\right]\\
&\qquad + y_i(\alpha_i)^{(2)}_j \hat{K}_{q^{(2)}_j} + 4y_{i,\xi_j}W_{jk}\hat{K}_{\xi_k}\Big)
\end{aligned} \tag{4.2.8}$$

where we have set $\rho = \lambda\sigma^{-1}$. We cannot split eq.(4.2.8) by equating exponentials of $\mu^{(2)}$, as two different roots α_i and α_j may have the same projection onto the 2-space.

Multiplying K by a constant does not affect the zero curvature condition in eq.(4.1.23), so we can always take the highest power of ρ appearing in K to be zero. Therefore we can always expand $\hat{K}$ in ρ as

$$\hat{K} = \sum_{s=0}^{\infty}\rho^{-s}k_s. \tag{4.2.9}$$

The k_s are matrices, and any of them may be zero. We do not know if this expansion terminates. We will assume that, like the bulk Lax pair, this defect matrix will consist of generators of the Lie algebra. More specifically, since it appears as part of the monodromy matrix, we would expect to be able to write it as an exponential or combination of exponentials of the generators. Expanding such an exponential in terms of ρ (which should appear in the exponent by comparison with the bulk monodromy matrix) we therefore expect that the matrices k_s will be some combination of generator matrices.

Substituting this expansion into the zero curvature relations in eqs.(4.2.7),(4.2.8) and equating powers of ρ gives a set of recursion relations,

$$\sqrt{2}\sqrt{n_i}|\alpha_i|\left[k_{s+1}, E_{\alpha_i}\right] = - x_i(\alpha_i)^{(2)}_j k_{s,q^{(2)}_j} + 4x_{i,\xi_j}W_{jk}k_{s,\xi_k}$$

$$+ x_i(\alpha_i)^{(1)}_j A_{jk} \left[k_s, H^{(1)}_k\right] + x_{i,q^{(2)}_j} \left[k_s, H^{(2)}_j\right] + x_i(\alpha_i)_j\{k_s, H_j\} \tag{4.2.10}$$

for $i = 0, \ldots, r$ and

$$\begin{aligned} - \sqrt{2}\sum_{i=0}^{r} \sqrt{n_i}|\alpha_i| e^{-(\alpha_i)^{(1)}_j A_{jk} q^{(1)}_k + (\alpha_i)^{(2)}_j \mu^{(2)}_j} \left(e^{(\alpha_i)_j q_j} k_s E_{-\alpha_i} - e^{-(\alpha_i)_j q_j} E_{-\alpha_i} k_s\right) \\ = \sum_{i=0}^{r} e^{-(\alpha_i)^{(1)}_j A_{jk} q^{(1)}_k + (\alpha_i)^{(2)}_j \mu^{(2)}_j} \Big(- y_i(\alpha_i)^{(1)}_j A_{jk} \left[k_{s+1}, H^{(1)}_k\right] + y_{i,q_j} \left[k_{s+1}, H_j\right] \\ + y_i(\alpha_i)^{(2)}_j k_{s+1,q^{(2)}_j} + 4y_{i,\xi_j} W_{jk} k_{s+1,\xi_k}\Big). \end{aligned} \tag{4.2.11}$$

We can now attempt to solve these relations, which would ensure zero curvature across any momentum conserving defect of the form given in eq.(2.2.26) in an ATFT. Beginning with $s = -1$ we have

$$0 = \sqrt{2}\sqrt{n_i}|\alpha_i| \left[k_0, E_{\alpha_i}\right] \tag{4.2.12}$$

for $i = 0, \ldots, r$ and

$$\begin{aligned} 0 = \sum_{i=0}^{r} e^{-(\alpha_i)^{(1)}_j A_{jk} q^{(1)}_k + (\alpha_i)^{(2)}_j \mu^{(2)}_j} \Big(- y_i(\alpha_i)^{(1)}_j A_{jk} \left[k_0, H^{(1)}_k\right] + y_{i,q_j} \left[k_0, H_j\right] \\ + y_i(\alpha_i)^{(2)}_j k_{0,q^{(2)}_j} + 4y_{i,\xi_j} W_{jk} k_{0,\xi_k}\Big). \end{aligned} \tag{4.2.13}$$

If k_0 is to commute with all simple root generators and the lowest weight root generator then by Schur's lemma it must be proportional to the identity matrix. This ensures the first $r + 1$ equations are satisfied. We will take k_0 to be a scalar multiple of the identity matrix (satisfying the final equation), and using the fact that K may be multiplied by a constant without affecting the defect zero curvature condition, set $k_0 = \mathbb{1}$. There may be some choices of k_0 which are dependent on $q^{(2)}$ and ξ and satisfy eq.(4.2.13), but it is certainly not obvious. No defects found thus far have contained auxiliary fields which couple only to other auxiliary fields, and if these is no ξ field vector then for eq.(4.2.13) to be satisfied we must have $k_{0,q^{(2)}_i} = 0$ and so k_0 will always be a scalar multiple of the identity matrix.

Now consider $s=0$. The recurrence relations give

$$\sqrt{2}\sqrt{n_i}|\alpha_i|\,[k_1, E_{\alpha_i}] = 2x_i(\alpha_i)_j H_j \tag{4.2.14}$$

for $i=0,\dots,r$ and

$$\begin{aligned}&-\sqrt{2}\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|e^{-(\alpha_i)_j^{(1)}A_{jk}q_k^{(1)}+(\alpha_i)_j^{(2)}\mu_j^{(2)}}\left(e^{(\alpha_i)_j q_j}-e^{-(\alpha_i)_j q_j}\right)E_{-\alpha_i}\\ &\quad=\sum_{i=0}^{r}e^{-(\alpha_i)_j^{(1)}A_{jk}q_k^{(1)}+(\alpha_i)_j^{(2)}\mu_j^{(2)}}\Big(-y_i(\alpha_i)_j^{(1)}A_{jk}\left[k_1,H_k^{(1)}\right]+y_{i,q_j}\left[k_1,H_j\right]\\ &\qquad\qquad+y_i(\alpha_i)_j^{(2)}k_{1,q_j^{(2)}}+4y_{i,\xi_j}W_{jk}k_{1,\xi_k}\Big),\end{aligned} \tag{4.2.15}$$

and we can immediately see that the first $r+1$ equations in eq.(4.2.14) are satisfied by

$$k_1 = -\frac{1}{\sqrt{2}}\sum_{j=0}^{r}\frac{1}{\sqrt{n_j}}|\alpha_j|x_j E_{-\alpha_j} \tag{4.2.16}$$

using the fact that a simple root plus the negative of a simple root is never a root and that the highest (lowest) weight root plus any positive (negative) root cannot be a root. The final equation, eq.(4.2.15), then becomes

$$\begin{aligned}&2\sum_{i=0}^{r}\sqrt{n_i}|\alpha_i|e^{-(\alpha_i)_k^{(1)}A_{kl}q_l^{(1)}+(\alpha_i)_k^{(2)}\mu_k^{(2)}}\left(e^{(\alpha_i)_k q_k}-e^{-(\alpha_i)_k q_k}\right)E_{-\alpha_i}\\ &=\sum_{i=0}^{r}\sum_{j=0}^{r}\frac{1}{\sqrt{n_j}}|\alpha_j|e^{-(\alpha_i)_k^{(1)}A_{kl}q_l^{(1)}+(\alpha_i)_k^{(2)}\mu_k^{(2)}}\Big(y_i(\alpha_i)_k^{(2)}x_{j,q_k^{(2)}}+4y_{i,\xi_k}W_{kl}x_{j,\xi_l}+x_j y_{i,q_k}(\alpha_j)_k\\ &\qquad\qquad-x_j y_i(\alpha_i)_k^{(1)}A_{kl}(\alpha_j)_l^{(1)}\Big)E_{-\alpha_j}\end{aligned} \tag{4.2.17}$$

where we have made use of eq.(A.0.1). Because the generators of the simple and lowest weight roots are linearly independent we can equate the coefficients of these matrices to give

$$\begin{aligned}2n_i\left(e^{(\alpha_i)_k q_k}-e^{-(\alpha_i)_k q_k}\right)&=\sum_{j=0}^{r}e^{(\alpha_i-\alpha_j)_k^{(1)}A_{kl}q_l^{(1)}+(\alpha_j-\alpha_i)_k^{(2)}\mu_k^{(2)}}\\ &\Big(y_j(\alpha_j)_k^{(2)}x_{i,q_k^{(2)}}+4y_{j,\xi_k}W_{kl}x_{i,\xi_l}+x_i y_{j,q_k}(\alpha_i)_k-x_i y_j(\alpha_j)_k^{(1)}A_{kl}(\alpha_i)_l^{(1)}\Big)\end{aligned} \tag{4.2.18}$$

for $i=0,\dots,r$. But this is identical to the set of differential equations appearing in eq.(3.3.3), which came from taking D and $\bar{D}$ to be of the form in eqs.(3.3.1),(3.3.2) then substituting these into the momentum conservation condition in eq.(2.2.33) to give a set of differential equations which must be satisfied by x_i and y_i if the defect is to be momentum conserving. We have not quite shown that momentum conservation is necessary for a system with a defect to have zero curvature, as we made the assumption that k_0 did not depend on ξ. We also have not shown that momentum conservation is a sufficient condition as this would require the recursion relations to be satisfied for all values of s. However, this highlights the link between momentum conservation and integrability, and for all defects found in section 3.3 their momentum conservation is necessary if they are to be integrable.

These first two terms indicate some sort of pattern of grading, with the n^{th} power of ρ in the expansion of $\hat{K}$ containing the product (or rather a sum of products) of n generators $E_{-\alpha_i}$ $(i = 0, \ldots, r)$. From eq.(A.0.4) we see that the generators of roots which are not simple or the lowest weight root can still be written as a sum of products of the generators of simple or lowest weight roots. This also implies some cyclicity, as by taking commutators of $E_{-\alpha_0}$ with $E_{-\alpha_i}$ $(i = 1, \ldots, r)$ we can eventually reach H. So the Cartan generators can be written as a sum of products of $1 + \sum_{i=1}^{r} n_i$ generators of negatives of simple roots and the generator associated with the highest weight root. So (from eq.(A.0.1)) the generators $E_{-\alpha_i}$ $(i = 0, \ldots, r)$ can be written as a sum of products of $2 + \sum_{i=1}^{r} n_i$ such generators. So if this grading pattern continues then the terms in the expansion in eq.(4.2.9) with $\rho^{-1-i-\sum n_i}$ are a rewriting of the terms with ρ^{-i}.

By inspection of the $s = 1$ recursion relations it appears that the grading described here will give the correct matrices from the commutators appearing in the recursion relation. However, actually calculating k_2 is too difficult, as we do not know anything about the root structure of the underlying Lie algebra and so do not know the exact form of the commutation relations for the generators. To actually calculate this defect zero curvature matrix we will need to consider specific ATFTs.

However, there is still some useful information about defects in ATFTs to be gleaned from these recursion relations if we consider what happens if the expansion for $\hat{K}$

terminates. Let us assume that for all $s > n$ we have $k_s = 0$. Then take $s = n$ for the recursion relations, giving

$$\begin{aligned} 0 = &- x_i(\alpha_i)^{(2)}_j k_{n,q^{(2)}_j} + 4x_{i,\xi_j} W_{jk} k_{n,\xi_k} \\ &+ x_i(\alpha_i)^{(1)}_j A_{jk} \left[k_n, H^{(1)}_k\right] + x_{i,q^{(2)}_j} \left[k_n, H^{(2)}_j\right] + x_i(\alpha_i)_j \{k_n, H_j\} \end{aligned} \tag{4.2.19}$$

for $i = 0, \ldots, r$ and

$$\sum_{i=0}^{r} \sqrt{n_i}|\alpha_i| e^{-(\alpha_i)^{(1)}_j A_{jk} q^{(1)}_k + (\alpha_i)^{(2)}_j \mu^{(2)}_j} \left(e^{(\alpha_i)_j q_j} k_n E_{-\alpha_i} - e^{-(\alpha_i)_j q_j} E_{-\alpha_i} k_n\right) = 0. \tag{4.2.20}$$

We will not solve these equations, but can use eq.(4.2.20) to get some information on the form of defects with zero curvature.

For the right hand side of eq.(4.2.20) to be zero the terms appearing there must either be equal to zero or proportional to another term, enabling cancellations to occur. For a term to disappear k_n must annihilate $E_{-\alpha_i}$ or vice versa. However, to know whether this happens and for which terms we need to know not just k_n but also what the underlying Lie algebra is and what representation we are using. We will therefore assume that this is never the case, and so every term in eq.(4.2.20) is non-zero. This assumption is acceptable as we are not trying to prove every defect with zero curvature must take a particular form. Instead we are looking for constraints which apply in certain cases which may be useful in finding momentum conserving defects for the E series ATFTs, which were not covered by the trial-and-error method used in section 3.3.

Every term in eq.(4.2.20) must cancel with at least one other term. First consider a cancellation between terms $k_n E_{-\alpha_i}$ and $k_n E_{-\alpha_j}$. Because k_n is only dependent on $q^{(2)}$ and ξ any dependence on $q^{(1)}$ and $\mu^{(2)}$ appearing in these two terms must match. From the exponentials appearing in these terms this requires

$$(\alpha_i)^{(1)}_k q^{(1)}_k - (\alpha_i)^{(1)}_k A_{kl} q^{(1)}_l + (\alpha_i)^{(2)}_k \mu^{(2)}_k = (\alpha_j)^{(1)}_k q^{(1)}_k - (\alpha_j)^{(1)}_k A_{kl} q^{(1)}_l + (\alpha_j)^{(2)}_k \mu^{(2)}_k. \tag{4.2.21}$$

As we have noted before A being antisymmetric means that $\mathbb{1} \pm A$ has complex eigenvalues which are all non-zero, so is invertible. Therefore requiring eq.(4.2.21) to hold gives $\alpha_i = \alpha_j$, so we cannot have a cancellation between two terms of the form $k_n E_{-\alpha_i}$. Next consider a cancellation between terms $E_{-\alpha_i} k_n$ and $E_{-\alpha_j} k_n$. This requires

$$-(\alpha_i)^{(1)}_k q^{(1)}_k - (\alpha_i)^{(1)}_k A_{kl} q^{(1)}_l + (\alpha_i)^{(2)}_k \mu^{(2)}_k = -(\alpha_j)^{(1)}_k q^{(1)}_k - (\alpha_j)^{(1)}_k A_{kl} q^{(1)}_l + (\alpha_j)^{(2)}_k \mu^{(2)}_k, \tag{4.2.22}$$

which again immediately gives $\alpha_i = \alpha_j$, and so no cancellations. So all cancellations must be between a term of the form $k_n E_{-\alpha_i}$ and another term of the form $E_{-\alpha_j} k_n$. This requires every root α_i to have another root α_j for which it satisfies

$$(\alpha_i)^{(1)}_k q^{(1)}_k - (\alpha_i)^{(1)}_k A_{kl} q^{(1)}_l + (\alpha_i)^{(2)}_k \mu^{(2)}_k = -(\alpha_j)^{(1)}_k q^{(1)}_k - (\alpha_j)^{(1)}_k A_{kl} q^{(1)}_l + (\alpha_j)^{(2)}_k \mu^{(2)}_k. \tag{4.2.23}$$

If the assumptions we have made about the $\hat{K}$ series terminating and the k_n matrix not annihilating any E_α operators hold (and for the Tzitzéica and D_4 defect matrices we find in the following sections they do hold) then we have some fairly restrictive constraints on the projections of the roots onto the 1-space and 2-space. Either the root α_i must have $(\alpha_i)^{(1)} = 0$, in which case the $k_n E_{\alpha_i}$ term is able to cancel with $E_{\alpha_i} k_n$, or there must be some other root α_j with $(\mathbb{1} + A)\alpha_i^{(1)} = (-\mathbb{1} + A)\alpha_j^{(1)}$ and $\alpha_i^{(2)} = \alpha_j^{(2)}$. By their projections onto the 2-space we should be able to find sets of roots whose projections onto the 1-space are linked.

For the A_r ATFTs found in [BCZ04a] there is no 2-space and these constraints give the relations between simple roots which were required for a type I defect to be momentum conserving. For the Tzitzéica defect in section 3.2.2 and the momentum conserving defects in A_r and C_r ATFTs found in section 3.3 there was no 1-space, so all roots have $(\alpha_i)^{(1)} = 0$. For the B_r defect also found in section 3.3 we can see that, for the roots given in eqs.(A.0.19),(A.0.36), the choice of 1-space and 2-space made in subsection 3.3.4 and $A = 0$, we have $(\alpha_0)^{(1)} = -(\alpha_1)^{(1)}$, $(\alpha_0)^{(2)} = (\alpha_1)^{(2)}$ and $(\alpha_i)^{(1)} = 0$ for all other roots. For the D_r defect found in section 3.3 with the roots given in eqs.(A.0.21),(A.0.36), the choice of 1-space and 2-space made in subsection 3.3.2 and $A=0$ we have $(\alpha_0)^{(1)} = -(\alpha_1)^{(1)}$, $(\alpha_0)^{(2)} = (\alpha_1)^{(2)}$, $(\alpha_{r-1})^{(1)} = -(\alpha_r)^{(1)}$, $(\alpha_{r-1})^{(2)} = (\alpha_r)^{(2)}$ and $(\alpha_i)^{(1)} = 0$ for all other roots. Whilst we have not proved anything definite the fact that the above constraints on the splitting of the root space into the 1-space and 2-space hold for these known momentum conserving defect certainly gives a possible direction for future calculations of E_6 defects.

