ALGEBRAIC SYNTHESIS METHODS FOR LINEAR MULTIVARIABLE CONTROL SYSTEMS

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DECLARATION

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ABSTRACT

The mathematical formulation of various control synthesis problems , (such as Decentralized Stabilization Problem , (DSP) , Total Finite Settling Time Stabilization for discrete time linear systems , (TFSTS) , Exact Model Matching Problem , (EMMP), Decoupling and Noninteracting Control Problems) , via the algebraic framework of Matrix Fractional Representation . (MFR) – i.e. the representation of the transfer matrices of the system as matrix fractions over the ring of interest – results to the study of matrix equations over rings , such as :

$$A \cdot X + B \cdot Y = C , (X \cdot A + Y \cdot B = C) \tag{1}$$
$$A \cdot X = B , (Y \cdot A = B) \tag{2}$$
$$A \cdot X \cdot B = C \tag{3}$$
$$A \cdot X + Y \cdot B = C , X \cdot A + B \cdot Y = C ,$$
$$A \cdot X \cdot B + C \cdot Y \cdot D = E \tag{4}$$

The main objective of this dissertation is to further investigate conditions for existence and characterization of certain types of solutions of equation (1) ; develop a unifying algebraic approach for solvability and characterization of solutions of equations (1) – (4), based on structural properties of the given matrices , over the ring of interest .

The standard matrix Diophantine equation (1) is associated with the TFSTS for discrete time linear systems and issues concerning the characterization of solutions according to the Extended McMillan Degree , (EMD) , (minimum EMD , or fixed EMD) , of the stabilizing controllers they define , are studied . A link between the issues in question and topological properties of certain families of solutions of (1) is established . Equation (1) is also studied in association with the DSP and Diagonal DSP (DDSP) , for continuous time linear systems . Conditions for characterizing block diagonal solutions of (1) , (which define decentralized stabilizing controllers) , are derived and a closed form description of the families of diagonal and two blocks diagonal decentralized stabilizing controllers is introduced .

The set of matrix equations (1) – (4) is assumed over the field of fractions of the ring of interest , $\mathfrak{R}$, (mainly a Euclidean Domain , (ED) , and thus a Principal Ideal Domain , (PID)) , and solvability as well as parametrization of solutions over $\mathfrak{R}$ is investigated under the unifying algebraic framework of extended non square matrix divisors , projectors and annihilators of the known matrices over $\mathfrak{R}$. In practice the ring of interest is either the ring of polynomials $\mathbf{R}[s]$, or the rings of proper $\mathbf{R}_{pr}(s)$ and especially proper and stable rational functions $\mathbf{R}_{\mathfrak{P}}(s)$. The importance of $\mathbf{R}_{\mathfrak{P}}(s)$ is highlighted early in the thesis and further computational issues arising from its structure as an ED are considered .

NOTATION AND ABBREVIATIONS

The following notation and abbreviations are used throughout this thesis unless otherwise is stated in the text :

– $\mathbb{N}$	: the set of natural numbers
– $\mathbb{Q}$	: the field of rational numbers
– $\mathbb{R}$	: the field of real numbers
– $\mathbb{C}$	: the field of complex numbers
– $\mathbb{C}_+$	: the right half plane of the complex numbers
– $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$	: the area of instability of linear , continuous time , control systems
– $\mathbb{R}[s]$	: the ring of polynomials
– $\mathbb{R}(s)$	: the field of rational functions
– $\mathbb{R}_{pr}(s)$	: the ring of proper rational functions
– $\mathbb{R}_{\mathcal{P}}(s)$	: the ring of proper and $\mathcal{P}$ stable rational functions
– $\mathbb{R}^{pxm}(s)$	: the set of *pxm* matrices with entries over $\mathbb{R}(s)$
– $\mathbb{R}_{pr}^{pxm}(s)$	: the set of *pxm* matrices with entries over $\mathbb{R}_{pr}(s)$
– $\mathbb{R}_{\mathcal{P}}^{pxm}(s)$	: the set of *pxm* matrices with entries over $\mathbb{R}_{\mathcal{P}}(s)$
– $\|\|\cdot\|\|_p$	: a norm function over the ring of polynomials
– ϱ_M	: a matrix metric defined over a set of matrices
– $\gamma_{\mathcal{P}}$	: the Euclidean degree of the Euclidean domain $\mathbf{R}_{\mathcal{P}}(s)$
– δ^*	: the extended McMillan degree
– $\underline{v}$	: the vector v
– $\mathcal{S}_A^r$	: row span of {A} over a field = row space of A over a field
– $\mathcal{S}_A^c$	: column span of {A} over a field = column space of A over a field
– $\mathcal{N}_r\{A\}$	: right null space of A
– $\mathcal{N}_l\{A\}$	: left null space of A
– $\mathcal{M}_A^r$	: row span {A} over a ring = row module

of A over a ring

- $\mathcal{M}^c_A$: column span {A} over a ring = column module of A over a ring
- $\widehat{\mathcal{M}}^r_A$: the maximum row module of A in $\mathcal{G}^r_A$
- $\widehat{\mathcal{M}}^c_A$: the maximum column module of A in $\mathcal{G}^c_A$
- $diag\{C_1, \dots, C_n\}$: a block diagonal matrix , with blocks C_i , $i = 1, \dots, n$
- $\mathcal{K}[x_1, \dots, x_n]$: the ring of polynomials in $x_1, \dots, x_n$ with coefficients in the field $\mathcal{K}$
- $\mathcal{V}(f_1, \dots, f_s)$: the affine variety by $f_1, \dots, f_s$, $f_i \in \mathcal{K}[x_1, \dots, x_n]$
- $>_{lex}$: the lexicographical order over $\mathbb{N}^n$
- A E_r B : the matrices A , B are right equivalent
- A E_l B : the matrices A , B are left equivalent
- A E B : the matrices A , B are equivalent
- BIBO : bounded input , bounded output
- CSP : centralized stabilization problem
- DSP : decentralized stabilization problem
- DDSP : diagonal decentralized stabilization problem
- DBRP : dead – beat response problem
- DDP : disturbance decoupling problem
- DDISP : disturbance decoupling with internal stability problem
- EMMP : exact model matching problem
- EMD : extended McMillan degree
- ED : Euclidean domain
- eld : extended left divisor
- erd : extended right divisor
- GCD : greatest common divisor
- gcerd : greatest common extended right divisor
- gceld : greatest common extended left divisor
- glrd : greatest left – right divisor
- geld : greatest extended left divisor
- gerd : greatest extended right divisor
- lrd : left – right divisor
- MDE : matrix Diophantine equation
- MDP : minimal design problem
- MIMO : many inputs , many outputs

- MFR : matrix fractional representation
- MFD : matrix fractional description
- NICP : noninteracting control problem
- NCISP : noninteracting control with internal stability problem
- PMDE : polynomial matrix Diophantine equation
- PID : principal ideal domain
- RP : regulator problem
- RPIS : regulator problem with internal stability
- $\Re$cp : $\Re$ column projector
- $\Re$rp : $\Re$ row projector
- $\Re$pra : $\Re$ prime right annihilator
- $\Re$pla : $\Re$ prime left annihilator
- $\Re$ri : $\Re$ right inverse
- $\Re$li : $\Re$ left inverse
- $\Re$mr : $\Re$ multiple of the rows
- $\Re$lmr : $\Re$ least multiple of the rows
- $\Re$cmr : $\Re$ common multiple of the rows
- $\Re$lcmr : $\Re$ least common multiple of the rows
- $\Re$mc : $\Re$ multiple of the columns
- $\Re$lmc : $\Re$ least multiple of the columns
- $\Re$cmc : $\Re$ common multiple of the columns
- $\Re$lcmc : $\Re$ least common multiple of the columns
- SEMMP : stable exact model matching problem
- SISO : single input , single output
- TFSTS : total finite settling time stabilization
- VDE : vector Diophantine equation

CHAPTER 1

INTRODUCTION

This dissertation is concerned with linear algebraic synthesis methods for linear, multivariable , time invariant , control systems and additional algebraic tools are developed on matrix divisors , projectors , annihilators , in order to achieve a unifying approach for solvability of certain types of matrix equations . It is well known that algebraically many control synthesis problems are reduced to the solution of , (sets of) , matrix equations such as :

$$A \cdot X + B \cdot Y = C \, , \, (X \cdot A + Y \cdot B = C) \tag{1.1}$$

$$A \cdot X = B \, , \, (Y \cdot A = B) \tag{1.2}$$

$$A \cdot X \cdot B = C \tag{1.3}$$

$$\sum_{i=1}^{n} A_i \cdot X_i \cdot B_i = C \tag{1.4}$$

where , A , B , A_i , B_i , C , X , Y , X_i , are matrices over the ring of interest , i.e. a given Euclidean domain , (ED) , or principal ideal domain , (PID) . The main aim of this thesis is to further investigate conditions for existence and characterization of special types of solutions of equations (1.1) ; develop a unifying algebraic approach for solvability and parametrization of solutions of equations (1.1) – (1.4) , based on the structural properties of a matrix over a PID . Recent work in this area is based on what is termed the Matrix Fractional Representation approach , (MFR) , to linear systems theory , [Des. 1] , [Sae. 2] , [Ant. 1] , [Vid. 1] , [Vid. 3] , [Vid. 4] , [Fra. 1] , [Ozg. 1] , [Bra. 1] , [Kal. 1] , [Kuc. 2] , [Var. 6] . The motivation to study matrices having elements in special rings , comes from the need to describe algebraically the familiar problems of stability , realizability and performance of linear systems .

From a control theory viewpoint the rings of importance are , $\mathbb{R}[s]$ – polynomials , $\mathbb{R}_{pr}(s)$ – própper rational functions , $\mathbb{R}_{\mathcal{P}}(s)$ – proper rational functions with no poles inside a prescribed region $\mathcal{P}$ of the complex plain . The structure of the set $\mathbb{R}_{\mathcal{P}}(s)$ has been investigated in [Var. 3] , [Var. 5] , [Vid. 4] , and structural as well as invariant aspects of it have been defined . Among the algebraic properties of $\mathbb{R}_{\mathcal{P}}(s)$, the one that makes it more interesting is that of the Euclidean ring or in other words , the existence of a Euclidean division . In [Vid. 4] , [Var. 5] , has been noticed that the pair of quotient and remainder of a Euclidean division in $\mathbb{R}_{\mathcal{P}}(s)$ is not characterized by a uniquely defined "Euclidean degree" , and the family of least possible "Euclidean degree" remainders is introduced . A quite tedious construction of this family based on the interpolation theorem of [You. 1] , is known , [Vid. 4] . An existence approach by using interpolation in a disc algebra has been introduced in [Vid. 4] . Further computational issues concerning the construction of more practical algorithms for the determination of the family in question are studied here . The role of $\mathbb{R}_{\mathcal{P}}(s)$ and $\mathbb{R}_{pr}(s)$ as the rings of interest in the case of linear , multivariable , continuous time , time invariant systems is taken over by $\mathbb{R}[s]$ in the case of linear , multivariable , discrete time , time invariant systems. The basic control schemes consisting of a precompensator , (or feedback compensator) and unity output feedback which are used to stabilize unstable plants , always lead to the study of a matrix Diophantine equation , (MDE) , of the type (1.1) over the ring of interest , ($\mathbb{R}_{\mathfrak{P}}(s)$ for continuous time , $\mathbb{R}[s]$ for discrete time linear systems) . In our study we associate the MDE (1.1) with the following two control synthesis problems :

i) The Total Finite Settling Time Stabilization , (TFSTS) , for discrete time linear systems .

ii) The Decentralized Stabilization Problem , (DSP) , for continuous time linear systems.

The TFSTS requires all the internal and external variables , (signals) , of the system to settle to a new steady – state after finite time from the application of a step change to its input and for every initial condition , [Kar. 1] , [Mil. 1] . The TFSTS comprises the Dead – Beat Response Problem , (DBRP) , i.e. the forcing of the state or output vector from any initial state to the origin in minimum time , [Ber. 1] , [Ise. 1] , [Kal. 1] , [Kuo. 1] , [Kuc. 1] – [Kuc. 8] , [Vid. 4] . The TFSTS and DBRP can be viewed as a type of Minimal Design Problems , (MDP) , i.e. as problems requiring the investigation of existence and parametrization of solutions of the corresponding MDE (1.1) , over $\mathbb{R}[s]$, which define stabilizing controllers with minimum number of finite and infinite poles , (minimum extended McMillan degree , (EMD)) , among the family of all stabilizing controllers . In our approach , in order to determine the required family of solutions of equation (1.1) , over $\mathbb{R}[s]$, we first focus on those solutions , (X , Y) , that correspond to column , (row) , reduced matrices $[X^T \vdots Y^T]^T$, ($[X \vdots Y]$) . We are motivated to do so by the fact that the EMD of a controller defined by a column , (row) , reduced solution of (1.1) , is equal to the sum of column , (row) , polynomial degrees of the corresponding matrices $[X^T \vdots Y^T]^T$, ($[X \vdots Y]$) , [Var. 5] , [Mil. 1] .

We prove that the solutions in question form a nonempty , dense but neither open nor closed subset of the family of solutions of (1.1) , (with C an arbitrary $\mathbb{R}[s]$ – unimodular matrix) , and thus the sum of minimum column , (row) , polynomial degrees of the corresponding matrices $[X^T \vdots Y^T]^T$, ($[X \vdots Y]$) , are more likely to serve as an upper bound rather than be equal to the minimum EMD of the corresponding controllers $X^{-1} \cdot Y$, ($Y \cdot X^{-1}$) . By transforming (1.1) to Vector Diophantine equations , (VDE) , over $\mathbb{R}[s]$, using the exterior product expressions of the rows , (columns) , columns , (rows) , of $[A \vdots B]$, ($[A^T \vdots B^T]^T$) , $[X^T \vdots Y^T]^T$, ($[X \vdots Y]$) , respectively and then expressing (1.1) and the corresponding VDEs via their Toeplitz matrix representations we can construct reliable bounds for the minimum EMD , i.e. the minimum EMD is bound between the sum of minimum column , (row) , polynomial degrees of $[X^T \vdots Y^T]^T$, ($[X \vdots Y]$) , and the minimum column , (row) , polynomial degree of the vector solutions of the VDE corresponding to (1.1) . A parametrization of the families of controllers corresponding to the upper and lower bounds is given .

A different stabilization problem is the DSP for continuous time linear systems . This problem is due to restrictions on the feedback compensator structure , which are often encountered in large scale systems . These systems have several local control stations ; each local compensator observes only the corresponding local outputs . Such decentralized control of systems results in a block diagonal compensator matrix scheme [San. 1] , [Gun. 1] , [Wan. 1] . Thus the DSP requires the stabilization of an unstable system by using a decentralized compensator and unity output feedback scheme . Wang and Davison , [Wan. 1] and Corfmat and Morse , [Cor. 1] , [Cor. 2] , have introduced synthesis methods for the design of stabilizing decentralized compensators . It has been derived that a necessary and sufficient condition for the existence of local control lows with dynamic compensation to stabilize a given system is that the system has no "fixed modes" , [Wan. 1] , over the region of instability . Further study of the problem has been done in [And. 1] , [And. 2] , [Vid. 3] , [Guc. 1] , [Ozg. 1] , [Kar. 2] , [Kar. 3] . In [Gun. 1] , the DSP is treated within the algebraic framework of Matrix Fractional Representation of the plant and controller transfer matrices over $\mathbb{R}_{\mathfrak{P}}(s)$. A solution of the DSP is constructed but a closed form parametrization of all decentralized stabilizing controllers is not given .

Our interest is to examine equation (1.1) in the algebraic framework already established and try to derive new results concerning the remaining open parametrization issues of the DSP . More precisely , if (A , B) denotes a coprime left Matrix Fractional Representation of the plant transfer matrix over $\mathbb{R}_{\mathfrak{P}}(s)$, T_i , are matrices formed by the p_i , m_i columns of the partitioning of A , B according to the number of local inputs – outputs respectively, then the parametrization of solutions of the DSP can be derived from the family of $\mathbb{R}_{\mathfrak{P}}(s)$ – left unimodular solutions , X_i , of the set of equations $T_i \cdot X_i = C_i$, $i = 1 , \dots , \kappa$ for which $[C_1 , \dots , C_\kappa]$ is $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular. In our study we show that the above parametrization requires the existence of a constructive method that enables us to generate the family of all $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular matrices of given dimensions , as well as , the families of $\mathbb{R}_{\mathfrak{P}}(s)$ – left , right unimodular matrices which complete given $\mathbb{R}_{\mathfrak{P}}(s)$ – left , right unimodular matrices to square $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular . Such methods are examined and a parametrization of solutions of the DSP is introduced.

The parameters are expressed in terms of upper , lower triangular unimodular matrices which must satisfy certain constraints . These constraints introduce a necessary and sufficient criterion that enables us to identify the admissible parameters . Although in the general case the family of qualifying parameters is not described in closed form , there are particular cases when this is possible . These cases are based on the property , [Vid. 4] , of the Smith forms of T_i over $\mathbb{R}_{\mathfrak{P}}(s)$ to be generic . A closed form description of the family of parameters is given in the case of two blocks decentralized stabilizing controllers .

A special case of decentralized stabilization of continuous time linear systems , for which a complete parametrization of stabilizing controllers can be achieved using a different approach , is the Diagonal Decentralized Stabilization problem , (DDSP) , [Kar. 2] , [Guc. 1] . In this special case , given a plant transfer matrix over $\mathbb{R}_{pr}^{pxp}(s)$, the problem is to determine a stabilizing compensator $C = diag\{c_1 , \dots , c_p\}$ over $\mathbb{R}_{pr}^{pxp}(s)$, such that the feedback system is stabilized by C . As in the case of the DSP the stability requirement may be expressed in terms of Matrix Fractional Representations of transfer matrices [Vid. 4] , and highlights the important role of "fixed modes" over the region of instability , [Wan. 1] , [And. 1] , [And. 2] . The existence and characterization of solutions of the DDSP is intimately related to systems that exhibit the property of cyclicity . After formulating the DDSP in a similar manner to the DSP , the construction of the family of all diagonal stabilizing controllers is reduced to determining what are termed mode T mutually stabilizing pairs . The existence of such pairs forms the base of a complete characterization of the family of diagonal stabilizing controllers . This characterization is essential , since it provides the means to define certain diagonal stabilizing controllers , such as proper , reliable , stable .

Notice that equation (1.1) is a special case of the more general equation (1.2) . Furthermore equation (1.2) is central to the formulation of the Exact Model Matching , (EMM) and Stable Exact Model Matching , (SEMM) , problems . The EMM requires the existence and characterization of proper solutions of (1.2) , when A , B are given matrices over $\mathbb{R}_{pr}(s)$, [Wol 1] , [Wol. 3] , [Var 5] , [Var. 6] , [For 1] . If the requirement that the solutions of (1.2) should be stable is added then we define the SEMM problem , [Wol. 3] [Sco 1] , [And. 3] , [Kuc. 9] , [Emr. 1] , [Kar. 5] , [Per. 1] . Equations (1.3) and (1.4) , (the last in the reduced form $A \cdot X + Y \cdot B = C$, $X \cdot A + B \cdot Y = C$) , appear in the formulation of a group of control synthesis problems known as Noninteracting , or Decoupling Control Problems . There are many different versions of such problems , depending on the control feedback configurations postulated . These are problems which require the existence and characterization of controllers that achieve certain outputs to be independent of certain inputs , or the transfer matrices of certain input – output channels to meet prespecified constraints , such as stability . Internal stability of the feedback scheme is quite often an additional requirement . We distinguish between the Disturbance Decoupling , (DDP) , and Disturbance Decoupling with Internal Stability , (DDISP) , Problems , [Aka. 1] , [Mor. 3] , [Ohm. 1] , [Ozg. 1] , [Ozg. 2] , [Sch. 1] , [Sto. 1] , [Wol. 4] , [Won. 1] , [Wil. 2] , [Tak. 1] ; the Noninteractive Control , (NICP) , and Noninteractive Control with Internal Stability , (NCISP) , Problems , [Aka. 1] , [Aka. 2] , [Bay. 1] , [Dsc. 1] , [Fal. 1] , [Ham. 1] , [Mrg. 1] , [Mor. 3] , [Wil. 1] , [Wol. 1] , [Won. 1]. Some additional problems to the above concerning especially equation (1.4) are the Regulator Problem , (RP) , and Regulator Problem with Internal Stability , (RPIS) , [Ben. 1] , [Chg. 1] , [Hau. 1] , [Kha. 1] , [Sae. 1] , [Sch. 2] , [Sch. 3] , [Wol. 5] , [Won. 1] , [Won. 2] , [Won. 3] . The first , second , fifth and sixth problems , i.e. DDP , DDISP , RP , RPIS are considered over a two vector channel , continuous time , linear system with feedback applying round the first channel . The first channel input – output is referred to as the control input – measured output , whereas the second one as the disturbance input – controlled output . For the third and forth problems , i.e. NICP , NCISP , a three vector channel , continuous time linear system with feedback applying around the first channel is postulated .

From establishing the existence of an intimate relation between certain control synthesis problems and matrix equations so far , the need for developing a unifying algebraic framework for treating these equations is motivated . In our attempt to do so the given matrices A , B , A_i , B_i , C , in (1.1) – (1.4) , are considered over the field of fractions of an arbitrary PID , whereas the unknown X , Y , X_i , are required to be over this PID . The approach of solving matrix equations within the same algebraic framework is based on the structural properties of matrices over PIDs . More precisely , if a matrix over a given PID , $\mathfrak{R}$, is considered , then certain algebraic tools over $\mathfrak{R}$ such as , greatest left – right divisors , nonsquare left – right divisors , projectors , annihilators , left – right inverses can be defined ; whereas if a matrix over the field of fractions of $\mathfrak{R}$ is given , an extension of the notions of common and least common multiplies of its rows , columns is introduced . Then the structural properties of a matrix over $\mathfrak{R}$ can be investigated via these algebraic tools . The solvability conditions and parametrization of solutions of (1.1) – (1.4) can be expressed in terms of greatest left – right divisors , projectors and left – right inverses, over the PID of interest $\mathfrak{R}$, of the given matrices along with parametric matrices over $\mathfrak{R}$.

The structure of this thesis and the organization of the material are developed as follows :

Chapter 2 is a survey of control synthesis problems and matrix equations that emerge in their mathematical formulation . In section 2.2 we briefly present the concept of stability of linear systems and the relation between the notions of internal and external stability . Stability is a very important requirement in all the control problems we deal with and in general it is an essential qualitative property of linear control systems , since there is great danger for an unstable system to "burst" as time goes to infinity . In sections 2.3 and 2.4 we review the classical control synthesis problems of Centralized and Decentralized Stabilization , the solution of which can be reduced to the study of solvability and characterization of solutions , (or special types of them) , of the standard matrix Diophantine equation (1.1) , over $\mathbf{R}_{\mathfrak{sp}}(s)$. In section 2.5 we review the Exact Model Matching and Stable Exact Model Matching Problems , central role in the formulation of which is played by the matrix equation (1.2) over $\mathbf{R}_{pr}(s)$, $\mathbf{R}_{\mathfrak{sp}}(s)$ respectively . In section 2.6 we switch to a group of control synthesis problems known as

Noninteracting , or Decoupling Control Problems . We present various case of them , (Disturbance Decoupling and Disturbance Decoupling with Internal Stability , Noninteracting Control . and Noninteracting Control with Internal Stability) , and we associate them with the solvability of the matrix equation (1.3) . Finally , in section 2.7 we consider the Regulator Problem and Regulator Problem with Internal Stability that gives rise to a special case of the matrix equation (1.4) , i.e. the equations $A \cdot X + Y \cdot B = C$, $X \cdot A + B \cdot Y = C$, $A \cdot X \cdot B + C \cdot Y \cdot E = F$) .

Chapter 3 is concerned with computational issues of the set of proper and $\mathfrak{P}$ stable rational functions , $\mathbb{R}_{\mathfrak{P}}(s)$. Our aim is to give an algorithmic construction of the family of least "Euclidean degree" remainders , bysteping the existing tedious one that can be found in [You. 1] . Our effort is based on the approach introduced in [Vid. 4] for the determination of the existence of a family of least "Euclidean degree" remainders . The construction of such a family is not presented there . More precisely , in section 3.2 the ring of proper and stable functions is introduced ; in section 3.3 a unique , modulo a real number of $\mathfrak{P}^c$, factorization for the elements of $\mathbb{R}_{\mathfrak{P}}(s)$ is introduced and in section 3.4 the Euclidean division as well as its non uniqueness of remainder is examined . The motivation for the use of unit interpolation in the following sections is given at the end .

In section 3.5 the interpolation by unit over $\mathbb{R}_{\mathfrak{P}}(s)$ is examined , by using the concept of the logarithm of an element of a Banach Algebra and introducing a special type of Banach algebra the Disc Algebra of symmetric analytic functions , which map a disc onto $\mathbb{C}$. Two approaches for the derivation of an interpolating unit over $\mathbb{R}_{\mathfrak{P}}(s)$ are given and lead to two algorithmic constructions of the least "Euclidean degree" family of remainders in section 3.6 . A comparison between the two methods gives the more efficient one . Finally , in section 3.7 a generalization of the Euclidean division between square matrices with entries proper and stable functions , [Vid. 4] , is presented . As an application of the knowledge of the family of least "Euclidean degree" remainders of a Euclidean division between two elements of $\mathbb{R}_{\mathfrak{P}}(s)$, the construction of the least number of unstable poles family of stabilizing controllers is described .

In Chapter 4 an alternative method for the computation of the greatest common divisor , (GCD) , of a set of polynomials is studied . The notions of common and GCDs of sets of polynomials are basic mathematical tools underlying the definitions and properties of concepts , such as multivariable zeroes , [Mac. 1] , decoupling zeroes , [Ros. 1] , of linear systems theory . These concepts are central in the computation of tools such as Smith forms , Hermit forms matrix divisors etc. of the algebraic systems theory , [Kai. 1] , [Kuc. 1] , etc . The computation of the GCD , f(s) , of a set of m polynomials of $\mathbb{R}[s]$, $\underline{p}(s)$, of a maximal degree δ , has attracted a lot of attention , [Bar. 1] , [Bar. 2] , [Kai. 1] , [Kar. 7] , [Kar. 8] , [Mit. 1] , [Mit. 2] , [Mit. 4] . The role of GCDs in the solution of problems of linear control theory is well established , [Kai. 1] . Various approaches for the computation of the GCD of $\underline{p}(s)$ have been established ; an analytical survey of the existing numerical methods can be found in [Mit. 2] , [Kar. 7] . Characterizations of the GCD in terms of standard results from linear systems theory and their relation to classical Matrix Pencil theory can be found in [Kar. 2] .

Our aim is to provide an alternative characterization for the GCD , f(s) , of a set of polynomials represented by the vector $\underline{p}(s)$, by expressing the relationship $\underline{p}(s)=\underline{q}(s)\cdot f(s)$ in terms of real matrices , (basis matrices (b.m.) P , Q of $\underline{p}(s)$, $\underline{q}(s)$ respectively) , and the Toeplitz representation of f(s) . This relates the GCD with the existence of a special Toeplitz base {W} of a subspace $\mathcal{V} \subseteq \mathcal{N}_r\{P\}$; this base has the additional property that the nonzero entries of W, (the matrix formed by {W}) , have a certain expression involving the coefficients of f(s) and $\mathcal{V}$ has the greatest possible dimension , ($\mathcal{V}$ may be $\mathcal{N}_r\{P\}$) , that the latter may happen . The above leads to the introduction of an algorithm which constructs the coefficients of the GCD as a tuple which belongs to a certain affine variety . The employment of Groebner bases , [Cox. 1], [Bec. 1] , [Sha. 1] [Har. 1] , is essential for the application of the algorithm .

In Chapter 5 we investigate structural properties of matrices over a PID , $\mathcal{R}$. The matrices are assumed to have entries over $\mathcal{R}$. These properties are used to generate algebraic tools that , (later on in Chapter 6) , will enable us to formulate a unifying framework to deal with solvability of matrix equations over $\mathcal{R}$. The existence and characterization of families of greatest left – right divisors , greatest extended (non square) left – right divisors , projectors , annihilators , left – right inverses over $\mathcal{R}$ is introduced . An extension of the notion of common , least common multiples of the rows , columns of a matrix over the field of fractions of $\mathcal{R}$ is also considered . The relation between these algebraic tools and the column , row $\mathcal{R}$ – modules , maximum $\mathcal{R}$ – modules of the matrices under investigation is established.

In Chapter 6 we tackle the very important issue of formulating a unifying approach for solving the matrix equations (1.1) – (1.4) over the PID of interest , $\mathcal{R}$. In our attempt to do so we use the results derived in Chapter 5 . The given matrices A , B , A_i B_i , C , in (1.1) – (1.2) are considered over the field of fractions , $\mathcal{F}$, of $\mathcal{R}$, whereas the unknown matrices X , Y , X_i are required to be over $\mathcal{R}$. Conditions for the existence as well as parametrization of solutions of the equations in question are provided in terms of greatest left – right divisors of the given matrices as well as parametric matrices over $\mathcal{R}$ Equations (1.2) , (1.3) are the most important in our study , since the remaining equations are special cases of them . The solutions of equation (1.4) for example are special type "block diagonal" solutions of (1.3) . The parametization over $\mathcal{R}$ of the families of solutions of the equations in question provided here are in closed form .

In Chapter 7 we consider equation (1.1) as it arises from the Total Finite Settling Time Stabilization and Dead – Beat Response Problems , for discrete time linear systems . Our main interest is to investigate equation (1.1) for solutions that define controllers with minimum extended McMillan degree , (EMD) . After an initial introduction and formulation of the problem in section 7.2 , parametrization issues for such stabilizing controllers are examined in section 7.3 . The importance of characterizing solutions of (1.1) that correspond to column , (row) , reduced matrices is established . We prove that those solutions of (1.1) , (with C an arbitrary polynomial unimodular matrix) , form a nonempty , dense , but neither open , nor closed subset of the set of solutions . The latter result implies that the sum of minimum column , (row), degrees that occur in the set of solutions of (1.1) is more likely to serve as an upper bound rather than be equal to the minimum EMD .

The approach employed for the parametrization of least column , (row) , degrees solutions of (1.1) is based on its Toeplitz matrix representation . This approach leads to a very simple algorithm involving only the computation of right , (left) , null spaces of real matrices . The construction of a lower bound for the minimum EMD takes place in section 7.7 . A method similar to the one used for the characterization of minimum column degrees is employed . Some additional issues , such as , the PI controller problem and fixed controllability index stabilizing controllers are studied as well .

Chapter 8 is concerned with the Decentralized Stabilization Problem , (DSP) , for multivariable , linear , continuous time , systems . Our aim in this chapter is to study alternative means of parametrization for the solutions of the DSP and try to provide closed form descriptions of the families of parameters in some cases . In section 8.2 we give a statement of the problem and present the mathematical framework for approaching it . If (D , N) denotes an $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime left MFD of the plant , T_i are the matrices formed from the p_i , m_i columns of the partitioning of D , N according to the number of local inputs – outputs respectively , then the parametrization of solutions of the DSP is derived from the set of left unimodular solutions , X_i , of the set of equations $T_i \cdot X_i = = U_i$, $i = 1 , \ldots , \kappa$, for which $[\, U_1 , \ldots , U_\kappa \,]$ is unimodular .

In our study we show that the above parametrization requires the existence of a constructive method that enables us to generate the family of all unimodular matrices of given dimensions , as well as the families of left , (right) unimodular matrices which complete given left , (right) , unimodular matrices to square unimodular ones . Such methods are examined in section 8.3 . The issue of interest in this chapter is introduced in section 8.4 . There , a parametrization of solutions of the DSP is introduced . The parameters are expressed in terms of upper , lower triangular matrices which must satisfy certain constraints . These constraints introduce a necessary and sufficient criterion that enables us to identify the admissible parameters . Although , in the general case , the family of qualifying parameters is not described in closed form there are particular cases when this is possible . These cases are based on the property , [Vid. 4] , of the Smith forms of T_i to be generic ; Then a closed form description of the family of parameters defined is given in section 8.5 .