We will now use these results to show that the momentum conserving Tzitzéica and D_4 defects found in sections 3.2.2 and 3.3.1 have a zero curvature representation.

4.2.1 Zero curvature for the Tzitzéica defect

The roots for Tzitzéica are given in eq.(3.2.21), the momentum conserving ATFT defect based on these roots in eq.(3.2.20) and the momentum conserving defect potential in eqs.(3.2.28),(3.2.29). The defect zero curvature conditions in eqs.(4.2.7),(4.2.8) then become

$$\begin{aligned}
2\sqrt{2}\rho\left[\hat{K},E_{\alpha_0}\right] =& e^{2f}(e^{q}+e^{-q})^2\left(\hat{K}_q-\{\hat{K},H\}+f_q\left[\hat{K},H\right]\right) \\
&+e^{2f}(e^{q}+e^{-q})(e^{q}-e^{-q})\left[\hat{K},H\right] \qquad (4.2.24)\\
2\rho\left[\hat{K},E_{\alpha_1}\right] =& e^{-f}\left(-\hat{K}_q+\{\hat{K},H\}-f_q\left[\hat{K},H\right]\right) \qquad (4.2.25)\\
\rho e^{-2f}\left(\hat{K}_q+f_q\left[\hat{K},H\right]\right) =& \sqrt{2}\left(e^{-2q}\hat{K}E_{-\alpha_0}-e^{2q}E_{-\alpha_0}\hat{K}\right) \qquad (4.2.26)\\
2\rho e^{f}\Big((e^{q}+e^{-q})\left(\hat{K}_q+f_q\left[\hat{K},H\right]\right)&+(e^{q}-e^{-q})\left[\hat{K},H\right]\Big)\\
=&-\left(e^{q}\hat{K}E_{-\alpha_1}-e^{-q}E_{-\alpha_1}\hat{K}\right), \qquad (4.2.27)
\end{aligned}$$

where eq.(4.2.8) has been split into two equations by equating powers of μ.

In order to solve eqs.(4.2.24)-(4.2.27) we will choose a representation, write down the generator matrices explicitly, then solve the matrix equations entry by entry. For notation we will take $e^n_{i,j}$ to denote an $n\times n$ matrix with zeroes everywhere except position (i,j), where the entry is 1. Our chosen representation is

$$H=\left(e^3_{1,1}-e^3_{3,3}\right) \qquad E_{\alpha_0}=e^3_{3,1} \qquad E_{\alpha_1}=\sqrt{2}\left(e^3_{1,2}+e^3_{2,3}\right) \qquad (4.2.28)$$

and we recall that $E_{-\alpha} = E_{\alpha}^{\dagger}$.

Using Maple to solve eqs.(4.2.24)-(4.2.27) as described then gives

$$\hat{K} = \begin{pmatrix} 1 - \frac{1}{4\sqrt{2}}\rho^{-3}e^{2q} & \frac{1}{2}\rho^{-2}e^{f}e^{q}(e^{q}+e^{-q}) & -\frac{1}{\sqrt{2}}\rho^{-1}e^{2f}(e^{q}+e^{-q})^{2} \\ -\frac{1}{\sqrt{2}}\rho^{-1}e^{-f} & 1 - \frac{1}{4\sqrt{2}}\rho^{-3} & \frac{1}{2}\rho^{-2}e^{f}e^{-q}(e^{q}+e^{-q}) \\ \frac{1}{4}\rho^{-2}e^{-2f} & -\frac{1}{\sqrt{2}}\rho^{-1}e^{-f} & 1 - \frac{1}{4\sqrt{2}}\rho^{-3}e^{-2q} \end{pmatrix}. \tag{4.2.29}$$

This matrix fits into the proposed form of $\hat{K}$ as a finite series in ρ. The structure of this matrix is identical to the Tzitzéica defect matrix found in [AAGZ11]. When writing $\hat{K}$ as given in eq.(4.2.29) in terms of the expansion in ρ given in eq.(4.2.9) one possible choice is

$$\begin{aligned} k_0 =& \mathbb{1} \\ k_1 =& -\frac{1}{\sqrt{2}}e^{2f}(e^{q}+e^{-q})^{2}E_{-\alpha_0} - \frac{1}{2}e^{-f}E_{-\alpha_1} \\ k_2 =& \frac{1}{2\sqrt{2}}e^{f}(e^{q}+e^{-q})\left(e^{q}E_{-\alpha_0}E_{-\alpha_1} + e^{-q}E_{-\alpha_1}E_{-\alpha_0}\right) + \frac{1}{8}e^{-2f}E_{-\alpha_1}E_{-\alpha_1} \\ k_3 =& -\frac{1}{8\sqrt{2}}\left(e^{2q}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_1} + E_{-\alpha_1}E_{-\alpha_0}E_{-\alpha_1} + e^{-2q}E_{-\alpha_1}E_{-\alpha_1}E_{-\alpha_0}\right). \end{aligned} \tag{4.2.30}$$

This fits into the grading hypothesised in the previous chapter, with k_s consisting of products of s generators. Because K appears as part of the monodromy matrix we would hope that $\hat{K}$ could be written as an exponential of generators, but so far such a form of eq.(4.2.29) has not been found. This is due to difficulties with the calculation (at least when carried out in Maple) and there is no proof that it is not possible.

The defect transport matrix satisfying eq.(4.1.23) is given by

$$K = e^{-\frac{1}{2}(p+q-2\mu)H}\hat{K}e^{\frac{1}{2}(p-q-2\mu)H}. \tag{4.2.31}$$

One interesting observation is that there is some additional gauge freedom to that already discussed for the bulk Lax pairs and the defect. Applying no transformations to the bulk Lax pair we can take $K \to e^{g(q)H}Ke^{-g(q)H}$, so $\hat{K} \to e^{g(q)H}\hat{K}e^{-g(q)H}$, to give

$$\hat{K} = \begin{pmatrix} 1-\frac{1}{4\sqrt{2}}\rho^{-3}e^{2q} & \frac{1}{2}\rho^{-2}e^{f+g}e^{q}(e^{q}+e^{-q}) & -\frac{1}{\sqrt{2}}\rho^{-1}e^{2f+2g}(e^{q}+e^{-q})^2 \\ -\frac{1}{\sqrt{2}}\rho^{-1}e^{-f-g} & 1-\frac{1}{4\sqrt{2}}\rho^{-3} & \frac{1}{2}\rho^{-2}e^{f+g}e^{-q}(e^{q}+e^{-q}) \\ \frac{1}{4}\rho^{-2}e^{-2f-2g} & -\frac{1}{\sqrt{2}}\rho^{-1}e^{-f-g} & 1-\frac{1}{4\sqrt{2}}\rho^{-3}e^{-2q} \end{pmatrix}. \quad (4.2.32)$$

This transformation obviously corresponds to making the field redefinition $\mu \to \mu+g(q)$, and so the defect matrix for defects with different definitions of the auxiliary fields are linked by this gauge transformation. The transformed matrix will also satisfy the zero curvature condition, but where before we had f in the defect equations of motion we will now have $f+g$.

4.2.2 Zero curvature for the D_4 defect

The roots for D_4 are given in eqs.(A.0.21),(A.0.36) and the momentum conserving defect Lagrangian in eq.(3.3.4). The two possible momentum conserving defect potentials are given in eqs.(3.3.10),(3.3.11) and eqs.(3.3.12),(3.3.13). Using the first defect potential in eqs.(4.2.7),(4.2.8) gives

$$\begin{aligned} 2\rho\left[\hat{K},E_{\alpha_0}\right] =& e^{f_{q_2}}\left(\hat{K}_{q_2}-\{\hat{K},H_1\}-\{\hat{K},H_2\}\right) \\ &+e^{f_{q_2}}f_{q_2q_2}\left[\hat{K},H_2\right]+e^{f_{q_2}}f_{q_2q_3}\left[\hat{K},H_3\right] && (4.2.33) \\ 2\rho\left[\hat{K},E_{\alpha_1}\right] =& e^{f_{q_2}}\left(\hat{K}_{q_2}+\{\hat{K},H_1\}-\{\hat{K},H_2\}\right) \\ &+e^{f_{q_2}}f_{q_2q_2}\left[\hat{K},H_2\right]+e^{f_{q_2}}f_{q_2q_3}\left[\hat{K},H_3\right] && (4.2.34) \\ \sqrt{2}\rho\left[\hat{K},E_{\alpha_2}\right] =& e^{-f_{q_2}+f_{q_3}}(e^{q_2}+e^{-q_2})\left(-\hat{K}_{q_2}+\hat{K}_{q_3}+\{\hat{K},H_2\}-\{\hat{K},H_3\}\right) \\ &+e^{-f_{q_2}+f_{q_3}}\left((-f_{q_2q_2}+f_{q_2q_3})(e^{q_2}+e^{-q_2})+e^{q_2}-e^{-q_2}\right)\left[\hat{K},H_2\right] \\ &+e^{-f_{q_2}+f_{q_3}}\left(-f_{q_2q_3}+f_{q_3q_3}\right)(e^{q_2}+e^{-q_2})\left[\hat{K},H_3\right] && (4.2.35) \\ 2\rho\left[\hat{K},E_{\alpha_3}\right] =& e^{-f_{q_3}}(e^{q_3}+e^{-q_3})\left(-\hat{K}_{q_3}+\{\hat{K},H_3\}-\{\hat{K},H_4\}\right) \\ &-e^{-f_{q_3}}f_{q_2q_3}(e^{q_3}+e^{-q_3})\left[\hat{K},H_2\right] \\ &+e^{-f_{q_3}}\left(-f_{q_3q_3}(e^{q_3}+e^{-q_3})+e^{q_3}-e^{-q_3}\right)\left[\hat{K},H_3\right] && (4.2.36) \\ 2\rho\left[\hat{K},E_{\alpha_4}\right] =& e^{-f_{q_3}}(e^{q_3}+e^{-q_3})\left(-\hat{K}_{q_3}+\{\hat{K},H_3\}+\{\hat{K},H_4\}\right) \end{aligned}$$

$$
\begin{aligned}
&+ -e^{-f_{q_3}} f_{q_2q_3}(e^{q_3}+e^{-q_3}) \left[\hat{K},H_2\right] \\
&+ e^{-f_{q_3}} \left(-f_{q_3q_3}(e^{q_3}+e^{-q_3}) + e^{q_3} - e^{-q_3}\right) \left[\hat{K},H_3\right] \qquad (4.2.37)
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(e^{-q_1-q_2}\hat{K}E_{-\alpha_0} - e^{q_1+q_2}E_{-\alpha_0}\hat{K} + e^{q_1-q_2}\hat{K}E_{-\alpha_1} - e^{-q_1+q_2}E_{-\alpha_1}\hat{K}\right) \\
&\quad = \rho e^{-f_{q_2}}\Big(-(e^{q_1}+e^{-q_1})(e^{q_2}+e^{-q_2})\hat{K}_{q_2} + (e^{q_1}-e^{-q_1})(e^{q_2}+e^{-q_2})\left[\hat{K},H_1\right] \\
&\qquad + \left(-f_{q_2q_2}(e^{q_1}+e^{-q_1})(e^{q_2}+e^{-q_2}) + (e^{q_1}+e^{-q_1})(e^{q_2}-e^{-q_2})\right)\left[\hat{K},H_2\right] \\
&\qquad - f_{q_2q_3}(e^{q_1}+e^{-q_1})(e^{q_2}+e^{-q_2})\left[\hat{K},H_3\right]\Big) \qquad (4.2.38) \\
&-2\sqrt{2}\left(e^{q_2-q_3}\hat{K}E_{-\alpha_2} - e^{-q_2+q_3}E_{-\alpha_2}\hat{K}\right) \\
&\quad = \rho e^{f_{q_2}-f_{q_3}}\Big((e^{q_3}+e^{-q_3})\left(\hat{K}_{q_2}-\hat{K}_{q_3}\right) + (f_{q_2q_2}-f_{q_2q_3})(e^{q_3}+e^{-q_3})\left[\hat{K},H_2\right] \\
&\qquad + \left((f_{q_2q_3}-f_{q_3q_3})(e^{q_3}+e^{-q_3}) + e^{q_3}-e^{-q_3}\right)\left[\hat{K},H_3\right]\Big) \qquad (4.2.39) \\
&-2\left(e^{q_3-q_4}\hat{K}E_{-\alpha_3} - e^{-q_3+q_4}E_{-\alpha_3}\hat{K} + e^{q_3+q_4}\hat{K}E_{-\alpha_4} - e^{-q_3-q_4}E_{-\alpha_4}\hat{K}\right) \\
&\quad = \rho e^{f_{q_3}}\Big((e^{q_4}+e^{-q_4})\hat{K}_{q_3} + f_{q_2q_3}(e^{q_4}+e^{-q_4})\left[\hat{K},H_2\right] \\
&\qquad + f_{q_3q_3}(e^{q_4}+e^{-q_4})\left[\hat{K},H_3\right] + (e^{q_4}-e^{-q_4})\left[\hat{K},H_4\right]\Big) \qquad (4.2.40)
\end{aligned}
$$

and using the second defect potential gives

$$
\begin{aligned}
2\rho\left[\hat{K},E_{\alpha_0}\right] =& e^{f_{q_2}}\left(-\hat{K}_{q_2} + \{\hat{K},H_1\} + \{\hat{K},H_2\}\right) \\
&- e^{f_{q_2}}f_{q_2q_2}\left[\hat{K},H_2\right] - e^{f_{q_2}}f_{q_2q_3}\left[\hat{K},H_3\right] \qquad (4.2.41) \\
2\rho\left[\hat{K},E_{\alpha_1}\right] =& e^{f_{q_2}}\left(\hat{K}_{q_2} + \{\hat{K},H_1\} - \{\hat{K},H_2\}\right) \\
&+ e^{f_{q_2}}f_{q_2q_2}\left[\hat{K},H_2\right] + e^{f_{q_2}}f_{q_2q_3}\left[\hat{K},H_3\right] \qquad (4.2.42) \\
\sqrt{2}\rho\left[\hat{K},E_{\alpha_2}\right] =& e^{-f_{q_2}+f_{q_3}}(e^{q_2}-e^{-q_2})\left(\hat{K}_{q_2} - \hat{K}_{q_3} - \{\hat{K},H_2\} + \{\hat{K},H_3\}\right) \\
&+ e^{-f_{q_2}+f_{q_3}}\left((f_{q_2q_2}-f_{q_2q_3})(e^{q_2}-e^{-q_2}) - e^{q_2} - e^{-q_2}\right)\left[\hat{K},H_2\right] \\
&+ e^{-f_{q_2}+f_{q_3}}(f_{q_2q_3}-f_{q_3q_3})(e^{q_2}-e^{-q_2})\left[\hat{K},H_3\right] \qquad (4.2.43) \\
2\rho\left[\hat{K},E_{\alpha_3}\right] =& e^{-f_{q_3}}(e^{q_3}-e^{-q_3})\left(\hat{K}_{q_3} - \{\hat{K},H_3\} + \{\hat{K},H_4\}\right) \\
&+ e^{-f_{q_3}}f_{q_2q_3}(e^{q_3}-e^{-q_3})\left[\hat{K},H_2\right] \\
&+ e^{-f_{q_3}}\left(f_{q_3q_3}(e^{q_3}-e^{-q_3}) - e^{q_3} - e^{-q_3}\right)\left[\hat{K},H_3\right] \qquad (4.2.44) \\
2\rho\left[\hat{K},E_{\alpha_4}\right] =& e^{-f_{q_3}}(e^{q_3}-e^{-q_3})\left(-\hat{K}_{q_3} + \{\hat{K},H_3\} + \{\hat{K},H_4\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- e^{-f_{q_3}} f_{q_2 q_3}(e^{q_3} - e^{-q_3}) \left[\hat{K}, H_2\right] \\
&+ e^{-f_{q_3}} \left(-f_{q_3 q_3}(e^{q_3} - e^{-q_3}) + e^{q_3} + e^{-q_3}\right) \left[\hat{K}, H_3\right]
\end{aligned} \tag{4.2.45}
$$

$$
\begin{aligned}
&- 2\left(e^{-q_1-q_2}\hat{K}E_{-\alpha_0} - e^{q_1+q_2}E_{-\alpha_0}\hat{K} + e^{q_1-q_2}\hat{K}E_{-\alpha_1} - e^{-q_1+q_2}E_{-\alpha_1}\hat{K}\right) \\
&\quad = \rho e^{-f_{q_2}}\Big((e^{q_1} - e^{-q_1})(e^{q_2} - e^{-q_2})\hat{K}_{q_2} - (e^{q_1} + e^{-q_1})(e^{q_2} - e^{-q_2})\left[\hat{K}, H_1\right] \\
&\qquad + \left(f_{q_2 q_2}(e^{q_1} - e^{-q_1})(e^{q_2} - e^{-q_2}) - (e^{q_1} - e^{-q_1})(e^{q_2} + e^{-q_2})\right)\left[\hat{K}, H_2\right] \\
&\qquad + f_{q_2 q_3}(e^{q_1} - e^{-q_1})(e^{q_2} - e^{-q_2})\left[\hat{K}, H_3\right]\Big)
\end{aligned} \tag{4.2.46}
$$

$$
\begin{aligned}
&- 2\sqrt{2}\left(e^{q_2-q_3}\hat{K}E_{-\alpha_2} - e^{-q_2+q_3}E_{-\alpha_2}\hat{K}\right) \\
&\quad = \rho e^{f_{q_2}-f_{q_3}}\Big((e^{q_3} - e^{-q_3})\left(\hat{K}_{q_2} - \hat{K}_{q_3}\right) + (f_{q_2 q_2} - f_{q_2 q_3})(e^{q_3} - e^{-q_3})\left[\hat{K}, H_2\right] \\
&\qquad + \left((f_{q_2 q_3} - f_{q_3 q_3})(e^{q_3} - e^{-q_3}) + e^{q_3} + e^{-q_3}\right)\left[\hat{K}, H_3\right]\Big)
\end{aligned} \tag{4.2.47}
$$

$$
\begin{aligned}
&- 2\left(e^{q_3-q_4}\hat{K}E_{-\alpha_3} - e^{-q_3+q_4}E_{-\alpha_3}\hat{K} + e^{q_3+q_4}\hat{K}E_{-\alpha_4} - e^{-q_3-q_4}E_{-\alpha_4}\hat{K}\right) \\
&\quad = \rho e^{f_{q_3}}\Big((e^{q_4} - e^{-q_4})\hat{K}_{q_3} + f_{q_2 q_3}(e^{q_4} - e^{-q_4})\left[\hat{K}, H_2\right] \\
&\qquad + f_{q_3 q_3}(e^{q_4} - e^{-q_4})\left[\hat{K}, H_3\right] + (e^{q_4} + e^{-q_4})\left[\hat{K}, H_4\right]\Big)
\end{aligned} \tag{4.2.48}
$$

where in both cases eq.(4.2.8) has been split into three equations by equating powers of μ.