Finally in chapter 9 we study a special case of Decentralized Stabilization , the

Diagonal Decentralized Stabilization , (DDSP) , Problem . The formulation of the problem is similar to the one in chapter 8 , but the approach employed for its solution is completely different , and results to a closed form parametrization of the desired stabilizing controllers . A statement of the problem and its consequent formulation are introduced in section 9.2 ; the notion of cyclicity is defined . Section 9.3 refers to an equivalent formulation of the problem which finally transforms it to the search for necessary and sufficient solvability conditions of a scalar Diophantine equation , over $\mathbb{R}_{\mathcal{P}}(s)$, the solutions of which must meet certain factorization constraints .

The actual necessary and sufficient solvability conditions for the problem are introduced in section 9.4 . The connection between the cyclicity property of the plant and the existence of diagonal stabilizing controllers is established . The parametrization of all stabilizing controllers is studied in section 9.5 . It is reduced to determining what are termed mode T mutually stabilizing pairs and the existence of such pairs forms the basis of a complete parametrization . The rest of the chapter deals with the determination of proper , reliable , stable stabilizing diagonal controllers by making use of the parametrization introduced in section 9.5 .

CHAPTER 2

SURVEY OF CONTROL SYNTHESIS PROBLEMS ASSOCIATED WITH MATRIX EQUATIONS OVER RINGS

2.1. INTRODUCTION

A great deal of research issues addressed in this thesis are motivated by the need of deriving conditions for the existence and characterization of solutions , (or special types of them) , of certain matrix equations . over the ring of interest , (in practice $\mathbb{R}[s]$, or $\mathbb{R}_{\mathfrak{P}}(s)$) . This chapter is a brief survey of control synthesis problems , (such as the centralized and decentralized stabilization problems , the model matching and exact model matching problems , the total finite settling time stabilization for discrete time systems , the decoupling and noninteracting control problems , the regulator problem) , the solution of which can be reduced to the solution of such matrix equations . A central requirement to all the problems we review here is the internal stability of the feedback system . Stability in general is a very important qualitative property of control systems , since an unstable system will "burst" as time approaches infinity . In literature [Won. 1] , [Vid. 4] , [Che. 1] , [Kai. 1] , [Ozg. 1] , [Kal. 1] , [Kuc. 2] , [Des. 1] , [Ros. 1] and references therein , one can find various concepts of stability such as , bounded input – bounded output (BIBO , or external) stability , stability in the sense of Lyapunov , asymptotic (or internal) stability , total stability . Following the approach of [Vid. 4] , [Che. 1] and [Kai. 1] , we concentrate in section 2.2 on the issue of internal and external stability , their interconnection and the properties a system should meet in order these two concepts to be equivalent .

The more general problem of centralized stabilization , (CSP) , [You. 1] , [Des. 1] , [Vid. 4] , [Kuc. 2] , i.e. the stabilization of an unstable plant using a precompensator , (or feedback compensator) , and unity output feedback scheme is presented in section 2.3 . The ring of proper and $\mathfrak{P}$ – stable rational functions , $\mathbb{R}_{\mathfrak{P}}(s)$, serves as the ring of interest . In this problem no restrictions on the input – output connections between controllers are required . The solution of the CSP is associated with the study of the standard matrix Diophantine equation , (MDE) :

$$A \cdot X + B \cdot Y = C \, , \, (X \cdot A + Y \cdot B = C) \qquad (2.1.1)$$

where (A , B) is a left , (right) , coprime matrix fractional description , (MFD) , of the plant transfer matrix and C an arbitrary unimodular matrix over the ring of interest . Later on , in chapter 7 , equation (2.1.1) will be associated with the ring of polynomials and certain issues concerning its solutions will be studied . Such polynomial MDEs arise from stabilization problems of discrete time linear systems , like the total finite settling time stabilization , (TFSTS) , and the dead – beat response , (DBR) , [Ber. 1] , [Ise. 1] , [Kal. 1] , [Kuo. 1] , [Kuc. 1] – [Kuc. 8] , [Vid. 4] , [Kar. 1] . Characterization of solutions of (2.1.1) according to the extended Mc Millan degree , (EMD) , of the controllers they define is an essential research issue . A problem of similar nature is the decentralized stabilization problem , (DSP) , [And. 1] , [And. 2] , [Cor. 1] , [Cor. 2] , [Won. 1] , [Wan. 1] , [Kar. 2] , [Ozg. 1] , only here the stabilizing controllers' transfer matrices must be of a block diagonal type . i.e. a well defined input output relationship between controllers must be maintained . The need for such type of stabilizing controllers especially appears in the stabilization of large scale systems with several control stations . The formulation of the DSP via the algebraic method of expressing the plant and controller transfer matrices as MFDs results to the need for existence and parametrization of a special type of solutions of (2.1.1) . The DSP and its formulation are presented in section 2.4 .

In section 2.5 the exact model matching , (EMMP) , and stable exact model matching , (SEMMP) problems are associated with the matrix equation :

$$A \cdot X = B \, , \, (Y \cdot A = B) \tag{2.1.2}$$

over , $\mathbb{R}_{pr}(s)$, or $\mathbb{R}_{\mathcal{P}}(s)$. The EMMP , [Wol. 1] , [Wol. 3] , [For. 1] , [Var. 6] , requires the existence and characterization of solutions of (2.1.2) over $\mathbb{R}_{pr}(s)$, where A , B are known matrices over $\mathbb{R}_{\mathcal{P}}(s)$. If the requirement that X , (Y) , should be stable is added then we define the SEMMP , [Wol. 3] , [Sco. 1] , [And. 3] , [Kuc. 9] , [Per. 1] , [Emr. 1] , [Kar. 5] .

In section 2.6 we switch to a type of problems that require one or more output vectors to be independent from one or more input vectors and are known as noninteracting or decoupling control problems . There are many different versions of such problems depending on the control feedback configurations postulated . In this section we distinguish between the disturbance decoupling , (DDP) , [Aka. 1] , [Mor. 3], [Ohm. 1] , [Sch. 1] , [Sto. 1] , [Wol. 4] , [Ozg. 1] , [Ozg. 2] , [Won. 1] , [Wil. 2] , [Tak. 1] and noninteracting control , (NICP) , [Aka. 1] , [Aka. 2] , [Bay. 1] , [Dsc. 1] , [Fal. 1] , [Ham. 1] , [Mrg. 1] , [Mor. 3] , [Wil. 1] , [Won. 1] , [Wol. 1] , [Ozg. 1] , with or without the internal stability requirement for the feedback system .

The DDP and DDP with internal stability , (DDISP) , are considered over a two vector channel system , with feedback applying around the first channel . The first channel input – output is referred to as the control input – measured output , where as the second channel one is referred to as the disturbance input – controlled output . The NICP and NICP with internal stability , (NCISP) , are considered over a three vector channel system , with feedback applying around the first channel . The solvability of all these problems is associated with the solvability and characterization of solutions of the matrix equation :

$$A \cdot X \cdot B = C \tag{2.1.3}$$

over $\mathbb{R}_{\mathcal{P}}(s)$. Finally a different type of problem associated with the same feedback configuration as the DDP and DDISP is the regulator problem and regulator problem with internal stability , (RP) , (RPIS) , respectively , [Ben. 1] , [Chg. 1] , [Hau. 1] , [Kha. 1] , [Sae. 1] , [Sch. 2] , [Sch. 3] , [Won. 1] , [Won. 2] , [Won. 3] , [Wol. 5] . The RP requires the parametrization of controllers that result to disturbance input – controlled output transfer matrices to be stable , where as in the case of RPIS the requirement that the controllers must internally stabilize the system is added . The solvability of this problems is reduced to the solvability and characterization of solutions of the matrix equations:

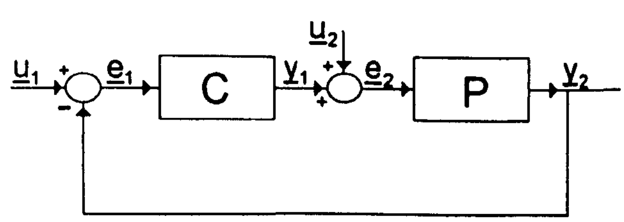
$$A \cdot X + Y \cdot B = C \, , \, X \cdot A + B \cdot Y = C \, , \, A \cdot X \cdot B + C \cdot Y \cdot D = E \qquad (2.1.4)$$

over $\mathbb{R}_{\mathcal{P}}(s)$. The matrix equations (2.1.1) – (2.1.4) derived in this chapter are treated later on in this thesis via a unifying algebraic framework established in chapter 6 .

2.2. THE CONCEPT OF STABILITY FOR LINEAR CONTROL SYSTEMS

Stability is a very important qualitative property of linear control systems , since every working system is designed to be closed loop stable . If a system is not closed loop stable , it is usually of no use as far as applications are concerned . In literature [Won. 1] , [Vid. 4] , [Che. 1] , [Kai. 1] , [Ozg. 1] , [Kal. 1] , [Kuc. 2] , [Des. 1] , [Ros. 1] and references therein , one can find various concepts of stability such as , bounded input – bounded output (BIBO , or external) stability , stability in the sense of Lyapunov , asymptotic (or internal) stability , total stability . But the two main concepts of stability that concern us here is external and internal stability . These are characterized by the external , (input – output) , internal , (state space) , descriptions of the system and under certain constraints , (stabilizability , detectability) , they are equivalent , [Kai. 1] , [Vid. 4] , [Ros. 1] , [Won. 1] .

More precisely , consider the standard feedback configuration associated with a precompensator and unity output feedback shown below :



where , $P \in \mathbb{R}_{pr}^{p x m}(s)$ represents the plant and $C \in \mathbb{R}_{pr}^{m x p}(s)$ the compensator transfer matrices respectively ; $\underline{u}_1$, $\underline{u}_2$ denote the externally applied inputs to the compensator and plant respectively ; $\underline{e}_1$, $\underline{e}_2$ denote the inputs to the compensator and plant respectively . The system under study is then described by :

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} - \begin{bmatrix} O & P \\ -C & O \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} , \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} C & O \\ O & P \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} \quad (2.2.1)$$

These system equations can be rewritten as :

$$\underline{e} = \underline{u} - F \cdot G \cdot \underline{e} \ , \ \underline{y} = G \cdot \underline{e} \quad (2.2.2)$$

where ,

$$\underline{e} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} , \underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} , \underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} , F = \begin{bmatrix} O & I \\ -I & O \end{bmatrix} , G = \begin{bmatrix} C & O \\ O & P \end{bmatrix} \quad (2.2.3)$$

It is easy to verify that $| I + F \cdot G | = | I + P \cdot C | = | I + C \cdot P |$.

***Definition (2.2.1) [Vid. 4]** : The system described by the set of equations (2.2.1) is well posed if $| I + F \cdot G |$ is nonzero as an element of $\mathbb{R}(s)$, i.e. if $| I + F \cdot G |$ is not identically zero for all $s \in \mathbb{C} \cup \{\infty\}$.* □

This condition is necessary and sufficient to ensure that (2.2.1) has a unique solution over $\mathbb{R}_{pr}^{(p+m)x(p+m)}(s)$ for $\underline{e}_1$, $\underline{e}_2$ corresponding to every $\underline{u}_1$, $\underline{u}_2$ of appropriate dimension If the system described by (2.2.1) is well posed then (2.2.1) can be solved for $\underline{e}_1$, $\underline{e}_2$; this gives :

$$\underline{e} = (I + F \cdot G)^{-1} \cdot \underline{u} \triangleq H(P,C) \cdot \underline{u} \quad (2.2.4)$$

where H(P,C) is the transfer matrix from $\underline{u}$ to $\underline{e}$. It is possible to obtain several equivalent expressions for H(P,C) . One of them may be proved to be :

$$H(P,C) = \begin{bmatrix} (I + P \cdot C)^{-1} & -P \cdot (I + C \cdot P)^{-1} \\ C \cdot (I + P \cdot C)^{-1} & (I + C \cdot P)^{-1} \end{bmatrix} = \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} \quad (2.2.5)$$

If we do not wish both $(I + P \cdot C)^{-1}$, $(I + C \cdot P)^{-1}$ to occur in (2.2.5) we can transform it by using the following matrix identities [Vid. 4] :

$$(I + P \cdot C)^{-1} = I - P \cdot (I + C \cdot P)^{-1} \cdot C \ , \ C \cdot (I + P \cdot C)^{-1} = (I + C \cdot P)^{-1} \cdot C \quad (2.2.6)$$

(2.2.6) holds true with P , C interchanged throughout as well . Thus H(P,C) takes the following two equivalent expressions :

$$H(P,C) = \begin{bmatrix} I - P\cdot(I + C\cdot P)^{-1}\cdot C & -P\cdot(I + C\cdot P)^{-1} \\ (I + C\cdot P)^{-1}\cdot C & (I + C\cdot P)^{-1} \end{bmatrix} \qquad (2.2.7)$$

$$= \begin{bmatrix} (I + P\cdot C)^{-1} & -(I + P\cdot C)^{-1}\cdot P \\ C\cdot(I + P\cdot C)^{-1} & I - C\cdot(I + P\cdot C)^{-1}\cdot P \end{bmatrix} \qquad (2.2.8)$$

of these the first involves only $(I + C\cdot P)^{-1}$ and the second only $(I + P\cdot C)^{-1}$.

Definition (2.2.2) [Vid. 4] *: The pair (P , C) is stable , if the system described by (2.2.1) is well posed and* $H(P,C) \in \mathbb{R}_{\mathcal{P}}^{(p+m)x(p+m)}(s)$. □

The condition for stability in definition (2.2.2) is symmetric in P and C ; thus (P , C) is stable if and only if (C , P) is stable . Consider now the transfer matrix from $\underline{u}$ to $\underline{y}$, W(P,C) . Then :

$$W(P,C) = G\cdot(I + F\cdot G)^{-1} \text{ and } \underline{y} = W(P,C)\cdot\underline{u} \qquad (2.2.9)$$

Lemma (2.2.1) [Vid. 4] *: W(P,C) is over* $\mathbb{R}_{\mathcal{P}}^{(p+m)x(p+m)}(s)$ *if and only if H(P,C) is over* $\mathbb{R}_{\mathcal{P}}^{(p+m)x(p+m)}(s)$. □

The above lemma justifies why stability for a pair (P , C) was defined is terms of H(P,C) and not W(P,C) ; both notions of stability are equivalent . We proceed now with the concepts of external , internal stability and their relationship .

Definition (2.2.3) [Kai. 1] , [Che. 1] *: The system described by the set of equations (2.2.1) is said to be externally , (BIBO) , stable if every bounded input* $\| \underline{u}(t) \| < M_1$, $-\infty < -T \le t \le \infty$ *, produces a bounded output* $\| \underline{y}(t) \| < M_2$, $-\infty < -T \le t \le \infty$ □

Remark (2.2.1) *: Definition (2.2.3) makes it clear that external stability refers to the external description of the system . It can be shown , [Kai. 1] , [Che. 1] , [Vid. 4] , that a system with external description given by (2.2.9) is externally stable if and only if the poles of W(P,C) have negative real parts .* □

Assume now that the state space equations of a realization of the system , described by (2.2.1) , is given by :

$$\begin{cases} \dot{\underline{x}} = A\cdot\underline{x} + B\cdot\underline{u} \ , \ \underline{x} \equiv \underline{x}(t) \ , \ \underline{u} \equiv \underline{u}(t) \\ \underline{y} = C\cdot\underline{x} + D\cdot\underline{u} \ , \ \underline{y} \equiv \underline{y}(t) \end{cases} \tag{2.2.10}$$

Definition (2.2.4) [Kai. 1] , [Che. 1] *: The system described by the set of equations (2.2.1) and a realization of it is given by (2.2.10) , is said to be internally , (asymptotically) , stable if the solutions of :*

$$\dot{\underline{x}} = A\cdot\underline{x} \ , \ \underline{x}(t_0) \equiv \underline{x}_0 \ , \ t \geq t_0 \tag{2.2.11}$$

tend towards zero as time approaches infinity , for arbitrary $\underline{x}_0$. □

Remark (2.2.2) *: Definition (2.2.4) makes it clear that internal stability refers to a realization of the system . It can be shown , [Kai. 1] , [Che. 1] , [Vid. 4] that if a system has a realization given by (2.2.10) then it is internally stable if and only if the eigenvalues of A have negative real parts .* □

The interconnection between external and internal stability is established next . Remarks (2.2.1) , (2.2.2) clearly yield that internal stability always imply external one , since the poles of the system transfer matrix form a subset of the set of eingevalues of the state space matrix A . The inverse though is not always true , since cancellations in the system transfer matrix may lead to the existence of unstable unobservable modes , (eigenvalues) , of A . The latter is illustrated in the following example :

Example (2.2.1) : Assume that a linear system has state space description given by :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot u \ , \ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$y = [\ 1 \ 1\ ] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.2.12}$$

Then ,

$$x_1 = e^{t}\cdot x_{10} + e^{t} * u \tag{2.2.13}$$

$$x_2 = (e^{-t} - e^{t})\cdot x_{10} + e^{-t}\cdot x_{20} + (e^{-t} - e^{t}) * u \tag{2.2.14}$$

$$y = e^{-t}\cdot(x_{10} + x_{20}) + e^{-t} * u \tag{2.2.15}$$

where , $f*u$ denotes the convolution of the functions f , u . While (2.2.15) implies that the system is externally stable , (2.2.13) and (2.2.14) imply that it is not internally stable . Furthermore we notice that the unstable eigenvalue of A , 1 , does not appear in (2.2.15) , i.e. is an unobservable mode , [Kai. 1] , whereas the stable one , -1 , does .

On the other hand , if we apply constant state feedback in (2.2.12) , described by :

$$v = u - [k_1 \; k_2] \cdot \underline{x} = u - \underline{k}^T \cdot \underline{x} \qquad (2.2.16)$$

the system is transformed to :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} k_1+1 & k_2 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot v \; , \; \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$y = [\; 1 \; 1 \;] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (2.2.17)$$

If we select $k_1 = -3$, $k_2 = 0$, then the system with state space description given by (2.2.17) is both externally and internally stable . Thus the original system in (2.2.12) has an unstable mode which can be shifted arbitrarily , i.e. the mode is controllable , [Kai. 1] . □

Example (2.2.1) has illustrated the effect the concepts of observable , unobservable , controllable , uncontrollable modes have to external and internal stability of a linear system . Simultaneously example (2.2.1) introduces the notions of detectability , stabilizability . In literature [Won. 1] , [Kai. 1] , [Ros. 1] , [Vid. 4] , [Kuc. 2] , one can find various definitions of detectability , stabilizability of a linear control system . The definition we state in the following is motivated by the observations of example (2.2.1) .

Definition (2.2.5) [Kai. 1] , [Won. 1] : i) *A system with state space description given in (2.2.10) is said to be stabilizable if all the uncontrollable egenvalues , (i.e. all the eigenvalues that can not be arbitrarily shifted by state feedback) , of the state matrix A are stable .*

ii) *A system with state space description given in (2.2.10) is said to be detectable if all the unobservable eigenvalues , (i.e. all the eigenvalues that do not appear as poles of the system transfer matrix) , of the state matrix A are stable .* □

Remark (2.2.3) *: It is clear that when a system is stabilizable , then it can be internally stabilized and thus become externally stable as well . On the other hand a detectable system which is externally stable is internally stable as well .* □

Theorem (2.2.1) [Kai. 1] , [Vid. 4] *: Let a system be described by the set of equations (2.2.1) . Then external stability is equivalent to internal stability , if and only if the state space realizations of both P , C are stabilizable and detectable .* □

2.3. CENTRALIZED STABILIZATION AND THE STANDARD MATRIX DIOPHANTINE EQUATION

Consider a well posed detectable and stabilizable control linear system described by the set of equations (2.2.1) , or equivalently (2.2.4) , (2.2.9) . The system is stable if and only if every element of W(P,C) , or equivalently H(P,C) belongs to $\mathbb{R}_{\mathcal{P}}(s)$, $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$. If $W(P,C) \equiv W$ and $H(P,C) \equiv H$ then :

$$W = G\cdot(I + F\cdot G)^{-1} = G\cdot H \Leftrightarrow G = W\cdot H^{-1} = W\cdot(adj H/|\,H\,|) \tag{2.3.1}$$

The last expression implies that every element of the matrix $G = diag\{C\ ,\ P\}$ and hence every element of C , P can be expressed as a ratio of two functions from $\mathbb{R}_{\mathcal{P}}(s)$. The latter has led to the development of an algebraic framework for solving stabilization problems , known as the matrix fraction description approach , (MFD) , [Vid. 1] , [Des. 1] , [Sae. 2] , [Ant. 1] , [Kal. 1] , [Kuc. 2] , [Fra. 1] , [Var. 3] and references therein . The most classical stabilization problem is the so called centralized stabilization problem , (CSP) , [You. 2] , [Des. 1] , [Vid. 4] , [Kuc. 2] , which requires the derivation of conditions for existence and characterization of stabilizing controllers for an unstable linear system . Within the algebraic framework of MFD approach the expression of P , C as coprime MFDs over $\mathbb{R}_{\mathcal{P}}(s)$ is important . Thus if :

$$P = D_1^{-1}\cdot N_1 = N_2\cdot D_2^{-1} \tag{2.3.2}$$
$$C = A_1^{-1}\cdot B_1 = B_2\cdot A_2^{-1} \tag{2.3.3}$$

with $(D_1\ ,\ N_1)$, $(A_1\ ,\ B_1)$ left coprime MFDs , $(D_2\ ,\ N_2)$, $(A_2\ ,\ B_2)$ right coprime MFDs of P , C over $\mathbb{R}_{\mathcal{P}}(s)$ respectively . By inserting (2.3.2) , (2.3.3) to (2.2.5) , H(P,C) is transformed to:

$$H(P,C) = \begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix}^{-1}\cdot\begin{bmatrix} D_1 & O \\ O & A_1 \end{bmatrix} = \begin{bmatrix} A_2 & O \\ O & D_2 \end{bmatrix}\cdot\begin{bmatrix} A_2 & N_2 \\ -B_2 & D_2 \end{bmatrix}^{-1} \tag{2.3.4}$$

Proposition (2.3.1) [Kai. 1] : *If $(D_1\ ,\ N_1)$, $(A_1\ ,\ B_1)$ left coprime MFDs , $(D_2\ ,\ N_2)$, $(A_2,\ B_2)$ right coprime MFDs of P , C over $\mathbb{R}_{\mathcal{P}}(s)$ respectively , then (2.3.4) defines a left , right coprime MFD of H(P,C) over $\mathbb{R}_{\mathcal{P}}(s)$.* □

Let a system described by the set of equations (2.2.10) , be free of "hidden modes" , i.e.

unobservable , uncontrollable eigenvalues of the state matrix A , and let the plant and compensator be expressed as in (2.3.2) , (2.3.3) . Then we state the following result :

Proposition (2.3.2) [Kuc. 1] : *The characteristic pole function of H(P,C) is given by the determinants of the denominator matrices :*

$$\begin{bmatrix} D_1 & N_1 \\ -B_1 & A_1 \end{bmatrix}, \begin{bmatrix} A_2 & N_2 \\ -B_2 & D_2 \end{bmatrix} \qquad (2.3.5)$$

multiplied by a unit of $\mathbb{R}_{\mathcal{P}}(s)$ *, or equivalently by :*

$$| D_1 \cdot A_2 + N_1 \cdot B_2 | , | A_1 \cdot D_2 + B_1 \cdot N_2 | \qquad (2.3.6)$$

modulo units of $\mathbb{R}_{\mathcal{P}}(s)$. □

Thus solvability of CSP is associated with the study of existence and characterization of solutions of the standard matrix Diophantine equations , [Vid. 4] , [Kuc. 2] , [You. 2] , [Des. 1] :

$$D_1 \cdot X + N_1 \cdot Y = U , X \cdot D_2 + Y \cdot N_2 = V \qquad (2.3.7)$$

where , (X , Y) must be right , left coprime pairs such that U , V are $\mathbb{R}_{\mathcal{P}}(s)$ unimodular Equations (2.3.7) have always a solution , since (D_1 , N_1) , (D_2 , N_2) are left , right coprime over $\mathbb{R}_{\mathcal{P}}(s)$; if (X_0 , Y_0) is a solution of (2.3.7) the family of solutions is given by :

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} + \begin{bmatrix} N_2 \\ -D_2 \end{bmatrix} \cdot L , [X , Y] = [X_0 , Y_0] + T \cdot [-N_1 , D_1] \qquad (2.3.8)$$

with L , T parametric matrices over $\mathbb{R}_{\mathcal{P}}(s)$. It has been proven , [Vid. 4] , that the determinants of the matrices X defined in (2.3.8) are generically nonzero and thus the pairs (X , Y) generically correspond to coprime MFDs over $\mathbb{R}_{\mathcal{P}}(s)$. In our study we concentrate to the investigation of conditions for the existence and characterization of special types of solutions of (2.3.7) in order to meet the constraints of the decentralized and diagonal decentralized stabilization problems , (DSP) , (DDSP) , as well as characterization of solutions of (2.3.7) , ((2.3.7) is assumed over the ring $\mathbb{R}[d]$, $d = z^{-1}$) , which define minimum extended McMillan degree , (EMD) , controllers so that the requirements of the total finite settling time stabilization , (TFSTS) , and dead - beat response , (DBRP) , problems , (for discrete time systems) , are satisfied .

The TFSTS requires all the internal and external variables , (signals) , of the system to settle to a new steady – state after finite time from the application of a step change to its input and for every initial condition , [Kar. 1] . The TFSTS comprises the dead – beat response problem , i.e. the forcing of the state or output vector from any initial state to the origin in minimum time , [Ber. 1] , [Ise. 1] , [Kal. 1] , [Kuo. 1] , [Kuc. 1] - [Kuc. 8] , [Vid. 4] . The TFSTS and DBRP can be viewed as a type of minimal design problems , (MDP) , because of the constraints imposed on the stabilizing controllers to have minimum number of finite and infinite poles , EMD , among the family of all stabilizing controllers . Additionally the DSP and DDSP are central in our study and much of our research effort has been devoted to them . The formulation of the DSP and DDSP as well as their interconnection to equation (2.3.7) is presented in the next section .

2.4. DECENTRALIZED STABILIZATION AND THE STANDARD MATRIX DIOPHANTINE EQUATION

Significantly different from the CSP is the DSP for continuous time linear systems . This problem is due to restrictions on the feedback compensator structure , which are often encountered in large scale systems . This systems have several local control stations ; each local compensator observes only the corresponding local outputs . Such decentralized control of systems results in a block diagonal compensator matrix scheme [San. 1] , [Gun. 1] , [Wan. 1] . Thus the DSP requires the stabilization of an unstable system by using a decentralized compensator and unity output feedback scheme . Wang and Davison , [Wan. 1] and Corfmat and Morse , [Cor. 1] , [Cor. 2] , have introduced synthesis methods for design of stabilizing decentralized compensators . It has been derived that a necessary and sufficient condition for the existence of local control laws with dynamic compensation to stabilize a given system is that the system has no "fixed modes" , [Wan. 1] , in the region of instability . Further study of the problem has been done in [And. 1] , [And. 2] , [Guc. 1] , [Ozg. 1] , [Kar. 3] . In [Gun. 1] , the DSP is treated within the algebraic framework of matrix fraction description of the plant and controller transfer matrices over $\mathbb{R}_{\mathfrak{sp}}(s)$.

A special case of decentralized stabilization of continuous time linear systems is the diagonal decentralized stabilization problem , (DDSP) , [Kar. 2] , [Guc. 1] . In this special case , given a plant transfer matrix over $\mathbb{R}_{pr}^{pxp}(s)$, the problem is to determine a stabilizing compensator C = $diag\{c_1 , \dots , c_p\}$ over $\mathbb{R}_{pr}^{pxp}(s)$, such that the feedback system is stabilized by C . As in the case of the DSP the stability requirement may be expressed in terms of matrix fraction descriptions of transfer matrices [Vid. 4] , and highlights the important role of "fixed modes" over the region of instability , [Wan. 1] . As it will be made clear in chapters 8 and 9 , the DDSP is considered separately from the DSP so that we are able to apply a different method of investigating issues , concerning the nature of stabilizing controllers , that can not be fully addressed via general DSP . The existence and characterization of solutions of the DDSP is intimately related to systems that exhibit the property of cyclicity . After formulating the DDSP in a similar manner to the DSP , the construction of the family of all diagonal stabilizing controllers is reduced to determining what are termed mode T mutually stabilizing pairs The existence of such pairs provides a base for addressing issues concerning the characterization and nature of the stabilizing controllers , (proper , reliable , stable controllers) .

The algebraic formulation of the DSP is following next . The same formulation applies in the case of the DDSP if $p = m = \kappa$, $p_i = m_i = 1$, $i = 1 , \dots , \kappa$. If $P \in \mathbb{R}_{pr}^{pxm}(s)$ is the transfer function of the plant , $C \in \mathbb{R}_{pr}^{mxp}(s)$ is the transfer function of the controller . Assume that P is $\mathcal{P}$ – stabilizable , $\mathcal{P}$ – detectable , with $\mathcal{P}^c$ the area of stability . If $\mathcal{P} = \mathbb{C}_+ \bigcup \{\infty\}$ and $\mathbb{R}_{\mathcal{P}}(s)$ denotes the ring of proper and $\mathcal{P}$ – stable functions consider an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of the plant $P = D^{-1} \cdot N$, where $D \in \mathbb{R}_{\mathcal{P}}^{pxp}(s)$, $N \in \mathbb{R}_{\mathcal{P}}^{pxm}(s)$ and (D_p , N_p) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair ; and let $C = \text{diag}\{C_1 , \dots , C_\kappa\} = N_c \cdot D_c^{-1}$ be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of the diagonal controller , where , $C_i = N_i \cdot D_i^{-1} \in \mathbb{R}_{\mathcal{P}}^{m_i x p_i}(s)$, $(i = 1 , 2 , \dots , \kappa , \sum_{i=1}^{\kappa} m_i = m , \sum_{i=1}^{\kappa} p_i = p)$, is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of C_i . Then $N_c = \text{diag}\{N_1 , \dots , N_\kappa\}$ and $D_c = \text{diag}\{D_1 , \dots , D_\kappa\}$. It is known that the controller internally stabilizes the feedback system if and only if there exists some $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular matrix U such that :

$$D\, D_c + N\, N_c = U \tag{2.4.1}$$

Partitioning D , N in terms of columns , (2.4.1) is expressed as :

$$[\, D^{p_1}, D^{p_2}, \dots , D^{p_\kappa} \,] \cdot \begin{bmatrix} D_1 & & O \\ & D_2 & \\ & & \ddots \\ O & & D_\kappa \end{bmatrix} + [\, N^{m_1}, N^{m_2}, \dots , N^{p_\kappa} \,] \cdot \begin{bmatrix} N_1 & & O \\ & N_2 & \\ & & \ddots \\ O & & N_\kappa \end{bmatrix} =$$

$$= [\, U_1 , U_2 , \dots , U_\kappa \,] \tag{2.4.2}$$

Or equivalently ,

$$[\, D^{p_i} \vdots N^{m_i} \,] \cdot \begin{bmatrix} D_i \\ N_i \end{bmatrix} = U_i \, , \, i = 1 , 2 , \dots , \kappa \tag{2.4.3}$$

where , $T_i = [\, D^{p_i} \vdots N^{m_i} \,] \in \mathbb{R}_{\mathcal{P}}^{px(p_i + m_i)}(s)$ are matrices defined by the plant and $X_i = [\, D_i^T , N_i^T \,]^T \in \mathbb{R}_{\mathcal{P}}^{(p_i + m_i)xp_i}(s)$ characterize the p_i input , m_i output local controllers .

The U_i are arbitrary matrices of $\mathbb{R}_{\mathcal{P}}^{p x p_i}(s)$, with the additional property that $U \triangleq [U_1 ; U_2 , \dots , U_\kappa]$ is $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular . The latter condition implies that U_i are left unimodular in $\mathbb{R}_{\mathcal{P}}^{p x p_i}(s)$. Parametrization issues and related topics of the DSP and DDSP are studied in chapters 8 and 9 .

2.5. MODEL MATCHING AND THE MATRIX EQUATION $A \cdot X = B$, $(Y \cdot A = B)$

Consider a well posed detectable and stabilizable control linear system described by the set of equations (2.2.1) , or equivalently (2.2.4) , (2.2.9) . If C_0 denotes a stabilizing controller for the system , then matrices D_1 , N_1 , A_1 , B_1 , D_2 , N_2 , A_2 , B_2 over $\mathbb{R}_{\mathcal{P}}(s)$ exist such that :

$$P = D_1^{-1} \cdot N_1 = N_2 \cdot D_2^{-1} \tag{2.5.1}$$

$$C_0 = A_1^{-1} \cdot B_1 = B_2 \cdot A_2^{-1} \tag{2.5.2}$$

with (D_1 , N_1) , (A_1 , B_1) left coprime MFDs , (D_2 , N_2) , (A_2 , B_2) right coprime MFDs of P , C_0 over $\mathbb{R}_{\mathcal{P}}(s)$ respectively and the following Bezout identity holds true :

$$\begin{bmatrix} A_1 & B_1 \\ -D_1 & N_1 \end{bmatrix} \cdot \begin{bmatrix} D_2 & -A_2 \\ N_2 & B_2 \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & I_p \end{bmatrix} \tag{2.5.3}$$

Multiplying (2.5.3) on the left and right by the $\mathbb{R}_{\mathcal{P}}(s)$ unimodular matrices :

$$\begin{bmatrix} I_m & W \\ O & I_p \end{bmatrix} , \begin{bmatrix} I_m & -W \\ O & I_p \end{bmatrix} \tag{2.5.4}$$

we obtain :

$$\begin{bmatrix} A_1 - W \cdot N_1 & B_1 + W \cdot N_1 \\ -N_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} D_2 & -(B_2 + D_2 \cdot W) \\ N_2 & A_2 - N_2 \cdot W \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & I_p \end{bmatrix} \tag{2.5.5}$$

Furthermore all the stabilizing controllers are given by :

$$C = [A_1 - W \cdot N_1]^{-1} \cdot [B_1 + W \cdot N_1] = [B_2 + D_2 \cdot W] \cdot [A_2 - N_2 \cdot W]^{-1} \tag{2.5.6}$$

Consider now the closed loop transfer matrix of the system from $\underline{u}_1$ to $\underline{y}_2$:

$$H_{\underline{u}_1\underline{y}_2}(s) \equiv T = [I_p + P\cdot C]^{-1}\cdot P\cdot C = P\cdot[I_m + C\cdot P]^{-1}\cdot C \in \mathbb{R}_{\mathcal{P}}^{p\times p}(s) \qquad (2.5.7)$$

Then we have the following result concerning T :

***Proposition (2.5.1) [Var. 5]** : T satisfies the following relations :*

$$T = N_2\cdot[B_1 + W\cdot D_1] \qquad (2.5.8)$$
$$I_p - T = [A_2 - N_2\cdot W]\cdot D_1 \qquad (2.5.9)$$

□

From proposition (2.5.1) it follows that the matrices $X = [B_1 + W\cdot D_1]$, $Y = [A_2 - N_2\cdot W]$ represent a pair of solutions to the matrix equations :

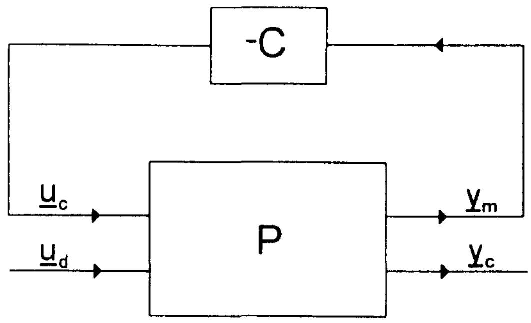
$$T = N_2\cdot X \qquad (2.5.10)$$
$$I_p - T = Y\cdot D_1 \qquad (2.5.11)$$

If the matrices T , N_2 , D_1 are all known then the problem of determining conditions under which the matrix equations (2.5.10) , (2..5.11) have solutions over $\mathbb{R}_{pr}(s)$, or $\mathbb{R}_{\mathcal{P}}(s)$ is known as the exact model matching , (EMMP) , or stable exact model matching , (SEMMP) , problem respectively and has been the subject of numerous investigations , [Wol. 1] , [Wol. 3] , [For. 1] , [Var. 6] , [Sco. 1] , [And.3] , [Kuc. 9] , [Per. 1] , [Kar. 5] . An additional constraint to the EMMP and SEMMP could be the characterization of proper, or proper and $\mathcal{P}$ stable solutions of (2.5.10) , (2.5.11) with minimum Mc Millan degree . These are known as the minimal design and stable minimal design problems associated with the model matching problem , [For. 1] , [Var. 6] , [Sco. 1] , [Wol. 3] .

In the next section we consider an other class of control synthesis problems known as noninteracting , or decoupling problems . These are problems associated with the matrix equation $A\cdot X\cdot B = C$.

2.6. DISTURBANCE DECOUPLING AND THE MATRIX EQUATION A·X·B = C

Some control problems in which a number of variables are made independent of one , or more other variables via feedback and/or feedforward compensation are known as noninteracting , or decoupling control problems . There are many different versions of noninteracting control problems in literature depending on the control configurations postulated . In the following sections we review noninteracting control problems the solvability of which is associated with the study of the matrix equation $A\cdot X\cdot B = C$. Such problems are the disturbance decoupling , (DDP) , [Aka. 1] , [Mor. 3] , [Ohm. 1] , [Sch. 1] , [Sto. 1] , [Wol. 1] , [Ozg. 1] , [Won. 1] , [Tak. 1] , and noninteracting control , (NICP) , [Aka. 1] , [Aka. 2] , [Bay. 1] , [Dsc. 1] , [Fal. 1] , [Ham. 1] , [Mrg. 1] , [Mor. 3] [Wil. 1] , [Won. 1] , [Wol. 1] , [Ozg. 1] , with or without the internal stability requirement for the feedback system . Consider a linear , multivariable , continuous time , time invariant , control system associated with the following feedback scheme :



where , $P \in \mathbb{R}_{pr}^{(p+q)x(m+n)}(s)$ represents the plant and $C \in \mathbb{R}_{pr}^{mxp}(s)$ the compensator transfer matrices respectively and :

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{2.6.1}$$

with , $P_{11} \in \mathbb{R}_{pr}^{pxm}(s)$, $P_{12} \in \mathbb{R}_{pr}^{pxn}(s)$, $P_{21} \in \mathbb{R}_{pr}^{qxm}(s)$, $P_{22} \in \mathbb{R}_{pr}^{qxn}(s)$ and P_{11} is strictly proper in order to avoid complications concerning the well defined nature of the feedback loop , when a feedback is applied . This model is widely used for various control problems , where it is either convenient , or necessary to distinguish between two types of inputs and outputs . The outputs that can be used as inputs to the controller and those with unwanted influences on the plant . Naturally , some outputs may be included in both channels if they are measurable , i.e. can be used to derive the controller . while at the same time its behavior needs to be changed . Similarly , a particular input may have unwanted influences on the plant and it may be suitable for control purposes , in which case it may be included in both channels . Motivated by application , the output vector $\underline{y}_m$ is called the measured output and $\underline{y}_c$ the controlled output , the input vector $\underline{u}_c$ is called the control input and $\underline{u}_d$ the disturbance input . Thus the first channel of the plant is the control channel around which the feedback is applied . The need to use a two – channel system model can also arise due to geographical separation of various subplants of the original plant as in the case of large scale plants . The plant transfer matrix can be represented in matrix fractions over $\mathbb{R}_{\mathfrak{P}}(s)$ as :

$$P = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \cdot Q_{11}^{-1} \cdot [\, R_1 \, , R_2 \,] + \begin{bmatrix} W & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{2.6.2}$$

where , $Z_1 \in \mathbb{R}_{\mathcal{P}}^{pxr}(s)$, $Z_2 \in \mathbb{R}_{\mathcal{P}}^{pxr}(s)$, $Q_{11} \in \mathbb{R}_{\mathcal{P}}^{rxr}(s)$, $R_1 \in \mathbb{R}_{\mathcal{P}}^{rxm}(s)$, $R_2 \in \mathbb{R}_{\mathcal{P}}^{rxn}(s)$. $W \in \mathbb{R}_{\mathcal{P}}^{pxm}(s)$, $W_{12} \in \mathbb{R}_{\mathcal{P}}^{pxn}(s)$, $W_{21} \in \mathbb{R}_{\mathcal{P}}^{qxm}(s)$, $W_{22} \in \mathbb{R}_{\mathcal{P}}^{qxn}(s)$, and Q_{11} nonsingular . We assume that this representation is bicoprime . If now the transfer matrix C of the controller is written in matrix fractions as :

$$C = Z_c \cdot Q_c^{-1} \cdot R_c \tag{2.6.3}$$

then it can be shown , [Ozg. 1] , that a resulting fractional representation for the transfer matrix between the disturbance input and the controlled output , P_{dc} , is given by :

$$P_{dc} = [\, Z_2 \, , \; -W_{21} \cdot Z_c \,] \cdot \begin{bmatrix} Q_{11} & R_1 \cdot Z_c \\ -R_c \cdot Z_1 & Q_c + R_c \cdot W \cdot Z_c \end{bmatrix}^{-1} \cdot \begin{bmatrix} R_2 \\ R_c \cdot W_{12} \end{bmatrix} + W_{22} \tag{2.6.4}$$

Given the bicoprime fraction representation of P_{11} by :

$$P_{11} = Z \cdot Q^{-1} \cdot R + W \tag{2.6.5}$$

matrices K , L , M , N , Q_l , R_l , P_r , Q_r , M_l , N_l , K_r , L_r over $\mathbb{R}_{\mathcal{P}}(s)$ exist such that (Q_l , R_l) are left coprime , (P_r , Q_r) are right coprime over $\mathbb{R}_{\mathcal{P}}(s)$ respectively and :

$$Z \cdot Q^{-1} = Q_l^{-1} \cdot R_l \, , \; Q^{-1} \cdot R = P_r \cdot Q_r^{-1} \tag{2.6.6}$$

$$\begin{bmatrix} K & -L \\ R_l & Q_l \end{bmatrix} \cdot \begin{bmatrix} Q & N_l \\ -Z & M_l \end{bmatrix} = I \tag{2.6.7}$$

$$\begin{bmatrix} Q & R \\ -L_r & K_r \end{bmatrix} \cdot \begin{bmatrix} M & -P_r \\ N & Q_r \end{bmatrix} = I \tag{2.6.8}$$

It can be proved , [Ozg. 1] , that the set of disturbance input , controlled output transfer matrices , P_{dc} , admissible for internal stability of the system is given by :

$$\mathcal{P}_{dc}^{is} = \Big\{ P_{dc}(X) = [Z_2 \, , \; -W_{21} \cdot (N \cdot N_l + Q_r \cdot X)] \cdot$$

$$\cdot \begin{bmatrix} Q_{11} & R_1 \cdot (N \cdot N_l + Q_r \cdot X) \\ -Z_1 & M_l + Z \cdot M \cdot N_l - Z \cdot P_r \cdot X \end{bmatrix}^{-1} \cdot \begin{bmatrix} R_2 \\ W_{12} \end{bmatrix} + W_{22} \, , \forall \, X \in \mathbb{R}_{\mathcal{P}}^{mxp}(s) \Big\} \tag{2.6.9}$$

or equivalently ,

$$\mathcal{P}^{is}_{dc} = \left\{ P_{dc}(X) = \bar{C}_1^{-1} \cdot (T \cdot \Theta_{12} + \Theta_{21} \cdot S - \Theta_{21} \cdot Q \cdot \Theta_{12} + \bar{C}_1 \cdot W_{22} \cdot \bar{D} - \Omega_{21} \cdot X \cdot \Omega_{12}) \cdot \bar{D}^{-1} \right. ,$$

$$\left. , \forall X \in \mathbb{R}^{mxp}_{\mathcal{P}}(s) \right\} \tag{2.6.10}$$

with ,

$$\Theta_{12} = K \cdot S - L \cdot W_{12} \cdot \bar{D} \tag{2.6.11}$$

$$\Theta_{21} = T \cdot N - \bar{C}_1 \cdot W_{21} \cdot N \tag{2.6.12}$$

$$\Omega_{12} = R_l \cdot S + Q_l \cdot W_{12} \cdot \bar{D} \tag{2.6.13}$$

$$\Omega_{21} = T \cdot P_r + \bar{C}_1 \cdot W_{21} \cdot Q_r \tag{2.6.14}$$

$$Z_2 \cdot C_1^{-1} = \bar{C}_1^{-1} \cdot T \ , \ D^{-1} R_2 = S \cdot \bar{D}^{-1} \tag{2.6.15}$$

$$C_1 = gcrd(Z_1 \ , Q_{11}) \ , \ Q_{11} = Q_1 \cdot C_1 \ , \ D = gcld(Q_1 \ , R_1) \qquad (2.6.11)$$

Some control problems in which the main objective is to decouple one or more outputs from one or more inputs , can be posed as follows :

Disturbance Decoupling Problem , (DDP) , [Ozg. 1] *: Consider the two channel system described by the set of equations (2.6.1) , (2.6.2) , (2.6.3) . Given the transfer matrix of the system P determine a controller C such that the disturbance input , controlled output transfer matrix , P_{dc} , given by :*

$$P_{dc} = P_{22} - P_{21} \cdot C \cdot (I + P_{11} \cdot C)^{-1} \cdot P_{12} \tag{2.6.16}$$

is identically zero . □

Disturbance Decoupling with Internal Stability Problem , (DDISP) , [Ozg. 1] *: Consider the two channel system described by the set of equations (2.6.1) , (2.6.2) , (2.6.3) . Given the transfer matrix of the system P determine a controller C such that in the closed loop system the pair (P_{11} , C) is internally stable and the disturbance input , controlled output transfer matrix , P_{dc} , given by :*

$$P_{dc} = P_{22} - P_{21} \cdot C \cdot (I + P_{11} \cdot C)^{-1} \cdot P_{12} \tag{2.6.17}$$

is identically zero . □

The decoupling objective $P_{dc} = 0$ accounts to making the controlled output $\underline{y}_c$ independent of the disturbance input $\underline{u}_d$. It is important to note that the dynamics with which the disturbance input $\underline{u}_d$ itself is generated has no relevance here . Our analysis so far implies that the DDP and DDISP can be transformed to the following equivalent problems :

i) The DDP can be seen as a general model matching problem , i.e. given to transfer matrix models $P_{12} \in \mathbb{R}_{pr}^{pxn}(s)$, $P_{21} \in \mathbb{R}_{pr}^{qxm}(s)$, and a reference model $P_{22} \in \mathbb{R}_{pr}^{qxn}(s)$, determine an in-between model $Y \in \mathbb{R}_{pr}^{mxp}(s)$, so that their cascade connection of transfer matrix $P_{21} \cdot Y \cdot P_{12}$ is identical with P_{22} . Furthermore , if

$$\Pi_{ij} = \begin{bmatrix} Q_{11} & R_j \\ -P_i & W_{ij} \end{bmatrix} , i , j = 1 , 2 \tag{2.6.18}$$

then :

***Theorem (2.6.1)** [Ozg. 1] : The DDP is solvable , if and only if there exists a solution $X \in \mathbb{R}_{pr}^{(r+m)x(r+p)}(s)$ satisfying equation :*

$$\Pi_{22} = \Pi_{21} \cdot X \cdot \Pi_{12} \tag{2.6.19}$$

□

ii) The solvability of the DDISP can be reduced to the existence of a matrix $X \in \mathbb{R}_{\mathfrak{P}}^{mxp}(s)$ for which $P_{dc}(X) = O$, i.e. determining X such that the elements of the set $\mathfrak{P}_{dc}^{is}$ are identically zero . A necessary and sufficient condition for the latter to happen is stated in the following proposition :

***Proposition (2.6.1)** [Ozg. 1] : The DDISP is solvable , if and only if there exists an $X \in \mathbb{R}_{\mathfrak{P}}^{mxp}(s)$ satisfying :*

$$\Omega_{21} \cdot X \cdot \Omega_{12} = T \cdot \Theta_{12} + \Theta_{21} \cdot S - \Theta_{21} \cdot Q \cdot \Theta_{12} + \bar{C}_1 \cdot W_{22} \cdot \bar{D} \tag{2.6.20}$$

□

An alternative condition for solvability of the DDISP is stated next . Consider the system matrices :

$$\bar{\Pi}_{12} = \begin{bmatrix} Q & S \\ -Z & W_{12} \cdot \bar{D} \end{bmatrix} , \bar{\Pi}_{21} = \begin{bmatrix} Q & R \\ -T & \bar{C}_1 \cdot W_{21} \end{bmatrix} , \bar{\Pi}_{22} = \begin{bmatrix} Q & S \\ -T & \bar{C}_1 \cdot W_{12} \cdot \bar{D} \end{bmatrix} \tag{2.6.21}$$

***Theorem (2.6.2)** [Ozg. 1] : The DDISP is solvable if and only if there exists an $\hat{X} \in \mathbb{R}_{\mathfrak{P}}^{(r+m)x(r+p)}(s)$ satisfying equation :*

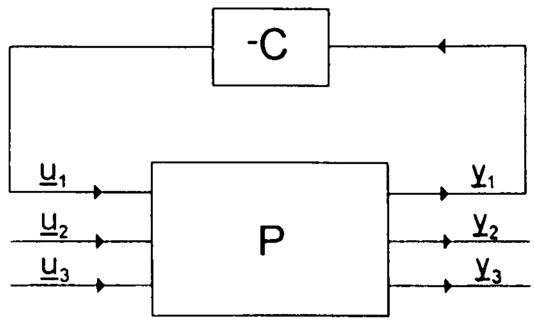
$$\bar{\Pi}_{22} = \bar{\Pi}_{21} \cdot \hat{X} \cdot \bar{\Pi}_{12} \tag{2.6.22}$$

□

It is clear from the above analysis that the matrix equation $A \cdot X \cdot B = C$ is central to the solvability of the DDP and DDISP .

2.7. NONINTERACTING CONTROL AND THE MATRIX EQUATION $A \cdot X \cdot B = C$

Consider a linear , multivariable , continuous time , time invariant , control system associated with the following feedback scheme :



where , $P \in \mathbb{R}_{pr}^{(p+q+s)x(m+n+l)}(s)$ represents the plant and $C \in \mathbb{R}_{pr}^{mxp}(s)$ the compensator transfer matrices respectively and :

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{2.7.1}$$

with , $P_{11} \in \mathbb{R}_{pr}^{pxm}(s)$, $P_{22} \in \mathbb{R}_{pr}^{qxn}(s)$, $P_{33} \in \mathbb{R}_{pr}^{sxl}(s)$ and P_{11} is strictly proper in order to avoid complications concerning the well defined nature of the feedback loop , when a feedback is applied . In terms of the matrix :

$$Y = C \cdot (I + P_{11} \cdot C)^{-1} \tag{2.7.2}$$

the resulting two channel plant has the input , output representation :

$$\begin{bmatrix} \underline{y}_2 \\ \underline{y}_3 \end{bmatrix} = \begin{bmatrix} P_{22} - P_{21} \cdot Y \cdot P_{12} & P_{23} - P_{21} \cdot Y \cdot P_{13} \\ P_{32} - P_{31} \cdot Y \cdot P_{12} & P_{33} - P_{31} \cdot Y \cdot P_{13} \end{bmatrix} \cdot \begin{bmatrix} \underline{u}_2 \\ \underline{u}_3 \end{bmatrix} = \begin{bmatrix} \hat{P}_{22} & \hat{P}_{23} \\ \hat{P}_{32} & \hat{P}_{33} \end{bmatrix} \cdot \begin{bmatrix} \underline{u}_2 \\ \underline{u}_3 \end{bmatrix} \tag{2.7.3}$$

Noninteracting Control Problem , (NICP) , [Ozg. 1] : Given the three channel plant in (2.7.1) , determine a controller C such that in the closed loop plant resulting from the application of the feedback control low $\underline{u}_1 = -C \cdot \underline{y}_1$, it holds that :

$$\hat{P}_{23} = P_{23} - P_{21} \cdot Y \cdot P_{13} = O, \ \hat{P}_{32} = P_{32} - P_{31} \cdot Y \cdot P_{12} = O \qquad (2.7.4)$$

□

Noninteracting Control Problem with Internal Stability , (NCISP) , [Ozg. 1] : *Given the three channel plant in (2.7.1) , determine a controller C such that the pair (P_{11} , C) is internally stable and in the closed loop plant resulting from the application of the feedback control low $\underline{u}_1 = -C \cdot \underline{y}_1$, it holds that :*

$$\hat{P}_{23} = P_{23} - P_{21} \cdot Y \cdot P_{13} = O, \ \hat{P}_{32} = P_{32} - P_{31} \cdot Y \cdot P_{12} = O \qquad (2.7.5)$$

□

Thus the resulting closed loop plant is required to be block diagonal with the same size of blocks as in the open loop plant from ($\underline{u}_2$, $\underline{u}_3$) to ($\underline{y}_2$, $\underline{y}_3$) , while assuring the stability of the feedback loop in the case of the NCISP , (for references on the two problems see sections 2.1 and 2.6) . Let the plant transfer matrix in (2.7.1) be written in bicoprime fraction representation over $\mathbb{R}_{\mathfrak{P}}(s)$ as :

$$P = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \cdot Q_{11}^{-1} \cdot [\, R_1 \, , R_2 \, , R_3] + \begin{bmatrix} W & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} \qquad (2.7.6)$$

where , $Q_{11} \in \mathbb{R}_{\mathfrak{P}}^{rxr}(s)$ is nonsingular . Let a bicoprime fraction representation of P_{11} given by :

$$P_{11} = Z \cdot Q^{-1} \cdot R + W \qquad (2.7.7)$$

and define the matrices K , L , M , N , Q_l , R_l , P_r , Q_r , M_l , N_l , K_r . L_r over $\mathbb{R}_{\mathfrak{P}}(s)$ exist such that :

$$\begin{bmatrix} K & -L \\ R_l & Q_l \end{bmatrix} \cdot \begin{bmatrix} Q & N_l \\ -Z & M_l \end{bmatrix} = I \, , \begin{bmatrix} Q & R \\ -L_r & K_r \end{bmatrix} \cdot \begin{bmatrix} M & -P_r \\ N & Q_r \end{bmatrix} = I \qquad (2.7.8)$$

If $P_{cr}(X)$, $Q_{cr}(X)$ denote the matrices :

$$P_{cr}(X) = N \cdot N_l + Q_r \cdot X \qquad (2.7.9)$$

$$Q_{cr}(X) = M_l + (Z \cdot M - W \cdot N) \cdot N_l - (Z \cdot P_r + W \cdot Q_r) \cdot X \qquad (2.7.10)$$

as X runs in $\mathbb{R}_{\mathfrak{P}}^{mxp}(s)$, it can be proved , [Ozg. 1] , that the set of closed loop transfer matrices from ($\underline{u}_2$, $\underline{u}_3$) to ($\underline{y}_2$, $\underline{y}_3$) admissible for internal stability is given by :

$$\mathcal{P}^{is}_{\underline{u}_{23}\underline{y}_{23}} = \left\{ \begin{bmatrix} \hat{P}_{22} & \hat{P}_{23} \\ \hat{P}_{32} & \hat{P}_{33} \end{bmatrix}(X) = \begin{bmatrix} Z_2 & -W_{21}\cdot P_{cr}(X) \\ Z_3 & -W_{31}\cdot P_{cr}(X) \end{bmatrix} \cdot \begin{bmatrix} Q_{11} & R_1\cdot P_{cr}(X) \\ -Z_1 & Q_{cr}(X)+W\cdot P_{cr}(X) \end{bmatrix}^{-1} \cdot \begin{bmatrix} R_2 & R_3 \\ W_{12} & W_{13} \end{bmatrix} + \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix}, \right.$$

$$\left. \forall\, X \in \mathbb{R}^{mxp}_{\mathcal{P}}(s) \right\} \qquad (2.7.11)$$

Let now :

$$C_1 = gcrd(Z_1\ ,\ Q_{11})\ ,\ Q_{11} = Q_1\cdot C_1\ ,\ D = gcld(Q_1\ ,\ R_1) \qquad (2.7.12)$$

$$Z_2\cdot C_1^{-1} = C_2^{-1}\cdot T_2\ ,\ Z_3\cdot C_1^{-1} = C_3^{-1}\cdot T_3\ ,\ D^{-1}\cdot R_2 = S_2\cdot D_2^{-1}\ ,\ D^{-1}\cdot R_3 = C_3^{-1}\cdot D_3 \qquad (2.7.13)$$

for some left coprime pairs $(C_2\ ,\ T_2)$, $(C_3\ ,\ T_3)$, right coprime pairs $(S_2\ ,\ D_2)$, $(S_3\ ,\ D_3)$ over $\mathbb{R}_{\mathcal{P}}(s)$. Define :

$$\Theta_{i1} = T_i\cdot M - C_i\cdot W_{i1}\cdot N\ ,\ \Omega_{i1} = T_i\cdot P_r + C_i\cdot W_{i1}\cdot Q_r\ ,\ i = 2\ ,\ 3 \qquad (2.6.12)$$

$$\Theta_{1j} = K\cdot S_j - L\cdot W_{1j}\cdot D_j\ ,\ \Omega_{1j} = R_l\cdot S_j + Q_l\cdot W_{1j}\cdot D_j\ ,\ j = 2\ ,\ 3 \qquad (2.6.13)$$

over $\mathbb{R}_{\mathcal{P}}(s)$. The latter can be used to give simpler definitions of the admissible off – diagonal closed loop transfer matrices , [Ozg. 1] :

$$\hat{P}_{23}(X) = C_2^{-1}\cdot(T_2\cdot\Theta_{13}+\Theta_{21}\cdot S_3 - \Theta_{21}\cdot Q\cdot\Theta_{13}+C_2\cdot W_{23}\cdot D_3 - \Omega_{21}\cdot X\cdot\Omega_{13})\cdot D_3^{-1} \qquad (2.7.14)$$

$$\hat{P}_{32}(X) = C_3^{-1}\cdot(T_3\cdot\Theta_{12}+\Theta_{31}\cdot S_2 - \Theta_{31}\cdot Q\cdot\Theta_{12}+C_3\cdot W_{32}\cdot D_2 - \Omega_{31}\cdot X\cdot\Omega_{12})\cdot D_2^{-1} \qquad (2.7.15)$$

We can now state some solvability conditions for the NICP and NCISP .

Theorem (2.7.1) [Ozg. 1] *: The NICP is solvable , if and only if there exists* $X \in \mathbb{R}_{pr}^{(r+m)x(r+p)}(s)$ *satisfying equations :*

$$P_{23} = P_{21}\cdot X\cdot P_{13}\ ,\ P_{32} = P_{31}\cdot X\cdot P_{12} \qquad (2.7.16)$$

□

Using the expressions (2.7.14) , (2.7.15) for admissible off – diagonal , closed loop transfer matrices , it is straightforward to state a similar result to theorem (2.7.1) for the NCISP :

Theorem (2.7.2) [Ozg. 1] : *The NCISP is solvable , if and only if there exists* $X \in \mathbb{R}_{\mathfrak{P}}^{(r+m)x(r+p)}(s)$ *satisfying equations :*

$$\Omega_{21} \cdot X \cdot \Omega_{13} = T_2 \cdot \Theta_{13} + \Theta_{21} \cdot S_3 - \Theta_{21} \cdot Q \cdot \Theta_{13} + C_2 \cdot W_{23} \cdot D_3 \qquad (2.7.17)$$

$$\Omega_{31} \cdot X \cdot \Omega_{12} = T_3 \cdot \Theta_{12} + \Theta_{31} \cdot S_2 - \Theta_{31} \cdot Q \cdot \Theta_{12} + C_3 \cdot W_{32} \cdot D_2 \qquad (2.7.18)$$

□

An alternative solvability condition for the NCISP is given next . Define the matrices :

$$\bar{\Pi}_{21} = \begin{bmatrix} Q & R \\ -T_2 & C_2 \cdot W_{21} \end{bmatrix}, \bar{\Pi}_{13} = \begin{bmatrix} Q & S_3 \\ -Z & W_{13} \cdot D_3 \end{bmatrix}, \bar{\Pi}_{23} = \begin{bmatrix} Q & S_3 \\ -T_2 & C_2 \cdot W_{23} \cdot D_2 \end{bmatrix} \qquad (2.7.19)$$

$$\bar{\Pi}_{31} = \begin{bmatrix} Q & R \\ -T_3 & C_3 \cdot W_{31} \end{bmatrix}, \bar{\Pi}_{12} = \begin{bmatrix} Q & S_2 \\ -Z & W_{12} \cdot D_2 \end{bmatrix}, \bar{\Pi}_{32} = \begin{bmatrix} Q & S_2 \\ -T_3 & C_3 \cdot W_{32} \cdot D_2 \end{bmatrix} \qquad (2.7.20)$$

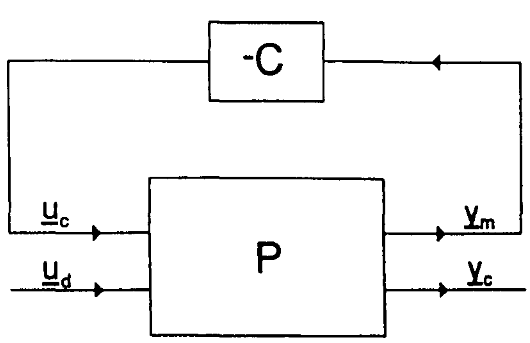
Theorem (2.7.3) [Ozg. 1] : *The DDISP is solvable , if and only if there exists an* $\widehat{X} \in \mathbb{R}_{\mathfrak{P}}^{(r+m)x(r+p)}(s)$ *satisfying equations :*

$$\bar{\Pi}_{23} = \bar{\Pi}_{21} \cdot \widehat{X} \cdot \bar{\Pi}_{13} \ , \ \bar{\Pi}_{32} = \bar{\Pi}_{31} \cdot \widehat{X} \cdot \bar{\Pi}_{12} \qquad (2.7.21)$$

□

2.8. THE REGULATOR PROBLEM AND THE MATRIX EQUATION A·X·B + C·Y·D = E

Consider a linear , multivariable , continuous time , time invariant , control system associated with the following feedback scheme :



where , $P \in \mathbb{R}_{pr}^{(p+q)x(m+n)}(s)$ represents the plant and $C \in \mathbb{R}_{pr}^{mxp}(s)$ the compensator transfer matrices respectively and :

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad (2.8.1)$$

with , $P_{11} \in \mathbb{R}_{pr}^{pxm}(s)$, $P_{12} \in \mathbb{R}_{pr}^{pxn}(s)$, $P_{21} \in \mathbb{R}_{pr}^{qxm}(s)$, $P_{22} \in \mathbb{R}_{pr}^{qxn}(s)$ and P_{11} is strictly proper in order to avoid complications concerning the well defined nature of the feedback loop , when a feedback is applied . For references on the regulator problem see section 2.1 .