Again in order to solve these matrix equations we must choose a representation of D_4. Using the same notation as in the Tzitzéica case we take

$$
H_1 = e^8_{1,1} - e^8_{2,2} \qquad H_2 = e^8_{3,3} - e^8_{4,4} \qquad H_3 = e^8_{5,5} - e^8_{6,6} \qquad H_4 = e^8_{7,7} - e^8_{8,8} \tag{4.2.49}
$$

$$
\begin{aligned}
&E_{\alpha_1} = e^8_{1,3} + e^8_{4,2} \qquad E_{\alpha_2} = e^8_{3,5} + e^8_{6,4} \qquad E_{\alpha_3} = e^8_{5,7} + e^8_{8,6} \qquad E_{\alpha_4} = e^8_{5,8} + e^8_{7,6} \\
&E_{\alpha_0} = e^8_{2,3} + e^8_{4,1}.
\end{aligned} \tag{4.2.50}
$$

Using this representation and the expansion of $\hat{K}$ in ρ given in eq.(4.2.9) we solve the matrix equations (4.2.33)-(4.2.40) for the first defect potential, given by eqs.(3.3.10),(3.3.11), to give

$$
\begin{aligned}
k_0 &= \mathbb{1} \\
k_1 &= - e^{f_{q_2}}\left(E_{-\alpha_0} + E_{-\alpha_1}\right) - \sqrt{2}e^{-f_{q_2}+f_{q_3}}(e^{q_2} + e^{-q_2})E_{-\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
&- e^{-fq_3}(e^{q_3}+e^{-q_3})\,(E_{-\alpha_3}+E_{-\alpha_4})\\
k_2 =\,& e^{2f_{q_2}}E_{-\alpha_0}E_{-\alpha_1}+\sqrt{2}e^{f_{q_3}}\left(e^{q_2}E_{-\alpha_0}E_{-\alpha_2}+e^{-q_2}E_{-\alpha_2}E_{-\alpha_0}\right)\\
&+\sqrt{2}e^{f_{q_3}}\left(e^{q_2}E_{-\alpha_1}E_{-\alpha_2}+e^{-q_2}E_{-\alpha_2}E_{-\alpha_1}\right)\\
&+\sqrt{2}e^{-f_{q_2}}(e^{q_2}+e^{-q_2})\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_3}+e^{-q_3}E_{-\alpha_3}E_{-\alpha_2}\right)\\
&+\sqrt{2}e^{-f_{q_2}}(e^{q_2}+e^{-q_2})\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_4}+e^{-q_3}E_{-\alpha_4}E_{-\alpha_2}\right)\\
&+e^{-2f_{q_3}}(e^{q_3}+e^{-q_3})^2E_{-\alpha_3}E_{-\alpha_4}\\
k_3 =& -\sqrt{2}e^{f_{q_2}+f_{q_3}}\left(e^{q_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}+e^{-q_2}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right)\\
&-\sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}\right)\\
&-\sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\right)\\
&-\sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_1}\right)\\
&-\sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\right)\\
&-\sqrt{2}e^{-f_{q_2}-f_{q_3}}(e^{q_2}+e^{-q_2})(e^{q_3}+e^{-q_3})\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}+e^{-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}\right)\\
k_4 =\,&2e^{2f_{q_3}}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}+2e^{-2f_{q_2}}(e^{q_2}+e^{-q_2})^2E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}\\
&+\sqrt{2}e^{f_{q_2}}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right)\\
&+\sqrt{2}e^{f_{q_2}}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right)\\
&+\sqrt{2}e^{-f_{q_3}}(e^{q_3}+e^{-q_3})\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\right)\\
&+\sqrt{2}e^{-f_{q_3}}(e^{q_3}+e^{-q_3})\left(e^{q_2+q_3}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\right)\\
k_5 =& -\sqrt{2}e^{f_{q_2}-f_{q_3}}(e^{q3}+e^{-q3})\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right)\\
&-2e^{f_{q_3}}\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}+e^{-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}\right)\\
&-2e^{f_{q_3}}\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4}+e^{-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}\right)\\
&-2e^{-f_{q_2}}(e^{q_2}+e^{-q_2})\left(e^{q_2}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}+e^{-q_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\right)\\
&-2e^{-f_{q_2}}(e^{q_2}+e^{-q_2})\left(e^{q_2}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}+e^{-q_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\right)\\
k_6 =\,&2e^{2q_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}\\
&+2e^{-2q_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\\
&+2e^{2q_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}\\
&+2e^{-2q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0} \\
&+ 2E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1} \\
&+ 2E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3} \\
&+ 2E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4}\Big).
\end{aligned} \tag{4.2.51}
$$

Solving eqs.(4.2.41)-(4.2.48) for the second defect potential, given by eqs.(3.3.12),(3.3.13), we have

$$
\begin{aligned}
k_0 =&\mathbb{1} \\
k_1 =&e^{f_{q_2}}\left(E_{-\alpha_0} - E_{-\alpha_1}\right) + \sqrt{2}e^{-f_{q_2}+f_{q_3}}(e^{q_2} - e^{-q_2})E_{-\alpha_2} \\
&+ e^{-f_{q_3}}(e^{q_3} - e^{-q_3})\left(E_{-\alpha_3} - E_{-\alpha_4}\right) \\
k_2 =&- e^{2f_{q_2}}E_{-\alpha_0}E_{-\alpha_1} + \sqrt{2}e^{f_{q_3}}\left(e^{q_2}E_{-\alpha_0}E_{-\alpha_2} - e^{-q_2}E_{-\alpha_2}E_{-\alpha_0}\right) \\
&- \sqrt{2}e^{f_{q_3}}\left(e^{q_2}E_{-\alpha_1}E_{-\alpha_2} - e^{-q_2}E_{-\alpha_2}E_{-\alpha_1}\right) \\
&+ \sqrt{2}e^{-f_{q_2}}(e^{q_2} - e^{-q_2})\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_3} - e^{-q_3}E_{-\alpha_3}E_{-\alpha_2}\right) \\
&- \sqrt{2}e^{-f_{q_2}}(e^{q_2} - e^{-q_2})\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_4} - e^{-q_3}E_{-\alpha_4}E_{-\alpha_2}\right) \\
&- e^{-2f_{q_3}}(e^{q_3} - e^{-q_3})^2E_{-\alpha_3}E_{-\alpha_4} \\
k_3 =&- \sqrt{2}e^{f_{q_2}+f_{q_3}}\left(e^{q_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2} - e^{-q_2}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right) \\
&+ \sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3} + e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}\right) \\
&- \sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_4} + e^{-q_2-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\right) \\
&- \sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3} + e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_1}\right) \\
&+ \sqrt{2}\left(e^{q_2+q_3}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4} + e^{-q_2-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\right) \\
&- \sqrt{2}e^{-f_{q_2}-f_{q_3}}(e^{q_2} - e^{-q_2})(e^{q_3} - e^{-q_3})\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4} - e^{-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}\right) \\
k_4 =&2e^{2f_{q_3}}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2} + 2e^{-2f_{q_2}}(e^{q_2} - e^{-q_2})^2E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2} \\
&- \sqrt{2}e^{f_{q_2}}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3} + e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right) \\
&+ \sqrt{2}e^{f_{q_2}}\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4} + e^{-q_2-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right) \\
&- \sqrt{2}e^{-f_{q_3}}(e^{q_3} - e^{-q_3})\left(e^{q2+q_3}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4} + e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\right) \\
&+ \sqrt{2}e^{-f_{q_3}}(e^{q_3} - e^{-q_3})\left(e^{q2+q_3}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4} + e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
k_5 =&\sqrt{2}e^{f_{q_2}-f_{q_3}}(e^{q3}-e^{-q3})\left(e^{q_2+q_3}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}+e^{-q_2-q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\right)\\
&+2e^{f_{q_3}}\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}-e^{-q_3}E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}\right)\\
&-2e^{f_{q_3}}\left(e^{q_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4}-e^{-q_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}\right)\\
&+2e^{-f_{q_2}}(e^{q_2}-e^{-q_2})\left(e^{q_2}E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}-e^{-q_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\right)\\
&-2e^{-f_{q_2}}(e^{q_2}-e^{-q_2})\left(e^{q_2}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}-e^{-q_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\right)\\
k_6 =&-2e^{2q_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}\\
&-2e^{-2q_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}\\
&-2e^{2q_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}\\
&-2e^{-2q_3}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}\\
&-2E_{-\alpha_0}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}\\
&-2E_{-\alpha_2}E_{-\alpha_2}E_{-\alpha_3}E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_1}\\
&-2E_{-\alpha_3}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_3}\\
&-2E_{-\alpha_4}E_{-\alpha_2}E_{-\alpha_0}E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_4}. \qquad (4.2.52)
\end{aligned}
$$

These solutions also fit into the proposed grading. We have not checked whether the solutions given here and in eq.(4.2.30) are representation independent.

The defect transport matrix satisfying eq.(4.1.23) is given by

$$
\begin{aligned}
K =&e^{-\frac{1}{2}((p_1+q_1)H_1+(p_2+q_2-2\mu_{2,t})H_2+(p_3+q_3-2\mu_{3,t})H_3+(p_4+q_4)H_4)}\hat{K}\\
&e^{\frac{1}{2}((p_1-q_1)H_1+(p_2-q_2-2\mu_{2,t})H_2+(p_3-q_3-2\mu_{3,t})H_3+(p_4-q_4)H_4)}. \qquad (4.2.53)
\end{aligned}
$$

Once again we have $K \to e^{g(q_2,q_3)_{q_2}H_2+g(q_2,q_3)_{q_3}H_3}Ke^{-g(q_2,q_3)_{q_2}H_2-g(q_2,q_3)_{q_3}H_3}$ taking the K matrix from that of the original defect to that of a defect which is the original defect with the auxiliary fields shifted by $\mu_2 \to \mu_2+g(q_2,q_3)_{q_2}$, $\mu_3 \to \mu_3+g(q_2,q_3)_{q_3}$.

The structure of these defect transport matrices is clearer if we write out the matrices in full. To do this we simplify the situation slightly by setting $f=0$, knowing that the above expression could immediately be used to restore the $e^{f_{q_{2,3}}}$ multipliers to their correct terms. We also take $\hat{K} \to \frac{1}{\sqrt{2}}\hat{K}$, which does not affect whether

K satisfies the zero curvature condition in eq.(4.1.23). We use $Q^{\pm}_{2,3}$ to denote the brackets $(e^{q_{2,3}} \pm e^{-q_{2,3}})$. The defect matrix for the defect with the first defect potential is

$$\hat{K} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{\rho^6} & -\frac{\sqrt{2}e^{q_2}Q_2^+}{\rho^5} & -\frac{1}{\sqrt{2}\rho} & \frac{e^{q_2+q_3}Q_3^+}{\rho^4} & \frac{e^{q_2}}{\rho^2} & -\frac{e^{q_2+q_3}}{\rho^3} & -\frac{e^{q_2+q_3}}{\rho^3} \\
\frac{\sqrt{2}}{\rho^6} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}e^{q_2}Q_2^+}{\rho^5} & -\frac{1}{\sqrt{2}\rho} & \frac{e^{q_2+q_3}Q_3^+}{\rho^4} & \frac{e^{q_2}}{\rho^2} & -\frac{e^{q_2+q_3}}{\rho^3} & -\frac{e^{q_2+q_3}}{\rho^3} \\
-\frac{1}{\sqrt{2}\rho} & -\frac{1}{\sqrt{2}\rho} & \frac{1}{\sqrt{2}}+\frac{\sqrt{2}e^{2q_2}}{\rho^6} & \frac{1}{\sqrt{2}\rho^2} & -\frac{e^{q_2+q_3}Q_3^+}{\rho^5} & -\frac{e^{q_2}}{\rho^3} & \frac{e^{q_2+q_3}}{\rho^4} & \frac{e^{q_2+q_3}}{\rho^4} \\
-\frac{\sqrt{2}e^{-q_2}Q_2^+}{\rho^5} & -\frac{\sqrt{2}e^{-q_2}Q_2^+}{\rho^5} & \frac{\sqrt{2}Q_2^{+2}}{\rho^4} & \frac{1}{\sqrt{2}}+\frac{\sqrt{2}e^{-2q_2}}{\rho^6} & -\frac{e^{q_3}Q_2^+Q_3^+}{\rho^3} & -\frac{Q_2^+}{\rho} & \frac{e^{q_3}Q_2^+}{\rho^2} & \frac{e^{q_3}Q_2^+}{\rho^2} \\
\frac{e^{-q_2}}{\rho^2} & \frac{e^{-q_2}}{\rho^2} & -\frac{Q_2^+}{\rho} & -\frac{e^{-q_2}}{\rho^3} & \frac{1}{\sqrt{2}}+\frac{\sqrt{2}e^{2q_3}}{\rho^6} & \frac{\sqrt{2}}{\rho^4} & -\frac{\sqrt{2}e^{q_3}}{\rho^5} & -\frac{\sqrt{2}e^{q_3}}{\rho^5} \\
\frac{e^{-q_2-q_3}Q_3^+}{\rho^4} & \frac{e^{-q_2-q_3}Q_3^+}{\rho^4} & -\frac{e^{-q_3}Q_2^+Q_3^+}{\rho^3} & -\frac{e^{-q_2-q_3}Q_3^+}{\rho^5} & \frac{Q_3^{+2}}{\sqrt{2}\rho^2} & \frac{1}{\sqrt{2}}+\frac{\sqrt{2}e^{-2q_3}}{\rho^6} & -\frac{Q_3^+}{\sqrt{2}\rho} & -\frac{Q_3^+}{\sqrt{2}\rho} \\
-\frac{e^{-q_2-q_3}}{\rho^3} & -\frac{e^{-q_2-q_3}}{\rho^3} & \frac{e^{-q_3}Q_2^+}{\rho^2} & \frac{e^{-q_2-q_3}}{\rho^4} & -\frac{Q_3^+}{\sqrt{2}\rho} & -\frac{\sqrt{2}e^{-q_3}}{\rho^5} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{\rho^6} \\
-\frac{e^{-q_2-q_3}}{\rho^3} & -\frac{e^{-q_2-q_3}}{\rho^3} & \frac{e^{-q_3}Q_2^+}{\rho^2} & \frac{e^{-q_2-q_3}}{\rho^4} & -\frac{Q_3^+}{\sqrt{2}\rho} & -\frac{\sqrt{2}e^{-q_3}}{\rho^5} & \frac{\sqrt{2}}{\rho^6} & \frac{1}{\sqrt{2}}
\end{pmatrix} \tag{4.2.54}$$

and for the second defect potential we have

$$\hat{K} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{\rho^6} & \frac{\sqrt{2}e^{q_2}Q_2^-}{\rho^5} & \frac{1}{\sqrt{2}\rho} & -\frac{e^{q_2+q_3}Q_3^-}{\rho^4} & \frac{e^{q_2}}{\rho^2} & -\frac{e^{q_2+q_3}}{\rho^3} & \frac{e^{q_2+q_3}}{\rho^3} \\
-\frac{\sqrt{2}}{\rho^6} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}e^{q_2}Q_2^-}{\rho^5} & -\frac{1}{\sqrt{2}\rho} & \frac{e^{q_2+q_3}Q_3^-}{\rho^4} & -\frac{e^{q_2}}{\rho^2} & \frac{e^{q_2+q_3}}{\rho^3} & -\frac{e^{q_2+q_3}}{\rho^3} \\
-\frac{1}{\sqrt{2}\rho} & \frac{1}{\sqrt{2}\rho} & 1-\frac{\sqrt{2}e^{2q_2}}{\rho^6} & -\frac{1}{\sqrt{2}\rho^2} & \frac{e^{q_2+q_3}Q_3^-}{\rho^5} & -\frac{e^{q_2}}{\rho^3} & \frac{e^{q_2+q_3}}{\rho^4} & -\frac{e^{q_2+q_3}}{\rho^4} \\
\frac{\sqrt{2}e^{-q_2}Q_2^-}{\rho^5} & -\frac{\sqrt{2}e^{-q_2}Q_2^-}{\rho^5} & \frac{\sqrt{2}Q_2^{+2}}{\rho^4} & \frac{1}{\sqrt{2}}-\frac{\sqrt{2}e^{-2q_2}}{\rho^6} & -\frac{e^{q_3}Q_2^-Q_3^-}{\rho^3} & \frac{Q_2^-}{\rho} & -\frac{e^{q_3}Q_2^-}{\rho^2} & \frac{e^{q_3}Q_2^-}{\rho^2} \\
\frac{e^{-q_2}}{\rho^2} & -\frac{e^{-q_2}}{\rho^2} & \frac{Q_2^-}{\rho} & \frac{e^{-q_2}}{\rho^3} & \frac{1}{\sqrt{2}}-\frac{\sqrt{2}e^{2q_3}}{\rho^6} & \frac{\sqrt{2}}{\rho^4} & -\frac{\sqrt{2}e^{q_3}}{\rho^5} & \frac{\sqrt{2}e^{q_3}}{\rho^5} \\
\frac{e^{-q_2-q_3}Q_3^-}{\rho^4} & -\frac{e^{-q_2-q_3}Q_3^-}{\rho^4} & \frac{e^{-q_3}Q_2^-Q_3^-}{\rho^3} & \frac{e^{-q_2-q_3}Q_3^-}{\rho^5} & -\frac{Q_3^{+2}}{\sqrt{2}\rho^2} & \frac{1}{\sqrt{2}}-\frac{\sqrt{2}e^{-2q_3}}{\rho^6} & -\frac{Q_3^-}{\sqrt{2}\rho} & \frac{Q_3^-}{\sqrt{2}\rho} \\
-\frac{e^{-q_2-q_3}}{\rho^3} & \frac{e^{-q_2-q_3}}{\rho^3} & -\frac{e^{-q_3}Q_2^-}{\rho^2} & -\frac{e^{-q_2-q_3}}{\rho^4} & \frac{Q_3^-}{\sqrt{2}\rho} & -\frac{\sqrt{2}e^{-q_3}}{\rho^5} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{\rho^6} \\
\frac{e^{-q_2-q_3}}{\rho^3} & -\frac{e^{-q_2-q_3}}{\rho^3} & \frac{e^{-q_3}Q_2^-}{\rho^2} & \frac{e^{-q_2-q_3}}{\rho^4} & -\frac{Q_3^-}{\sqrt{2}\rho} & \frac{\sqrt{2}e^{-q_3}}{\rho^5} & -\frac{\sqrt{2}}{\rho^6} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \tag{4.2.55}$$

With these defect contributions to the Lax pair which give zero curvature if and only if the equations of motion for a momentum conserving D_4 defect are satisfied we have made a step towards proving the integrability of the general momentum conserving defects found in chapter 2 and the defects in specific ATFTs given in chapter 3. In both the Tzitzéica and D_4 case momentum conservation gave sufficient constraints on the defect for the generation of an infinite number of conserved quantities. It is very likely that in all cases momentum conservation is necessary for integrability.

Chapter 5

Solitons and defects in affine Toda field theories

5.1 Introduction

A soliton is a localised structure within a field theory which moves with constant velocity and retains its form over time, even after interactions with other solitons. They appear as solutions to many integrable systems including the Kortweg-de Vries equation [GGKM67; Hir71], the nonlinear Schrödinger equation [ZM74; FT86] and, most importantly for us, the ATFTs [Hal94]. Well-studied due to interest in their stability and soliton-soliton interactions, they appear in various physical models [SCM73]. For an overview of solitons in integrable systems see [FT86; For90].

An integrable soliton appears as a solution to the equations of motion of an integrable field theory, and is stable due to a cancellation of (reinforcing) nonlinear effects and dispersive effects, and the existence of an infinite number of conserved charges. A topological soliton is stable due to possessing a topological charge. The solitons in ATFTs are both integrable and topological, with the field taking different values as $x \to \pm\infty$ (corresponding to occupying different vacua of U) and their topological charge is given by $u(t,\infty) - u(t,-\infty)$. Consider the ATFT potential in eq.(3.1.1) with $m = \beta = 1$. As $x \to \pm\infty$, so $u \to$ constant, we see that in order for U to have multiple minima, and so support solitons, u must take a complex value. The minima of U occur when $u = 2\pi i w$, where w is an element of the weight lattice of the underlying Lie algebra.

Soliton solutions for the A_r ATFTs were found in [Hol92], with these solitons having real mass and energy despite the fields being complex. Their topological charges lie in the weight lattice of the fundamental representations of A_r. The static single soliton solutions for all other ATFTs were found in [MM93], with folding utilised to give the solitons in non simply laced ATFTs. Investigations into the topological charges of these solitons were made in [McG94a] and [McG94b]. A survey of solitons in ATFTs may be found in [Hal94]. These solitons were found using the Hirota bilinear method [Hir71], which involves finding a bilinear form of the equations of motion and allows very easy construction of multi-soliton solutions. Using this method the single solitons are given by

$$u = -\sum_{i=0}^{n} \alpha_i \ln \tau_i \tag{5.1.1}$$

where the τ functions are dependent on $E = e^{\sqrt{\lambda}(\cosh(\theta)x - \sinh(\theta)t)+c}$. A particular set of τ functions will specify a particular soliton. θ is the rapidity of the soliton, λ is a constant which may take different values for different solitons and c is some constant dictating the position and topological charge of the soliton. In [McG94a; McG94b] it was found that for each soliton the possible topological charges will all be of the form $2\pi i(w_i + \alpha)$, where w_i is a particular fundamental weight and α may be any root. This soliton can then be said to be associated with the corresponding simple root α_i. There are always as many species of solitons as there are simple roots and no soliton is associated with the α_0 root. The possible topological charges of a particular soliton all lie in a fundamental representation of the Lie algebra, but do not necessarily fill it.

For the D_4 ATFT there are three solitons with $\lambda = 2$ which are associated with the outer nodes on the Dynkin diagram and one soliton with $\lambda = 6$ which is associated with the central node on the Dynkin diagram. The τ functions of these solitons are

$$\lambda = 2: \qquad \tau_0 = \tau_1 = 1 + E \quad \tau_3 = \tau_4 = 1 - E \quad \tau_2 = 1 + E^2 \tag{5.1.2}$$

$$\tau_0 = \tau_3 = 1 + E \quad \tau_1 = \tau_4 = 1 - E \quad \tau_2 = 1 + E^2 \tag{5.1.3}$$

$$\tau_0 = \tau_4 = 1 + E \quad \tau_1 = \tau_3 = 1 - E \quad \tau_2 = 1 + E^2 \tag{5.1.4}$$

$$\lambda = 6: \qquad \tau_0 = \tau_1 = \tau_3 = \tau_4 = 1 + E \quad \tau_2 = 1 - 4E + E^2. \tag{5.1.5}$$

These solitons are generally taken to be zero as $x \to -\infty$ and their topological charge as $x \to \infty$, but if a different value of u is required as $x \to \pm\infty$ we can shift u (in such a way that the bulk equations of motion are left invariant), giving a soliton with the same topological charge but different values of u as $x \to \pm\infty$ to the above expressions. For the soliton in eq.(5.1.2) the topological charges are such that it is associated with α_1, for eq.(5.1.3) α_3, for eq.(5.1.4) α_4 and for eq.(5.1.5) α_2. There is an orthogonal transformation (an outer automorphism of the Dynkin diagram) which permutes the solitons associated with the outer nodes. This transformation will also permute the $\alpha_{1,3,4}.u$ terms appearing in the bulk potential U in eq.(3.1.1).

The interactions of solitons and defects have been investigated in the sine-Gordon [BCZ04b], A_r [BCZ04a; CZ09b] and Tzitzéica [CZ09a] cases. In both the quantum and linearised classical cases the type I defects were found to be purely transmitting [DMS94b; KL99; BCZ04b], so here we are considering purely transmitting defects and expect the soliton to be delayed by the defect. The delay experienced by a soliton passing through a defect is found by taking u to be a soliton dependent on E and v to be a soliton dependent on zE, where z is the delay. The defect equations are then solved for z. Evidently z will modify the constant c, shifting the soliton and possibly changing its topological charge. If $z = 0$ or $z \to \pm\infty$ then all t and x dependence is removed from the v field leaving it constant, with the soliton appearing to be absorbed by the defect. The topological charge of the defect is given by $v(t, 0) - u(t, 0)$, and in the cases where the soliton changes topological charge when transmitted through the defect or is absorbed by the defect the topological charge of the defect before and after the interaction will be different.

In the sine-Gordon case there is a single soliton, associated with the α_1 root, and depending on the value of c this may have a topological charge of $2\pi i$ (a soliton) or $-2\pi i$ (an antisoliton). For a soliton interacting with a type I defect z may take a range of values which are completely determined by the rapidity of the incoming soliton and the defect parameter σ. z may be such that a soliton emerges, that an antisoliton emerges, or, if z is 0 or ∞, that no soliton emerges and v is constant. In the A_r case there are two different type I defects and the delay of the α_i soliton passing through one defect is the same as the delay of the α_{r+1-i} soliton passing through the other defect. Again the delay factor z is dependent only on the soliton rapidity and defect parameter and the defect may absorb a soliton or change its topological charge [BCZ04a; CZ09b].

This absorption or changing of topological charge is particularly interesting, and highlights the link between defects and Bäcklund transformations. By taking the Bäcklund transformation of a system and setting one of the fields to be constant it is possible to solve the equations for the other field to give the one soliton solution.

In this chapter we aim to give some flavour of how the generalised type II defects we have found in chapter 3 interact with solitons. We will only consider the D_4 defects. In section 5.2 we find the minima of the two possible defect potentials, which limits the possible soliton configurations which may appear on either side of these defects. We then show that soliton solutions to the D_4 ATFT with a type II defect exist and calculate the soliton delays resultant from interactions with the defect in section 5.3 and consider the behaviour of the auxiliary fields at the defect in section 5.4.

5.2 Minima of the D_4 defect potential

We will consider defects with the two different defect potentials found in section 3.3.1. They are given in eqs.(3.3.10),(3.3.11), referred to as D^+, $\bar{D}^+$, and (3.3.12),(3.3.13), referred to as D^-, $\bar{D}^-$. We begin with $f = 0$ in both cases, where f is the arbitrary function which may be introduced to the defect potential by a redefinition of the auxiliary fields.

To investigate the minima of the defect potential we will need to know the possible values the fields can take at the defect as $t \to \pm\infty$, since they must be the solitons given in eqs.(5.1.1),(5.1.2)-(5.1.5). We will always take $u(\infty, 0) = 0$ and then $u(-\infty, 0)$ and $v(\pm\infty, 0)$ may be a $2\pi i$ multiple of any weight of D_4, as given in eq.(A.0.47). So for $t \to \pm\infty$ we will always have $e^{u_i} = \pm 1$, $e^{v_i} = \pm 1$ and so (recalling $p_i = \frac{1}{2}(u_i + v_i)$, $q_i = \frac{1}{2}(u_i - v_i)$) $e^{p_i} = \pm 1, \pm i$, $e^{q_i} = \pm 1, \pm i$ as the possible values for the exponentials of the fields appearing in the defect potential.

Firstly consider the minima of the potential given by $F = D^+ + \bar{D}^+$. From $F_{p_1} = 0$ we have $e^{p_1} = \pm 1$, from $F_{p_2} = 0$ we have $e^{q_2} + e^{-q_2} \neq 0$ so $e^{q_2} = \pm 1$, then using this we have that $F_{q_1} = 0$ gives $e^{q_1} = \pm 1$. $F_{p_3} = 0$ gives $e^{q_3} + e^{-q_3} \neq 0$ and $e^{p_4} + e^{-p_4} \neq 0$ so $e^{q_3} = \pm 1$ and $e^{p_4} = \pm 1$, $F_{q_4} = 0$ gives $e^{q_4} = \pm 1$ and then $F_{q_2} = 0$, $F_{q_3} = 0$ and $F_{p_4} = 0$ are automatically satisfied. None of the signs are correlated and $F_{p_{2,3}} = 0$ and $F_{\mu_{2,3}} = 0$ are not yet satisfied. Solving $F_{p_2} + F_{\mu_2} = 0$ and $F_{p_3} + F_{\mu_3} = 0$ gives $e^{\mu_2} = \pm(\pm 2)^{\frac{2}{3}}$ and $e^{\mu_3} = \pm(\pm 2)^{\frac{1}{3}}$, where the $\pm$ within the brackets are correlated. Finally $F_{p_2} = 0$ and $F_{p_3} = 0$ give $e^{p_2} = \pm 1$ and $e^{p_3} = \pm 1$. By our choice of values for the bulk fields the $\pm$ signs appearing within the expressions for the auxiliary fields are completely determined. So for any particular choice of values for the bulk fields there are three possible values which the exponentials of the auxiliary fields may take, arising from the powers of $\frac{1}{3}$.

Although we do not know the exact values taken by e^{p_i} and e^{q_i} (as this would involve specifying c in the soliton solutions so that the topological charge of the solitons was set) we do have $p_i = \pi i n_i$ and $q_i = \pi i m_i$, where n_i and m_i are integers. Because we assume $u \to 0$ as $t \to \infty$ the expression $u_i = p_i + q_i = \pi i(n_i + m_i)$ gives $m_i = -n_i$, and so $v_i = p_i - q_i = 2\pi i n_i$ as $t \to \infty$. Looking at the fundamental weights given in eq.(A.0.47) this means that we must have the far left value of the v field being a $2\pi i$ multiple of a weight in the root lattice shifted by 0, w_1 or w_2. A solution with $u = 0$ and $v = 2\pi i w_{3,4}$ as $t \to \infty$ (so the v soliton has been shifted from one given in eqs.(5.1.2)-(5.1.5) by $2\pi i w_{3,4}$) cannot exist for this defect as it does not lie in a minimum of the defect potential. The final topological charge of this defect will be a $2\pi i$ multiple of a weight in the root lattice shifted by 0, w_1 or w_2. If we assume that the soliton does not change species as it passes through the defect then the topological charge (and so value of the field as $x \to \infty$ or $t \to -\infty$) of the u soliton will be $2\pi i(w_i + \alpha)$ and the value of the v soliton as $x \to \infty$ or $t \to -\infty$ may be $2\pi i(w_i + \beta)$, $2\pi i(w_i + w_1 + \beta)$ or $2\pi i(w_i + w_2 + \beta)$ where α and β are roots. So the initial topological charge of the defect will be $v - u = \beta - \alpha(+2\pi i w_{1,2})$. The topological charge of this defect must always lie in either the root lattice or the root lattice shifted by $2\pi i w_{1,2}$ as $t \to \pm\infty$.

Now consider the minima of the potential given by $F = D^- + \bar{D}^-$. From $F_{p_1} = 0$ we have $e^{p_1} = \pm i$, from $F_{p_2} = 0$ we have $e^{q_2} - e^{-q_2} \neq 0$ so $e^{q_2} = \pm i$, then using this we have $F_{q_1} = 0$ giving $e^{q_1} = \pm i$. $F_{p_3} = 0$ gives $e^{q_3} - e^{-q_3} \neq 0$ and $e^{p_4} - e^{-p_4} \neq 0$ so $e^{q_3} = \pm i$ and $e^{p_4} = \pm i$, $F_{q_4} = 0$ gives $e^{q_4} = \pm i$ and then $F_{q_2} = 0$, $F_{q_3} = 0$ and $F_{p_4} = 0$ are automatically satisfied. Again none of the $\pm$ are correlated and $F_{p_{2,3}} = 0$ are not yet satisfied. Solving $F_{p_2} + F_{\mu_2} = 0$ and $F_{p_3} + F_{\mu_3}$ gives $e^{\mu_2} = \pm(\pm 2i)^{\frac{2}{3}}$ and $e^{\mu_3} = \pm(\pm 2i)^{\frac{1}{3}}$, where the $\pm$ within the brackets are correlated. Finally $F_{p_2} = 0$ and $F_{p_3} = 0$ give $e^{p_2} = \pm i$ and $e^{p_3} = \pm i$. Again the choice of values for the bulk fields completely determines the $\pm$ signs which appear in the expressions for the auxiliary fields. For any particular choice of values for the bulk fields there are again three possible values which the exponentials of the auxiliary fields may take, arising from the power of $\frac{1}{3}$.