Regulator Problem with Internal Stability , (RPIS) , [Ozg. 1] : *Given the two channel plant introduced in (2.8.1) , determine a compensator C such that , in the closed loop system the pair (P_{11} , C) is internally stable and the disturbance input to controlled output transfer matrix , P_{dc} , given by :*

$$P_{dc} = P_{22} - P_{21} \cdot C \cdot (I + P_{11} \cdot C)^{-1} \cdot P_{12} \qquad (2.8.2)$$

is over $\mathbb{R}_{\mathfrak{P}}^{qxn}(s)$. □

The regulator objective $P_{dc} \in \mathbb{R}_{\mathfrak{P}}^{qxn}(s)$, ensures that the closed loop system is bounded input , bounded output stable . Thus , if the regulator objective is achieved , then (in time domain) for all inputs $\underline{u}_d$ generated by stable dynamics , the output $\underline{y}_c$ will asymptotically approach zero . The flexibility in choosing the area of stability $\mathfrak{P}^c$ allows us to consider continuous time , as well as , discrete time systems and also to adjust the speed of convergence to zero of state and output variables in the closed loop system . Recall from section 2.6 that if a bicoprime fraction representation of the plant , P , controller , C , transfer matrices is given by :

$$P = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \cdot Q_{11}^{-1} \cdot [\, R_1 \, , \, R_2 \,] + \begin{bmatrix} W & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \qquad (2.8.3)$$

$$C = Z_c \cdot Q_c^{-1} \cdot R_c \qquad (2.8.4)$$

respectively as well as a bicoprime fraction representation of P_{11} is :

$$P_{11} = Z \cdot Q^{-1} \cdot R + W \qquad (2.8.5)$$

then the set of disturbance input to controlled output transfer matrices , P_{dc} , admissible for internal stability of the closed loop system is given by :

$$\mathfrak{P}_{dc}^{is} = \Big\{ P_{dc}(X) = [Z_2 \, , \; -W_{21} \cdot (N \cdot N_l + Q_r \cdot X)] \cdot$$

$$\cdot\begin{bmatrix} Q_{11} & R_l\cdot(N\cdot N_l+Q_r\cdot X) \\ -Z_1 & M_l+Z\cdot M\cdot N_l-Z\cdot P_r\cdot X\end{bmatrix}^{-1}\cdot\begin{bmatrix} R_2 \\ W_{12}\end{bmatrix}+W_{22}\ ,\forall\ X\in\mathbb{R}_{\mathcal{P}}^{mxp}(s)\Big\} \tag{2.8.6}$$

or equivalently ,

$$\mathcal{P}_{dc}^{is}=\Big\{\ P_{dc}(X)=\bar{C}_1^{-1}\cdot(T\cdot\Theta_{12}+\Theta_{21}\cdot S-\Theta_{21}\cdot Q\cdot\Theta_{12}+\bar{C}_1\cdot W_{22}\cdot\bar{D}-\Omega_{21}\cdot X\cdot\Omega_{12})\cdot\bar{D}^{-1}\ ,$$

$$,\ \forall\ X\in\mathbb{R}_{\mathcal{P}}^{mxp}(s)\Big\} \tag{2.8.7}$$

with K , L , M , N , Q_l , R_l , P_r , Q_r , M_l , N_l , K_r , L_r , Θ_{12} , Θ_{21} , Ω_{21} , Ω_{12} , $\bar{C}_1$, T , S $\bar{D}$ defined in section 2.6 . The RPIS can now be reduced to determining a matrix $X\in\mathbb{R}_{\mathcal{P}}^{mxp}(s)$ such that :

$$P_{dc}(X)\in\mathbb{R}_{\mathcal{P}}^{qxn}(s) \tag{2.8.8}$$

***Proposition (2.8.1) [Ozg. 1]** : Consider the matrix equation :*

$$\Omega_{21}\cdot X\cdot\Omega_{12}+\bar{C}_1\cdot Y\cdot\bar{D}=T\cdot\Theta_{12}+\Theta_{21}\cdot S-\Theta_{21}\cdot Q\cdot\Theta_{12} \tag{2.8.9}$$

***i)** RPIS is solvable , if and only if there exist matrices $X\in\mathbb{R}_{\mathcal{P}}^{mxp}(s)$, $Y\in\mathbb{R}_{\mathcal{P}}^{qxn}(s)$ satisfying (2.8.9) .*

***ii)** The set of all solutions of the RPIS is given by :*

$$\mathcal{S}_c^{rpis}=\{\ C_{cr}(X)\ :\ X\in\mathbb{R}_{\mathcal{P}}^{mxp}(s)\ \text{and}\ (X\ ,\ Y)\ \text{satisfies (2.8.9) for some}\ Y\in\mathbb{R}_{\mathcal{P}}^{qxn}(s)\ \} \tag{2.8.10}$$

where , $C_{cr}(X)=P_{cr}(X)\cdot Q_{cr}^{-1}(X)$ and

$$\begin{bmatrix} Q_{cr}(X) \\ P_{cr}(X)\end{bmatrix}=\begin{bmatrix} M_l+(Z\cdot M-W\cdot N)\cdot N_l & -(Z\cdot P_r+W\cdot Q_r) \\ N\cdot N_l & Q_r\end{bmatrix}\cdot\begin{bmatrix} I \\ X\end{bmatrix} \tag{2.8.11}$$

***iii)** The set of admissible transfer matrices P_{dc} for the RPIS is given by :*

$$\mathcal{P}_c^{rpis}=\{\ Y+W_{22}\ :\ Y\in\mathbb{R}_{\mathcal{P}}^{qxn}(s)\ \text{and}\ (X\ ,\ Y)\ \text{satisfies (2.8.9)}\ \} \tag{2.8.12}$$

□

The next result improves proposition (2.8.1) by eliminating the matrices K , L , M , N , (that occur in Θ_{12} , Θ_{21} , Ω_{21} , Ω_{12}) , from the solvability conditions . Let :

$$\bar{\Pi}_{12} = \begin{bmatrix} Q & S \\ -Z & W_{12}\cdot\bar{D} \end{bmatrix}, \bar{\Pi}_{21} = \begin{bmatrix} Q & R \\ -T & \bar{C}_1\cdot W_{21} \end{bmatrix}, \bar{\Pi}_{22} = \begin{bmatrix} Q & S \\ -T & \bar{C}_1\cdot W_{12}\cdot\bar{D} \end{bmatrix} \tag{2.8.13}$$

$$\bar{\Gamma}_1 = \begin{bmatrix} I & O \\ O & \bar{C}_1 \end{bmatrix}, \bar{\Delta} = \begin{bmatrix} I & O \\ O & \bar{D} \end{bmatrix}$$

***Theorem (2.8.1) [Ozg. 1]** : The RPIS is solvable , if and only if :*

***i)** The system (Z_1 , Q_{11} , R_1 , W) is free of unstable input , output decoupling zeroes .*

***ii)** There exist matrices $\bar{X} \in \mathbb{R}_{\mathcal{P}}^{(r+m)x(r+p)}(s)$, $\bar{Y} \in \mathbb{R}_{\mathcal{P}}^{(r+q)x(r+n)}(s)$ satisfying :*

$$\bar{\Pi}_{21}\cdot\bar{X}\cdot\bar{\Pi}_{12} + \bar{\Gamma}_1\cdot\bar{Y}\cdot\bar{\Delta} = \bar{\Pi}_{22} \tag{2.8.14}$$

□

The following results refer to the solvability of the RPIS in terms of bilateral matrix equations . Let :

$$\Pi_{12}^{C_1} = \begin{bmatrix} D\cdot Q & R_2 \\ -Z & W_{12} \end{bmatrix}, \Pi_{21}^{D_1} = \begin{bmatrix} Q_0\cdot C & R_0 \\ -Z_2 & W_{21} \end{bmatrix} \tag{2.8.15}$$

$$\Gamma_1 = \begin{bmatrix} C_1 & O \\ O & I \end{bmatrix}, \Delta = \begin{bmatrix} D_1 & O \\ O & I \end{bmatrix} \tag{2.8.16}$$

with $C_1 = gcrd(Z, Q_{11})$, $D_1 = gcld(Q_{11}, R)$ and (R_0 , Q_0) such that if $Q_{11} = D_1\cdot Q_2$, $C = gcrd(Z_1, Q_2)$, then $R_1 = D_1\cdot R_0$, $Q_2 = Q_0\cdot C$.

***Theorem (2.8.2) [Ozg. 1]** : The RPIS is solvable , if and only if :*

***i)** The system (Z_1 , Q_{11} , R_1 , W) is free of unstable input , output decoupling zeroes .*

***ii)** There exist matrices $X^0 \in \mathbb{R}_{\mathcal{P}}^{(r+p)x(r+n)}(s)$, $Y^0 \in \mathbb{R}_{\mathcal{P}}^{(r+n)x(r+p)}(s)$ satisfying :*

$$X^0\cdot\Pi_{12}^{C_1} + \Gamma_1\cdot Y^0 = I \tag{2.8.17}$$

***iii)** There exist matrices $X_0 \in \mathbb{R}_{\mathcal{P}}^{(r+m)x(r+q)}(s)$, $Y_0 \in \mathbb{R}_{\mathcal{P}}^{(r+q)x(r+m)}(s)$ satisfying :*

$$\Pi_{21}^{D_1}\cdot X_0 + Y_0\cdot\Delta_1 = I \tag{2.8.18}$$

□

It is clear that the solvability of the RPIS is associated with the matrix equation $A\cdot X\cdot B + C\cdot Y\cdot D = E$.

2.9. CONCLUSIONS

A survey of control synthesis problems , the solvability of which is associated with the solvability and characterization of solutions , (or special types of them) , of certain matrix equations over the ring of interest has been presented in this chapter . Central to all these problems has been the concept of stability of a linear system . A brief account of stability and especially the constraints imposed on a system so that external stability is equivalent to internal stability has been introduced in section 2.2 . The first problem reviewed has directly risen from the concept of stability itself , and it is the centralized stabilization problem , (CSP) . This problem has been associated with the standard matrix Diophantine equation over the ring of interest , $\mathbb{R}_{\mathfrak{P}}(s)$, or $\mathbb{R}[s]$, and the study of special types of solutions of it have been related to the total finite settling time stabilization , (TFSTS) , and dead – beat response , (DBRP) , problems . The case of imposing restrictions on the stabilizing controllers structure has been presented next . These structural constraints lead to the formulation of the decentralized stabilization problem , (DSP) , and to the investigation for special block diagonal structured solutions of the standard matrix Diophantine equation associated with the CSP .

In section 2.5 the model matching problem has been presented and formulated via the matrix equations $A \cdot X = B$, $Y \cdot A = B$. The latter matrix equation is fundamental to the study of many other matrix equations and central to the model matching problem . Problems that require the independence of certain outputs from certain inputs have been also reviewed . The disturbance decoupling and noninteracting control problems have been formulated and their solvability has been shown to be related to the matrix equation $A \cdot X \cdot B = C$. Finally the bilateral matrix equations $A \cdot X + Y \cdot B = C$, $X \cdot A + B \cdot Y = C$ and their generalization $A \cdot X \cdot B + C \cdot Y \cdot D = E$ have been presented and associated with the solvability of the regulator problem with internal stability requirement for the closed loop feedback system .

CHAPTER 3

THE RING OF PROPER AND STABLE RATIONAL FUNCTIONS : COMPUTATIONAL ISSUES

3.1. INTRODUCTION

Problems of linear systems theory , such as , stability and performance of linear multivariable control systems , have motivated the study of matrices having elements in special rings that describe in an algebraic sense these properties . Stability and to a certain extent the performance of a control system , for example , can be characterized by absence of poles from its transfer function matrix from a prescribed symmetric – with respect to a real axis – region Ω of the finite complex plane .

The algebraic structure of the set $\mathbb{R}_{\mathfrak{P}}(s)$ of proper rational functions which have no poles inside a region $\mathfrak{P} = \Omega \cup \{\infty\}$, $(\Omega \subset \mathbb{C})$ has been examined initially by Morse , [Mor. 1] . Subsequently Hung and Anderson , [Hun. 1] , showed that with an appropriately defined "degree" function the set $\mathbb{R}_{\mathfrak{P}}(s)$ has the structure of a Euclidean ring , [Var. 2] , [Var. 5] , [Var. 6] . This important result has been the basis for the subsequent work of Vidyasagar , [Vid. 1] , Francis and Vidyasagar , [Fra. 1] , Desoer , Liu , Murrey and Saeks , [Des. 1] , Saeks and Murrey , [Sae. 1] , Vidyasagar , Schneider and Francis , [Vid. 2] , Vidyasagar and Viswanadham , [Vid. 3] , Francis and Vidyasagar , [Fra. 2] , Saeks and Murrey , [Sae. 2] , on "fractional representation" of proper rational matrices and their use to analysis and synthesis problems. The detailed structure of the set $\mathbb{R}_{\mathfrak{P}}(s)$ has been thoroughly investigated in [Var. 3] , [Var. 5] .

Among the algebraic properties of $\mathbb{R}_{\mathfrak{P}}(s)$, the one that plays crucial role in our study is that of the Euclidean ring , or in other words , the existence of a Euclidean division . This division helps to specify the family of stabilizing controllers with the least number of unstable poles among the family of all stabilizing controllers of an unstable , linear , time invariant , multivariable control system , as well as it can be generalized , [Vid. 4], in the case of square matrices with entries in $\mathbb{R}_{\mathfrak{P}}(s)$.

In [Vid. 4] and [Var. 3] has been noticed that the pair of quotient and remainder of a Euclidean division in $\mathbb{R}_{\mathfrak{P}}(s)$ is not characterized by a uniquely defined "Euclidean degree" and the family of least possible "Euclidean degree" remainders is introduced . A quite tedious construction of this family by using the interpolation theorem of [You. 1] , as well as an existence approach by using interpolation in a Disc Algebra can be found in [Vid. 4] . Our aim in this chapter is to give an algorithmic construction of the family of least "Euclidean degree" remainders and present its powerful involvement in the construction of the family of least number unstable poles stabilizing controllers of an unstable , linear , time invariant , MIMO , system . More precisely , in section 3.2 the ring of proper and stable functions is introduced ; in section 3.3 a unique modulo a real number of $\mathfrak{P}^c$ factorization for the elements of $\mathbb{R}_{\mathfrak{P}}(s)$ is introduced and in section 3.4 the Euclidean division as well as its non uniqueness of remainder is examined . The motivation for the use of unit interpolation in the following sections is given at the end . In section 3.5 the interpolation by unit in $\mathbb{R}_{\mathfrak{P}}(s)$ is examined , by using the concept of the logarithm of an element of a Banach algebra and introducing a special type of Banach algebra the Disc Algebra of symmetric analytic functions , which map a Disc onto $\mathbb{C}$. Two approaches for the construction of an interpolating unit in $\mathbb{R}_{\mathcal{P}}(s)$ are given and lead to two algorithmic constructions of the least "Euclidean degree" family of remainders in section 3.6 . A comparison between the two methods gives the more efficient one .

Finally , in section 3.7 a generalization of the Euclidean division between square matrices with entries proper and stable functions is introduced . As an application of the knowledge of the family of least "Euclidean degree" remainders of a Euclidean division between two elements of $\mathbb{R}_{\mathcal{P}}(s)$, the construction of the least number unstable poles family of stabilizing controllers is described .

3.2. THE RING OF PROPER AND STABLE FUNCTIONS

Let $\mathbb{R}[s]$ be the ring of polynomials with real coefficients and $\mathbb{R}(s)$ the field of rational functions $t(s) = n(s)/d(s)$, with $n(s)$, $d(s) \in \mathbb{R}[s]$, $d(s) \neq 0$, $s \in \mathbb{C} \cup \{\infty\}$. Given a rational function $t(s) = n(s)/d(s)$ with $n(s)$, $d(s)$ coprime ; it can be written :

$$t(s) = \left(\frac{1}{s}\right)^{q_\infty} \frac{n_1(s)}{d_1(s)} \text{ , with } \deg(n_1(s)) = \deg(d_1(s)) \tag{3.2.1}$$

with :

$$q_\infty = \deg(d(s)) - \deg(n(s)) \tag{3.2.2}$$

Definition (3.2.1) *: Given a rational function t(s) in the form (3.2.1) :*

i) t(s) is called proper if $q_\infty \geq 0$.

ii) t(s) is called strictly proper if $q_\infty > 0$.

iii) If t(s) as well as its multiplicative inverse are proper then t(s) is called biproper. □

Let $\mathbb{C}$ be the field of complex numbers . Assume $\mathbb{P}$ a symmetric subset of $\mathbb{C}$ which excludes at least one point $\alpha \in \mathbb{R}$. Regarding a $t(s) \in \mathbb{R}(s)$ it can be factorized as follows :

$$t(s) = \frac{n_{\mathbb{P}}(s)}{d_{\mathbb{P}}(s)} \frac{n_{\mathbb{P}^c}(s)}{d_{\mathbb{P}^c}(s)} \tag{3.2.3}$$

with $n_{\mathbb{P}}(s)$, $d_{\mathbb{P}}(s)$ coprime polynomials in $\mathbb{R}[s]$ with their zeros not outside $\mathbb{P}$, $n_{\mathbb{P}^c}(s)$, $d_{\mathbb{P}^c}(s)$ coprime polynomials in $\mathbb{R}[s]$ with their zeros outside $\mathbb{P}$ and let $\mathcal{P} = \mathbb{P} \cup \{\infty\}$.

Definition (3.2.2) *: A rational function t(s) in $\mathbb{R}(s)$ is called $\mathcal{P}$-stable , if all the zeros of its denominator are outside $\mathbb{P}$ and $q_\infty \geq 0$.* □

$$\text{Let } \mathbb{R}_{\mathcal{P}}(s) = \{\, t(s) \in \mathbb{R}(s) : t(s) \text{ is } \mathcal{P}\text{-stable} \} \tag{3.2.4}$$

If addition and multiplication of two functions in $\mathbb{R}_{\mathcal{P}}$ are defined pointwise then it is known [Hun. 1] , [Kar. 1] that $\mathbb{R}_{\mathcal{P}}(s)$ is an integral domain .

Definition (3.2.3) *: An integral domain* $\mathcal{R}$ *is said to be a Euclidean Domain (or Ring) if there exists a function* γ *(the Euclidean Valuation or Degree) such that the following conditions are satisfied :*

i) $\gamma : \mathcal{R}-\{0\} \to \mathbb{Z}_{\geq 0}$, ($\mathbb{Z}_{\geq 0}$ *the set of nonnegative integers)* *(3.2.5)*

ii) *For all a , b in* $\mathcal{R}-\{0\}$ $\gamma(a \cdot b) \geq \gamma(a)$ *(3.2.6)*

iii) *For all a , b in* $\mathcal{R}$ *with* $b \neq 0$ *there exist elements q , r in* $\mathcal{R}$ *(the quotient and remainder respectively) such that :*

$$a = b\,q + r \tag{3.2.7}$$

where either $r = 0$ *or else* $\gamma(r) < \gamma(b)$. □

Let $t(s) \in \mathbb{R}_{\mathcal{P}}(s)$. Then by (3.2.3) and definition (3.2.2) t(s) can be factorized as follows:

$$t(s) = n_{\mathbf{P}}(s)\, \frac{n_{\mathcal{P}^c}(s)}{d_{\mathcal{P}^c}(s)} \tag{3.2.8}$$

Define now the function $\gamma_{\mathcal{P}} : \mathbb{R}_{\mathcal{P}}(s) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that:

$$\gamma_{\mathcal{P}} = \begin{cases} \deg\big(d_{\mathcal{P}^c}(s)\big) - \deg\big(n_{\mathcal{P}^c}(s)\big), & \text{if } t(s) \neq 0 \\ \infty & \text{, if } t(s) = 0 \end{cases} \tag{3.2.9}$$

Our next step is to define a Euclidean Division in $\mathbb{R}_{\mathcal{P}}(s)$ and show that $\mathbb{R}_{\mathcal{P}}(s)$ is a Euclidean Domain with $\gamma_{\mathcal{P}}$ serving as a Euclidean Valuation (Degree) . In order to proceed so we have to present a procedure for factorization in $\mathbb{R}_{\mathcal{P}}(s)$, [Kar. 1] , [Var. 1] , [Var. 2] .

3.3. FACTORIZATION IN THE RING OF PROPER AND STABLE FUNCTIONS

Consider a t(s) in $\mathbb{R}_{\mathcal{P}}(s)$. It can always be factorized as in (3.2.8) . By (3.2.2) , (3.2.8) (3.2.9) is implied that :

$$q_{\infty} = \deg\big(d_{\mathcal{P}^c}(s)\big) - \deg\big(n_{\mathbf{P}}(s) \cdot n_{\mathcal{P}^c}(s)\big) =$$

$$= \deg\big(d_{\mathcal{P}^c}(s)\big) - \deg\big(n_{\mathcal{P}^c}(s)\big) - \deg\big(n_{\mathbf{P}}(s)\big) \overset{(3.2.9)}{=}$$

$$= \gamma_{\mathfrak{P}}(t(s)) - \deg\left(n_{\mathbb{P}}(s)\right) \tag{3.3.1}$$

By (3.3.1) we take :

$$q = \gamma_{\mathfrak{P}}(t(s)) = q_{\infty} + \deg\left(n_{\mathbb{P}}(s)\right) \tag{3.3.2}$$

Now by (3.2.8) and for $\alpha > 0$, $t(s)$ can be written as :

$$t(s) = \frac{n_{\mathbb{P}}(s)}{(s+\alpha)^q} \cdot \left\{ \frac{n_{\mathfrak{P}^c}(s)}{d_{\mathfrak{P}^c}(s)} \cdot (s+\alpha)^q \right\} \tag{3.3.3}$$

with both $u(s) = \{(n_{\mathfrak{P}^c}(s)/d_{\mathfrak{P}^c}(s)) \cdot (s+\alpha)^q\}$ and its multiplicative inverse $\mathfrak{P}$ – stable .

***Definition (3.3.1)** : Let t(s) in* $\mathbb{R}_{\mathfrak{P}}(s)$ *and* $t^{-1}(s)$ *its multiplicative inverse . If both t(s) and* $t^{-1}(s)$ *are* $\mathfrak{P}$*-stable then t(s) is called a unit in* $\mathbb{R}_{\mathfrak{P}}(s)$. □

Denote $d' = \deg\left(n_{\mathbb{P}}(s)\right)$. By (3.3.2) , (3.3.3) is implied that :

$$t(s) = \frac{n_{\mathbb{P}}(s)}{(s+\alpha)^{d'}} \; \frac{1}{(s+\alpha)^{q_\infty}} \, u(s) \tag{3.3.4}$$

By (3.3.4) and by factorizing $n_{\mathbb{P}}(s)$ into irreducible factors over $\mathbb{R}[s]$ as :

$$n_{\mathbb{P}}(s) = \kappa\, (s + l_1)^{n_1} \cdot \ldots \cdot (s + l_\nu)^{n_\nu} \cdot (s^2 + b_1 s + c_1)^{n'_1} \cdot \ldots \cdot (s^2 + b_\rho\, s + c_\rho)^{n'_\rho} \tag{3.3.5}$$

we have that: $t(s) =$

$$= \kappa \cdot \left[\frac{(s+l_1)}{(s+\alpha)}\right]^{n_1} \cdots \left[\frac{(s+l_\nu)}{(s+\alpha)}\right]^{n_\nu} \cdot \left[\frac{(s^2+ b_1\, s+ c_1)}{(s+\alpha)^2}\right]^{n'_1} \cdots \left[\frac{(s^2+ b_\rho\, s+ c_\rho)}{(s+\alpha)^2}\right]^{n'_\rho} \cdot \left[\frac{1}{(s+\alpha)^{q_\infty}}\right] \cdot$$

$$\cdot u(s) = \left[p_1(s)\right]^{n_1} \cdots \left[p_\nu(s)\right]^{n_\nu} \cdot \left[p'_1(s)\right]^{n'_1} \cdots \left[p'_\rho(s)\right]^{n'_\rho} \cdot \left[p_*(s)\right]^{q_\infty} \cdot u(s) \;,\; -\alpha \in \mathfrak{P}^c, \text{ real} \tag{3.3.6}$$

The uniqueness of factorization of $n_{\mathbb{P}}(s)$ implies that the one in (3.3.6) is also a unique factorization of $t(s)$ over $\mathbb{R}_{\mathfrak{P}}(s)$, modulo α and units . The elements $p_i(s)$, $p'_j(s)$, $p_*(s)$ with $i \in \{1 , \ldots , \nu\}$, $j \in \{1 , \ldots , \rho\}$ are the primes of $t(s)$. By (3.2.9) we observe that :

$$\begin{cases} \gamma_{\mathfrak{P}}(p_i(s)) = 1 \,,\; i \in \{1 , \ldots , \nu\} \\ \gamma_{\mathfrak{P}}(p'_j(s)) = 2 \,,\; j \in \{1 , \ldots , \rho\} \\ \gamma_{\mathfrak{P}}(p_*(s)) = 1 \end{cases} \tag{3.3.7}$$

We also observe that :

$$\gamma_{\mathfrak{P}}(t_1(s) \cdot t_2(s)) = \gamma_{\mathfrak{P}}(t_1(s)) + \gamma_{\mathfrak{P}}(t_2(s)) \tag{3.3.8}$$

(something we shall prove later in proposition (3.4.1)) . By (3.3.6) , (3.3.7) , (3.3.8) we have that :

$$\gamma_{\mathcal{P}}(t(s)) = \sum_{i=1}^{\nu} n_i + \sum_{j=1}^{\rho} 2\, n'_j + q_\infty \tag{3.3.9}$$

which reveals that $\gamma_{\mathcal{P}}$ expresses the total number of zeros of t(s) in $\mathcal{P}$.

Example (3.3.1) : Let $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$. $t(s) = \dfrac{(s-1)\,(s^2+2s+2)}{(s+1)^4}$, then according to factorization (3.2.6) t(s) is written as :

$$t(s) = \frac{(s-1)}{(s+1)}\,\frac{(s^2+2s+2)}{(s+1)^2}\,\frac{1}{(s+1)} .$$

□

We are presenting now the procedure for carrying out Euclidean division between two elements of $\mathbb{R}_{\mathcal{P}}(s)$, [Hun. 1] , [Kar. 1] .

Proposition (3.3.1) *: Let t(s) in* $\mathbb{R}_{\mathcal{P}}(s)$, $-\alpha \in \mathcal{P}^c$, *real and let us denote by* $w = (1/(s+\alpha))$. *Then t(s) may be expressed as :*

$$t(s) = t'_\alpha(w)\, u_\alpha(s) \tag{3.3.10}$$

where $u_\alpha(s)$ *is a unit in* $\mathbb{R}_{\mathcal{P}}(s)$ *and* $t'_\alpha(w)$ *is a polynomial in* $\mathbb{R}[w]$ *such that* $deg(t'_\alpha(w)) = \gamma_{\mathcal{P}}(t(s))$.

Proof

For any α , such that $-\alpha \in \mathcal{P}^c$, real by (3.3.4) we may write :

$$t(s) = \frac{n_{\mathbf{P}}(s)}{(s+\alpha)^{d'}}\,\frac{1}{(s+\alpha)^{q_\infty}}\,u_\alpha(s) =$$

$$= t'_\alpha(s)\, u_\alpha(s) \tag{3.3.11}$$

Given that $w = (1/(s+\alpha))$, then $s = ((1-\alpha w)/w)$; substituting s in $t'_\alpha(s)$ we have :

$$t'_\alpha(s) = t'_\alpha\left(\frac{1-\alpha w}{w}\right) = w^{d'}\; n_{\mathbf{P}}\left(\frac{1-\alpha w}{w}\right) w^{q_\infty} \tag{3.3.12}$$

If $n_{\mathbf{P}}(s) = \left(a_{d'}\, s^{d'} + \ldots + a_0\right)$ then :

$$n_{\mathbf{P}}\left(\frac{1-\alpha w}{w}\right) = \frac{1}{w^{d'}}\left[a_{d'}\,(1-\alpha w)^{d'} + \ldots + a_0 w^{d'}\right] = \frac{1}{w^{d'}}\, n'_{\mathbf{P}}(w) \tag{3.3.13}$$

where $n'_{\mathbf{P}}(w)$ polynomial in $\mathbb{R}[w]$ with $deg(n'_{\mathbf{P}}(w)) = deg(n_{\mathbf{P}}(s)) = d'$ (3.3.14)

By (3.3.11) , (3.3.12) , (3.3.13) , (3.3.14) is implied that :

$$t(s) = w^{d'} \frac{1}{w^{d'}} n'_{\mathbb{P}}(w)\, w^{q_\infty} u_\alpha(s) = n'_{\mathbb{P}}(w)\, w^{q_\infty} u_\alpha(s) = t'_\alpha(w)\, u_\alpha(s)$$

with : $\deg(t'_\alpha(w)) = \deg(n'_{\mathbb{P}}(w) \cdot w^{q_\infty}) = \deg(n'_{\mathbb{P}}(w)) + \deg(w^{q_\infty}) = d' + q_\infty \overset{(3.2.2)}{=} \gamma_{\mathcal{P}}(t(s))$

□

Remark (3.3.1) *: The transformation* $w = (1/(s+\alpha))$ *maps* $\mathcal{P}$ *onto* $\mathcal{P}_w$ *which is a subset of the* $w-$*plane . If* $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$ *the transformation* $w = (1/(s+\alpha))$ *maps* $\mathcal{P}$ *onto* $\mathcal{P}_w \cup \{0\}$ *which is a closed circle in the* $w-$*plane with centre* $\left((1/2\alpha)\ ,\ 0\right)$ *and radius* $\left(1/2\alpha\right)$ *. If* $\mathcal{P} \subset \mathbb{C}_+ \cup \{\infty\}$ *the above mentioned transformation maps* $\mathcal{P}$ *onto a closed subset of* $\mathcal{P}_w \cup \{0\}$ *.* □

Remark (3.3.2) *: The primes of* $\mathbb{R}_{\mathcal{P}}(s)$ *are transformed under the transformation* $s = ((1-\alpha\ w)/w)$ *into irreducible factors of the polynomials in* $\mathbb{R}[w]$ *with zeros inside* $\mathcal{P}_w$ *. Hence :*

$$p_*(s) = \frac{1}{(s+\alpha)} = w$$

$$p_i(s) = \frac{(s+l)}{(s+\alpha)} = (l-\alpha)\ w+1$$

$$p'_j(s) = \frac{(s^2 + b\ s + c)}{(s+\alpha)^2} = (\alpha^2 - \alpha\ b + c)\ w^2 + (b-2\ \alpha)\ w + 1$$

□

Definition (3.3.2) *: Let* $t_1(s)$ *,* $t_2(s)$ *be two functions in* $\mathbb{R}_{\mathcal{P}}(s)$ *. We say that* $t_1(s)$ *divides* $t_2(s)$ *if there exists a* $t_3(s)$ *in* $\mathbb{R}_{\mathcal{P}}(s)$ *such that* $t_2(s) = t_1(s) \cdot t_3(s)$ *.* □

Proposition (3.3.2) : If $t_1(s)$ ***,*** $t_2(s) \in \mathbb{R}_{\mathcal{P}}(s)$ ***then*** $t_1(s)$ ***divides*** $t_2(s)$ ***, if and only if the set of zeros of*** $t_1(s)$ ***in*** $\mathcal{P}$ ***is a subset of the set of zeros of*** $t_2(s)$ ***in*** $\mathcal{P}$ ***.***

Proof

(⇒) If $t_1(s)$ divides $t_2(s)$ and we factorize $t_1(s)$ and $t_2(s)$ as in (3.3.6) then all the primes of $t_1(s)$ are also primes of $t_2(s)$ and so the zeros of $t_1(s)$ in $\mathcal{P}$, which are the zeros of its primes , are also zeros of $t_2(s)$ and the necessary condition has been proved .

(⇐) Denote by $\mathcal{Z}_1$, $\mathcal{Z}_2$ the two sets of zeros in $\mathcal{P}$ of $t_1(s)$ and $t_2(s)$ respectively . Let $\mathcal{Z}_1 \subset \mathcal{Z}_2$ then by (3.3.6) , (3.3.9) the set of primes of $t_2(s)$ contains the set of primes of $t_1(s)$ and so there exists $t_3(s) \in \mathbb{R}_{\mathcal{P}}(s)$ such that $t_2(s) = t_1(s) \cdot t_3(s)$, where $t_3(s)$ contains the primes of $t_2(s)$ which differ from the ones of $t_1(s)$ as well as the product $u_2(s) \cdot u_1^{-1}(s)$, where $u_1(s)$, $u_2(s)$ are the units of $t_1(s)$, $t_2(s)$ respectively as they come out from (3.3.6) and $u_1^{-1}(s)$ the multiplicative inverse of $u_1(s)$. □

Proposition (3.3.3) *: Let $t'(w)$ in $\mathbb{R}[w]$, $-\alpha \in \mathcal{P}^c$, real and $\mathcal{P}_w$ be the region of the w-plane defined as the mapping of $\mathcal{P}$ under the transformation $w = (1/(s+\alpha))$. The rational function defined as $t(s) = t'(1/(s+\alpha))$ belongs to $\mathbb{R}_{\mathcal{P}}(s)$ and $\gamma_{\mathcal{P}}(t(s)) \leq \leq deg(t'(w))$. Further more $\gamma_{\mathcal{P}}(t(s))$ is equal to the total number of zeros of $t'(w)$ in $\mathcal{P}_w$.*

Proof

Let $t'(w) = a_d\, w^d + \ldots + a_0$. Then :

$$t(s) = t'\left(\frac{1}{(s+\alpha)}\right) = \frac{1}{(s+\alpha)^d}\left[a_d + \ldots + a_0\,(s+\alpha)^d\right] = \frac{n(s)}{(s+\alpha)^d}$$

and thus $t(s) \in \mathbb{R}_{\mathcal{P}}(s)$. The maximum number of zeros of n(s) is d . Given that n(s) has zeros in $\mathcal{P}^c$ it follows that : $\gamma_{\mathcal{P}}(t(s)) \leq d$. By (3.3.3) we have $t(s) = =\left(n_{\mathbb{P}}(s)/(s+\alpha)^{d'}\right)\cdot u(s)$, where u(s) is a unit , $d' = \gamma_{\mathcal{P}}(t(s))$ and $n_{\mathbb{P}}(s)$ has no zeros outside $\mathbb{P}$ then by proposition (3.3.1) and remark (3.3.1) $\left(n_{\mathbb{P}}(s)/(s+\alpha)^{d'}\right)$ yields under the transformation $s = ((1-\alpha\, w)/w)$ a polynomial p[w] in $\mathbb{R}[w]$ with all its zeros in $\mathcal{P}_w$ and of degree $\gamma_{\mathcal{P}}(t(s))$. □

3.4. EUCLIDEAN DIVISION IN THE RING OF PROPER AND STABLE FUNCTIONS

In the following we introduce a Euclidean division algorithm over the ring of proper and $\mathcal{P}$ stable functions.