Again we have not specified c in the soliton solutions, so do not know the topological charge of the solitons or the exact values of the bulk fields, but we do have $p_i = \frac{1}{2}\pi i(2n_i + 1)$ and $q_i = \frac{1}{2}\pi i(2m_i + 1)$, where n_i and m_i are integers. With $u = 0$ as $t \to \infty$ the expression $u_i = p_i + q_i = \pi i(n_i + m_i + 1)$ gives $m_i = -n_i - 1$, and so $v_i = p_i - q_i = \pi i(2n_i + 1)$. Looking at the fundamental weights in eq.(A.0.47) this means that we must have v a $2\pi i$ multiple of a weight in the root lattice shifted by w_3 or w_4. A solution with $u = 0$ and $v = 0, 2\pi i w_{1,2}$ as $t \to \infty$ cannot exist for this defect. The final topological charge of the defect must be a $2\pi i$ multiple of a weight in the root lattice shifted by w_3 or w_4. Again as we assume that the soliton does not change species as it passes through the defect, so the value of the u field as $t \to -\infty$ will be $2\pi i(w_i + \alpha)$ and the value of the v field will be $2\pi i(w_i + w_3 + \beta)$ or $2\pi i(w_i + w_4 + \beta)$, where α and β are roots. So the initial topological charge of the defect will be $v - u = \beta - \alpha + 2\pi i w_{3,4}$. The topological charge of this defect must always lie in the root lattice shifted by $2\pi i w_{3,4}$.

So the possible topological charges for the two defect potentials separate into the two distinct parts of the weight lattice described above. We do not know if these lattices are filled by the topological charges of the defects. This analysis of the minima makes sense if we recall our findings regarding the effect of shifts of the bulk fields on the defect potential in section 3.3.1. There the potential as given in eqs.(3.3.10),(3.3.11) was labelled with D^+ and $\bar{D}^+$ and the potential as given in eqs.(3.3.12),(3.3.13) was labelled with D^- and $\bar{D}^-$. Then a shift of $v \to v + 2\pi i w_1$ gave $\sigma \to -\sigma$ for D^+, $\bar{D}^+$ and D^-, $\bar{D}^-$, a shift of $v \to v + 2\pi i w_2$ left D^+, $\bar{D}^+$ and D^-, $\bar{D}^-$ unchanged, a shift of $v \to v + 2\pi i w_3$ gave $D^+ \to D^-$, $\bar{D}^+ \to \bar{D}^-$ and $D^- \to D^+$, $\bar{D}^- \to \bar{D}^+$ and finally a shift of $v \to v + 2\pi i w_4$ gave $D^+ \to D^-$, $\bar{D}^+ \to \bar{D}^-$ and $D^- \to D^+$ with $\sigma \to -\sigma$ in both cases as well. The effects of these shifts on the defect potentials will dictate their effect on the soliton solutions. If we start with a defect with potential D^+, $\bar{D}^+$ and a solution such that $u, v = 0$ as $t \to \infty$ then for any shift written in terms of the fundamental weights acting on v we can use these results to determine the effects of the shift on the soliton delay and whether such a bulk soliton configuration is allowed for the D^+, $\bar{D}^+$ or D^-, $\bar{D}^-$ defect potential. For example, we take a bulk soliton solution such that $u, v = 0$ as $t \to \infty$. This is a solution for a defect with the D^+, $\bar{D}^+$ defect potential but not the D^-, $\bar{D}^-$ one. Making the shift $v \to v + 2\pi i w_3$ takes $D^+, \bar{D}^+ \leftrightarrow D^-, \bar{D}^-$, and so this new soliton configuration should have the same soliton delay but now be a solution for a defect with the D^-, $\bar{D}^-$ potential, not the D^+, $\bar{D}^+$ potential. As another example, we could instead make the shift $v \to v + 2\pi i w_1$. We would expect the new bulk soliton configuration to still be a solution for the defect with the D^+, $\bar{D}^+$ potential and not the D^-, $\bar{D}^-$ potential, and because this bulk field shift also gave $\sigma \to -\sigma$ we would expect the soliton delay to be the same as before except for $\sigma \to -\sigma$.

5.3 Soliton delays

We will now consider soliton solutions to a D_4 ATFT containing a defect. The soliton on the left of the defect will be one of the four given in eqs.(5.1.2)-(5.1.5) dependent on $E = e^{\sqrt{\lambda}(\cosh(\theta)x - \sinh(\theta)t)+c}$ with $u \to 0$ as $t \to \infty$. The soliton on the right of the defect will be given by the same equation as the soliton on the left and have $v \to 0$ as $t \to \infty$, but with E replaced by zE. z is the soliton delay and we will solve the defect equations for z.

We will first consider the defect equations (2.2.34)-(2.2.39) with D^+ and $\bar{D}^+$ as the defect potential and $f = 0$. Any redefinitions of the auxiliary fields, which are what this function f corresponds to, will change the values of the μ fields themselves, and the positions of the minima of the defect potentials with respect to the μ fields, but should not affect the behaviour of the bulk fields, or the delays experienced by the solitons. For the soliton to the right of the defect we take $v \to 0$ as $t \to \infty$. We introduce the constant $\rho = 2^{\frac{1}{6}}\sigma e^{\theta}$, where σ is the defect parameter and θ is the soliton rapidity, and note that it is different from the constant ρ which was introduced in the previous chapter. For the soliton given in eq.(5.1.2) (that associated with simple root α_1) there are three possible delays. These are

$$z_1 = \frac{(\rho-1)(\rho+e^{\frac{1}{3}\pi i})}{(\rho+1)(\rho-e^{\frac{1}{3}\pi i})} \quad z_2 = \frac{(\rho-1)(\rho-e^{\frac{2}{3}\pi i})}{(\rho+1)(\rho+e^{\frac{2}{3}\pi i})} \quad z_3 = \frac{(\rho+e^{\frac{1}{3}\pi i})(\rho-e^{\frac{2}{3}\pi i})}{(\rho-e^{\frac{1}{3}\pi i})(\rho+e^{\frac{2}{3}\pi i})}. \quad (5.3.1)$$

For the soliton given in eq.(5.1.3) (that associated with simple root α_3) and the soliton given in eq.(5.1.4) (that associated with simple root α_4) the three possible delays are

$$z_1 = \frac{1+\rho}{1-\rho} \qquad z_2 = \frac{e^{\frac{1}{3}\pi i}-\rho}{e^{\frac{1}{3}\pi i}+\rho} \qquad z_3 = \frac{e^{\frac{2}{3}\pi i}+\rho}{e^{\frac{2}{3}\pi i}-\rho}. \qquad (5.3.2)$$

Finally for the soliton given in eq.(5.1.5) (that associated with simple root α_2) the delays are

$$z_1 = \frac{(\rho - i)(\rho - e^{\frac{1}{6}\pi i})}{(\rho + i)(\rho + e^{\frac{1}{6}\pi i})} \quad z_2 = \frac{(\rho + i)(\rho + e^{\frac{5}{6}\pi i})}{(\rho - i)(\rho - e^{\frac{5}{6}\pi i})} \quad z_3 = \frac{(\rho + e^{\frac{1}{6}\pi i})(\rho - e^{\frac{5}{6}\pi i})}{(\rho - e^{\frac{1}{6}\pi i})(\rho + e^{\frac{5}{6}\pi i})}. \tag{5.3.3}$$

As expected the defect equations with D^- and $\bar{D}^-$ can never be satisfied for this soliton configuration.

Now consider the soliton configuration where $u \to 0$ and $v \to 2\pi i w_1$ as $t \to \infty$ and the defect with potential given by D^+ and $\bar{D}^+$. For the soliton associated with α_1 we have

$$z_1 = \frac{(\rho + 1)(\rho - e^{\frac{1}{3}\pi i})}{(\rho - 1)(\rho + e^{\frac{1}{3}\pi i})} \quad z_2 = \frac{(\rho + 1)(\rho + e^{\frac{2}{3}\pi i})}{(\rho - 1)(\rho - e^{\frac{2}{3}\pi i})} \quad z_3 = \frac{(\rho - e^{\frac{1}{3}\pi i})(\rho + e^{\frac{2}{3}\pi i})}{(\rho + e^{\frac{1}{3}\pi i})(\rho - e^{\frac{2}{3}\pi i})}, \tag{5.3.4}$$

for the soliton associated with α_3 or α_4 we have

$$z_1 = \frac{1 - \rho}{1 + \rho} \qquad z_2 = \frac{e^{\frac{1}{3}\pi i} + \rho}{e^{\frac{1}{3}\pi i} - \rho} \qquad z_3 = \frac{e^{\frac{2}{3}\pi i} - \rho}{e^{\frac{2}{3}\pi i} + \rho}, \tag{5.3.5}$$

and for the soliton associated with α_2 we have

$$z_1 = \frac{(\rho + i)(\rho + e^{\frac{1}{6}\pi i})}{(\rho - i)(\rho - e^{\frac{1}{6}\pi i})} \quad z_2 = \frac{(\rho - i)(\rho - e^{\frac{5}{6}\pi i})}{(\rho + i)(\rho + e^{\frac{5}{6}\pi i})} \quad z_3 = \frac{(\rho - e^{\frac{1}{6}\pi i})(\rho + e^{\frac{5}{6}\pi i})}{(\rho + e^{\frac{1}{6}\pi i})(\rho - e^{\frac{5}{6}\pi i})}. \tag{5.3.6}$$

These are simply the delays for the previous configuration, as seen in eqs.(5.3.1)-(5.3.3), with $\rho \to -\rho$. For the defect potential given by D^- and $\bar{D}^-$ there are again no possible soliton solutions for this particular soliton configuration.

For the soliton configuration where $u \to 0$ and $v \to 2\pi i w_2$ as $t \to \infty$ the defect with defect potential given by D^+ and $\bar{D}^+$ again gives the delays in eqs.(5.3.1)-(5.3.3) and the defect with defect potential given by D^- and $\bar{D}^-$ again has no solutions.

For the soliton configuration where $u \to 0$ and $v \to 2\pi i w_3$ as $t \to \infty$ the defect with defect potential given by D^+ and $\bar{D}^+$ now has no soliton solutions and the defect with defect potential given by D^- and $\bar{D}^-$ gives the delays in eqs.(5.3.1)-(5.3.3).

Finally, for the soliton configuration where $u \to 0$ and $v \to 2\pi i w_4$ as $t \to \infty$ the defect with defect potential given by D^+ and $\bar{D}^+$ has no soliton solutions and the defect with defect potential given by D^- and $\bar{D}^-$ gives the delays in eqs.(5.3.4)-(5.3.6).

Recall that in section 3.3.1 we considered the effect of shifts in the bulk fields on the defect potential. Comparing the results there to the soliton delays given here for various soliton configurations in which the bulk v field has been shifted, we see that the changes to the delays seen here correspond exactly to the changes in the defect potential seen in section 3.3.1 when the v field is shifted by various fundamental weights. All of these results fit exactly with our predictions made when considering how the defect potential is affected by shifts of the bulk fields.

For each soliton configuration there are three possible delays. As there are three possible minima for the auxiliary fields for every soliton configuration we expect that the choice of minimum will dictate the delay. Taking $\rho \to e^{\frac{2}{3}\pi i}\rho$ cycles between the delays and corresponds to the redefinition $\sigma \to e^{\frac{2}{3}\pi i}\sigma$ on the defect parameter. Considering the defect potential, if this redefinition is applied to $D^+ + \bar{D}^+$ or $D^+ - \bar{D}^-$ then the potential remains invariant if and only if we make some redefinition of the auxiliary fields such that $e^{\mu_2} \to e^{-\frac{2}{3}\pi i}e^{\mu_2}$ and $e^{\mu_3} \to e^{\frac{2}{3}\pi i}e^{\mu_3}$. In the previous section we saw that for every field configuration there would be three possible values for μ_2 and μ_3 which corresponded to minima of the defect potential, and these field redefinitions would move us from one minimum to another. This confirms that the delay factor experienced by a particular soliton will be dependent on which minimum the auxiliary fields began in. We expect the equations of motion for $\mu_{2,3}$ to be dependent on z, then requiring these fields to take a particular value as $t \to -\infty$ will set the value of z.

We do not expect the value of f to affect the delays experienced by the solitons, only the equations of motion for the auxiliary fields. As a small check that this is the case we also calculated the soliton delays when

$$f = \int^{q_2} \mathrm{d}q_2' \ln\left(e^{q_2'} + e^{-q_2'}\right) + \int^{q_3} \mathrm{d}q_3' \ln\left(e^{q_3'} + e^{-q_3'}\right) \tag{5.3.7}$$

rather than $f = 0$ for the defect with defect potential given by D^+ and $\bar{D}^+$ and with the field configuration $u \to 0$ and $v \to 0$ as $t \to \infty$, and when

$$f = \int^{q_2} \mathrm{d}q_2' \ln\left(e^{q_2'} - e^{-q_2'}\right) + \int^{q_3} \mathrm{d}q_3' \ln\left(e^{q_3'} - e^{-q_3'}\right) \tag{5.3.8}$$

rather than $f = 0$ for the defect with defect potential given by D^- and $\bar{D}^-$ and with the field configuration $u \to 0$ and $v \to 2\pi i w_3$ as $t \to \infty$. In both of these cases the soliton delays were found to be those in eqs.(5.3.1)-(5.3.3).

The values of ρ which correspond to a pole or a zero in the soliton delay give the defect parameter and soliton rapidity which lead to the soliton being absorbed by the defect. Depending on the value z takes the defect can also alter the topological charge of the soliton. Although changes in the topological charge of solitons are allowed we have not considered the possibility for changes in the species of soliton. A defect cannot transform a soliton into another soliton with a different value of λ, as this would require the delay to be x and t dependent. However, it may be possible for a defect to change the species of a soliton to another with the same λ value. For the D_4 solitons with $\lambda = 2$, as given in eqs.(5.1.2)-(5.1.4), there is an orthogonal transformation which permutes the $\alpha_{1,3,4}$ solitons. We suggest that taking this transformation and applying it to the v fields appearing in the standard D_4 defect (recalling that orthogonal transformations of the bulk fields and invertible transformations of the auxiliary fields do not fundamentally change the system) will give a defect which, while no longer having the standard (kinetic) defect Lagrangian of $u_i v_{i,t} + 2\mu_i^{(2)}(u_{i,t}^{(2)} - v_{i,t}^{(2)})$, is still momentum conserving and can take, for example, an incoming α_1 soliton and transform it to an α_3 soliton. However, the complicated form of such a defect means that this calculation of soliton delay has not yet been carried out. It may also be that the allowed minima of the defect are such that this combination of bulk fields cannot be a solution to the system as a whole.

In section 2.3 we saw that carrying out the Lorentz transformation in eqs.(2.3.35),(2.3.36) (where (t', x') are the initial coordinates and (t, x) are the boosted coordinates) on a system containing a stationary momentum conserving defect with defect parameter σ gave a defect moving with velocity $\tanh(\eta)$, which is momentum conserving if its defect parameter is $e^{-\eta}\sigma$. Applying the same coordinate transformation to a defect with rapidity θ gives $E = e^{\cosh(\theta)x'-\sinh(\theta)t'+c} \rightarrow e^{\cosh(\theta+\eta)x-\sinh(\theta+\eta)t+c}$, so a soliton with rapidity $\theta+\eta$. Recalling that $\rho = 2^{\frac{1}{6}}\sigma e^{\theta}$ is the only parameter which appears in the expressions for the soliton delays we see that this Lorentz boost will not affect the delays experienced by solitons passing through defects.

5.4 Auxiliary field behaviour

The auxiliary field behaviour is considered here for the defect with defect potential given by D^+ and $\bar{D}^+$ and the bulk fields such that $u \to 0$, $v \to 0$ as $t \to \infty$. The behaviour of the auxiliary fields in all other cases should be given by the redefinitions of the auxiliary fields specified in section 3.3.1 when considering the effects of shifts of the bulk fields on the defect potential.

For the soliton given in eq.(5.1.2) (that associated with simple root α_1) the auxiliary fields are given by

$$\begin{aligned}
e^{\mu_2} =& 2^{\frac{2}{3}}\rho^{-1}\Big(E^2+1\Big)^{-\frac{1}{2}}\Big(z^2E^2+1\Big)^{-\frac{1}{2}} \\
&\Big(\Big(z^2(z+1)E^4+8z^2E^3-(z+1)(z^2-8z+1)E^2+8zE+z+1\Big)\rho^3 \\
&\quad +(z-1)\Big(z^2E^4-(z^2-4z+1)E^2+1\Big)\Big) \\
&\Big((z-1)\Big(zE^2+(z+1)E+1\Big)\rho^3 \\
&\quad +\Big(z(z+1)E^2+(z-1)^2E+z+1\Big)\Big)^{-1} \qquad (5.4.1) \\
e^{\mu_3} =& 2^{\frac{1}{3}}\rho\Big(E-1\Big)^{-1}\Big(zE-1\Big)^{-1} \\
&\Big(z^2E^4-z(z+1)E^3+(z^2+1)E^2-(z+1)E+1\Big) \\
&\Big((z-1)\Big(zE^2+(z+1)E+1\Big)\rho^3+\Big(z(z+1)E^2+(z-1)^2E+z+1\Big)\Big) \\
&\Big((z+1)\Big(z^2E^4+(z^2-4z+1)E^2+1\Big)\rho^3 \\
&\quad +(z-1)\Big(z^2E^4+(z^2+1)E^2+1\Big)\Big)^{-1} \qquad (5.4.2)
\end{aligned}$$

with E as given above. For the soliton in eq.(5.1.3) the auxiliary fields are

$$e^{\mu_2} = 2^{\frac{2}{3}}\rho^{-1}(z-1)(z+1)^{-1}(zE^2+1)(E^2+1)^{-\frac{1}{2}}(z^2E^2+1)^{-\frac{1}{2}} \tag{5.4.3}$$

$$e^{\mu_3} = 2^{\frac{1}{3}}\rho(z+1)(z-1)^{-1}(zE^2-1)(z^2E^2-1)^{\frac{1}{2}}(E^2-1)^{-\frac{1}{2}} \tag{5.4.4}$$

and likewise for the soliton given by eq.(5.1.4). Finally for the soliton in eq.(5.1.5) the auxiliary fields are given by

$$\begin{aligned} e^{\mu_2} =& 2^{\frac{5}{3}}\rho^2(z-1)^{-1}(E+1)^{-4}(zE+1)^{-4}\Big(E^2-4E+1\Big)^{\frac{1}{2}}\Big(z^2E^2-4zE+1\Big)^{\frac{1}{2}} \\ & \Big(z^2E^4 - z(z+1)E^3 + (z^2-8z+1)E^2 - (z+1)E + 1\Big) \\ & \Big(z^2E^4 - z(z+1)E^3 + 6zE^2 - (z+1)E + 1\Big)^{-1} \\ & \Big(z^6(z^2+4z+1)E^{12} + 2z^4(2z^4+z^3+12z^2+z+2)E^{10} \\ & \quad - 4z^4(z+1)(z^2-14z+1)E^9 \\ & \quad + z^2(z^6+16z^5-10z^4+76z^3-10z^2+16z+1)E^8 \\ & \quad + 4z^2(z+1)(z^4+5z^3+24z^2+5z+1)E^7 \\ & \quad + 2z(z^6+8z^5-11z^4+256z^3-11z^2+8+1)E^6 \\ & \quad + 4z(z+1)(z^4+5z^3+24z^2+5z+1)E^5 \\ & \quad + (z^6+16z^5-10z^4+76z^3-10z^2+16z+1)E^4 \\ & \quad - 4z(z+1)(z^2-14z+1)E^3 \\ & \quad + 2(2z^4+z^3+12z^2+z+2)E^2 + z^2 + 4z + 1\Big) \\ & \Big(\sqrt{3}\Big(z^3(z+1)E^6 + z(z+1)(z^2+z+1)E^4 + 16z^2E^3 \\ & \qquad + (z+1)(z^2+z+1)E^2 + z + 1\Big)\rho^3 \\ & \quad + (z-1)\Big(z^3E^6 - z(z^2-5z+1)E^4 \\ & \qquad + 4z(z+1)E^3 - (z^2-5z+1)E^2 + 1\Big)\Big)^{-1} \end{aligned} \tag{5.4.5}$$

$$\begin{aligned} e^{\mu_3} =& 2^{-\frac{2}{3}}\rho^{-2}(z-1)(E+1)^3(zE+1)^3\Big(E^2-4E+1\Big)^{-1}\Big(z^2E^2-4zE+1\Big)^{-1} \\ & \Big(z^2E^4 - z(z+1)E^3 + 6zE^2 - (z+1)E + 1\Big) \\ & \Big(\sqrt{3}\Big(z^3(z+1)E^6 + z(z+1)(z^2+z+1)E^4 + 16z^2E^3 \\ & \qquad + (z+1)(z^2+z+1)E^2 + z + 1\Big)\rho^3 \end{aligned}$$

$$
\begin{aligned}
&+ (z-1)\Big(z^3E^6 - z(z^2-5z+1)E^4 \\
&\qquad + 4z(z+1)E^3 - (z^2-5z+1)E^2 + 1\Big)\Big) \\
&\Big(z^6(z^2+4z+1)E^{12} + 2z^4(2z^4+z^3+12z^2+z+2)E^{10} \\
&- 4z^4(z+1)(z^2-14z+1)E^9 \\
&+ z^2(z^6+16z^5-10z^4+76z^3-10z^2+16z+1)E^8 \\
&+ 4z^2(z+1)(z^4+5z^3+24z^2+5z+1)E^7 \\
&+ 2z(z^6+8z^5-11z^4+256z^3-11z^2+8z+1)E^6 \\
&+ 4z(z+1)(z^4+5z^3+24z^2+5z+1)E^5 \\
&+ (z^6+16z^5-10z^4+76z^3-10z^2+16z+1)E^4 \\
&- 4z(z+1)(z^2-14z+1)E^3 \\
&+ 2(2z^4+z^3+12z^2+z+2)E^2 + z^2+4z+1\Big)^{-1}
\end{aligned}
\tag{5.4.6}
$$

We will consider the behaviour of these solutions as $t \to -\infty$, so $E \to \infty$. From our analysis of the minima of the defect potential we have $e^{\mu_2} = \pm(\pm 2)^{\frac{2}{3}}$ and $e^{\mu_3} = \pm(\pm 2)^{\frac{1}{3}}$. Depending on the values of these $\pm$ signs, as set by the bulk fields, the possible values for the auxiliary fields are $e^{\mu_2} = \pm 2^{\frac{2}{3}} e^{\frac{4}{3}\pi i n_2}$ and $e^{\mu_3} = \pm 2^{\frac{2}{3}} e^{\frac{2}{3}\pi i n_3}$ or $e^{\mu_2} = \pm 2^{\frac{2}{3}} e^{\frac{2}{3}\pi i n_2}$ and $e^{\mu_2} = \pm 2^{\frac{2}{3}} e^{\frac{1}{3}\pi i n_3}$, where $n_{2,3}$ are integers.

First consider the expressions for the auxiliary fields when a soliton associated with the α_1 root passes through the defect, as given in eqs.(5.4.1),(5.4.2). When $E \to \infty$ these become

$$
e^{\mu_2} = 2^{\frac{2}{3}} \rho^{-1} \frac{(z+1)\rho^3 + z - 1}{(z-1)\rho^3 + z + 1} \tag{5.4.7}
$$

$$
e^{\mu_3} = 2^{\frac{1}{3}} \rho \frac{(z-1)\rho^3 + z + 1}{(z+1)\rho^3 + z - 1}. \tag{5.4.8}
$$

The three possible delays for this soliton-defect configuration are given in eq.(5.3.1). By substituting these into the above expression we see that if we are to have delay z_1 then the initial values of the auxiliary fields must be $e^{\mu_2} = 2^{\frac{2}{3}} e^{-\frac{2}{3}\pi i}$ and $e^{\mu_3} = 2^{\frac{1}{3}} e^{\frac{2}{3}\pi i}$, if we are to have delay z_2 then the auxiliary fields must take initial values of

$e^{\mu_2} = 2^{\frac{2}{3}} e^{\frac{2}{3}\pi i}$ and $e^{\mu_3} = 2^{\frac{1}{3}} e^{-\frac{2}{3}\pi i}$ and if we are to have delay z_3 then they must take values $e^{\mu_2} = 2^{\frac{2}{3}}$ and $e^{\mu_3} = 2^{\frac{1}{3}}$.

For a soliton associated with either the α_3 or α_4 root the auxiliary fields expressions become

$$e^{\mu_2} = 2^{\frac{2}{3}} \rho^{-1} \frac{z+1}{z-1} \tag{5.4.9}$$

$$e^{\mu_3} = 2^{\frac{1}{3}} \rho \frac{z+1}{z-1} \tag{5.4.10}$$

as $t \to -\infty$. The three possible soliton delays are given in eq.(5.3.2). For a delay of z_1 the auxiliary fields must have initial values $e^{\mu_2} = 2^{\frac{2}{3}}$ and $e^{\mu_3} = 2^{\frac{1}{3}}$, for a delay of z_2 they must have initial values of $e^{\mu_2} = 2^{\frac{2}{3}} e^{\frac{2}{3}\pi i}$ and $e^{\mu_3} = 2^{\frac{1}{3}} e^{-\frac{2}{3}\pi i}$ and for a delay of z_3 they must have initial values of $e^{\mu_2} = 2^{\frac{2}{3}} e^{-\frac{2}{3}\pi i}$ and $e^{\mu_3} = 2^{\frac{1}{3}} e^{\frac{2}{3}\pi i}$.

Finally for a soliton associated with α_2 passing through a defect the auxiliary field expressions become

$$e^{\mu_2} = 2^{\frac{5}{3}} \rho^2 \frac{z^2+4z+1}{(z-1)(\sqrt{3}(z+1)\rho^3 + z - 1)} \tag{5.4.11}$$

$$e^{\mu_2} = 2^{-\frac{2}{3}} \rho^{-2} \frac{(z-1)(\sqrt{3}(z+1)\rho^3 + z - 1)}{z^2+4z+1} \tag{5.4.12}$$

as $t \to -\infty$ and the three possible soliton delays are given in eq.(5.3.3). For a delay of z_1 the auxiliary fields must have the initial values $e^{\mu_2} = 2^{\frac{2}{3}} e^{\frac{2}{3}\pi i}$ and $e^{\mu_3} = 2^{\frac{1}{3}} e^{-\frac{2}{3}\pi i}$, for a delay of z_2 they must have the initial values $e^{\mu_2} = 2^{\frac{2}{3}} e^{-\frac{2}{3}\pi i}$ and $e^{\mu_3} = 2^{\frac{1}{3}} e^{\frac{2}{3}\pi i}$ and for a delay of z_3 they must have the initial values $e^{\mu_2} = 2^{\frac{2}{3}}$ and $e^{\mu_3} = 2^{\frac{1}{3}}$.

These early values which the auxiliary field must take fit into our previous analysis of the minima of the defect potential.

Chapter 6

Conclusion

In this thesis we have sucessfully expanded the Lagrangian defect picture by generalising the type II defects first seen in [CZ09a]. From chapter 2 we have the general form any momentum conserving defect (up to the restrictions on the defect couplings described in that chapter) must take. We were also able to show that the equations of motion of such a defect can always be modified to give a Bäcklund transformation for the bulk theory. In chapter 3 we were able to find momentum conserving defects in the A_r, B_r, C_r and D_r ATFTs using the conditions found in the previous chapter. Type I A_r ATFTs had been found previously in [BCZ04b; BCZ04a] and type II C_r ATFTs had been found via folding of the type I A_r defects in [Rob14b], but the B_r and D_r defects are new. The modifications of the defect equations which give a Bäcklund transformation may be applied to all the defects found in this chapter, giving new Bäcklund transformations for the theories. In chapter 4 we gave some thought to the integrability of a system with a defect. Only the existence of an infinite number of conserved quantities was considered, and it was found that momentum conservation was likely necessary for the existence of these conserved quantities. Some possible restrictions on the splitting of the fields into those which couple as a type I defect and those which couple as a type II defect were found, but these were not proved to be necessary. A new transport matrix across the defect which ensured zero curvature, and so an infinite number of conserved quantities, was found for defects in the D_4 ATFT. In chapter 5 solitons passing from the bulk theory on the left of a D_4 defect to the bulk theory on the right were investigated. The delays experienced by the soliton during such an interaction were found and the behaviour of the auxiliary fields was analysed. As in previous cases the solitons experienced a delay and in some cases a change of topological charge, with the delay set be the rapidity of the incoming soliton and the defect parameter [BCZ04b; BCZ04a; CZ09b; CZ09a], and like in the Tzitzéica case the initial values taken by the auxiliary fields also affected the soliton delays.

This work has confirmed previous results (the squeezed sine-Gordon defects found in [CZ09a] and the C_3 defects found in [Rob14b]), provided new energy and momentum conserving defects, and gives us a framework which will hopefully cover all defects in ATFTs. The fact that all defects satisfying the conditions given in chapter 2 can be used to give a Bäcklund transformation suggests that these momentum conserving defects are integrable, as well as being an interesting observation in its own right. The explicit calculations for transmission of solitons through the D_4 defect also strongly suggest that it is an integrable system. The most compelling evidence for the integrability of these momentum conserving defects is the zero curvature of the Tzitzéica and D_4 defects.

The obvious next step is to attempt to find defects in the remaining exceptional simply laced ATFTs (E_6, E_7, E_8). In principle these are the only remaining cases it is necessary to solve, as the folding procedure for defects in [Rob14b] can then be used to find momentum conserving defects for all non-simply laced ATFTs. These E series momentum conserving defects have not been found so far due to the difficultly of finding appropriate 2-space. It may also be that a non-zero A matrix or ξ vector field is required. However we have no systematic way of finding the 1-space and 2-space splitting, A matrix or ξ field required for a momentum conserving defect and this is a difficult task to complete by trial and error alone. Chapter 4 did give us some likely constraints on the 1-space and 2-space splittings which give an integrable defect, but we have not yet been able to apply these conditions correctly to the E

series. The D_4 defect found should fold to the G_2 defect, but this folding has not yet been successfully carried out.

Another significant area for further study would be defects in quantum ATFTs. The type I defects have been investigated in the quantum case in [BCZ05; CZ07; Ump08; CZ09b; CZ10] and the type II defects in [CZ11; Rob15]. With the information about the soliton-defect interactions for the D_4 defect given in chapter 5 it should be possible to construct a quantum scattering matrix for this defect.

Appendix A

Simple roots and generators of Lie algebras

This appendix is intended to establish some of the notation and properties of Lie algebras and their representations, roots and weights used in chapters 3 and 4. For a more complete set of notes on this area see [Cah84; Geo99; Sam90].

The matrices which form a basis of a representation of a semi-simple Lie algebra obey the following commutation relations.

$$[H_j, E_\alpha] = (\alpha)_j E_\alpha \tag{A.0.1}$$

$$E_\alpha = E_{-\alpha}^\dagger \tag{A.0.2}$$

$$[E_\alpha, E_{-\alpha}] = \frac{2}{|\alpha|^2}(\alpha)_j H_j \tag{A.0.3}$$

$$[E_\alpha, E_\beta] = n_{\alpha\beta} E_{\alpha+\beta} \qquad \text{if} \quad \alpha+\beta \in \text{roots} \tag{A.0.4}$$

$$[E_\alpha, E_\beta] = 0 \qquad \text{if} \quad \alpha+\beta \notin \text{roots}, 0. \tag{A.0.5}$$

Here subscripts are used to identify the different generator matrices. A subscript outside a bracket denotes an element of the bracketed vector. The matrices H_i are the Cartan matrices and form a basis of the Cartan subalgebra, the largest set of mutually commuting (and so mutually diagonalisable) matrices in the representation of the algebra. The matrix E_α is an eigenvector of all Cartan matrices H_i under the commutator. The vector α is known as a root and the element $(\alpha)_i$ is the eigenvalue of E_α with H_i. For eq.(A.0.4) we must be able to determine whether a certain vector is a root or not.

For any two roots α and β there will be a chain of roots $\beta + p\alpha, \ldots, \beta, \ldots, \beta - m\alpha$ which must satisfy

$$m - p = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}. \tag{A.0.6}$$

If α is a root then the only multiples of α which are also roots are α, 0 and $-\alpha$, so the chain of roots can have a maximum of four elements. A chain with more than four elements would give other multiples of roots as also being roots. So $m - p$ can take integer values between -3 and 3. The angle between two roots is given by

$$\cos^2\theta = \frac{\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}, \tag{A.0.7}$$

and so from this and eq.(A.0.6) we see that there is a restricted set of values which angles between roots can take.

We can take some of the roots to be an ordered basis of the root space. A positive root is a root whose first non-zero coefficient when written in this basis is positive. The simple roots are defined as the positive roots which cannot be written as the sum of two positive roots and are labelled $\alpha_1, \alpha_2, \ldots, \alpha_r$. They form a basis of the root space. All positive roots can be written as a sum of simple roots and all negative roots can be written as a sum of negatives of simple roots. So we can use eq.(A.0.6) to check whether sums of simple roots are in fact a root, and so build up all the roots written as sums of simple roots. Note that $\alpha_i - \alpha_j$ (where $\alpha_{i,j}$ are simple roots) cannot be a root as then either $\alpha_i - \alpha_j$ or $\alpha_j - \alpha_i$ would be a positive roots, and so we would have $\alpha_i = (\alpha_i - \alpha_j) + \alpha_j$ or $\alpha_j = (\alpha_j - \alpha_i) + \alpha_i$ giving a simple root written as a sum of positive roots.