Theorem (3.4.1) *: Let $t_1(s)$, $t_2(s) \in \mathbb{R}_{\mathcal{P}}(s)$, $t_2(s) \neq 0$ and let $w = (1/(s+\alpha))$, $-\alpha \in \mathcal{P}^c$, real . If $t_i(w) = t'_{i\alpha}(w)\, u_{i\alpha}(w)$, $i = 1$, 2 are (mod α) factorizations of $t_1(s)$, $t_2(s)$, where $t'_{i\alpha}(w) \in \mathbb{R}[w]$, $u_{i\alpha}(w)$ units in $\mathbb{R}_{\mathcal{P}}(s)$ and $\gamma_{\mathcal{P}}(t_i(s)) = deg\,(t'_{i\alpha}(w))$, then :*

i) *There exist polynomials $q'_\alpha(w)$, $r'_\alpha(w) \in \mathbb{R}[w]$ such that $t'_{1\alpha}(w) = t'_{2\alpha}(w)\cdot q'_\alpha(w) + r'_\alpha(w)$ and either $r'_\alpha(w) = 0$ or else $deg\,(r'_\alpha(w)) \leq deg(t'_{2\alpha}(w))$.*
ii) *The rational functions $q_\alpha(s)$, $r_\alpha(s) \in \mathbb{R}_{\mathcal{P}}(s)$ defined by :*

$$q_\alpha(s) = u_{1\alpha}(s)\cdot\left[u_{2\alpha}(s)\right]^{-1}\cdot q'_\alpha\left(\frac{1}{(s+\alpha)}\right)$$

$$r_\alpha(s) = u_{1\alpha}(s)\cdot r'_\alpha\left(\frac{1}{(s+\alpha)}\right)$$

satisfy the Euclidean division conditions for $t_1(s)$, $t_2(s)$:

$$t_1(s) = t_2(s)\ q_\alpha(s) + r_\alpha(s)$$

and either $r_\alpha(s) = 0$ *or else* $\gamma_{\mathcal{P}}(r_\alpha(s)) < \gamma_{\mathcal{P}}(t_2(s))$.

Proof

The modulus α factorization of $t_1(s)$, $t_2(s)$ has been established by proposition (3.3.1) and for the polynomials $t'_{1\alpha}(w)$, $t'_{2\alpha}(w)$ of the part *i*) of the theorem we know from the theory of polynomials that there exist unique $q'_\alpha(w)$, $r'_\alpha(w)$ such that :

$$t'_{1\alpha}(w) = t'_{2\alpha}(w)\ q'_\alpha(w) + r'_\alpha(w) \tag{3.4.1}$$

and either $r'_\alpha(w) = 0$ or else $\deg(r'_\alpha(w)) < \deg(t'_{2\alpha}(w))$. By multiplying both sides of (3.4.1) by $u_{1\alpha}(s)$ and by setting $w = (1/(s+\alpha))$ we have the following identity :

$$t_1(s) = t'_{1\alpha}(w)\cdot u_{1\alpha}(s) = \left\{u_{1\alpha}(s)\cdot[u_{2\alpha}(s)]^{-1}\cdot q'_\alpha\left(\frac{1}{(s+\alpha)}\right)\right\}\cdot\left\{u_{2\alpha}(s)\cdot t'_{2\alpha}\left(\frac{1}{(s+\alpha)}\right)\right\} + \\ + u_{1\alpha}(s)\cdot r'_\alpha\left(\frac{1}{(s+\alpha)}\right)$$

or ,

$$t_1(s) = t_2(s)\cdot q_\alpha(s) + r_\alpha(s)$$

By proposition (3.3.2) $q_\alpha(s)$, $r_\alpha(s) \in \mathbb{R}_{\mathcal{P}}(s)$ and $\gamma_{\mathcal{P}}(r_\alpha(s)) < \deg(r'_\alpha(w))$. Given that :

$$\deg(r'_\alpha(w)) < \deg(t'_{2\alpha}(w)) = \gamma_{\mathcal{P}}(t_2(s))$$

it follows that $\gamma_{\mathcal{P}}(r_\alpha(s)) < \gamma_{\mathcal{P}}(t_2(s))$. □

Now we can return to the last statement of section 1 that $\gamma_{\mathcal{P}}$ serves as a Euclidean Valuation for $\mathbb{R}_{\mathcal{P}}(s)$.

***Proposition (3.4.1)** : The function* $\gamma_{\mathcal{P}}$ *as it was defined by (3.2.9) is a Euclidean Valuation for* $\mathbb{R}_{\mathcal{P}}(s)$.

Proof

By definition of $\gamma_{\mathcal{P}}$ in (3.2.9) condition (3.2.5) is satisfied . Consider now $t_1(s)$, $t_2(s) \in \mathbb{R}_{\mathcal{P}}(s)$, then by (3.2.8) we take :

$$t_1(s) = n'_{\mathbf{P}}(s)\ \frac{n'_{\mathcal{P}^c}(s)}{d'_{\mathcal{P}^c}(s)}\ ,\quad t_2(s) = n_{\mathbf{P}}(s)\ \frac{n_{\mathcal{P}^c}(s)}{d_{\mathcal{P}^c}(s)}$$

$$\gamma_{\mathcal{P}}(t_1(s)\cdot t_2(s)) = \deg(d'_{\mathcal{P}^c}(s)) + \deg(d_{\mathcal{P}^c}(s)) - \deg(n'_{\mathcal{P}^c}(s)) - \deg(n_{\mathcal{P}^c}(s))$$

$$= \gamma_{\mathcal{P}}(t_1(s)) + \gamma_{\mathcal{P}}(t_2(s))$$

and condition (3.2.6) is satisfied . By theorem (3.4.1) condition (3.2.7) is satisfied . □

Corollary (3.4.1) *: By proposition (3.4.1) we conclude that* $\mathbb{R}_{\mathcal{P}}(s)$ *is a Euclidean Ring .* □

Proposition (3.4.2) *: Let* $\mathcal{R}$ *be a Euclidean Ring . The quotient and the remainder of (3.2.7) , (in definition (3.2.3)) are uniquely defined , if and only if :*

$$\gamma(e+g) \leq max\{\gamma(e) , \gamma(g)\} \qquad (3.4.2)$$

$\forall\ e , g \in \mathcal{R}$, *[Bur. 1] .*

Proof

($\Rightarrow$) Let the quotient and remainder of the Euclidean division of any two elements of $\mathcal{R}$ be uniquely defined . And let (3.4.2) does not hold true , namely $\gamma(e+g) > max\{\gamma(e) , \gamma(g)\}$. Then for e , $(e+g) \in \mathcal{R}$ we take :

$$e = (e+g)\ 0 + e \quad \text{and } \gamma(e) < \gamma(e+g)$$
$$e = (e+g)\ 1 - g \quad \text{and } \gamma(-g) = \gamma(g) < \gamma(e+g)$$

Hence we take two quotients and two remainders for the Euclidean division of e by (e+g) which is a contradiction ; and (3.4.2) holds true .

($\Leftarrow$) Let (3.4.2) holds true $\forall$ e , $g \in \mathcal{R}$. And let e, $g \in \mathcal{R}$ for which :

$$e = g\ q + r\ , (r = 0 \text{ or } \gamma(r) < \gamma(g))$$
$$e = g\ q' + r', (r' = 0 \text{ or } \gamma(r') < \gamma(g))$$

Then $r' - r = g\ (q - q')$ and by (3.1.6) $\gamma(g) \leq \gamma(r' - r) < max\{\gamma(r') , \gamma(-r)\} < \gamma(g)$. This is a contradiction , so r′ must be equal to r and hence q′ must be equal to q . □

Remark (3.4.1) *: When* $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$ *, we consider two functions* $t_1(s) = =(-(2s+1)/(s+1))$ *,* $t_2(s) = ((s+2)/(s+1))$ *. Both* $t_1(s)$ *,* $t_2(s)$ *are units in* $\mathbb{R}_{\mathcal{P}}(s)$ *so* $\gamma(t_1(s)) = \gamma(t_2(s)) = 0$ *, whereas* $\gamma(t_1(s) + t_2(s)) = 1$ *. By proposition (3.4.2) we conclude that the quotient and remainder of the Euclidean division in* $\mathbb{R}_{\mathcal{P}}(s)$ *are not uniquely defined , [Kar. 1] , [Vid. 3] , [Vid. 4] . Similar arguments can be stated when* $\mathcal{P} \subset \mathbb{C}_+ \cup \{\infty\}$. □

Because stability for SISO , lumped , linear systems is studied over the extended right half plane of the complex numbers (or subsets of it) in the following we assume $\mathcal{P} \subseteq$

$\subseteq \mathbb{C}_+ \cup \{\infty\}$. Especially we study the case of $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$; since everything we state for $\mathcal{P}$ holds for the subsets of $\mathcal{P}$ as well (by using remark (3.3.1)) . Proposition (3.4.2) and remark (3.4.1) imply that the quotient and the remainder of the Euclidean division in $\mathbb{R}_{\mathcal{P}}(s)$ are not uniquely defined . What follows is the presentation of an algorithm for the construction of the class of the minimum possible Euclidean degree remainders in $\mathbb{R}_{\mathcal{P}}(s)$. Consider two functions $t_1(s)$ and $t_2(s)$ coprime and take the Euclidean division of $t_1(s)$ by $t_2(s)$:

$$t_1(s) = t_2(s)\, q(s) + r(s) \tag{3.4.3}$$

We can equivalently write $t_1(s) - r(s) = t_2(s)\, q(s)$ and if s_i denotes a zero of $t_2(s)$ over $\mathcal{P}$, with m_i its multiplicity then :

$$\frac{d^j}{(ds)^j}\,(t_1(s_i) - r(s_i)) = 0 \quad , \quad j = 0 , \dots , m_i - 1 \tag{3.4.4}$$

and if we factorize r(s) as in section 3.3 , namely $r(s) = r_\alpha(s)\, u(s)$ (where u(s) is a unit in $\mathbb{R}_{\mathcal{P}}(s)$) then it is implied by (3.4.3) that $(t_1(s_i)\, u^{-1}(s_i))^{(j)} = (r_\alpha(s_i))^{(j)}$, $j = 0 , \dots ,$ $m_i - 1$. Further more we can take:

$$(u^{-1}(s_i))^{(j)} = \frac{r_\alpha(s_i)^{(j)} - \left\{ \sum_{\kappa=0}^{j-1} \binom{j}{\kappa}\, t_1^{(j-\kappa)}(s_i)\;\; u^{-1}(s_i)^{(\kappa)} \right\}}{t_1(s_i)} \;, \quad j = 0 , \dots , m_i - 1 \tag{3.4.5}$$

where $u^{-1}(s_i)^{(0)} = u^{-1}(s_i) = r_\alpha(s_i)/t_1(s_i)$, $i = 1 , \dots , n$, which clearly implies that the search for a least Euclidean degree remainder of the division (3.4.3) is connected with the existence of a unit in $\mathbb{R}_{\mathcal{P}}(s)$, u(s) , such that :

i) $r(s) = r_\alpha(s)\, u(s)$.

ii) (3.4.5) holds true and $r_\alpha(s)$ has the least possible Euclidean degree , (since $\gamma_{\mathcal{P}}(r(s)) =$ $= \gamma_{\mathcal{P}}(r_\alpha(s)))$.

The Euclidean degree of $r_\alpha(s)$ is equal to the number of its zeros in $\mathcal{P}$. Condition (3.4.5) motivates the investigation of the existence of a unit in $\mathbb{R}_{\mathcal{P}}(s)$, u(s) which satisfies given interpolation constraints , [Vid. 4] , [You. 1] . We do so in the next section .

3.5. INTERPOLATION BY UNIT IN THE RING OF PROPER AND STABLE FUNCTIONS

Suppose that $S = \{s_1 , \dots , s_n\}$ is a set of points in $\mathcal{P}$, $M = \{m_1 , \dots , m_n\}$ is a corresponding set of positive integers and $R = \{r_{ij} , j = 0 , \dots , m_i - 1 , i = 1 , \dots , n \}$

is a respective set of complex numbers . We are interested in finding whether or not there exists a unit in $\mathbb{R}_{\mathcal{P}}(s)$ that satisfies the interpolation constraints :

$$\frac{d^j}{(ds)^j} u(s_i) = r_{ij} \ , \ j = 0 \ , \ldots \ , m_i - 1 \ , \ i = 1 \ , \ldots \ , n \qquad (3.5.1)$$

We observe that if s_i is real r_{ij} is real since :

$$\bar{r}_{ij} = \overline{u^{(j)}(s_i)} = u^{(j)}(\bar{s}_i) = u^{(j)}(s_i) = r_{ij}$$

***Theorem (3.5.1) [Vid. 4]** : Let σ_1 , ... , σ_l be distinct nonnegative extended real numbers (that means that at most one of the σ_i can be infinity) and let s_{l+1} , ... , s_n be distinct complex numbers with positive real part . Let $S = \{\sigma_1 , \ldots , \sigma_l , s_{l+1} , \ldots , s_n\}$, $M = \{m_1 , \ldots , m_n\}$ a corresponding set of positive integers and let $R = \{r_{ij} , j = 0 , \ldots , m_i - 1 , i = 1, \ldots , n\}$ be a set of complex numbers with r_{ij} real whenever $j = 0 , \ldots , m_i - 1 , i = 1 , \ldots , l$ and $r_{i0} \neq 0$ for all i (since $r_{i0} = u(s_i) \neq 0$ because u(s) is a unit in $\mathbb{R}_{\mathcal{P}}(s)$) . Under these conditions there exists a unit u(s) in $\mathbb{R}_{\mathcal{P}}(s)$ satisfying the conditions (3.4.1) if and only if the numbers r_{10} , ... , r_{l0} are all of the same sign .* □

In order to prove this theorem we have to introduce the concept of the logarithm of an element of a Banach algebra as well as to state a few essential definitions and results, [Vid. 4] .

***Definition (3.5.1)** : A pair ($\mathcal{B}$, $\|\cdot\|$) is a Banach algebra if :*
i) ($\mathcal{B}$, $\|\cdot\|$) is a Banach space .
ii) $\mathcal{B}$ is an algebra over the real or complex field .
iii) $\forall$ a , b in $\mathcal{B} \Rightarrow \|a \cdot b\| \leq \|a\| \cdot \|b\|$. □

$\mathcal{B}$ is commutative if $a \cdot b = b \cdot a$ for all a , b in $\mathcal{B}$ and has an identity if there is an element 1 in $\mathcal{B}$ such that $1 \cdot b = b \cdot 1 = b$ for all b in $\mathcal{B}$. An element a in $\mathcal{B}$ is a unit of $\mathcal{B}$ if there exists a b in $\mathcal{B}$ such that a b = b a = 1 . The set of units of a commutative Banach algebra U is nonempty since $1 \in U$.

***Definition (3.5.2)** : For each f in $\mathcal{B}$, the element $exp(f) = \sum_{i=0}^{\infty} \frac{f^i}{i!}$ is well defined . An element f in $\mathcal{B}$ is said to have a logarithm , if there exists a g in $\mathcal{B}$ such that f = exp(g)* □

***Remark (3.5.1)** : If $f \in \mathcal{B}$ has a logarithm g then $f \cdot exp(-g) = 1$. So that f must necessarily be a unit of $\mathcal{B}$. Thus only units can have logarithm .* □

What follows holds $\forall$ α real and outside $\mathcal{P}$; so for convenience during the operations we select $\alpha = 1$, $-1 \in \mathbb{R}$ and $-1 \notin \mathfrak{P}$. By remark (3.3.1) we take that $\mathfrak{P}_w = \mathfrak{D}$ where $\mathfrak{D}$ is the closed circle of the w−plane with centre ((1/2) , 0) and radius (1/2) . In the following we state some well known facts from Real and Complex Analysis , which will be used in the proves of lemmas (3.5.1) , (3.5.2) , [Ahl. 1] , [Con. 1] , [Rud. 1] , [Neg. 1].

Definition (3.5.3) *: Let* $\mathcal{G}$ *be a set ,* $(\mathcal{Y}, d)$ *a metric space and* $f_n : \mathcal{G} \to \mathcal{Y}$ *a sequence of functions .* (f_n) *is said to be uniformly Cauchy if* $\forall \epsilon > 0$, $\exists N \in \mathbb{N} : \forall n , m \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon$, $\forall x \in \mathcal{G}$. □

Proposition (3.5.1) *: Let* $\mathcal{G}$ *be a set ,* $(\mathcal{Y}, d)$ *a complete metric space and* $f_n : \mathcal{G} \to \mathcal{Y}$ *a uniformly Cauchy sequence of functions . Then there exists a function* $f : \mathcal{G} \to \mathcal{Y}$ *, such that* $f_n \to f$ *uniformly .* □

Proposition (3.5.2) *: Let* $(\mathcal{S}, \rho)$ *,* $(\mathcal{Y}, d)$ *be metric spaces ,* $f_n : \mathcal{S} \to \mathcal{Y}$ *continuous functions for* $n = 1, 2, \ldots$ *,* $f : \mathcal{S} \to \mathcal{Y}$ *function such that* $f_n \to f$ *uniformly (over* $\mathcal{S}$*) . Then f is a continuous function .* □

Proposition (3.5.3) *: Let* $c : [a, b] \to \mathbb{C}$ *be a curve with length ,* $f_n : c([a, b]) \to \mathbb{C}$ *continuous functions for* $n = 1, 2, \ldots$ *, and* $f : c([a, b]) \to \mathbb{C}$ *such that* $f_n \to f$ *uniformly (over* $c([a, b])$*) . Then f is continuous on* $c([a, b])$ *and* $\int_c f_n \to \int_c f$ *as* $n \to \infty$. □

Theorem (3.5.2) (Cauchy − Goursat) *: Let* Ω *be an open set in* $\mathbb{C}$ *,* Δ *a closed triangle such that* $\Delta \subset \Omega$ *and* $f : \Omega \to \mathbb{C}$ *analytic function . Then* $\int_{\partial\Delta} f(z)\, dz = 0$. □

Theorem (3.5.3) (Morera) *: Let* Ω *be an open set in* $\mathbb{C}$ *, and* $f : \Omega \to \mathbb{C}$ *continuous function such that* $\int_{\partial\Delta} f(z)\, dz = 0$ *for all the closed triangles* $\Delta \subset \Omega$ *. Then f is analytic in* Ω . □

Definition (3.5.4) *: Let* $\mathcal{T} \subset \mathbb{C}$ *and* (f_n) *, f functions defined on* $\mathcal{T}$ *with images in* $\mathbb{C}$ *. The sequence* (f_n) *is said to converge to f uniformly on compact subsets of* $\mathcal{T}$ *, if for every compact subset* $\mathcal{K}$ *of* $\mathcal{T}$ *and for all the* $\epsilon > 0$ *there exists a natural number N , (dependent on* $\mathcal{K}$ *and* ϵ*) , such that :* $| f_n(z) - f(z) | < \epsilon \ \forall n \geq N$ *and* $\forall z \in \mathcal{K}$ □

Remark (3.5.2) *: If* Ω *is an open subset of* $\mathbb{C}$ *,* (f_n) *is a sequence of continuous complex functions defined on* Ω *, and* $f_n \to f$ *uniformly over the compact subsets of* $\mathbb{C}$ *, then f is continuous in* Ω . □

Theorem (3.5.4) (Convergence of Weierstrass) *: Let* Ω *be an open subset of* $\mathbb{C}$ *,* (f_n) *is a sequence of analytic complex functions defined on* Ω *,* $f : \Omega \to \mathbb{C}$ *and* $f_n \to f$ *uniformly* *over the compact subsets of* Ω . *Then* f *is analytic in* Ω .

Proof

From remark (3.5.2) f is continuous in Ω . Let Δ be a closed triangle in Ω . Then Δ is compact and (from proposition (3.5.3) , theorem (3.5.2)) $\int_{\partial\Delta} f(z)\, dz = \lim_{n\to\infty} \int_{\partial\Delta} f_n(z)\, dz =$ $=0$. From theorem (3.5.3) is implied that f is analytic in Ω . □

Theorem (3.5.5) (Maximum modulus theorem) *: Let* Ω *be an open and connected subset of* $\mathbb{C}$ *and* $f : \Omega \to \mathbb{C}$ *analytic no constant function . Then* $|f|$ *has no maximum value in* Ω . □

Lemma (3.5.1) *: Let the set* $\mathcal{A}$ *consists of all the continuous function mapping* $\mathfrak{D}$ *into the complex numbers which have the additional property that they are analytic in the interior of* $\mathfrak{D}$. *If addition and multiplication of two functions are defined pointwise , then* $\mathcal{A}$ *becomes a commutative Banach algebra with identity over the complex field , with the norm* $\|\cdot\|$ *as* $\|f\| = \sup \{|f(w)| \text{ for all } w \text{ in } \partial\mathfrak{D}\}$ *(from the maximum modulus theorem (3.5.5)) .*

Proof

$(\mathcal{A}, \|\ \|)$ is a Banach algebra over the complex field if definition (3.5.1) is satisfied .

i) $(\mathcal{A}, \|\ \|)$ must be a Banach space or equivalently a complete metric space . Using the norm $\|\ \|$ as defined in lemma (3.5.1) a metric $d : \mathcal{A} x \mathcal{A} \to \mathbb{R}^+ \cup \{0\}$ can be created , such that :

$$d(f, g) = \| f - g \| = max \{ | f(w) - g(w) | , \text{ for all w in } \partial\mathfrak{D} \}$$

Clearly d(f , g) is a metric for $\mathcal{A}$. So $(\mathcal{A}, \|\ \|)$ is a metric space . Consider now a sequence of functions f_n of $\mathcal{A}$ which is Cauchy , namely :

$$\forall \epsilon > 0 , \exists N \in \mathbb{N} : \forall n , m \geq N \Rightarrow \| f_n - f_m \| < \epsilon$$

or,

$$\mu = max \{ | f_n(w) - f_m(w) | , \text{ for all w in } \partial\mathfrak{D} \} < \epsilon$$

Then , (since for all w in $\mathfrak{D}$, $| f_n(w) - f_m(w) | < \mu$) ,

$$\forall w \in \mathfrak{D} \text{ and } \forall \epsilon > 0 , \exists N \in \mathbb{N} : \forall n , m \geq N \Rightarrow | f_n(w) - f_m(w) | < \epsilon$$

or equivalently $\| f_n - f_m \| < \epsilon$. So , (f_n) satisfies definition (3.5.3). Because $(\mathbb{C}, |\ |)$ is a complete metric space , by using proposition (3.5.1) , f_n converges uniformly to an f over $\mathfrak{D}$, (and over any compact subset of it) ; f belongs to $\mathcal{A}$ because f is continuous , (proposition (3.5.2) , remark (3.5.2)) and analytic , (theorem (3.5.4)) . As a result $(\mathcal{A}, \|\ \|)$ is a complete metric space .

ii) $\mathcal{A}$ must be an algebra over the complex field . Let the three – tuple $(\mathcal{A}, +, \odot)$ with f+g and $f\odot g$ the pointwise addition and multiplication of the functions of $\mathcal{A}$. Then $(\mathcal{A}, +, \odot)$ is a commutative ring with identity the constant function of $\mathcal{A}$, I : $\mathfrak{D} \to \mathbb{C}$ and $I(w) = 1$. Let the three – tuple $(\mathcal{A}, +, \cdot)$ with f+g the pointwise addition of the functions of $\mathcal{A}$ and $z\cdot f : \mathbb{C}x\mathcal{A} \to \mathcal{A}$, such that $(z\cdot f)(w) = z\cdot f(w)$, $\forall\ w \in \mathfrak{D}$. Then $(\mathcal{A}, +, \cdot)$ is a vector space over the complex field ; additionally if f , g belong to $\mathcal{A}$ and z in $\mathbb{C}$ then $z\cdot(f\odot g) = (z\cdot f)\odot g = f\odot(z\cdot g)$ because for all the w in $\mathfrak{D}$, $z\cdot(f\odot g)(w) = z\cdot f(w)\cdot g(w)$.

iii) $\| f\odot g\| \leq \| f \|\cdot\| g \|$, for all f , g in $\mathcal{A}$. Indeed :

$$\| f\odot g\| = max\{\ | f(w)\cdot g(w) |\ , \text{for all w in } \partial\mathfrak{D}\ \} =$$

$$= max\{\ | f(w) |\cdot| g(w) |\ , \text{for all w in } \partial\mathfrak{D}\ \} \leq \{max\{\ | f(w) |\ , \text{for all w in } \partial\mathfrak{D}\ \}\cdot$$

$$\cdot max\{\ | g(w) |\ , \text{for all w in } \partial\mathfrak{D}\ \}\ \} = \| f \|\cdot\| g \| \qquad \square$$

Lemma (3.5.2) *: Let* $\mathcal{A}_s$ *denote the subset of* $\mathcal{A}$ *consisting of all the symmetric functions i.e.*

$$\mathcal{A}_s = \{\ f \in \mathcal{A} : \overline{f}(\overline{w}) = f(w)\ ,\ \forall\ w \in \mathfrak{D}\} \qquad (3.5.2)$$

Then $\mathcal{A}_s$ *is a commutative Banach algebra with identity over the real field .*

Proof

$\mathcal{A}_s$ is a subset of $\mathcal{A}$. Following the same steps as in the proof of lemma(3.5.1) it is shown that :

i) $(\mathcal{A}_s, \|\ \|)$ is a Banach space . Every Cauchy sequence (f_n) of $\mathcal{A}_s$ is a Cauchy sequence for $\mathcal{A}$, so as in lemma(3.5.1) , *i)* , $f_n \to f$ uniformly and f is analytic . For the sequence (f_n) , (3.5.2) implies that $\overline{f}_n(\overline{w}) = f_n(w)$ for all w in $\mathfrak{D}$. Consider now f(w) ;

$f(w) = \lim_{n\to\infty} f_n(w) = \lim_{n\to\infty} \overline{f}_n(\overline{w}) = \overline{\lim_{n\to\infty} f_n(\overline{w})} = \overline{f}(\overline{w})$, hence f belongs to $\mathcal{A}_s$.

ii) , ***iii)*** Are straight forward , because $\mathcal{A}_s$ is a subset of $\mathcal{A}$. □

Proposition (3.5.4) *: Given f(s) in* $\mathbb{R}(s)$ *define* $g(w) = f((1-w)/w)$ *. Since the bilinear transformation* $w = (1/(s+1))$ *maps* $\mathfrak{P}$ *onto the disc* $\mathfrak{D}$ *, we have that g(w) is a rational function belonging to* $\mathcal{A}_s$ *, if and only if* $f(s) \in \mathbb{R}_{\mathfrak{P}}(s)$ *.*

Proof

i) Let f(s) in $\mathbb{R}_{\mathfrak{P}}(s)$; f is defined on $\mathfrak{P}$ so the domain of $g(w) = f((1-w)/w)$ is $\mathfrak{D}$.

ii) f(s) in $\mathbb{R}_{\mathcal{P}}(s)$ is a rational function so g(w) = f((1−w)/w) is a rational function .
iii) f(s) in $\mathbb{R}_{\mathcal{P}}(s)$ and g(w) = f((1−w)/w) rational over $\mathcal{D}$. g(w) is analytical in the interior of $\mathcal{D}$ since f(s) is analytical in the interior of $\mathcal{P}$. Since g(w) is rational it is either a polynomial or a fraction of polynomials , (with real coefficients) , so from the properties of the conjugate symbol we take $\bar{g}(\bar{w}) = g(w)$.
i') If g(w) is a rational , analytic in the interior of $\mathcal{D}$ and symmetric function of $\mathcal{A}_s$ then f(s) = g(1/(s+1)) is defined over $\mathcal{P}$ (as the image of $\mathcal{D}$ within the transformation w = =(1/(s+1))) .
ii') f(s) is rational since g(w) is rational and f(s) = g(1/(s+1)) .
iii') f(s) is proper , namely the limit of f(s) as s tends to infinity is finite , since if it was not then the limit of g(w) , as w = (1/(s+1)) tends to zero would be infinity . But g(w) is continuous and defined over the compact set $\mathcal{D}$, so | g(w) | is bounded over $\mathcal{D}$. The maximum of | g(w) | is taken on the border of $\mathcal{D}$. Since $0 \in \partial\mathcal{D}$, and | g(0) | is infinity that is a contradiction and thus f(s) is proper .
iv') f(s) is a $\mathcal{P}$-stable function since if f(s) had a pole , s_0 , inside $\mathcal{P}$, then $f(s_0)$ would be infinity . Thus g(w) would be infinity at the w_0 which is the image of s_0 , within the transformation $w_0 = (1/(s_0+1))$. But g(w) is continuous and defined over the compact set $\mathcal{D}$, so | g(w) | is bounded over $\mathcal{D}$. Hence f(s) is a $\mathcal{P}$-stable function . □

***Proposition (3.5.5)** : Whenever f(s) is a unit in* $\mathbb{R}_{\mathcal{P}}(s)$ *then g(w) = f((1−w)/w) is a rational unit in* $\mathcal{A}_s$ *and vice versa .*

Proof
(⇒) Let f(s) be a unit in $\mathbb{R}_{\mathcal{P}}(s)$, then g(w) = f((1−w)/w) is a rational function of $\mathcal{A}_s$ (proposition(3.5.4)) and since for all s in $\mathcal{P}$, f(s) is no zero that implies that for all w in $\mathcal{D}$, $g(w) \neq 0$ and $g^{-1}(w) = (1/g(w))$ is defined over $\mathcal{D}$. Indeed $g \odot g^{-1} = I$ – (I as in (lemma(3.5.1) , *ii*) – since $(g \odot g^{-1})(w) = g(w) \cdot g^{-1}(w) = 1 = I(w)$ for all w in $\mathcal{D}$.
(⇐) Let g(w) be a rational unit of $\mathcal{A}_s$. Then f(s) = g(1/(s+1)) is an element of $\mathbb{R}_{\mathcal{P}}(s)$ – (proposition(3.5.4)) – and f(s) is no zero for all s in $\mathcal{P}$, (since g(w) is no zero for all w in $\mathcal{D}$) . Since g(w) is a unit then (1/g(w)) is no zero for all w in $\mathcal{D}$ and hence (1/f(s)) is no zero for all s in $\mathcal{P}$. The above two results mean that neither the numerator nor the denominator of f(s) can be zero for s in $\mathcal{P}$. Hence f(s) is a unit in $\mathbb{R}_{\mathcal{P}}(s)$. □

In theorem (3.5.1) it is investigated the existence of a condition under which a unit of $\mathbb{R}_{\mathcal{P}}(s)$ satisfies certain interpolation constraints . After having introduced the disc algebra of $\mathcal{A}_s$ and proposition(3.5.5) it is sensible to establish theorem(3.5.1) in its equivalent form ; that is , to establish an equivalent to the condition of theorem(3.5.1) under which a rational unit of $\mathcal{A}_s$ satisfies certain interpolation constraints , [Vid. 4] .