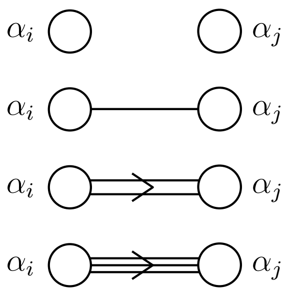
For our purposes all Lie algebras are completely characterised by their Cartan matrix or Dynkin diagram, which encodes the inner products between the simple roots. The

Cartan matrix is an $r \times r$ matrix with entries given by

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \tag{A.0.8}$$

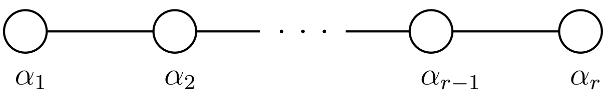
where α_i $(i = 0, \ldots, r)$ are the simple roots. Obviously the diagonal entries will be 2. Considering eq.(A.0.6), the fact that $\alpha_i - \alpha_j$ (where α_i and α_j are simple roots) is not a root means that A_{ij}, $i \neq j$ can only take values 0, -1, -2, -3. These values correspond to different inner products between the simple roots. The Schwartzchild inequality $\langle a, b \rangle^2 \leq \langle a, a \rangle \langle b, b \rangle$ also gives limits on A_{ij}. It has equality when a is proportional to b, but if a and b are simple roots we can never have equality (as they are linearly independent), and therefore $A_{ij} A_{ji} < 4$.

The information in the Cartan matrix can be represented on a Dynkin diagram. For every simple root we place a dot or node. Then the i^{th} and j^{th} nodes are connected by a number of lines equal to $A_{ij} A_{ji}$. If $A_{ij} \neq A_{ji}$ then $\langle \alpha_i, \alpha_i \rangle \neq \langle \alpha_j, \alpha_j \rangle$ and we add a direction to the lines pointing towards the shortest root. So, all our possibilities (taking into account the restrictions on A_{ij} detailed in the previous paragraph) are

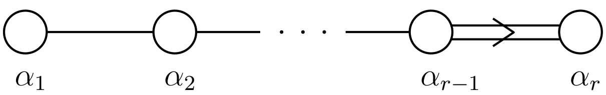


$$A_{ij} = 0 \qquad A_{ji} = 0$$
$$A_{ij} = -1 \qquad A_{ji} = -1$$
$$A_{ij} = -2 \qquad A_{ji} = -1$$
$$A_{ij} = -3 \qquad A_{ji} = -1.$$

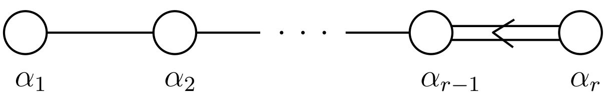
In using these to build Dynkin diagrams it can be found that there is a fairly limited number. Below we give the Dynkin diagrams of all semi-simple Lie algebras.

$A_r :$ 

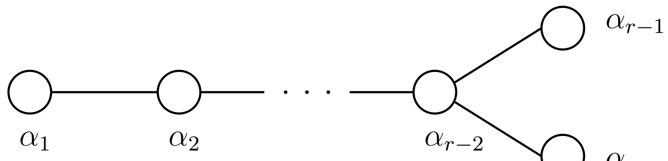
(A.0.9)

$B_r :$ 

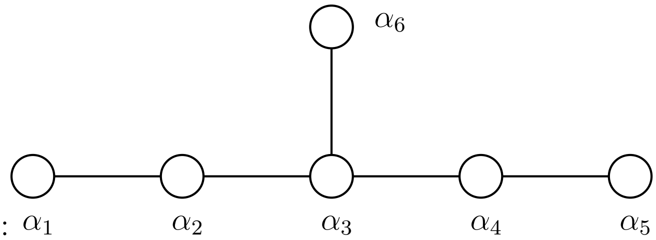
(A.0.10)

$C_r :$ 

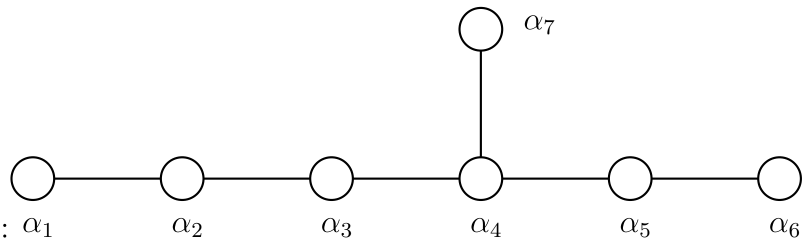
(A.0.11)

D_r :


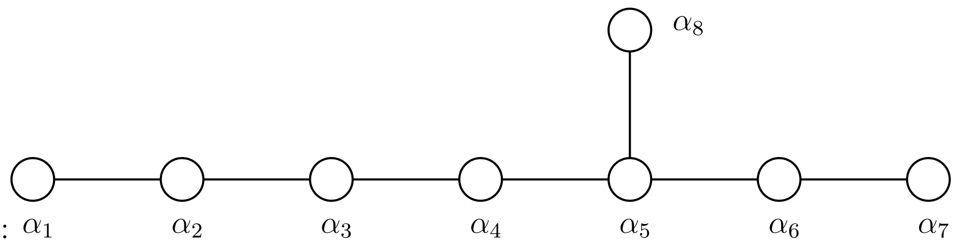
(A.0.12)

E_6 :


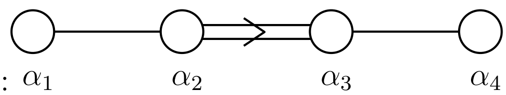
(A.0.13)

E_7 :


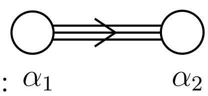
(A.0.14)

E_8 :


(A.0.15)

F_4 :


(A.0.16)

G_2 :


(A.0.17)

For small r some of the A, B, C and D diagrams are isomorphic to each other so to avoid these overlaps we have A_r for $r \geq 1$, B_r for $r \geq 2$, C_r for $r \geq 3$ and D_r for $r \geq 4$. The diagrams with only single lines are called simply laced, and all their simple roots are the same length. The following roots satisfy the inner product relations encapsulated within the above diagrams.

$$A_r: \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \alpha_{r-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \alpha_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \tag{A.0.18}$$

$$B_r: \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \alpha_{r-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \alpha_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{A.0.19}$$

$$C_r: \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \alpha_{r-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \alpha_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 2 \end{pmatrix} \tag{A.0.20}$$

$$D_r: \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \alpha_{r-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \alpha_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{A.0.21}$$

$$E_6: \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_6 = \frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \tag{A.0.22}$$

$$E_7: \quad \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_7 = \frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \\ -1 \\ -1 \end{pmatrix} \tag{A.0.23}$$

$$E_8: \quad \alpha_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \alpha_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \qquad \alpha_8 = \frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \end{pmatrix} \tag{A.0.24}$$

$$F_4: \quad \alpha_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \alpha_4 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \tag{A.0.25}$$

$$G_2: \quad \alpha_1 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \tag{A.0.26}$$

These are the simple roots in (one of) their simplest possible forms, with the exact choice of vectors dictated by trying to set as many entries to zero in as many simple roots as possible. Because the Cartan matrices and Dynkin diagrams only give information about the angles between roots and their relative lengths we can act on these vectors with any orthogonal transformation or overall scaling and still have the simple roots of the same Lie algebra. We have chosen to take the first simple root to have length $\sqrt{2}$. Note that for the A_r simple roots the vectors are length $r+1$ as they are far simpler to write this way. For A_1 we will simply take $\alpha_1 = 1$.

It is possible to use these simple roots to construct a Toda field theory, a conformal field theory without soliton solutions. For *affine* Toda field theory we base the field theory on the simple roots encoded in the affine Dynkin diagrams. The affine Dynkin diagrams have an extra node, and so there is an extra root being treated like the simple roots. This extra "simple" root is the lowest weight root, denoted by α_0 and given by $\alpha_0 = -\sum_{i=1}^{r} n_i\alpha_i$ where α_i are the simple roots and n_i are some numbers which are characteristic of the algebra. These n_is are

$$A_r: \quad n_i = 1 \qquad i = 1, \ldots, r \tag{A.0.27}$$

$$B_r: \quad n_1 = 1 \quad n_i = 2 \qquad i = 2, \ldots, r \tag{A.0.28}$$

$$C_r: \quad n_i = 2 \quad n_r = 1 \qquad i = 1, \ldots, r-1 \tag{A.0.29}$$

$$D_r: \quad n_1 = 1 \quad n_{r-1} = 1 \quad n_r = 1 \quad n_i = 2 \qquad i = 2, \ldots, r-2 \tag{A.0.30}$$

$$\begin{aligned} E_6: \quad & n_1 = 1 \quad n_2 = 2 \quad n_3 = 3 \quad n_4 = 2 \\ & n_5 = 1 \quad n_6 = 2 \end{aligned} \tag{A.0.31}$$

$$\begin{aligned} E_7: \quad & n_1=1 \quad & n_2=2 \quad & n_3=3 \quad & n_4=4 \\ & n_5=3 \quad & n_6=2 \quad & n_7=2 \end{aligned} \tag{A.0.32}$$

$$\begin{aligned} E_8: \quad & n_1=2 \quad & n_2=3 \quad & n_3=4 \quad & n_4=5 \\ & n_5=6 \quad & n_6=4 \quad & n_7=2 \quad & n_8=3 \end{aligned} \tag{A.0.33}$$

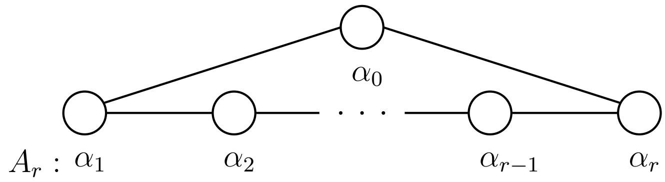
$$F_4: \quad n_1=2 \quad n_2=3 \quad n_3=4 \quad n_4=2 \tag{A.0.34}$$

$$G_2: \quad n_1=2 \quad n_2=3 \tag{A.0.35}$$

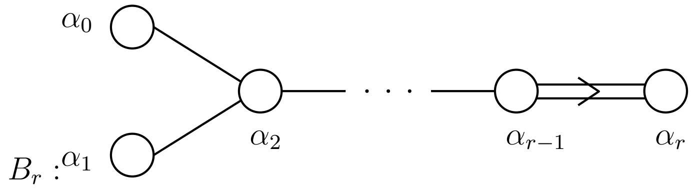
and so the lowest weight α_0 roots for the various algebras are

$$A_r: \quad \alpha_0 = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad B_r: \quad \alpha_0 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad C_r: \quad \alpha_0 = \begin{pmatrix} -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad D_r: \quad \alpha_0 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$E_6: \quad \alpha_0 = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \end{pmatrix} \quad E_7: \quad \alpha_0 = \frac{1}{3}\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \end{pmatrix} \quad E_8: \quad \alpha_0 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$F_4: \quad \alpha_0 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad G_2: \quad \alpha_0 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}} \end{pmatrix}. \tag{A.0.36}$$

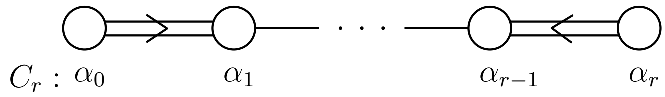
When the α_0 root is included in the Dynkin diagrams we have the affine Dynkin diagrams,



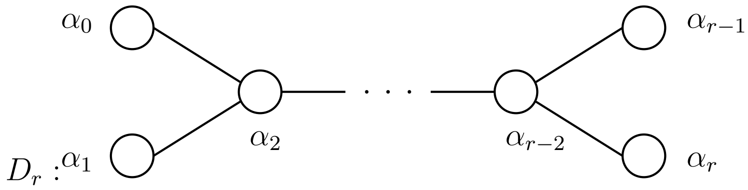
(A.0.37)



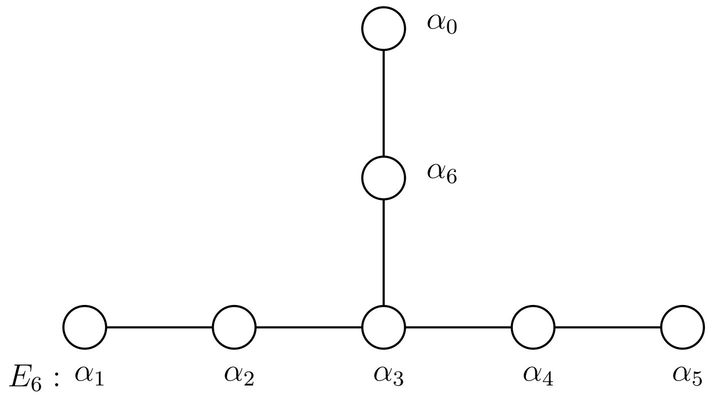
(A.0.38)



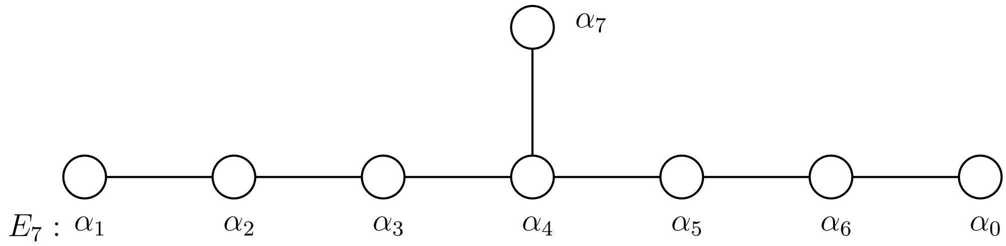
(A.0.39)



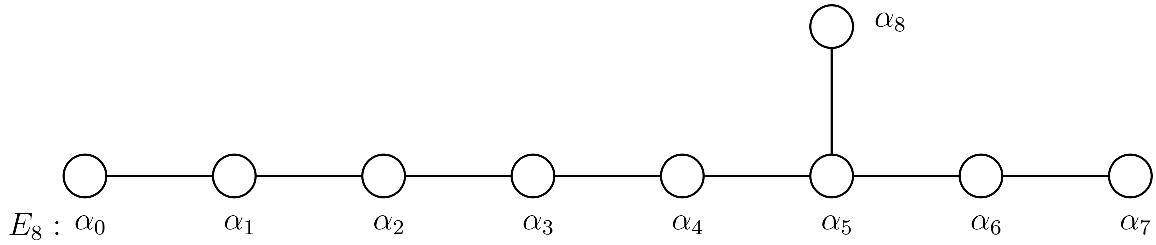
(A.0.40)



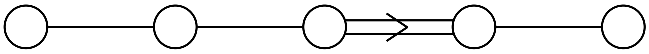
(A.0.41)



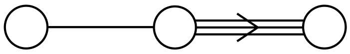
(A.0.42)



(A.0.43)



$$F_4 : \alpha_0 \qquad \alpha_1 \qquad \alpha_2 \qquad \alpha_3 \qquad \alpha_4 \tag{A.0.44}$$



$$G_2 : \alpha_0 \qquad \alpha_1 \qquad \alpha_2 . \tag{A.0.45}$$

Because the B_2 and C_2 affine Dynkin diagrams are isomorphic but their form is more akin to the pattern followed by the C_r diagrams we choose to take B_r for $r \geq 3$ and C_r for $r \geq 2$, which differs from the non-affine convention. Often the semi-simple Lie algebras are denoted by a capital letter and their affine versions are denoted by the same letter with a tilde. Here we just take the un-tilded letter to mean the affine version, as that is all we will be working with from now on. Also note that while we are using the affine simple roots we are still using the non-affine, finite dimensional generators as defined by the commutation relations in eqs.(A.0.1)-(A.0.5).

The roots are the eigenvectors of the H_i matrices in the adjoint representation, defined by $\mathrm{ad}X(Y) = [X, Y]$. Weights are eigenvectors of H_i in any representation. All possible roots form a root lattice and all possible weights form a weight lattice which contains the root lattice. The weights also obey eq.(A.0.6) but with the root β replaced by a weight. The fundamental weights w_j satisfy $\langle \alpha_i, w_j \rangle = \delta_{ij}$, with w_i being the fundamental weight associated to the simple root α_i.

The fundamental weights of A_3 are

$$w_1 = \frac{1}{4}\begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix} \qquad w_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \qquad w_3 = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \tag{A.0.46}$$

and the fundamental weights of D_4 are

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad w_3 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \qquad w_4 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} . \tag{A.0.47}$$

Bibliography

[AAGZ11] A. R. Aguirre, T. R. Araujo, J. F. Gomes and A. H. Zimerman. 'Type-II Bäcklund Transformations via Gauge Transformations'. In: *JHEP* 12 (2011), p. 056. `arXiv:1110.1589 [nlin-si]`.

[ABT00] N. V. Alexeeva, I. V. Barashenkov and G. P. Tsironis. 'Impurity-Induced Stabilization of Solitons in Arrays of Parametrically Driven Nonlinear Oscillators'. In: *Phys. Rev. Lett.* 84 (14 Apr. 2000), pp. 3053–3056. `arXiv:patt-sol/9905009 [patt-sol]`.

[AD12a] J. Avan and A. Doikou. 'Liouville integrable defects: The non-linear Schrodinger paradigm'. In: *JHEP* 01 (2012), p. 040. `arXiv:1110.4728 [hep-th]`.

[AD12b] J. Avan and A. Doikou. 'The sine-Gordon model with integrable defects revisited'. In: *JHEP* 11 (2012), p. 008. `arXiv:1205.1661 [hep-th]`.

[AFZ79] A. Arinshtein, V. Fateyev and A. Zamolodchikov. 'Quantum S-matrix of the (1 + 1)-dimensional Toda chain'. In: *Phys. Lett. B* 87.4 (1979), pp. 389–392.

[AKNS73] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur. 'Nonlinear-Evolution Equations of Physical Significance'. In: *Phys. Rev. Lett.* 31 (2 July 1973), pp. 125–127.

[BB17] R. Bristow and P. Bowcock. 'Momentum conserving defects in affine Toda field theories'. In: *JHEP* 2017.5 (2017), pp. 1–24. `arXiv:1612.03002 [hep-th]`.

[BBT03] O. Babelon, D. Bernard and M. Talon. *Introduction to classical integrable systems*. Cambridge University Press, 2003.

[BCDS90] H. Braden, E. Corrigan, P. Dorey and R. Sasaki. 'Affine Toda field theory and exact S-matrices'. In: *Nucl. Phys. B* 338.3 (1990), pp. 689–746.

[BCDS91] H. Braden, E. Corrigan, P. Dorey and R. Sasaki. 'Multiple poles and other features of affine Toda field theory'. In: *Nucl. Phys. B* 356.2 (1991), pp. 469–498.

[BCZ04a] P. Bowcock, E. Corrigan and C. Zambon. 'Affine Toda field theories with defects'. In: *JHEP* 01 (2004), p. 056. `arXiv:hep-th/0401020 [hep-th]`.

[BCZ04b] P. Bowcock, E. Corrigan and C. Zambon. 'Classically integrable field theories with defects'. In: *Int. J. Mod. Phys.* A19S2 (2004), pp. 82–91. `arXiv:hep-th/0305022 [hep-th]`.

[BCZ05] P. Bowcock, E. Corrigan and C. Zambon. 'Some aspects of jump-defects in the quantum sine-Gordon model'. In: *JHEP* 08 (2005), p. 023. `arXiv:hep-th/0506169 [hep-th]`.

[Bog76] O. I. Bogoyavlensky. 'On perturbations of the periodic Toda lattice'. In: *Comm. Math. Phys.* 51.3 (1976), pp. 201–209.

[Bow17] P. Bowcock. 'Private communication'. 2017.

[Bri17] R. Bristow. 'Integrability of generalised type II defects in affine Toda field theory'. 2017. `arXiv:1709.03927 [hep-th]`.

[BSS93] A. Y. Boldin, S. S. Safin and R. A. Sharipov. 'On an old article of Tzitzeica and the inverse scattering method'. In: *J. Math. Phys.* 34.12 (1993), pp. 5801–5809.

[Cah84] R. N. Cahn. *Semi-simple Lie algebras and their representations*. The Benjamin/Cummings Publishing Company, Inc., 1984.

[Cau08] V. Caudrelier. 'On a systematic approach to defects in classical integrable field theories'. In: *Int. J. Geom. Methods Mod. Phys.* 5 (2008), pp. 1085–1108.
`arXiv:0704.2326 [math-ph]`.

[Cau15] V. Caudrelier. 'Multisymplectic approach to integrable defects in the sine-Gordon model'. In: *J. Phys.* A48.19 (2015), p. 195203.
`arXiv:1411.5171 [math-ph]`.

[CDS93] E. Corrigan, P. E. Dorey and R. Sasaki. 'On a generalized bootstrap principle'. In: *Nucl. Phys.* B408 (1993), pp. 579–599.
`arXiv:hep-th/9304065 [hep-th]`.

[CFG02] O. A. Castro-Alvaredo, A. Fring and F. Gohmann. 'On the absence of simultaneous reflection and transmission in integrable impurity systems'. 2002.
`arXiv:hep-th/0201142 [hep-th]`.

[CK15] V. Caudrelier and A. Kundu. 'A multisymplectic approach to defects in integrable classical field theory'. In: *JHEP* 02 (2015), p. 088.
`arXiv:1411.0418 [math-ph]`.

[CMR04] V. Caudrelier, M. Mintchev and E. Ragoucy. 'The quantum non-linear Schrodinger model with point-like defect'. In: *J. Phys.* A37 (2004), pp. L367–L376.
`arXiv:hep-th/0404144 [hep-th]`.

[CMRS05] V. Caudrelier, M. Mintchev, E. Ragoucy and P. Sorba. 'Reflection and transmission quantum Yang-Baxter equations'. In: *J. Phys. A* 38.15 (2005), p. 3431.
arXiv:hep-th/0412159 [hep-th].

[CP16] E. Corrigan and R. Parini. 'Type I integrable defects and finite-gap solutions for KdV and sine-Gordon models'. In: *a* a.a (2016), a.
arXiv:1612.06904 [hep-th].

[CZ04] E. Corrigan and C. Zambon. 'Aspects of sine-Gordon solitons, defects and gates'. In: *J. Phys.* A37 (2004), p. L471.
arXiv:hep-th/0407199 [hep-th].

[CZ06] E. Corrigan and C. Zambon. 'Jump-defects in the nonlinear Schrodinger model and other non-relativistic field theories'. In: *Nonlinearity* 19 (2006), pp. 1447–1469.
arXiv:nlin/0512038 [nlin-si].

[CZ07] E. Corrigan and C. Zambon. 'On purely transmitting defects in affine Toda field theory'. In: *JHEP* 07 (2007), p. 001.
arXiv:0705.1066 [hep-th].

[CZ09a] E. Corrigan and C. Zambon. 'A new class of integrable defects'. In: *J. Phys.* A42 (2009), p. 475203.
arXiv:0908.3126 [hep-th].

[CZ09b] E. Corrigan and C. Zambon. 'Comments on defects in the a(r) Toda field theories'. In: *J. Phys.* A42 (2009), p. 304008.
arXiv:0902.1307 [hep-th].

[CZ10] E. Corrigan and C. Zambon. 'A transmission matrix for a fused pair of integrable defects in the sine-Gordon model'. In: *J. Phys.* A43 (2010), p. 345201.
arXiv:1006.0939 [hep-th].

[CZ11] E. Corrigan and C. Zambon. 'Integrable defects in affine Toda field theory and infinite dimensional representations of quantum groups'. In: *Nucl. Phys.* B848 (2011), pp. 545–577. `arXiv:1012.4186 [hep-th]`.

[Dav77] A. S. Davydov. 'Solitons and energy transfer along protein molecules'. In: *J. Theor. Biol.* 66 (1977), p. 379.

[DB76] R. K. Dodd and R. K. Bullough. 'Bäcklund transformations for the sine-Gordon equations'. In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 351.1667 (1976), pp. 499–523.

[DMS94a] G. Delfino, G. Mussardo and P. Simonetti. 'Scattering theory and correlation functions in statistical models with a line of defect'. In: *Nucl. Phys.* B432 (1994), pp. 518–550. `arXiv:hep-th/9409076 [hep-th]`.

[DMS94b] G. Delfino, G. Mussardo and P. Simonetti. 'Statistical models with a line of defect'. In: *Phys. Lett.* B328 (1994), pp. 123–129. `arXiv:hep-th/9403049 [hep-th]`.

[Doi12] A. Doikou. 'Defects in the discrete non-linear Schrodinger model'. In: *Nucl. Phys.* B854 (2012), pp. 153–165. `arXiv:1106.1602 [hep-th]`.

[Doi15] A. Doikou. 'Jumps and twists in affine Toda field theories'. In: *Nucl. Phys.* B893 (2015), pp. 107–121. `arXiv:1407.7777 [hep-th]`.

[Doi16] A. Doikou. 'Classical integrable defects as quasi Bäcklund transformations'. In: *Nucl. Phys.* B911 (2016), pp. 212–230. `arXiv:1603.04688 [hep-th]`.

[Dor91] P. Dorey. 'Root systems and purely elastic S-matrices'. In: *Nucl. Phys. B* 358.3 (1991), pp. 654–676.

[Dor92] P. E. Dorey. 'Root systems and purely elastic S-matrices (II)'. In: *Nucl. Phys. B* 374.3 (1992), pp. 741–761. arXiv:hep-th/9110058 [hep-th].

[FG80] A. P. Fordy and J. Gibbons. 'Integrable nonlinear Klein-Gordon equations and Toda lattices'. In: *Comm. Math. Phys.* 77.1 (1980), pp. 21–30.

[For90] A. P. Fordy, ed. *Soliton theory: A survey of results*. Manchester University Press, 1990.

[FT86] L. D. Faddeev and L. A. Takhtajan. *Hamiltonian method in the theory of solitons*. Springer-Verlag, 1986.

[G L93] E. M. W. G. Lenz P. Meystre. 'Nonlinear atom optics'. In: *Phys. Rev. Lett.* 71 (1993), p. 3271.

[Geo99] H. Georgi. *Lie algebras in particle physics*. Westview Press, 1999.

[GGKM67] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura. 'Method for Solving the Korteweg-deVries Equation'. In: *Phys. Rev. Lett.* 19 (19 Nov. 1967), pp. 1095–1097.

[Hal94] R. A. Hall. 'Affine Toda solitons and fusing rules'. PhD thesis. Durham University, 1994.

[Hir71] R. Hirota. 'Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons'. In: *Phys. Rev. Lett.* 27 (18 Nov. 1971), pp. 1192–1194.

[Hol92] T. J. Hollowood. 'Solitons in affine Toda field theories'. In: *Nucl. Phys.* B384 (1992), pp. 523–540. arXiv:hep-th/9110010 [hep-th].

[Jun+14] N. R. Jungwirth, Y. Y. Pai, H. S. Chang, E. R. MacQuarrie, K. X. Nguyen and G. D. Fuchs. 'A single-molecule approach to ZnO defect studies: Single photons and single defects'. In: *J. Appl. Phys.*

116.4 (2014), p. 043509.
`arXiv:1402.1773 [cond-mat]`.

[KL99] R. Konik and A. LeClair. 'Purely transmitting defect field theories'. In: *Nucl. Phys.* B538 (1999), pp. 587–611.
`arXiv:hep-th/9703085 [hep-th]`.

[Lam67] G. Lamb. 'Propagation of ultrashort optical pulses'. In: *Phys. Lett. A* 25.3 (1967), pp. 181–182.

[Lax68] P. D. Lax. 'Integrals of Nonlinear Equations of Evolution and Solitary Waves'. In: *Commun. Pure Appl. Math.* 21 (1968), pp. 467–490.

[LOT93] H. C. Liao, D. Olive and N. Turok. 'Topological solitons in Ar affine Toda theory'. In: *Phys. Lett. B* 298.1 (1993), pp. 95–102.

[McG94a] W. A. McGhee. 'The topological charges of the a-n(1) affine Toda solitons'. In: *Int. J. Mod. Phys.* A9 (1994), pp. 2645–2666.
`arXiv:hep-th/9307035 [hep-th]`.

[McG94b] W. A. McGhee. 'On the topological charges of the affine toda solitons'. PhD thesis. Durham University, 1994.

[Mik79] A. V. Mikhailov. 'Integrability of the two-dimensional generalization of Toda chain'. In: *JETP Letters* 30 (1979), p. 414.

[Miu76] R. M. Miura, ed. *Bäcklund transformations, the inverse scattering method, solitons, and their applications.* Springer-Verlag, 1976.

[MM93] N. J. MacKay and W. A. McGhee. 'Affine Toda solitons and automorphisms of Dynkin diagrams'. In: *Int. J. Mod. Phys.* A8 (1993). [Erratum: Int. J. Mod. Phys.A8,3830(1993)], pp. 2791–2807.
`arXiv:hep-th/9208057 [hep-th]`.

[MOP81] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov. 'Two-dimensional generalized Toda lattice'. In: *Comm. Math. Phys.* 79.4 (1981), pp. 473–488.

[MP80] B. M. McCoy and J. H. H. Perk. 'Two-Spin Correlation Functions of an Ising Model with Continuous Exponents'. In: *Phys. Rev. Lett.* 44 (13 Mar. 1980), pp. 840–844.

[MRS02] M. Mintchev, E. Ragoucy and P. Sorba. 'Scattering in the presence of a reflecting and transmitting impurity'. In: *Phys. Lett.* B547 (2002), pp. 313–320.
`arXiv:hep-th/0209052 [hep-th]`.

[MRS03] M. Mintchev, E. Ragoucy and P. Sorba. 'Reflection transmission algebras'. In: *J. Phys.* A36 (2003), p. 10407.
`arXiv:hep-th/0303187 [hep-th]`.

[Nim83] J. Nimmo. 'A bilinear Bäcklund transformation for the nonlinear Schrödinger equation'. In: *Phys. Lett. A* 99.6 (1983), pp. 279–280.

[OT83a] D. Olive and N. Turok. 'The symmetries of Dynkin diagrams and the reduction of Toda field equations'. In: *Nucl. Phys. B* 215.4 (1983), pp. 470–494.

[OT83b] D. Olive and N. Turok. 'Algebraic structure of Toda systems'. In: *Nucl. Phys. B* 220.4 (1983), pp. 491–507.

[OT85] D. Olive and N. Turok. 'Local conserved densities and zero-curvature conditions for Toda lattice field theories'. In: *Nucl. Phys. B* 257 (1985), pp. 277–301.

[OT86] D. Olive and N. Turok. 'The Toda lattice field theory hierarchies and zero-curvature conditions in Kac-Moody algebras'. In: *Nucl. Phys. B* 265.3 (1986), pp. 469–484.

[PS96] S. Pratik Khastgir and R. Sasaki. 'Non-canonical folding of Dynkin diagrams and reduction of affine Toda theories'. In: *Prog. Theor. Phys.* 95 (1996), pp. 503–518.
`arXiv:hep-th/9512158 [hep-th]`.

[Rob14a] C. Robertson. 'Defect fusing rules in affine Toda field theory'. In: *J. Phys.* A47.48 (2014), p. 485205.
`arXiv:1408.1960 [hep-th]`.

[Rob14b] C. Robertson. 'Folding defect affine Toda field theories'. In: *J. Phys.* A47 (2014), p. 185201.
`arXiv:1304.3129 [hep-th]`.

[Rob15] C. Robertson. 'On defects in affine Toda field theory'. PhD thesis. Durham University, 2015.

[RS02] C. Rogers and W. K. Schief. *Bäcklund and Darboux transformations: Geometry and modern applications in soliton theory.* Cambridge University Press, 2002.

[RV04] A. V. Rybin and I. P. Vadeiko. 'Manipulation of optical solitons in Bose-Einstein condensates'. In: *J. Opt. B* 6.10 (2004), p. 416.
`arXiv:quant-ph/0407142 [quant-ph]`.

[Sam90] H. Samelson. *Notes on Lie algebras.* Springer-Verlag, 1990.

[SCM73] A. C. Scott, F. Y. F. Chu and D. W. McLaughlin. 'The soliton: A new concept in applied science'. In: *Proceedings of the IEEE* 61.10 (Oct. 1973), pp. 1443–1483.

[Sem83] M. A. Semenov-Tian-Shansky. 'What is a classical r-matrix?' In: *Funct. Anal. Appl.* 17 (1983). [Funkt. Anal. Pril.17N4,17(1983)], pp. 259–272.

[Skl80] Sklyanin. 'On complete integrability of the Landau-Lifshitz equation'. In: *Leningrad preprint LOMI* (1980).

[Sky61] T. H. R. Skyrme. 'Particle States of a Quantized Meson Field'. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 262.1309 (1961), pp. 237–245.

[Tod70] M. Toda. ‘Waves in nonlinear lattice’. In: *Prog. Theor. Phys. Supplement* 45 (1970), pp. 174–200.

[Ump08] J. Umpleby. ‘Complex sine-Gordon theory: Solitons, defects and boundaries’. PhD thesis. Durham University, 2008.

[Ust98] A. V. Ustinov. ‘Solitons in Josephson junctions’. In: *Physica D Nonlinear Phenomena* 123 (Nov. 1998), pp. 315–329.

[WE73] H. D. Wahlquist and F. B. Estabrook. ‘Bäcklund Transformation for Solutions of the Korteweg-de Vries Equation’. In: *Phys. Rev. Lett.* 31 (23 Dec. 1973), pp. 1386–1390.

[Wil81] G. Wilson. ‘The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras’. In: *Ergod. Theor. Dynam. Syst.* 1 (1981), p. 361.

[ZF71] V. E. Zakharov and L. D. Faddeev. ‘Korteweg-de Vries equation: A completely integrable Hamiltonian system’. In: *Funct. Anal. Appl.* 5.4 (1971), pp. 280–287.

[ZM74] V. Zakharov and S. Manakov. ‘On the complete integrability of a nonlinear Schrödinger equation’. In: *Theoretical and Mathematical Physics(Russian Federation)* 19.3 (June 1974), pp. 551–559.

[ZS72] V. E. Zakharov and A. B. Shabat. ‘Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Waves in Nonlinear Media’. In: *Soviet Journal of Experimental and Theoretical Physics* 34 (1972), p. 62.

[ZZ79] A. B. Zamolodchikov and A. B. Zamolodchikov. ‘Factorized S matrices in two-dimensions as the exact solutions of certain relativistic quantum field models’. In: *Annals Phys.* 120 (1979), pp. 253–291.