Suppose $\{s_1 , \dots , s_n\}$ is a set of points in $\mathcal{P}$ and $\{m_1 , \dots , m_n\}$ is a corresponding set of positive integers and $\{r_{ij}, j = 0, \dots, m_i-1, i = 1, \dots, n\}$ is a corresponding set of complex numbers . The objective is to determine a unit u(s) in $\mathbb{R}_{\mathfrak{P}}(s)$ such that :

$$\frac{d^j}{(ds)^j} u(s_i) = r_{ij} , j = 0 , \dots , m_i-1 , i = 1 , \dots , n \tag{3.5.2}$$

Consider the transformation $w = (1/(s+1))$. If u(s) is unit of $\mathbb{R}_{\mathfrak{P}}(s)$ satisfying the constraints (3.5.2) , then by proposition(3.5.5) the function $f : \mathfrak{D} \to \mathbb{C}$ with $f(w) = u((1-w)/w)$ be a rational unit in A_s which satisfies the equivalent to the (3.5.2) constraints :

$$f^{(j)}(w_i) = q_{ij} , j = 0 , \dots , m_i-1 , i = 1 , \dots , n \tag{3.5.3}$$

where ,

$$w_i = \frac{1}{s_i+1}$$

$$q_{i0} = f(w_i) = u\left(\frac{1-w_i}{w_i}\right) = u(s_i) = r_{i0}$$

$$q_{ij} = f^{(j)}(w_i) = u^{(j)}\left(\frac{1-w_i}{w_i}\right) \frac{(-1)^j \cdot j!}{w_i^{j+1}} = u^{(j)}(s_i) \frac{(-1)^j \cdot j!}{w_i^{j+1}} = r_{ij} \frac{(-1)^j \cdot j!}{w_i^{j+1}} ,$$

$j = 1 , \dots , m_i-1 , i = 1 , \dots , n$.

The w_i are real whenever s_i are real $i = 1 , \dots , l$. Now we have transformed the problem to an equivalent one of constructing a rational unit $f(w) \in A_s$ which satisfies the constraints (3.5.3) .

Theorem (3.5.6) [Vid. 4] *: Given elements $w_1 , \dots , w_n$ of $\mathfrak{D}$, positive integers $m_1 , \dots , m_n$ and complex numbers q_{ij} , $j = 0 , \dots , m_i-1$, $i = 1 , \dots , n$ suppose $w_1 , \dots , w_l$ are real and $w_{l+1} , \dots , w_n$ are nonreal . Suppose also that q_{ij} is real for all j whenever w_i is real . Under these conditions , there exists a rational unit f(s) of A_s satisfying (3.5.3) , if and only if $q_{10} , \dots , q_{l0}$ are all of the same sign .* □

Before we prove this theorem – and furthermore its equivalent(3.5.1) – we have to introduce some useful lemmata .

Lemma (3.5.3) *: If $f(w) \in A_s$, then it is a unit , if and only if $f(w) \neq 0$ for all w in $\mathfrak{D}$.*

Proof

(⇒) If $f(w) \in A_s$ and is a unit then there exists $g(w) \in A_s$ such that $f \cdot g = g \cdot f = 1$ or $f(w) \cdot g(w) = 1$ for all w in A_s . So $g(w) = (1/f(w))$ which implies $f(w) \neq 0$ for all w in $\mathfrak{D}$.

(⇐) If $f(w) \in A_s$ and $f(w) \neq 0$ for all w in $\mathfrak{D}$ then (1/f(w)) is defined and $\overline{(1/f(\bar{w}))} = (1/f(w))$, hence $(1/f(w)) \in A_s$ and $f(w) \cdot (1/f(w)) = 1$, so f(w) is a unit . □

Lemma (3.5.4) *: Let* $h(w) \in A_s$ *. If* $\|h\| < 1$ *then* $1+h$ *is a unit in* A_s *.*

Proof

$$\|h\| < 1 \Rightarrow 1 - \|h\| > 0 \Rightarrow 1 - \sup\{\ |h(w)| \text{ for all w in } \partial\mathfrak{D}\} > 0$$

Consider now the $|\ h(w) + 1\ |\ \forall\ w \in \mathfrak{D}$, then :

$$\{\ |\ h(w) + 1\ | \geq |\ 1 - |\ h(w)\ |\ | \Leftrightarrow |\ h(w) + 1\ | > 1 - |\ h(w)\ |\ \}\ \forall\ w \in \mathfrak{D}$$

Thus
$$|\ h(w) + 1\ | > 1 - \sup\{\ |\ h(w)\ | \text{ for all w in } \partial\mathfrak{D}\}\ \forall\ w \in \mathfrak{D}$$

and
$$\inf\{\ |\ h(w) + 1\ |\ \ \forall\ w \in \mathfrak{D}\ \} > 1 - \|h\| > 0 \tag{3.5.4}$$

By (3.5.4) we take that $h(w) + 1 \neq 0$ and so by lemma(3.5.3) $1+h(w)$ is a unit . □

Lemma (3.5.5) *: Let* $h(w) \in A_s$ *,* $h(w)$ *be a unit and* $f(w) \in A_s$ *. If* $\|h-f\| < \frac{1}{\|h^{-1}\|}$ *then* $f(w)$ *is also a unit .*

Proof

$$\|h-f\| < \frac{1}{\|h^{-1}\|} \Rightarrow \|1-h^{-1}f\| < 1$$

By lemma(3.5.4) the proof implies that the function $(h^{-1}f-1)+1$ is a unit , or $h^{-1}f$ is a unit and because h^{-1} is a unit f is also a unit . □

Lemma (3.5.6) *: Let* $f(w)$ *be a polynomial of degree* n *in* A_s *,* $w_1, \dots, w_\rho$ *be its distinct roots with multiplicities* $m_1, \dots, m_\rho$ *respectively and* $\sum_{i=1}^{\rho} |\ m_i\ | = n$ *. Then* $\forall w_0 \in \mathfrak{D} \Rightarrow |f^{(j)}(w_0)| < |\ \alpha\ |\ n^j$ *, where* α *is the coefficient of the n power of* $f(w)$ *.*

Proof

From the hypothesis of the lemma we can express f(w) as follows :

$$f(w) = \alpha \cdot \prod_{i=1}^{\rho} (w - w_i)^{m_i} \tag{3.5.5}$$

We know that $|\ (w_0 - w_i)^{m_i}\ | \leq 1$ and then :

$$|\ \alpha\ | \cdot \prod_{i=1}^{\rho} |\ (w_0 - w_i)^{m_i}\ | < |\ \alpha\ | \tag{3.5.6}$$

Consider now :

$$f'(w) = \alpha \cdot \left[\sum_{i=1}^{\rho} \left\{ m_i \cdot (w-w_i)^{m_i - 1} \cdot \prod_{\substack{j=1 \\ j \neq i}}^{\rho} (w-w_j)^{m_j} \right\} \right] \tag{3.5.7}$$

Then :

$$| f'(w) | = | \alpha | \cdot \left| \left[\sum_{i=1}^{\rho} \left\{ m_i \cdot (w-w_i)^{m_i-1} \cdot \prod_{\substack{j=1 \\ j \neq i}}^{\rho} (w-w_j)^{m_j} \right\} \right] \right| \leq$$

$$\leq \left[\sum_{i=1}^{\rho} \left\{ | \alpha | \cdot | m_i | \cdot | (w_0-w_i)^{m_i-1} | \cdot \prod_{\substack{j=1 \\ j \neq i}}^{\rho} | (w_0-w_j)^{m_j} | \right\} \right] \tag{3.5.8}$$

Because $| \alpha | \cdot | (w_0-w_i)^{m_i-1} | \cdot \prod_{\substack{j=1 \\ j \neq i}}^{\rho} | (w_0-w_j)^{m_j} | \leq | \alpha |$ and by (3.5.8) we take :

$$| f'(w_0) | \leq | \alpha | \cdot \sum_{i=1}^{\rho} m_i = | \alpha | \cdot n \tag{3.5.9}$$

We proceed now for the $| f''(w_0) |$. Since we know that $f'(w)$ is a polynomial of degree $n-1$, then following the same steps for $f'(w)$ as we did for $f(w)$ we have that :

$$| f''(w_0) | \leq | \alpha | \cdot n \cdot (n-1) \tag{3.5.10}$$

After j finite steps we take that :

$$| f^{(j)}(w_0) | < | \alpha | \cdot n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-j+1) \tag{3.5.11}$$

If $j \geq n$, then $| f^{(j)}(w_0) | = 0$ and (3.5.11) holds . Hence (3.5.11) holds for all $j \in \mathbb{N}$. By (3.5.11) it is obvious that $| f^{(j)}(w_0) | < | \alpha | n^j$. □

***Remark (3.5.3)** : If f(w) is a polynomial in A_s a method for the estimation of $|| f ||$, using the maximum modulus theorem is given as follows . $|| f || = \sup \{ | f(w)| \text{ for all } w \text{ in } \partial D \}$, in our case :*

$$|| f || = \sup \{ |f(0.5 (1+ e^{i\theta}))| \text{ for all arguments } \theta \}$$

Proceeding we have $|| f || = \sup \{| f(0.5 (1+ [\cos(\theta) + i \sin(\theta)])) | \text{ for all arguments } \theta\}$. Observe now that $| f(0.5 (1+ [\cos(\theta) + i \sin(\theta)])) |$ is a real function of θ and we can find its maximum by studying the change of sign of its second order derivative at θ where its first order derivative vanishes . Because $| f(0.5 (1+ [\cos(\theta) + i \sin(\theta)])) |$ is continuous over the closed disc $\mathfrak{D}$ its maximum value serves as its supremum . □

***Remark (3.5.4)** : Using remark(3.5.3) we can estimate that $|| \exp(g) || \leq \exp || (g) ||$ where $g(w)$ is a polynomial in A_s . By definition(3.5.2) $\exp(g) = \sum_{i=0}^{\infty} (g^i/i!)$, hence exp is a continuous , analytic and bounded function over $\mathfrak{D}$ and from the maximum modulus theorem it takes its maximum , which serves as its supremum , at a $w \in \partial\mathfrak{D}$.*

Consider :

$$\left| \sum_{i=0}^{\infty} \frac{g(w)^i}{i!} \right| \leq \sum_{i=0}^{\infty} \frac{| g(w) |^i}{i!} \; , \; \forall w \in \partial \mathfrak{D}$$

Hence :

$$sup \left\{ \left| \sum_{i=0}^{\infty} \frac{g(w)^i}{i!} \right| , \forall w \in \partial \mathfrak{D} \right\} \leq \left\{ \sum_{i=0}^{\infty} \frac{sup| g(w) |^i}{i!} \right\} \; \forall w \in \partial \mathfrak{D}$$

$$|| \, exp(g) \, || \leq \left\{ \sum_{i=0}^{\infty} \frac{|| \, g \, ||^i}{i!} \right\}$$

Which implies that :

$$|| \, exp(g) \, || \leq exp(\, || \, g || \,) \tag{3.5.12}$$

By (3.5.12) we obtain :

$$\frac{1}{|| \, exp(g) \, ||} \geq \frac{1}{exp \, ||(g) \, ||}$$

□

***Remark** (3.5.5) : In the following (proof of theorem(3.5.6)) we shall need to approach the function* $exp(g(w))$ *, where* $g(w)$ *is a polynomial in* $\mathcal{A}_s$ *, sufficiently close by a polynomial* $p(w) = a_0 + a_1 w + \ldots + a_t w^t$ *. This approach can by achieved by two ways:*

***i)**Express* $exp(g(w))$ *as* $\sum_{i=0}^{\infty} g(w)^i/i!$ *and approach by terms of* $g(w)^i/i!$ *, namely* $p(w) = \sum_{i=0}^{t} g(w)^i/i!$

***ii)**Express* $exp(g(w))$ *as a power series about (0.5 , 0) , [Apo. 1] :*

$$exp(g(w)) = exp(0.5) \sum_{i=0}^{\infty} \frac{(g(w) - 0.5)^i}{i!}$$

and rearrange in terms of increasing w^i *. Then the approach is achieved by polynomials of the form :*

$$p(w) = a_0 + a_1 \, w + \ldots + a_t \, w^t$$

where $p(w)$ *consist of the first t terms of* $exp(g(w))$ *. In the proof that follows we shall present the first approach . In the subsequent remark(3.5.6) we shall present the same procedure for the second approach and in example(3.6.1) we shall compare the two methods .* □

Proof of theorem(3.5.6)

(⇒) If f is a rational unit of A_s satisfying the constraints (3.5.3) , then by lemma(3.5.3) $f(w) \neq 0$, $\forall w \in \mathfrak{D}$ and so f(w) does not change sign for all w in [0 , 1] . If f(w) had at least one sign change or two elements of [0 , 1] , w_i , w_{i+1} with $f(w_i) \cdot f(w_{i+1}) < 0$ then the continuity of f(w) would imply that w′ in [0 , 1] exists such that : $w_i < w' < w_{i+1}$ and

$f(w')=0$, which is not true since $f(w)$ is a unit and has no roots in $\mathcal{D}$. So for the real $w_1, \ldots, w_l$ of the hypothesis of theorem(3.5.2) we have that $f(w_1), \ldots, f(w_l)$ have the same sign and so $q_{10}=f(w_1), \ldots, q_{l0}=f(w_l)$ have the same sign .

($\Leftarrow$) This step of the proof is constructive for the rational unit $f(w)$ of A_s which satisfies the constraints (3.5.3) . Suppose $q_{10}, \ldots, q_{l0}$ have the same sign . We can assume without loss of generality that all these numbers are positive , or otherwise we have the equivalent problem of finding a rational unit of A_s , f , which satisfies the equivalent to the (3.5.3) interpolation constraints :

$$f^{(j)}(w_i) = -q_{ij} \, , \; j=0, \ldots, m_i-1 \, , \; i=1, \ldots, n \tag{3.5.13}$$

It is first shown that an $h(w) \in A_s$ not necessarily rational satisfying (3.5.3) can be constructed . If we construct a function $g(w) \in A_s$ satisfying :

$$\frac{d^j}{(dw)^j} \exp(g(w)) \Big|_{w=w_i} = q_{ij} \, , \; j=0, \ldots, m_i-1 \, , \; i=1, \ldots, n \tag{3.5.14}$$

then $h(w)=\exp(g(w))$ is a unit of A_s satisfying (3.5.3) . Since $w_1, \ldots, w_l$ are real $q_{10}, \ldots, q_{l0}$ are real and positive so that the Log q_{i0} always exists and it is real for $i=1, \ldots, l$, then (3.5.14) can be expressed as :

$$g(w_i) = \text{Log}\, q_{i0} \, , \; i=1, \ldots, n$$

$$g'(w_i) = \frac{q_{i1}}{q_{i0}} \, , \; q_{i0} \neq 0 \tag{3.5.15}$$

$$g''(w_i) = \left\{ \frac{q_{i1}-[g'(w_i)]^2}{q_{i0}} \right\}$$

and so on for $j=1, \ldots, m_i-1$, $i=1, \ldots, n$. Thus the original interpolation problem has been reduced to one of constructing a function $g(w) \in A_s$ – not required to be a unit – satisfying the interpolation constraints (3.5.15) . Such a function g(w) can be constructed to be the interpolation polynomial which satisfies (3.5.15) ; in other words :

$$g(w) = \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} g^{(\kappa)}(w_i) \frac{(w-w_i)^{\kappa}}{\kappa!} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(w-w_j)^{m_j}}{(w_i-w_j)^{m_j}} \tag{3.5.16}$$

and g(w) belongs to A_s since the polynomials belong to A_s . So we have constructed a unit $h(w)=\exp(g(w))$ satisfying the conditions (3.5.3) . Now we construct a rational unit in A_s , f(w) which satisfies (3.5.3) . First we would like to make the following remark . It is well known from analysis that for all $\epsilon>0$ there exists a polynomial p(w) , such that $\| h-p \| < \epsilon$. Consider now the polynomials $\psi(w)$, $\phi(w)$ such that $\psi(w)$

interpolates the same q_{ij} as h(w) and $\phi(w) = y \cdot \prod_{i=1}^{n} (w-w_i)^{m_i}$. Then we can write :

$$h(w) - \psi(w) = \phi(w) \cdot \pi(w)$$

since $(h(w_i) - \psi(w_i))^{(j)} = 0$, $\forall i = 1 , \ldots , n$, $j = 1 , \ldots , m_i - 1$. Furthermore :

$$p(w) = \phi(w) \cdot \pi_1(w) + v(w)$$

Then as p(w) tends to h(w) , it is implied that $\pi(w)$ tends to $\pi_1(w)$ and $v(w)$ tends to $\psi(w)$. So $\forall \epsilon > 0$ we can find a p(w) such that $\| \psi - v \| < \epsilon$. From the above mentioned we can create an $v(w)$ such that $| \alpha | \leq \omega$, where α is the highest degree coefficient of $\psi(w) - v(w)$ and ω is a given positive real number . Algorithmically this can be achieved as follows .

Step 1 : Start with some p(w) approaching h(w) and after dividing p(w) by $\phi(w)$ take the difference $\psi(w) - v(w)$ and check $| \alpha |$. If $| \alpha | \leq \omega$ then stop , else approach h(w) by a new $p_1(w)$ such that $\| h - p_1 \| < \| h - p \|$.

Step 2 : Divide $p_1(w)$ by $\phi(w)$ this time . Since $p_1(w)$ is a better approach for h(w) , from the one of step 1 , $v_1(w)$ – the new remainder – is a better approach for $\psi(w)$. So , $\| \psi - v_1 \|$ is closer to zero now than $\| \psi - v \|$. That means , that the coefficients of $\psi(w) - v_1(w)$ are closer to zero than the ones of $\psi(w) - v(w)$ and hence $| \alpha_1 | < | \alpha |$, where α_1 is the highest degree coefficient of $\psi(w) - v_1(w)$. If $| \alpha_1 | \leq \omega$ then stop , else approach h(w) by a new $p_2(w)$ such that $\| h - p_2 \| < \| h - p_1 \|$.

Step 3 : Repeat step 2 for $p_2(w)$ and $\phi(w)$.

This algorithm will eventually create a $v_n(w)$ with $| \alpha_n | \leq \omega$. It will take finite number of steps because when $| \alpha | \geq \omega$ the difference $| \alpha | - \omega$ is finite . Proceeding now with the proof let :

$$d = \sum_{i=1}^{n} | m_i | \ , \ \lambda = min \left\{ | w_i - w_j | , \forall i , j = 1 , \ldots , n \right\} ,$$

$$m = max \ \{ m_1 , \ldots , m_n \} \tag{3.5.17}$$

Assume that :

$$\epsilon = \frac{\lambda^{(n-1)\, m}}{(\lambda^{(n-1)\, m} + b) \exp(\| -g \|)} , \text{ with } b = \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i - 1} d^{\kappa} .$$

Now we construct a polynomial p(w) over $\mathfrak{D}$ such that :

$$\| h - p \| < \epsilon \tag{3.5.18}$$

By definition(3.5.2) $h(w) = \sum_{i=0}^{\infty} \frac{g(w)^i}{i!}$. By remark (3.5.6) we can find $c = \| g \|$ and let $p(w) = \sum_{i=0}^{t} \frac{g(w)^i}{i!}$. It is enough to find the appropriate t such that (3.5.18) holds true . By (3.5.18) we take :

$$\| h - p \| = \sup \left\{ \left| \sum_{i=0}^{\infty} \frac{g(w)^i}{i!} - \sum_{i=0}^{t} \frac{g(w)^i}{i!} \right| , \forall\, w \in \partial\mathfrak{D} \right\}$$

$$= \sup \left\{ \left| \sum_{i=t+1}^{\infty} \frac{g(w)^i}{i!} \right| , \forall\, w \in \partial\mathfrak{D} \right\}$$

$$\leq \left\{ \left| \sum_{i=t+1}^{\infty} \frac{c^i}{i!} \right| \right\} = \left\{ \left| \sum_{i=0}^{\infty} \frac{c^i}{i!} - \sum_{i=0}^{t} \frac{c^i}{i!} \right| \right\} \tag{3.5.19}$$

In order to estimate a t such that (3.5.18) holds true , it is enough to find a t such that :

$$\left\{ \left| \sum_{i=0}^{\infty} \frac{c^i}{i!} - \sum_{i=0}^{t} \frac{c^i}{i!} \right| \right\} \leq \epsilon \tag{3.5.20}$$

then by (3.5.19) we can verify that this t leads to a $p(w)$ that satisfies (3.5.18) . By (3.5.20) , we have :

$$\left\{ \left| e^c - \sum_{i=0}^{t} \frac{c^i}{i!} \right| \right\} \leq \epsilon \tag{3.5.21}$$

and since ϵ is a finite not varying number , after finite number of steps a t which satisfies (3.5.21) can be found . Now denote :

$$\phi(w) = y \cdot \prod_{i=1}^{n} (w - w_i)^{m_i}$$

where y is an arbitrary real number . Dividing the polynomial $p(w)$ by $\phi(w)$:

$$p(w) = \phi(w)\, \pi(w) + \upsilon(w) \tag{3.5.22}$$

and
$$p^{(j)}(w_i) = \upsilon^{(j)}(w_i) \ , \ j = 0 , \dots , m_i - 1 \ , \ i = 1 , \dots , n \tag{3.5.23}$$

We also assume the polynomial $\psi(w)$ which interpolates q_{ij} , $j = 0 , \dots , m_i - 1$, $i = 1 , \dots , n$ namely :

$$\psi^{(j)}(w_i) = h^{(j)}(w_i) = q_{ij} \ , \ j = 0 , \dots , m_i - 1 \ , \ i = 1 , \dots , n \tag{3.5.24}$$

with degree of both $\phi(w)$, $\upsilon(w)$ less or equal than d−1 .

Set :
$$f(w) = p(w) - \upsilon(w) + \psi(w) \tag{3.5.25}$$

$f(w)$ is a rational function in A_s and by (3.5.23) , (3.5.24) , (3.5.25) it is implied that :

$$q_{ij} = f^{(j)}(w_i) = \psi^{(j)}(w_i) = h^{(j)}(w_i) \quad , j = 0 , \ldots , m_i-1 , i = 1 , \ldots , n \tag{3.5.26}$$

Consider now :

$$|| h - f|| = || h-p + \upsilon-\psi || < || h-p || + ||\upsilon-\psi || \tag{3.5.27}$$

Observe that (3.5.23) , (3.5.24) for $j = 0$ imply :

$$| \upsilon(w_i)-\psi(w_i) | = | p(w_i)-h(w_i) | \quad , i = 1 , \ldots , n$$

which gives by (3.5.17) :

$$| \upsilon(w_i)-\psi(w_i) | < \epsilon \tag{3.5.28}$$

whereas $\deg(\upsilon(w)-\psi(w)) = \mu \leq d-1$. By lemma(3.5.6) it is implied that :

$$| (\upsilon(w_i)-\psi(w_i))^{(j)} | < | \alpha | \cdot \mu^j < | \alpha | \cdot d^j , j = 0 , \ldots , m_i- 1 \tag{3.5.29}$$

If $| \alpha |$ is greater than ϵ , then we can increase t to a t' in (3.5.20) such that if we follow the algorithm described in steps $1-3$, an α' corresponding to $\upsilon'(w)-\psi(w)$, with $| \alpha' |$ less than ϵ can be constructed and thus :

$$| (\upsilon'(w_i)-\psi(w_i))^{(j)} | < | \alpha' | \cdot \mu^j < \epsilon \cdot d^j , j = 0 , \ldots , m_i- 1 \tag{3.5.30}$$

Consider now the polynomial Q(w) which has the properties :

$$Q^{(j)}(w_i) = (\upsilon'(w_i)-\psi(w_i))^{(j)} , j = 0 , \ldots , m_i-1 , i = 1 , \ldots , n \tag{3.5.31}$$

and $\deg(Q(w)) = d-1$. Since Q(w) is an interpolation polynomial and the amplitude of the values of interpolation is greater than the degrees of Q(w) and $(\upsilon'(w) -\psi(w))$ as well as by (3.5.31) it is implied that :

$$Q(w) = (\upsilon'(w) -\psi(w)) \tag{3.5.32}$$

By (3.5.6) the form of Q(w) is given as :

$$\sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} \left[(\upsilon'(w_i)-\psi(w_i))^{(\kappa)}(w_i)\right] \frac{(w-w_i)^{\kappa}}{\kappa!} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(w-w_j)^{m_j}}{(w_i-w_j)^{m_j}} \tag{3.5.33}$$

Consider now the :

$$|| \upsilon'-\psi || = \sup \left\{ | \upsilon'(w) -\psi(w) | , \forall\ w \in \partial\mathfrak{D}\right\}$$

$$= \sup \left\{ | Q(w) | , \forall\ w \in \partial\mathfrak{D}\right\} \tag{3.5.34}$$

$$| Q(w) | = \left| \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} \left[(v'(w_i)-\psi(w_i))^{(\kappa)}(w_i)\right] \frac{(w-w_i)^{\kappa}}{\kappa!} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(w-w_j)^{m_j}}{(w_i-w_j)^{m_j}} \right|$$

$$\leq \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} | (v'(w_i)-\psi(w_i))^{(\kappa)}(w_i) | \frac{| w-w_i |^{\kappa}}{\kappa!} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{| w-w_j |^{m_j}}{| w_i-w_j |^{m_j}}$$

and by (3.5.16) , (3.5.30) , $| w-w_i | \leq 1$ it is implied that :

$$| Q(w) | \leq \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} \epsilon\, d^{\kappa} \frac{1}{\lambda^{(n-1)\, m}} = \epsilon \frac{1}{\lambda^{(n-1)\, m}} \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} d^{\kappa} = \epsilon \frac{1}{\lambda^{(n-1)\, m}}\, b \tag{3.5.35}$$

Relation (3.5.35) holds for all $w \in \mathfrak{D}$ and thus :

$$\sup \left\{ | Q(w) | , \forall\, w \in \partial\mathfrak{D} \right\} \leq \epsilon \frac{1}{\lambda^{(n-1)\, m}}\, b \tag{3.5.36}$$

By (3.5.35) and (3.5.36) it is implied that :

$$|| v'-\psi || \leq \epsilon \frac{1}{\lambda^{(n-1)\, m}}\, b \tag{3.5.37}$$

(3.5.17) , (3.5.27) , (3.5.37) imply that :

$$|| h-f|| < || h-p || + ||v'-\psi || < \epsilon + \epsilon \frac{1}{\lambda^{(n-1)\, m}}\, b \tag{3.5.38}$$

and (3.5.17) , (3.5.38) yield :

$$|| h - f|| < \frac{1}{\exp|| -g ||} \tag{3.5.39}$$

By remarks(3.5.3) , (3.5.4) it is then implied that :

$$|| (\exp(g))^{-1}|| = || \exp(-g) || \leq \exp(||-g ||)$$

or ,

$$\frac{1}{|| \exp(-g) ||} \geq \frac{1}{\exp(||-g ||)} \tag{3.5.40}$$

By lemma(3.5.5) and relations (3.5.39) , (3.5.40) ⇒

$$|| h - f|| < \frac{1}{|| \exp(-g) ||} = \frac{1}{|| h^{-1} ||} \tag{3.5.41}$$

Thus f(w) is a rational unit in A_s which – by (4.26) – satisfies the interpolation constraints (3.5.3) . □

***Remark (3.5.6)** : We shall present here the method of constructing a rational unit of $\mathcal{A}_s$,* *which satisfies the interpolation constraints (3.5.3) by using part ii) of remark(3.5.5) . First , we follow the same steps as in the proof of theorem (3.5.6) to construct a unit* $exp(g(w))$ *in* $\mathcal{A}_s$ *, which satisfies the interpolation constraints (3.5.3) . Then we expand* $exp(g(w))$ *as a power series about (0.5 , 0) :*

$$exp(g(w)) = exp(g(0.5)) \sum_{i=0}^{\infty} \frac{g^{(i)}(0.5)}{i!} (g(w) - g(0.5))^i$$

and rearrange in terms of increasing w^i *. Then we approach* $exp(g(w))$ *by a polynomial of the form :*

$$p(w) = a_0 + a_1 w + \dots + a_t w^t$$

where $p(w)$ *consists of the first t , t = 1 , 2 , ... , terms of* $exp(g(w))$ *. Then we divide* $p(w)$ *as in (3.5.22) and construct the polynomial* $f(w)$ *as in (3.5.25) . Using remark(3.5.3) we calculate :*

$$\| exp(g(w)) - p(w) + v(w) - \psi(w) \|$$

which it is required to be less than or equal to $1/exp(\| - g(w) \|)$ *. If it is not then we take more terms of* $exp(g(w))$ *in* $p(w)$ *and repeat the above process until :*

$$\| exp(g(w)) - p(w) + v(w) - \psi(w) \| \leq 1/exp(\| - g(w) \|)$$

The algorithm takes finite number of steps to complete since , as we have pointed out in the proof of theorem(3.5.6) , as $p(w) \underset{t\to\infty}{\to} exp(g(w))$ *,* $v(w) \underset{t\to\infty}{\to} \psi(w)$ *and thus :*

$$\| exp(g(w)) - p(w) + v(w) - \psi(w) \| \underset{t\to\infty}{\to} 0$$

$f(w)$ *is a unit in* $\mathcal{A}_s$ *, since* $1/exp(\| - g(w) \|)$ *is less than* $1/\| exp(-g(w)) \|$ *, (remark(3.5.4) and lemma(3.5.5)) . As it will be demonstrated in example(3.6.1) , this algorithm is faster than the one described in the proof of theorem(3.5.6) .* □

The proof of theorem(3.5.1) is a consequence of the proof of theorem(3.5.6) bearing in mind the transformation of constraints (3.5.2) to (3.5.3) . The inverse transormation from (3.5.3) to (3.5.2) is also possible .

Proof of theorem (3.5.1)

($\Leftarrow$) If r_{i0} , $i = 1 , \dots , l$ does not change sign then the same happens with q_{i0} – (3.5.2) , (3.5.3)) – and by theorem(3.5.6) a polynomial unit f(w) of A_s exists that satisfies (3.5.3). Furthermore f(1/(s+1)) = u(s) is a rational unit in $\mathbb{R}_{\mathcal{P}}(s)$ – propositions(3.5.4) , (3.5.5) – and satisfies the interpolation constraints of (3.5.2) .

($\Rightarrow$) If f(s) is a unit in $\mathbb{R}_{\mathcal{P}}(s)$ then the r_{i0} , $i = 1 , \ldots , l$ must not change sign else we could find s_i in $\mathcal{P}$ such that $f(s_i) = 0$ which is not true . □

3.6. CONSTRUCTION OF THE CLASS OF MINIMUM DEGREE REMAINDERS

Let v(s) , g(s) be two rational , proper and $\mathcal{P}$-stable functions . Consider the Euclidean Division of v(s) by g(s) as it was defined by theorem(3.4.1) . It is well known by proposition(3.3.3) and remarks(3.4.2) , (3.4.3) that there does not always exist a unique pair of quotient and remainder for a Euclidean division . Thus it is interesting to investigate classes of remainders with least Euclidean degree , [Vid. 4] , [Var. 5] . In what follows we shall show that the least possible Euclidean degree which the remainder of a Euclidean division may have is equal to the number of the sign changes of the dividend at the extended , real , positive , in ascending order positioned zeros of the divisor . Namely the sign changes in the set $\{v(s_i) , i = 1 , \ldots , l\}$, with s_i the real positive finite and infinite zeros of g(s).

Theorem (3.6.1) *: Let v(s) , g(s) be two coprime functions of* $\mathbb{R}_{\mathcal{P}}(s)$ *and* $\gamma_{\mathcal{P}}(g(s)) = n$, $\{s_1 , \ldots , s_n\}$ *the zeros of g(s) in* $\mathcal{P}$ *with multiplicity* $\{m_1 , \ldots , m_n\}$ *respectively and* $\{s_1, \ldots , s_l\}$ *are extended , real , nonnegative , in ascending order . Then the least possible degree of the remainder of the Euclidean Division of v(s) by g(s) is* ν *the number of sign changes in* $\{v(s_1) , \ldots , v(s_l)\}$ *and a representative of the class of remainders of such a degree is given by the form :*

$$r(s) = \prod_{i=1}^{\nu} \frac{(s-b_i)}{(s+1)} u^{-1}(s) \qquad (3.6.1)$$

where b_i *are in* $\mathbb{R}^+ \cup \{0\}$ *and* $s_i < b_i < s_{i+1}$ *whenever* $v(s_i) \cdot v(s_{i+1}) < 0$, $i = 1 , \ldots , l$, $\nu \leq l$, $u^{-1}(s)$ *is a unit in* $\mathbb{R}_{\mathcal{P}}(s)$.

Proof

Let $v(s) = g(s) \cdot q(s) + \tau(s)$ is the Euclidean Division of v(s) by g(s) with q(s) , $\tau(s)$ the quotient and the remainder respectively . Then $\tau(s) = v(s) - g(s) \cdot q(s)$ and $\gamma_{\mathcal{P}}(\tau(s)) =$ $= \gamma_{\mathcal{P}}(v(s) - g(s) \cdot q(s))$. Now we consider the set :

$$\{v(s_i) - g(s_i) \cdot q(s_i)\} = \{v(s_i)\} , i = 1 , \ldots , l$$

It contains ν sign changes so :

$$\tau(s) = v(s) - g(s) \cdot q(s)$$

has at least ν roots in $\mathcal{P}$ and thus by (3.3.9) , $\gamma_{\mathcal{P}}(\tau(s)) \geq \nu$. Now we construct an r(s) such that :

$$\gamma_{\mathcal{P}}(\tau(s)) = \nu.$$

Let:

$$\tau(s) = \prod_{i=1}^{\nu} \frac{(s-b_i)}{(s+1)} .$$

b_i are real positive and $s_i < b_i < s_{i+1}$, whenever $v(s_i)\cdot v(s_{i+1}) < 0$, $i = 1 , \ldots , l$, ($\tau(s)=$ $= 1$ if $\nu = 0$) . If we find a unit u(s) in $\mathbb{R}_{\mathcal{P}}(s)$ such that :

$$(v(s_i)\cdot u(s_i))^{(j)} = \tau(s_i)^{(j)},\ j = 0 , \ldots , m_i-1 , i = 1 , \ldots , n \tag{3.6.2}$$

Then the function $V(s) = v(s)\cdot u(s)-\tau(s)$ vanishes at the zeros of g(s) in $\mathcal{P}$ as well as at their multiplicities , so the zeros of g(s) in $\mathcal{P}$ are also zeros of V(s) , in $\mathcal{P}$, and from proposition(3.3.2) there exists t(s) in $\mathbb{R}_{\mathcal{P}}(s)$ such that :

$$V(s) = g(s)\ t(s) \Leftrightarrow v(s)\ u(s)-\tau(s) = g(s)\ t(s) \Leftrightarrow v(s)\ u(s) = g(s)\ t(s) + \tau(s)$$

$$v(s) = g(s)\ t(s)\ u^{-1}(s) + \tau(s)\ u^{-1}(s) \Leftrightarrow \gamma_{\mathcal{P}}(\tau(s)\cdot u^{-1}(s)) = \nu$$

And we have constructed the class of remainders $\tau(s)\cdot u^{-1}(s)$ with the possible minimum degree ν . Now we must construct a unit u(s) in $\mathbb{R}_{\mathcal{P}}(s)$ such that (3.6.2) holds . First we consider the values $(\tau(s_i)/v(s_i))$, $i = 1 , \ldots , n$, which are real and do not change sing $\forall i = 1 ,\ldots, l$. By (3.6.2) and the type of Leibnintz for the $j^{\underline{th}}$ order derivative of the product of two functions we have:

$$\sum_{\kappa=0}^{j} \binom{j}{\kappa} v^{(j-\kappa)}(s_i)\ u(s_i)^{(\kappa)} = \tau(s_i)^{(j)}$$

or

$$u(s_i)^{(j)} = \frac{\tau(s_i)^{(j)} - \left\{ \sum_{\kappa=0}^{j-1} \binom{j}{\kappa} v^{(j-\kappa)}(s_i)\ u(s_i)^{(\kappa)} \right\}}{v(s_i)} \tag{3.6.3}$$

$j = 0 , \ldots , m_i-1$, $i = 1 , \ldots , n$, where $u(s_i)^{(0)} = u(s_i) = \dfrac{\tau(s_i)}{v(s_i)}$. Set :

$$r_{ij} = u(s_i)^{(j)},\ j = 0 , \ldots , m_i-1 , i = 1 , \ldots , n \tag{3.6.4}$$

By (3.6.3) and the theorem(3.5.1) it is possible to construct a unit u(s) in $\mathbb{R}_{\mathcal{P}}(s)$ such that the interpolation constraints (3.6.4) hold , since r_{i0} do not change sing $\forall\ i = 1 , \ldots , l$. This construction is possible by using the algorithmic interpretation of the proof of theorem(3.5.6) . In remark(3.6.1) we give the algorithmic interpretation corresponding to the method of remark(3.6.6) :

Algorithm for the construction of a unit of $\mathbb{R}_{\mathcal{P}}(s)$ for which (3.6.4) holds

Step 1 : Set ν the number of sign changes in $\{v(s_1), \dots, v(s_l)\}$

Step 2 : Set

$$\tau(s) = \prod_{i=1}^{\nu} \frac{(s-b_i)}{(s+1)},$$

b_i real and $s_i < b_i < s_{i+1}$, whenever $v(s_i)\cdot v(s_{i+1}) < 0$, $i = 1, \dots, l$ or $\tau(s) = 1$ if $\nu = 0$.

Step 3 : Set $r_{ij} = u(s_i)^{(j)}$, $j = 0, \dots, m_i-1$, $i = 1, \dots, n$, where :

$$u(s_i)^{(0)} = u(s_i) = \frac{\tau(s_i)}{v(s_i)},$$

$$u(s_i)^{(j)} = \frac{\tau(s_i)^{(j)} - \left\{ \sum_{\kappa=0}^{j-1} \binom{j}{\kappa} v^{(j-\kappa)}(s_i)\, u(s_i)^{(\kappa)} \right\}}{v(s_i)}$$

Step 4 : Set

$$w_i = \frac{1}{s_i+1}, \; i = 1, \dots, n$$

Step 5 : Set

$$q_{ij} = r_{ij} \frac{(-1)^j \cdot j!}{w_i^{j+1}}, \; j = 0, \dots, m_i-1, \; i = 1, \dots, n$$

Step 6 : If $q_{i0} < 0$ then set $q'_{ij} = -q_{ij}$ and follow the construction for these q'_{ij}.

Step 7 : Factorize g(s) as in (3.3.6) and set $s = ((1-w)/w)$ in its non unit part, (use the types of remark(3.3.2)). This results to a polynomial $\phi(w)$.

Step 8 : Solve the equation :

$$\frac{d^j}{(dw)^j} \exp(d(w)) \Big|_{w=w_i} = q_{ij}$$

with respect to $d^{(j)}(w_i)$ – $d(w)$ polynomial – and set $a_{ij} = d^{(j)}(w_i)$, $j = 0, \dots, m_i-1$, $i = 1, \dots, n$. Thus a $d(w)$ that interpolates the values a_{ij} can be constructed.

Step 9 : Set $h(w) = \exp(d(w))$, (a non rational unit of $\mathcal{A}_s$).

Step 10 : Set $d' = \sum_{i=1}^{n} m_i$, $\lambda = min\left\{ |w_i - w_j|, \forall\, i, j = 1, \dots, n \right\}$,

$m = max\{m_1, \dots, m_n\}$.

Step 11 : Set

$$\epsilon = \frac{\lambda^{(n-1)\,m}}{(\lambda^{(n-1)\,m} + b)\exp(\ ||\ -d\ ||\ )} \quad .$$

with $b = \sum_{i=1}^{n} \sum_{\kappa=0}^{m_i-1} (d')^{\kappa}$, $||\ -d\ || = ||\ d\ || = c$, the norm of $d(w)$.

Step 12 : Estimate a t such that $\left\{ \left| \ e^{c} - \sum_{i=0}^{t} \frac{c^i}{i!} \right| \right\} \leq \epsilon$ and set $p(w) = \sum_{i=0}^{t} \frac{d(w)^i}{i!}$.

Step 13 : Divide $p(w)$ by $\phi(w)$ as $p(w) = \phi(w)\ \pi(w) + \upsilon(w)$

Step 14 : Construct the polynomial $\psi(w)$ which interpolates q_{ij} , $j = 0 , \ldots , m_i-1$, $i = 1 , \ldots , n$.

Step 15 : Set α the highest degree coefficient of $(\psi(w) - \upsilon(w))$. If $|\ \alpha\ | > \epsilon$ then repeat steps 12 , 13 for $t' > t$, until $|\ \alpha\ | \leq \epsilon$.

Step 16 : Set $f(w) = p(w) - \upsilon(w) + \psi(w)$.

Step 17 : If step 6 has been used then substitute $f(w)$ by $-f(w)$ in the following.

Theorems(3.5.1) , (3.5.6) , propositions(3.5.4) , (3.5.5) imply that $u(s) = f(1/(s+1))$ is a unit in $\mathbb{R}_{\mathcal{P}}(s)$ satisfying (3.6.4) and by (3.6.3) , (3.6.4) $\Rightarrow$ (3.6.2) holds true . □

Remark (3.6.1) : The method introduced in remark(3.5.6) can be algorithmically interpreted as follows : Steps 1 through 9 remain the same as above .

Algorithm for the implementation of remark(3.5.6)

Step 10 : Set $\epsilon = 1/\exp(c)$, with $c = ||\ d\ || = ||-d\ ||$, the norm of $d(w)$.

Step 11 : Expand $\exp(d(w))$ as a power series about the point $(0.5\ ,\ 0)$.

Step 12 : Is same as step 14 in the proof of theorem(3.6.1) .

Step 13 : For $t = 0$, set $p(w) = a_0 + a_1\, w + \ldots + a_t\, w^t$, the first t terms of the expansion of $\exp(d(w))$.

Step 14 : Is same as step 13 in the proof of theorem (3.6.1)

Step 15 : Calculate the norm $||\ \exp(d(w)) - p(w) + \upsilon(w) - \psi(w)\ ||$, (use remark(3.5.3)) .

Step 16 : If $\| \exp(d(w)) - p(w)+v(w) - \psi(w) \| \leq \epsilon$ then go to step 17 , else go to step 13 and set $t = t+1$, then repeat steps 14 , 15 , 16 until the inequality is true .

Step 17 : Is same as step 16 in the proof of theorem(3.6.1) .

Step 18 : Is same as step 17 in the proof of theorem(3.6.1) . □

Remark (3.6.2) *: Let $f(w)$ be a rational unit of $\mathcal{A}_s$ such that $u(s) = f(1/(s+1))$ satisfies (3.6.4) . Such an $f(w)$ can be found by using either the first or the second algorithm described above . A natural number t corresponding to $f(w)$ exists and is constructed either in steps 12 , 15 of the first algorithm or steps 13 , 16 of the second . For all $t' > t$ set $p_{t'}(w)$ to be :*

$$\text{either } \sum_{i=0}^{t'} \frac{d(w)^i}{i!} \text{ , or } a_0 + a_1 w + \ldots + a_{t'} w^{t'}$$

according to steps 12 , 15 of the first algorithm or steps 13, 16 of the second . For the $g(w)$ of step 7 in both algorithms and each $p_{t'}(w)$, set $v_{t'}(w)$ to be the remainder of the Euclidean division of $p_{t'}(w)$ by $g(w)$. For the $\psi(w)$ of step 14 in both algorithms set $f_{t'}(w) = p_{t'}(w) - v_{t'}(w) + \psi(w)$. The family $\mathfrak{F}$ of all rational units of $\mathcal{A}_s$, $f(w)$, such that $u(s) = f(1/(s+1))$ satisfies (3.6.4) is parametrized by the above mentioned procedure . As a result the family $\mathfrak{U}$ of units , $u(s)$, that satisfy (3.6.4) is parametrized by $\mathfrak{F}$ via the transformation $w \to 1/(s+1)$. Finally , if s_1 , ... , s_l are extended , real , nonnegative zeros of the divisor $g(s)$ and ν the number of sign changes in $\{v(s_1)$, ... , $v(s_l)\}$ the family of least Euclidean degree remainders , $\mathfrak{R}$, of the Euclidean division of $v(s)$ by $g(s)$ is parametrized by :

$$\mathfrak{R} = \left\{ r(s) = \prod_{i=1}^{\nu} \frac{(s-b_i)}{(s+1)} u^{-1}(s) \text{ , } \forall\ b_i \in \mathbb{R}^+ \cup \{0\} \text{ and } s_i < b_i < s_{i+1} \text{ if } v(s_i) \cdot v(s_{i+1}) < 0 \text{ ,} \right.$$
$$\left. i = 1 \text{ , } \ldots \text{ , } l \text{ , } u(s) = f(1/(s+1)) \text{ , } \forall\ f(w) \in \mathfrak{F} \right\}$$ □

Example (3.6.1) : Let $v(s) = ((s-2)/(s+1))$ and $g(s) = (((s-1)^2 s)/(s+1)^3)$ and so the zeros of $g(s)$ in $\mathcal{P}$ are $s_1 = 1$ and $s_2 = 0$ with multiplicities $m_1 = 2$ and $m_2 = 1$ respectively . First we use the algorithm of the theorem(3.6.1) :

Step 1 : The number of sign changes in $\{v(s_1) , v(s_2)\} = \{ -0.5 , -2 \}$ is 0 thus $\nu = 0$.

Step 2 : $\tau(s) = 1$.

Step 3 : $r_{10} = -2$, $r_{20} = -0.5$, $r_{11} = -3$.

Step 4 : $w_1 = 0.5$, $w_2 = 1$.

Step 5 : $q_{10} = -4$, $q_{20} = -0.5$, $q_{11} = 12$.

Step 6 : $q'_{10} = 4$, $q'_{20} = 0.5$, $q'_{11} = -12$.

Step 7 : $g(w) = (-4)\, w^3 + 8\, w^2 - 5\, w + 1$.

Step 8 : The polynomial $d(w) = 2.30685 - 0.682234\, w - 2.31777\, w^2$, interpolates the values $a_{10} = \text{Log}(4)$, $a_{20} = \text{Log}(0.5)$, $a_{11} = (q'_{11}/q'_{10}) = -3$.

Step 9 : $h(w) = \exp(d(w))$.

Steps 10 , 11 : $d' = 3$, $\lambda = 0.5$, $m = 2$, $c = 2.56283$, $b = 5$, $\epsilon = 0.00367077$.

Steps 12 : $t = 10$, $p(w) = \sum_{i=0}^{10} \frac{d(w)^i}{i!}$.

Step 13 : Dividing $p(w)$ by $\phi(w)$ the remainder $v(w) = 9.99995\, w^2 - 21.999\, w + 12.5$.

Step 14 : The polynomial $\psi(w) = 12.5 - 22\, w + 10\, w^2$ interpolates the values $q'_{10} = 4$, $q'_{20} = 0.5$, $q'_{11} = -12$.

Step 15 : $|\alpha| = 0.00005 < \epsilon$.

Step 16 : $f(w) = p(w) - v(w) + \psi(w) =$

$$= 10.0425 - 6.85058\, w - 20.9377\, w^2 + 15.3397\, w^3 + 21.6445\, w^4 - 17.1466\, w^5 -$$
$$- 14.7701\, w^6 + 12.7249\, w^7 + 7.47859\, w^8 - 7.00523\, w^9 - 3.00314\, w^{10} + 3.00128\, w^{11} +$$
$$+ 1.00387\, w^{12} - 1.00002\, w^{13} - 0.288763\, w^{14} + 0.253267\, w^{15} + 0.0703417\, w^{16} -$$
$$- 0.048267\, w^{17} - 0.0127835\, w^{18} + 0.00362908\, w^{19} + 0.00123292\, w^{20}$$

Now we study the same example in view of the algorithm in remark(3.6.1) .

Steps 1 through 9 are the same as above .

Step 10 : $\epsilon = 0.0770863$, $c = 2.56283$.

Step 11 : The expansion can be done using a mathematical package (ie. Mathematica , etc) .

Step 12 : Is the same as step 14 above .

Step 13 : For t=14 , take as p(w) the first 14 terms of step 11 .

Step 14 : $v(w) = 9.99853\, w^2 - 21.9981\, w + 12.4994$.

Step 15 : $\|\exp(d(w)) - p(w) + v(w) - \psi(w)\| = 0.0209831$.

Step 16 : $\|\exp(d(w)) - p(w) + v(w) - \psi(w)\| < \epsilon$.

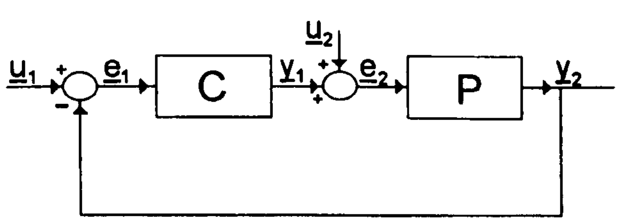
Step 17 : $f(w) = p(w) - v(w) + \psi(w) =$

$$= 10.0376 - 6.83549\, w - 20.9018\, w^2 + 15.2007\, w^3 + 21.5539\, w^4 - 16.6366\, w^5 -$$
$$- 14.6357\, w^6 + 11.6296\, w^7 + 7.32686\, w^8 - 5.50139\, w^9 - 2.8155\, w^{10} +$$
$$+ 1.65489\, w^{11} + 0.782572\, w^{12} - 0.242195\, w^{13} - 0.117545\, w^{14}$$

We can clearly see that the second algorithm gives a less degree unit than the unit of the first one . This is due to the approaching of exp(d(w)) by terms of $d(w)^i/i!$ which employ in p(w) all the terms of the polynomial $d(w)^i$, something not always necessary . In other words we may need only the few first terms of $d(w)^i$ and not all of them so that p(w) will approach exp(d(w)) as close as required . And finally $u(s) = -f(1/(s+1))$ is the unit which interpolates the values $r_{10} = -2$, $r_{20} = -0.5$, $r_{11} = -3$. By theorem(3.6.1) a least degree remainder of the Euclidean Division of v(s) by g(s) is $u^{-1}(s)$, while corollary(3.6.1) implies that the class of all least degree remainders of the Euclidean Division of v(s) by g(s) is $\mathfrak{R} = \{\, u(s) = -f_t^{-1}(1/(s+1)) \text{ , } \forall f(w) \in \mathfrak{F} \,\}$. □

3.7. CLOSED – LOOP STABILITY AND MATRIX EUCLIDEAN DIVISION .

Consider the standard feedback configuration associated with a lumped , linear , time invariant (continuous – time) system :



Suppose that P , C $\in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$, (where $\mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ is the ring of matrices with entries in $\mathbb{R}_{\mathcal{P}}(s)$) . Let (N_p , D_p) , $(\tilde{D}_p , \tilde{N}_p)$ be any $\mathbb{R}_{\mathcal{P}}(s)$ – right coprime, ($\mathbb{R}_{\mathcal{P}}(s)$ – left coprime) , factorization of P and let (N_c , D_c) , $(\tilde{D}_c , \tilde{N}_c)$ be any $\mathbb{R}_{\mathcal{P}}(s)$ – right coprime , ($\mathbb{R}_{\mathcal{P}}(s)$ – left coprime) , factorization of C . Under these conditions the problem of feedback stabilization leads to the following equivalent statements , [Vid. 4] :

i) The pair (P , C) is stable .

ii) The matrix $\widetilde{N}_c N_p + \widetilde{D}_c D_p$ is unimodular , $(\widetilde{N}_c N_p + \widetilde{D}_c D_p = I)$ (3.7.1)

iii) The matrix $\widetilde{N}_p N_c + \widetilde{D}_p D_c$ is unimodular , $(\widetilde{N}_p N_c + \widetilde{D}_p D_c = I)$ (3.7.2)

The parametrization of all stabilizing controllers or equivalently the construction of the 'family of solutions of equation $\widetilde{N}_c N_p + \widetilde{D}_c D_p = I$, $(\widetilde{N}_p N_c + \widetilde{D}_p D_c = I)$, is given by the set :

$$\mathcal{S}(P) = \{(Y - R\,\widetilde{N}_p)^{-1}\cdot(X + R\,\widetilde{D}_p) : R \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s)) \; , \; |\, Y - R\,\widetilde{N}_p \,| \neq 0\}$$

$$= \{(\widetilde{X} + D_p S)\cdot(\widetilde{Y} - N_p S)^{-1} : S \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s)) \; , \; |\, \widetilde{Y} - N_p S \,| \neq 0\} \qquad (3.7.3)$$

Many times it is essential to be able to select the elements of $\mathcal{S}(P)$ with the least possible number of unstable poles . The number n of unstable poles of a stabilizing controller from $\mathcal{S}(P)$ is given by :

$$n = \gamma_{\mathcal{P}}(|\, Y - R\,\widetilde{N}_p \,|) = \gamma_{\mathcal{P}}(|\, \widetilde{Y} - N_p S \,|) \qquad (3.7.4)$$

where , $\gamma_{\mathcal{P}}$ as in (3.2.9) and proposition(3.4.1) . Hence , the least possible number m of unstable zeros of the elements of $\mathcal{S}(P)$ is given by :

$$m = min\ \{ \gamma_{\mathcal{P}}(|\, Y - R\,\widetilde{N}_p \,|) : R \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s)) \; , \; |\, Y - R\,\widetilde{N}_p \,| \neq 0\}$$

$$= min\ \{ \gamma_{\mathcal{P}}(|\, \widetilde{Y} - N_p S \,|) : S \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s)) \; , \; |\, \widetilde{Y} - N_p S \,| \neq 0\} \qquad (3.7.5)$$

The expressions $|\, Y - R\,\widetilde{N}_p \,|$, or $|\, (\widetilde{Y} - N_p S) \,|$, in (3.7.5) motivate the study of the following problem, [Vid. 4] .

Problem *: Given A , B* $\in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ *, (where* $\mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ *is the ring of matrices with entries in* $\mathbb{R}_{\mathcal{P}}(s)$*) , with A square and A , B right coprime , (the matrix* $[\,A^T \vdots B^T\,]^T$ *is full column rank for all the finite s in* $\mathcal{P}$ *and the* $\lim_{s\to\infty} [\,A^T \vdots B^T\,]^T$ *is a full column rank matrix as well) , over what elements of* $\mathbb{R}_{\mathcal{P}}(s)$ *does* $|\,A + R\,B\,|$ *vary .* □

Theorem (3.7.1) *: Suppose A , B* $\in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ *are right coprime and A is square . Let* $\alpha = = |\,A\,|$ *and b denote the greatest common divisor of all the entries of B . Then the sets* $\{\,\alpha + r\,b : r \in \mathbb{R}_{\mathcal{P}}(s)\,\}$ *and* $\{\,|\,A + R\,B\,| : R \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))\,\}$ *are equal . As a consequence :*

$$min \{ \gamma_{\mathcal{P}}(| A+R\ B |) : R \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s)) \} = min \{ \gamma_{\mathcal{P}}(\alpha + r\ b) : r \in \mathbb{R}_{\mathcal{P}}(s) \} = I(\alpha , b)$$

where , $\gamma_{\mathcal{P}}$ *was defined in (3.2.9) .* □

Remark (3.7.1) *: The first part of the theorem means that , if any element f in* $\mathbb{R}_{\mathcal{P}}(s)$ *can be expressed as* $\alpha + r\ b$ *for some* $r \in \mathbb{R}_{\mathcal{P}}(s)$*, then there exists an R in* $\mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ *such that* $f = | A+R\ B |$ *and conversely.* □

In order to derive the number m of (3.7.5) , we set $\alpha = | Y |$ and b is the g.c.d. of the elements of $\tilde{N}_p$. Then by using the algorithm described in section3.6 the family of remainders $\mathcal{R}$ with least Euclidean degree d , of the division between α and b can be constructed . By theorem(3.7.1) the number m of (3.7.5) is equal to d and the parametric matrices R can be found by the knowledge of the family of quotients Q corresponding to $\mathcal{R}$, [Vid. 4] . Using theorem(3.7.1) we can expand Euclidean division for the square matrices A , B in $\mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$.

Corollary(3.7.1) *: Suppose A , B* $\in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ *are both square , with* $| B | \neq 0$ *. Then there exists and* $R \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$ *such that :*

$$\gamma_{\mathcal{P}}(| A+R\ B |) < \gamma_{\mathcal{P}}(| B |) \qquad (3.7.1)$$

Proof

If $| A | = 0$, (3.7.1) is satisfied with R = 0 , so suppose $| A | \neq 0$. Let F be a greatest common right divisor of A , B and let $A = A_1 \cdot F$, $B = B_1 \cdot F$. Let $a_1 = | A_1 |$ and b_1 denote the greatest common divisor of all the elements of B_1 . Then theorem(3.7.1) implies that , for some $R \in \mathcal{M}(\mathbb{R}_{\mathcal{P}}(s))$:

$$\gamma_{\mathcal{P}}(| A+R\ B |) = \gamma_{\mathcal{P}}(| F |) + \gamma_{\mathcal{P}}(| A_1+R\ B_1|) = \gamma_{\mathcal{P}}(| F |) + I(a_1 , b_1) <$$

$$< \gamma_{\mathcal{P}}(| F |) + \gamma_{\mathcal{P}}(b_1) \leq \gamma_{\mathcal{P}}(| F |) + \gamma_{\mathcal{P}}(| B_1 |) = \gamma_{\mathcal{P}}(| B |)$$

This completes the proof . □

3.8. CONCLUSIONS

The very important – for stabilization of unstable control systems – Euclidean Domain of proper and $\mathbb{C}_+$ stable rational functions , $\mathbb{R}_{\mathcal{P}}(s)$, ($\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$) has been considered in this chapter . A detailed analysis of a method for introducing unique – modulo $\alpha \in \mathbb{R}^-$ – factorization and hence a definition for exact division between two elements of $\mathbb{R}_{\mathcal{P}}(s)$ has been described . The important property of non uniqueness of Euclidean remainder in the Euclidean division in $\mathbb{R}_{\mathcal{P}}(s)$ leads to the need of characterization of the various families of remainders according to invariant characteristics as for example is the number of zeros in $\mathcal{P}$. The need for constructing the family of least Euclidean degree remainders of the Euclidean division in $\mathbb{R}_{\mathcal{P}}(s)$, has implied the transformation of this problem to the construction of a rational unit over the Disc Algebra of symmetric analytic functions which map the Disc $\overline{((0 , \frac{1}{2}) , \frac{1}{2})}$ into the complex numbers , under certain interpolation constraints . A description of this Disc Algebra has been made and an interconnection between its units and the units of $\mathbb{R}_{\mathcal{P}}(s)$ has been given . An algorithmic construction of the required unit has been introduced and the family of least possible Euclidean degree remainders has been constructed . The knowledge of the least degree family of remainders in $\mathbb{R}_{\mathcal{P}}(s)$ has been used in the last section of chapter 3 for the estimation of least unstable poles stabilizing controllers . An extension of the Euclidean division in matrices over $\mathbb{R}_{\mathcal{P}}(s)$ has been provided .

CHAPTER 4

THE GREATEST COMMON DIVISOR OF A SET OF POLYNOMIALS : A GRÖBNER BASES APPROACH

4.1. INTRODUCTION

The notion of the common divisor of a set of polynomials of $\mathbb{R}[s]$ is the basic mathematical tool underlying the definitions and properties of concepts , such as multivariable zeros , [Mac. 1] , decoupling zeros , [Ros. 1] , of Linear Systems theory . This concept is central in the computation of tools such as Smith forms , Hermit forms matrix divisors etc. of the Algebraic Systems theory , [Kai. 1] , [Kuc. 1] , etc . The computation of the Greatest Common Divisor (GCD) , f(s) , of a set of m polynomials of $\mathbb{R}[s]$, $\underline{p}(s)$, of a maximal degree δ , has attracted a lot of attention , [Bar. 1] , [Bar. 2] , [Kai. 1] , [Kar. 7] , [Kar. 8] , [Mit. 1] , [Mit. 2] , [Mit. 4] . The role of GCD in the solution of problems of Linear Control theory is well established , [Kai. 1] . Various approaches for the computation of the GCD of $\underline{p}(s)$ have been established ; an analytical survey of the existing numerical methods can be found in [Mit. 2] , [Kar. 7] . Characterizations of the GCD in terms of standard results from Linear Systems theory and their relation to classical Matrix Pencil theory can be found in [Kar. 2] . Our aim is to provide an alternative characterization for the GCD , f(s) , of a set of polynomials represented by the vector $\underline{p}(s)$, by expressing the relationship $\underline{p}(s) = \underline{q}(s)\cdot f(s)$ in terms of real matrices , (basis matrices (b.m.) P , Q of $\underline{p}(s)$, $\underline{q}(s)$ respectively) , and the Toeplitz representation of f(s) . This relates the GCD with the existence of a special Toeplitz base $\mathcal{W} = \{W\}$ of a subspace $\mathcal{V} \subseteq \mathcal{N}_r\{P\}$; this base has the additional property that the nonzero entries of W, (the matrix formed by {W}) , have a certain expression involving the coefficients of f(s) and $\mathcal{V}$ has the greatest possible dimension , ($\mathcal{V}$ may be $\mathcal{N}_r\{P\}$) , that the latter may happens . The above leads to the introduction of an algorithm which constructs the coefficients of the GCD as a tuple which belongs to a certain affine variety . The employment of Groebner bases , [Cox. 1] , [Bec. 1] , [Har. 1] [Sha. 1] , is essential for the application of this algorithm .

4.2. STATEMENT OF THE PROBLEM - PRELIMINARY RESULTS

Let $\underline{p}(s) \in \mathbb{R}^m[s]$, $\partial\{\ \underline{p}(s)\ \} = \delta$ and express $\underline{p}(s)$ as :

$$\underline{p}(s) = [\ \underline{p}_0 , \underline{p}_1 , \dots , \underline{p}_\delta\ ]\cdot\underline{e}_\delta(s) = P\cdot\underline{e}_\delta(s) \tag{4.2.1}$$

$$\underline{e}_\delta(s) = [\ 1 , s , \dots , s^\delta\ ]^T$$

where , $P \in \mathbb{R}^{mx(\delta+1)}$ is the <u>basis matrix</u> (b.m.) of $\underline{p}(s)$.

Problem : *Let $\underline{p}(s)$, $\underline{q}(s) \in \mathbb{R}^m[s]$ and let us assume that :*

$$\underline{p}(s) = \underline{q}(s) \cdot f(s) \tag{4.2.2}$$

where , $f(s) = f_0 + f_1 \cdot s + \cdots + f_\kappa \cdot s^\kappa \in \mathbb{R}[s]$. The problem that arises is to express relationship (4.2.2) as an equivalent relationship with real matrices and thus provide alternative means for characterizing the GCD of polynomials . □

If $P = [\, \underline{p}_0 , \underline{p}_1 , \dots , \underline{p}_\delta \,]$, $Q = [\, \underline{q}_0 , \underline{q}_1 , \dots , \underline{q}_d \,]$ are the b.ms of $\underline{p}(s)$, $\underline{q}(s)$, then :

$$\underline{p}(s) = (\, \underline{q}_0 + \underline{q}_1 s + \cdots + \underline{q}_d s^d \,) \cdot (\, f_0 + f_1 s + \cdots + f_\kappa s^\kappa \,)$$

or ,

$$\begin{aligned} \underline{p}_0 &= \underline{q}_0 f_0 \\ \underline{p}_1 &= \underline{q}_0 f_1 + \underline{q}_1 f_0 \\ &\vdots \\ \underline{p}_\delta &= \underline{q}_d f_\kappa \end{aligned} \tag{4.2.3}$$

or ,

$$\begin{bmatrix} \underline{p}_0 \\ \underline{p}_1 \\ \vdots \\ \vdots \\ \underline{p}_\delta \end{bmatrix} = \begin{bmatrix} \underline{q}_0 & \underline{0} & \cdots & \underline{0} \\ \underline{q}_1 & \underline{q}_0 & \ddots & \vdots \\ \vdots & \underline{q}_1 & \ddots & \underline{0} \\ \underline{q}_d & \vdots & \ddots & \underline{q}_0 \\ \underline{0} & \underline{q}_d & & \underline{q}_1 \\ \vdots & \ddots & \ddots & \vdots \\ \underline{0} & \cdots & \underline{0} & \underline{q}_d \end{bmatrix} \cdot \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_\kappa \end{bmatrix} \tag{4.2.4}$$

Relationship (4.2.4) is the Toeplitz representation of (4.2.2) , or (4.2.3) and it is referred to as <u>Composite Toeplitz representation</u> . An equivalent form to (4.2.4) is given below :

$$[\, \underline{p}_0 , \underline{p}_1 , \dots , \underline{p}_d , \dots , \underline{p}_\delta \,] = [\, \underline{q}_0 , \underline{q}_1 , \dots , \underline{q}_d , \underline{0} , \dots , \underline{0} \,] \cdot \begin{bmatrix} f_0 & f_1 & \cdots & f_\kappa & 0 & \cdots & 0 \\ 0 & f_0 & f_1 & \cdots & f_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & \ddots & f_0 & f_1 & \cdots & f_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & f_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_0 \end{bmatrix}$$

(4.2.5)

or ,

$$P = [\, Q : O \,]\, T_\delta(f) \tag{4.2.6}$$

where ,

$$T_\delta(f) \triangleq \begin{bmatrix} f_0 & f_1 & \cdots & f_\kappa & 0 & \cdots & 0 \\ 0 & f_0 & f_1 & \cdots & f_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & \ddots & f_0 & f_1 & \cdots & f_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & f_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_0 \end{bmatrix}$$

where , $T_\delta(f) \in \mathbb{R}^{(\delta+1)x(\delta+1)}$ is referred to as the <u>δ – Toeplitz representation</u> of $f(s) \in \mathbb{R}[s]$ with $\kappa = \partial\{ f(s) \} \le \delta$. We shall denote by $\mathbb{R}_\kappa[s]$ the set of all polynomials of maximal degree κ and by $\mathbb{R}^0_\kappa[s]$ the subset of $\mathbb{R}_\kappa[s]$ such that for all $f(s) \in \mathbb{R}^0_\kappa[s]$, $f(0) \neq 0$; this subset will be referred to as a <u>regular</u> subset of $\mathbb{R}_\kappa[s]$. If $f(s) = f_0 + f_1 \cdot s + \cdots + {}+f_\kappa \cdot s^\kappa \in \mathbb{R}^0_\kappa[s]$ and denote by $T_\delta(f)$ the δ – Toeplitz representation of $f(s)$, $\delta \ge \kappa$, where :

$$T_\delta(f) \triangleq \begin{bmatrix} f_0 & f_1 & \cdots & f_\kappa & 0 & \cdots & 0 \\ 0 & f_0 & f_1 & \cdots & f_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & \ddots & f_0 & f_1 & \cdots & f_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & f_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_0 \end{bmatrix} \in \mathbb{R}^{(\delta+1)x(\delta+1)} \qquad (4.2.7)$$

We shall denote by $\mathcal{T}_\delta$ the set of all matrices of the $T'_\delta(f)$ type :

$$T'_\delta(f) \triangleq \begin{bmatrix} f_0 & f_1 & \cdots & f_\kappa & f_{\kappa+1} & \cdots & f_\delta \\ 0 & f_0 & f_1 & \cdots & f_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & f_{\kappa+1} \\ \vdots & & \ddots & f_0 & f_1 & \cdots & f_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & f_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_0 \end{bmatrix} \in \mathbb{R}^{(\delta+1)x(\delta+1)} \qquad (4.2.8)$$

Clearly $| \, T'_\delta(f) \, | \neq 0$ and :

Lemma (4.2.1) *: The set $\mathcal{T}_\delta$, under the multiplication of matrices , is an abelian group with $I_{\delta+1}$ as identity .*

Proof

It is trivial to verify the properties of the abelian group ; we shall prove the existence of an inverse for all the elements of $\mathcal{T}_\delta$. Let $T'_\delta(f)$ be an element of $\mathcal{T}_\delta$ (as in (4.2.8)) , then by using induction we shall prove that there exists an element of $\mathcal{T}_\delta$, $T'_\delta(f)^{-1}$ such that , $T'_\delta(f) \cdot T'_\delta(f)^{-1} = I_{\delta+1}$. For $\delta = 0$, $T'_\delta(f)$ has the form $T'_0(f) = [\ f_0\ ]$ and clearly the matrix $T'_0(f)^{-1} = [\ (1/\ f_0)\ ]$ belongs to $\mathcal{T}_0$ and $T'_0(f) \cdot T'_0(f)^{-1} = I_1 = 1$. For $\delta = 1$, $T'_\delta(f)$ has the form :

$$T'_1(f) = \begin{bmatrix} f_0 & f_1 \\ 0 & f_0 \end{bmatrix}$$

and clearly the matrix :

$$T'_1(f)^{-1} = \begin{bmatrix} (1/f_0) & (-f_1/f_0^2) \\ 0 & (1/f_0) \end{bmatrix}$$

belongs to $\mathcal{T}_1$ and $T'_1(f) \cdot T'_1(f)^{-1} = I_2$. Let now suppose that for $\delta = n$ the hypothesis holds true , we shall prove it for $\delta = n+1$. Let :

$$T'_n(f)^{-1} \triangleq \begin{bmatrix} g_0 & g_1 & \cdots & g_\kappa & g_{\kappa+1} & \cdots & g_n \\ 0 & g_0 & g_1 & \cdots & g_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & g_{\kappa+1} \\ \vdots & & \ddots & g_0 & g_1 & \cdots & g_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & g_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & g_0 \end{bmatrix} \in \mathbb{R}^{(n+1)x(n+1)}$$

be the inverse element of $T'_n(f)$. Set as $T'_{n+1}(f)^{-1}$ the matrix :

$$T'_{n+1}(f)^{-1} = \begin{bmatrix} g_0 & g_1 & \cdots & g_\kappa & g_{\kappa+1} & \cdots & g_n & b \\ 0 & g_0 & g_1 & \cdots & g_\kappa & \ddots & \vdots & g_n \\ \vdots & \ddots & \ddots & \ddots & & \ddots & g_{\kappa+1} & \vdots \\ \vdots & & \ddots & g_0 & g_1 & \cdots & g_\kappa & g_{\kappa+1} \\ \vdots & & & \ddots & \ddots & \ddots & \vdots & g_\kappa \\ \vdots & & & & \ddots & \ddots & g_1 & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & g_0 & g_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & g_0 \end{bmatrix} \in \mathbb{R}^{(n+2)x(n+2)}$$

with :

$$b = \frac{-\sum_{i=1}^{n+1} f_i \cdot g_{n-i+1}}{f_0}$$

Clearly $T'_{n+1}(f)^{-1}$ belongs to $\mathcal{T}_{n+1}$ and $T'_{n+1}(f) \cdot T'_{n+1}(f)^{-1} = I_{n+2}$. □

The group $(\mathcal{T}_\delta , \cdot)$ will be simply denoted by $\mathcal{T}_\delta$; using the properties of this group we have that (4.2.6) may be expressed as :

$$[\,Q \vdots O\,] = P\ \widetilde{T}_\delta(f) \tag{4.2.9}$$

where , $\widetilde{T}_\delta(f) = T_\delta^{-1}(f)$.

Remark (4.2.1) *: Condition (4.2.9) may be seen as the reverse of the condition defined by (4.2.2) and thus it is equivalent to an extraction of a divisor from $\underline{p}(s)$ polynomial vector . The extracted divisor is defined as the polynomial corresponding to the matrix $\widetilde{T}_\delta(f) = T_\delta^{-1}(f)$. It is clear that the extracted divisor becomes a gcd , if and only if the number of zero columns in $[\,Q \vdots O\,]$ is the maximal possible that can be extracted by $\widetilde{T}_\delta(f)$ type of transformations , the inverse of which corresponds to a polynomial .* □

In the following remark we state some useful results for the later development of the topic .

Remark (4.2.2) *: If $f(s) = f_0 + f_1 \cdot s + \cdots + f_\kappa \cdot s^\kappa \in \mathbb{R}^0_\kappa[s]$, then without loss of generality we can assume that $f_0 = 1$. Then a δ – Toeplitz representation , $T_\delta(f)$, of $f(s)$, $\delta \geq \kappa$, is given as in (4.2.7) , where $f_0 = 1$. If we take $T_\delta^{-1}(f)$ this is an upper triangular Toeplitz matrix in $\mathcal{T}_\delta$ and even more its elements are of the type $\left(\sum_\alpha c_\alpha\ f^\alpha\right)_{i,j}$, where $\alpha = (\alpha_1 , \alpha_2 , \ldots , \alpha_\kappa) \in \mathbb{A} \subset \mathbb{N}^\kappa$, $\mathbb{A}$ finite , $f^\alpha = f_1^{\alpha_1} \cdots f_\kappa^{\alpha_\kappa}$, and c_α real constants . If we fix a $\delta \geq \kappa$, then all the elements of $\mathcal{T}_\delta$., the inverse of which corresponds to $f(s)$ for $\kappa = 0 , 1 , \ldots , \delta$ must have elements of the type $\left(\sum_\alpha c_\alpha\ f^\alpha\right)_{i,j}$. If we fix κ as well and find the inverse $T_\delta^{-1}(f)$, of the $T_\delta(f)$, δ – Toeplitz representation , then we can find the inverse of the of the δ – Toeplitz representation of $1 + f_1 \cdot s + \cdots + f_{\kappa-1} \cdot s^{\kappa-1}$ by simply setting $f_\kappa = 0$ in $T_\delta^{-1}(f)$. On the same token we can characterize the elements of $\mathcal{T}_\delta$, the inverse of which corresponds to a polynomial $f(s) \in \mathbb{R}^0_\kappa[s]$.* □

Remark (4.2.3) *: Let $f(s) = 1 + f_1 \cdot s + \cdots + f_\kappa \cdot s^\kappa \in \mathbb{R}^0_\kappa[s]$ be the gcd of a set of polynomials then the family of gcds is given by $\mathfrak{D} = \{\ u \cdot f(s) , u \in \mathbb{R} - \{0\}\ \}$ and hence the parametrization of f_i , $i = 1 , 2 , \ldots , \kappa$ is given by $g_i = u \cdot f_i$.* □

Remark (4.2.4) *: If $P \in \mathbb{R}^{m x (\delta+1)}$ is the b.m. of a set of polynomials with rank $P = \rho$, the greatest number of columns of P that can be annihilated is $\tau = \delta + 1 - \rho$. Hence , τ*

is the upper bound for the degree of the gcd of the set of polynomials and is achieved when an element $\widetilde{T}_\delta(f)$ of $\mathcal{T}_\delta$ exists such that $[\ Q \vdots O_\tau\ ] = P\cdot\widetilde{T}_\delta(f)$ and the inverse of $\widetilde{T}_\delta(f)$ is the r – Toeplitz representation of a polynomial $f(s)$ with deg τ. □

Remark (4.2.1) implicitly connects the existence of elements of $\mathcal{T}_\delta$, such that (4.2.6) holds true , to the investigation of the right null space of P for bases of Toeplitz type , the elements of which satisfy certain conditions . In the next section the notion of scalar annihilating Toeplitz bases is introduced and their contribution to the construction of the family of gcds of a set of polynomials is investigated.

Note : If W denotes a full column rank matrix , $\mathcal{W}$, or {W} , will denote the base formed by the columns of W and vice versa .

4.3. SCALAR ANNIHILATING TOEPLITZ BASES

In the following we state a condition for the existence of matrices $\widetilde{T}_\delta(f)$ such that (4.2.9) holds true , (with $O \in \mathbb{R}^{mxr}$, $r \leq \tau = \delta+1-rank\{\ P\ \}$) . This condition is connected to the bases $\mathcal{W}$ of $\mathcal{V} \subseteq \mathcal{N}_r\{\ P\ \}$. More precisely if $\mathcal{W}_i$ denotes a base of $\mathcal{V}_i \subseteq \mathcal{N}_r\{\ P\ \}$, with $rank\{\mathcal{W}_i\} = i$ and $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$, $i = 1, 2, \ldots, \tau$, then :

Proposition (4.3.1) : *Let $P = [\ \underline{p}_0, \underline{p}_1, \ldots, \underline{p}_\delta\ ] \in \mathbb{R}^{mx(\delta+1)}$, with rank $P = \rho$. Then a matrix $\widetilde{T}_\delta(f)$, such that (4.2.9) holds true (with $O \in \mathbb{R}^{mxr}$, $1 \leq r \leq \tau$,) , exists if and only if there exists a base $\mathcal{W}_i$ of $\mathcal{V}_i \subseteq \mathcal{N}_r\{\ P\ \}$, for $i = r$, such that it has the following form :*

$$W_r = \begin{bmatrix} w_{\delta-r} & w_{\delta-r+1} & \cdots & w_\delta \\ w_{\delta-r-1} & w_{\delta-r} & \ddots & w_{\delta-1} \\ \vdots & w_{\delta-r-1} & \ddots & \vdots \\ w_0 & \vdots & \ddots & \vdots \\ \vdots & w_0 & & w_{\delta-r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_0 \end{bmatrix} \in \mathbb{R}^{(\delta+1)xr} \qquad (4.3.1)$$

where , w_0 is non zero .

Proof

(⇒) Let a matrix $\widetilde{T}_\delta(f)$, such that (4.2.9) holds true (with $O \in \mathbb{R}^{mxr}$, $1 \leq r \leq \tau$) , exists

then $\widetilde{T}_\delta(f)$ is a matrix as in (4.2.8) . Since the maximal number of columns that $\widetilde{T}_\delta(f)$ annihilates is r , if we select the last r columns of $\widetilde{T}_\delta(f)$, an $\mathbb{R}^{(\delta+1)xr}$ matrix is formed and is denoted as W_r . This matrix is of the form (4.3.1) , has its low rxr part invertible and $O = P \cdot W_r$. Hence $\mathcal{W}_r$ is a base for $\mathcal{V}_r \subseteq \mathcal{N}_r\{ P \}$ of the form (4.3.1) .
($\Leftarrow$) Let a base $\mathcal{W}_r$ of $\mathcal{V}_r \subseteq \mathcal{N}_r\{ P \}$ of the form (4.3.1) exist , then we form the matrix :

$$\widetilde{T}_\delta(f) = \begin{bmatrix} w_0 & w_1 & \cdots & w_\kappa & w_{\kappa+1} \cdots & w_\delta \\ 0 & w_0 & w_1 & \cdots & w_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & w_{\kappa+1} \\ \vdots & & \ddots & w_0 & w_1 & \cdots & w_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & w_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & w_0 \end{bmatrix} \in \mathbb{R}^{(\delta+1)x(\delta+1)}$$

$\widetilde{T}_\delta(f)$ is of the form (4.2.8) and $[\, Q \,\vdots\, O \,] = P \cdot \widetilde{T}_\delta(f)$, with $O \in \mathbb{R}^{mxr}$, $1 \le r \le \tau$. □

***Definition (4.3.1)** : A base $\mathcal{W}_r$ of the type (4.3.1) will be called an r – scalar annihilating Toeplitz base (r.s.a.t.b.) , or r – annihilating base (r.a.b.) for simplicity .* □

***Remark (4.3.1)** : The condition of proposition(4.3.1) is necessary and sufficient as far as the annihilation of columns of the b.m. P in (4.2.9) is concerned , but as the example below illustrates , it is only necessary when it comes to the estimation of the gcd of the set of polynomials with b.m. P .* □

Example (4.3.1) : Let $\underline{p}(s) = [\, s^4 - 1 \,,\, s^4 - s^3 + 2\, s^2 - s - 1 \,]^T$, then the basis matrix of $\underline{p}(s)$ is :

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2 & -1 & 1 \end{bmatrix}$$

$\delta = 4$, *rank* P $= 2$, $\tau = \delta + 1 - rank\ P = 3$. Clearly the set of polynomials has as its gcd the $(s-1)$. If we try to find the family of gcds of $\underline{p}(s)$ using proposition(4.2.1) , first we must find an 1 – annihilating Toeplitz base $\mathcal{W} \in \mathbb{R}^{5x1}$, for some $\mathcal{V} \subset \mathcal{N}_r\{P\}$, with its (5 , 1) element nonzero . Then $\mathcal{W}$ can generate a Toeplitz matrix $\widetilde{T}_4(f)$, which annihilates the last column of P . If the condition of proposition(4.3.1) is sufficient then $\widetilde{T}_4^{-1}(f)$ must be a Toeplitz matrix corresponding to a first degree polynomial of the form $u \cdot (s-1)$, $u \in \mathbb{R} - \{0\}$. A base $\mathcal{B}$ for $\mathcal{N}_r\{ P \}$ is given by :

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{bmatrix}^T$$

and for $W = [\ 1\ 0\ 0\ 0\ 1\ ]^T$ we see that the (5 , 1) element is nonzero and hence the Toeplitz matrix generated by W is :

$$\widetilde{T}_4(f) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the inverse of $\widetilde{T}_4(f)$ is :

$$\widetilde{T}_4^{-1}(f) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which clearly does not correspond to a first degree polynomial . But if we try a second base of $\mathcal{N}_r\{\ P\ \}$, let say $\mathcal{G}$, given by :

$$G = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \end{bmatrix}^T$$

Then the Toeplitz matrix $\widetilde{T}_4(f)$ generated by 1 – annihilating base $C = [-1\ ,\ -1\ ,\ -1\ ,\ -1\ ,\ -1\ ]^T$ is :

$$\widetilde{T}_4(f) = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

which annihilates the last column of P and its inverse is $\widetilde{T}_4^{-1}(f)$

$$\widetilde{T}_4^{-1}(f) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

which clearly corresponds to the polynomial $(s-1)$. Hence the condition of proposition (4.3.1) is necessary (else no annihilating Toeplitz matrix would exist at all) , but not sufficient . □

Example (4.3.1) leads us to impose further restrictions on the form of the r.a.b. of proposition (4.3.1) . Let $\underline{p}(s) \in \mathbb{R}^m[s]$, with b.m. $P = [\, \underline{p}_0 , \underline{p}_1 , \dots , \underline{p}_\delta \,] \in \mathbb{R}^{m x(\delta+1)}$, *rank* $P = \rho$, $\tau = \delta+1-\rho$. Let $f(s) = 1 + f_1 \cdot s + \cdots + f_\kappa \cdot s^\kappa \in \mathbb{R}^0_\kappa[s]$ and $T_\delta(f)$ be its δ – Toeplitz representation as in (4.2.7) . Consider $T_\delta^{-1}(f)$, then by remark(4.2.2) its elements are $\left(\sum_\alpha c_\alpha f^\alpha\right)_{i,j}$, $i = 1 , \dots , \delta+1$, $j = i , i+1 , \dots , \delta+1$.

Proposition (4.3.2) *: f(s) is a gcd of* $\underline{p}(s)$ *if and only if there exists a base* $\mathcal{W}_\kappa$ *of a* $\mathcal{V}_\kappa \subseteq \mathcal{N}_r\{ P \}$ *such that :* $\mathcal{W}_\kappa$ *is a* κ *– annihilating base and its elements are given by* $\left(\sum_\alpha c_\alpha f^\alpha\right)_{i,j}$, $i = 1 , 2 , \dots , \delta+1$, $j = \delta-\kappa+1 , \delta-\kappa+2 , \dots , \delta+1$ *with* κ *the greatest possible* $(\kappa \le \tau)$. □

Comment *: The proposition in other words states that if* κ , $\kappa \le \tau$, *is the greatest possible for which an element* $\widetilde{T}_\delta(f)$ *of* $\mathcal{T}_\delta$ *satisfies* $[\, Q \vdots O_\kappa \,] = P \cdot \widetilde{T}_\delta(f)$ *and* $T_\delta(f) = \widetilde{T}_\delta^{-1}(f)$ *is a* δ *– Toeplitz representation for a polynomial f(s) of degree* κ *then f(s) is a gcd and vice versa .* □

Proof of proposition(4.3.2)

(⇒) If f(s) is a gcd of $\underline{p}(s)$ then by simply following the steps (4.2.1) – (4.2.9) of section 4.2 we reach the equation :

$$[\, Q : O_\kappa \,] = P \, \widetilde{T}_\delta(f) \tag{4.3.2}$$

where , $\widetilde{T}_\delta(f) = T_\delta^{-1}(f)$ and $T_\delta(f)$ is the δ – Toeplitz representation of f(s) . By remark (4.2.2) the elements of $\widetilde{T}_\delta(f) \equiv T_\delta^{-1}(f)$ are $\left(\sum_\alpha c_\alpha f^\alpha\right)_{i,j}$, $i = 1 , \dots , \delta+1$, $j = i , i+1 , \dots , \delta+1$. Inspection of equation (4.3.2) leads to the conclusion that the matrix W_κ formed by the last κ columns of $\widetilde{T}_\delta(f)$ forms a base $\mathcal{W}_\kappa$ of a $\mathcal{V}_\kappa \subseteq \mathcal{N}_r\{ P \}$, such that, $\mathcal{W}_\kappa$ is a κ – annihilating base and its elements are given by $\left(\sum_\alpha c_\alpha f^\alpha\right)_{i,j}$, $i = 1 , 2 , \dots , \delta+1$, $j=$

$= \delta - \kappa + 1$, $\delta - \kappa + 2$, ... , $\delta + 1$ and because f(s) is a gcd κ is the greatest possible $(\kappa \leq \tau)$.

($\Leftarrow$) Let $\mathcal{W}_\kappa$ be a base of a $\mathcal{V}_\kappa \subseteq \mathcal{N}_r\{P\}$, such that : $\mathcal{W}_\kappa$ is a κ – annihilating base and its elements are given by $\left(\sum_\alpha c_\alpha f^\alpha\right)_{i,j}$, $i = 1 , 2 , \ldots , \delta + 1$, $j = \delta - \kappa + 1 , \delta - \kappa + 2 , \ldots , \delta + 1$, with κ the greatest possible , $\kappa \leq \tau$. Then the last column of W_κ generates a Toeplitz matrix $\widetilde{T}_\delta(f)$ the inverse of which corresponds to the δ – Toeplitz representation of a κ degree polynomial f(s) , (remark(4.2.2)) . For the $\widetilde{T}_\delta(f)$ equation (4.3.2) holds true. If we follow the reverse steps (4.2.9) – (4.2.2) we conclude that f(s) is a common divisor of the set of polynomials $\underline{p}(s)$ and hence it divides the gcd of $\underline{p}(s)$, let say t(s), (deg t(s) = d) . But we already know that a necessary condition for t(s) to be a gcd is the existence of a Toeplitz matrix $\widetilde{T}_\delta(t)$ which satisfies equation $[\ Q : O_d\ ] = P \cdot \widetilde{T}_\delta(t)$ and its inverse is the δ – Toeplitz representation of t(s) . Since κ is the greatest possible, $\kappa \leq \tau$, for which such a $\widetilde{T}_\delta(t)$ exists it is implied that $d = \kappa$. Hence , from the polynomial division in $\mathbb{R}[s]$ we conclude that $t(s) = u \cdot f(s)$, $u \in \mathbb{R} - \{0\}$. Thus , by remark(4.2.3) f(s) is a gcd for the set of polynomials $\underline{p}(s)$. □

Now we can reexamine example(4.3.1) and explain why the base $\mathcal{W}$ failed to give us the gcd , whereas base $\mathcal{C}$ did not . Since the gcd of the set of polynomials $\underline{p}(s) = [\ s^4 - 1, s^4 - s^3 + 2\ ,\ s^2 - s - 1\ ]^T$ is the polynomial f(s) = 1 – s , we need an δ – Toeplitz representation $T_4(f)$ of (1 – s) the inverse of which $T_4^{-1}(f)$ annihilates the last column of P′ . Generally the elements of $\mathcal{T}_4$, the inverse of which is an δ – Toeplitz representation of a polynomial $(1+f_1\ s)$ must have elements of the type $\left((-1)^j f_1^j\right)_{ij}$, $i = 0 , 1 , \ldots , 4$, $j = i , i + 1, \ldots , 4$. Hence , in order a first degree polynomial to be a gcd of $\underline{p}(s)$ a base $\mathcal{C}$ of a subset of $\mathcal{N}_r\{\ P\ \}$ must exist such that $\mathcal{C} = [\ f_1^4\ ,\ -f_1^3\ ,\ f_1^2\ ,\ -f_1\ ,\ 1\ ]^T$. Since $f_1 = -1$ is implied that $C = -C = [-1\ ,\ -1\ ,\ -1\ ,\ -1\ ,\ -1\ ]^T$, whereas for all $f_1 \in \mathbb{R} - \{0\}$, $W = [\ 1\ 0\ 0\ 0\ 1\ ]^T$ is not of the form $[\ f_1^4\ ,\ -f_1^3\ ,\ f_1^2\ ,\ -f_1\ ,\ 1\ ]^T$. According to proposition(4.3.2) we should examine the cases $\kappa = 3$ and $\kappa = 2$, $\kappa \leq 3$, first and then the case $\kappa = 1$. This examination employs methods based on Algebraic Geometry and will be discussed later on after the presentation of the main mathematical results required , has been completed .

In the following we give a method for the characterization of r – annihilating bases of a space $\mathcal{N}_r\{A\}$, $A \in \mathbb{R}^{m x n}$, as they where introduced in proposition(4.3.1) and definition(4.3.1) without the additional constrain of proposition(4.3.2) about the type of their elements . This characterization is useful when the gcd of a set of polynomials with b.m. A has degree , d , equal to $dim\mathcal{N}_r\{A\}$, (remark(4.2.4) , proposition(4.3.1)) , and is much more easier than the one described in section 4.5 when $d \leq dim\mathcal{N}_r\{A\}$. Let $A \in \mathbb{R}^{m x n}$, $dim\ \mathcal{N}_r\{A\} = r$. Then A can be considered as the b.m. of a set of m polynomials $\underline{p}(s)$, with $\deg(\underline{p}(s)) = n$. Let $\mathcal{W}_r$ be a base of $\mathcal{N}_r\{A\}$. If $\mathcal{W}_r$ is an r – annihilating base for $\mathcal{N}_r\{A\}$ then $A \cdot W_r = O_r$, W_r is of the form (4.3.1) and its lowest rxr part is full rank . Then we can write :

$$
W_r = \begin{bmatrix} w_{11} w_{12} & \cdots & w_{1r} \\ w_{21} w_{22} & \cdots & w_{2r} \\ \vdots \quad \vdots & & \vdots \\ w_{j1} w_{j2} & \cdots & w_{jr} \\ 1 \quad 0 & \cdots & 0 \\ \vdots \quad 1 & \ddots & \vdots \\ \vdots \quad \vdots & \ddots & 0 \\ 0 \quad 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{nxr} \tag{4.3.3}
$$

where , $j = n - r$. Then all the r – annihilating bases $\mathcal{G}_r$ of $\mathcal{N}_r\{A\}$ are constructed by multiplying W_r by the full rank matrix :

$$
U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{r-2} & u_{r-1} \\ 0 & u_0 & \cdots & u_{r-3} & u_{r-2} \\ \vdots & \vdots & & \vdots & u_{r-3} \\ 0 & 0 & \cdots & u_0 & \vdots \\ 0 & 0 & \cdots & 0 & u_0 \end{bmatrix}
$$

and $G_r = W_r \cdot U$. Alternatively , we may express G_r as :

$$
G_r = \begin{bmatrix} w_{11}u_0 & \sum_{i=1}^{2} w_{1i}u_{2-i} & \cdots\cdots & \sum_{i=1}^{r-1} w_{1i}u_{r-1-i} & \sum_{i=1}^{r} w_{1i}u_{r-i} \\ w_{21}u_0 & \sum_{i=1}^{2} w_{2i}u_{2-i} & \cdots\cdots & \sum_{i=1}^{r-1} w_{2i}u_{r-1-i} & \sum_{i=1}^{r} w_{2i}u_{r-i} \\ \vdots & \vdots & & \vdots & \vdots \\ w_{j1}u_0 & \sum_{i=1}^{2} w_{ji}u_{2-i} & \cdots\cdots & \sum_{i=1}^{r-1} w_{ji}u_{r-1-i} & \sum_{i=1}^{r} w_{ji}u_{r-i} \\ u_0 & u_1 & \cdots\cdots & u_{r-2} & u_{r-1} \\ 0 & u_0 & \cdots\cdots & u_{r-3} & u_{r-2} \\ \vdots & \vdots & & \vdots & u_{r-3} \\ 0 & 0 & \cdots\cdots & u_0 & \vdots \\ 0 & 0 & \cdots\cdots & 0 & u_0 \end{bmatrix} \tag{4.3.4}
$$

Among the relations between the elements of G_r we regard those which express the u_κ , $\kappa = 1 , 2 , \ldots , r-1$, (u_0 can be an arbitrary nonzero element of $\mathbb{R}$) , which are :

$$u_\kappa = w_{(j-\kappa+1)1}\, u_0 \;,\; \kappa = 1 , 2 , \ldots , j$$
$$u_{j+1} = \sum_{i=1}^{2} w_{1i}\, u_{2-i}$$
$$u_{j+2} = \sum_{i=1}^{3} w_{1i}\, u_{3-i} \tag{4.3.5}$$
$$\vdots$$
$$u_{r-1} = \sum_{i=1}^{r-j} w_{1i}\, u_{r-j-i}$$

or equivalently ,

$$u_\kappa = \begin{cases} w_{(j-\kappa+1)1}\, u_0 \;,\; \kappa = 1 , 2 , \ldots , j \\ \sum_{i=1}^{\kappa-j+1} w_{1i}\, u_{\kappa-j-i+1} \;,\; \kappa = j+1 , j+2 , \ldots , r-1 \end{cases} \tag{4.3.6}$$

It is clear that when $j \geq r-1$, $u_\kappa = w_{(j-\kappa+1)1}\, u_0$, $\kappa = 1 , 2 , \ldots , r-1$. By (4.3.5) , or (4.3.6) it is obvious that we can write :

$$u_\kappa = c_\kappa\, u_0 \;,\; \kappa = 1 , 2 , \ldots , r-1 \tag{4.3.7}$$

where , c_κ is a sum of products of elements from the first column and row of W_r and is easy to calculate from (4.3.6) . Hence , an r – annihilating base $\mathcal{G}_r$ of $\mathcal{N}_r\{A\}$ is expressed as a multiple of base $\mathcal{W}_r$ by :

$$U = u_0 \cdot \begin{bmatrix} 1 & c_1 & \cdots & c_{r-2} & c_{r-1} \\ 0 & 1 & \cdots & c_{r-3} & c_{r-2} \\ \vdots & \vdots & & \vdots & c_{r-3} \\ 0 & 0 & \cdots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = u_0 \cdot C \tag{4.3.8}$$

Then the characterization of all the r – annihilating bases of $\mathcal{N}_r\{A\}$, $\mathcal{D}_r$ are given by the relation $D_r = W_r \cdot C \cdot u_0$, with u_0 an arbitrary nonzero real number .

Remark (4.3.2) *: Proposition(4.3.2) clearly states that the existence of r – annihilating Toeplitz bases of $\mathcal{N}_r\{A\}$ with a special type of elements $\left(\sum_\alpha c_\alpha f^\alpha\right)_{i,j}$ is related to the* *existence of a gcd* $f(s) = 1 + f_1 \cdot s + \cdots + f_r \cdot s^r \in \mathbb{R}^0_r[s]$ *of* $\underline{p}(s)$ *with* $f^\alpha = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ *as in remark(4.2.2) . The knowledge of the last column of such a base is enough for the generation of the whole base . Hence , the question arising is for which* $f_1 , \ldots , f_r$ *real* $f_r \neq 0$ *a column vector :*

$$\underline{v} = [\, (\sum_\alpha c_\alpha f^\alpha)_1 , \ldots , (\sum_\alpha c_\alpha f^\alpha)_n \,]^t \qquad (4.3.9)$$

belongs to $\mathcal{N}_r\{A\}$ *and generates an r.a.b. Or , in other wards for which* $f_1 , \ldots , f_r$ *real* $f_r \neq 0$ *the system of equations :*

$$A \cdot \underline{v} = \underline{0} \qquad (4.3.10)$$

holds true . If the system (4.3.10) has no desirable solution then a gcd of degree r does not exist and the next step is to examine the existence of a gcd of degree $r-1$ *. This investigation is similar to the one for the case of degree r apart from the fact that now we set* $f_r = 0$ *in (4.3.9) and (4.3.10) and the result (if any) will be the characterization of* $(r-1)$ *– annihilating Toeplitz bases of a* $\mathcal{V}_{(r-1)} \subset \mathcal{N}_r\{A\}$ *. On the same token we can examine the cases of degree* $i = (r-j) , \ldots , 1$ *, whenever the cases degree* $i = r , \ldots , (r-j+1)$ *fail to give a gcd ,* $j = 0 , \ldots , (r-1)$ *.* □

The solution of (4.3.10) under the constraints $\underline{v}$ as in (4.3.9) and $f_1 , \ldots , f_i$ real $f_i \neq 0$, $i = r , \ldots , 1$, will be examined in section 5 . First some necessary mathematical results from the theory of Varieties and Ideals is presented .

4.4. MATHEMATICAL PRELIMINARIES

The set of equations (4.3.10) as described in remark(4.3.2) forms a system of nonlinear equations over $\mathbb{R}$ and thus the study of solution of (4.3.10) requires results from algebraic geometry . An introduction to the main concepts of Algebraic Geometry required is given in th following . Further details can be found in [Cox. 1] , [Bec. 1] , [Har. 1] , [Sha. 1] .

Definition (4.4.1) *: A monomial in* $x_1 , \ldots , x_n$ *is a product of the form* $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *, where all of the exponents* $\alpha_1 , \ldots , \alpha_n$ *are non negative integers . The total degree of this monomial is the sum* $\sum_{i=1}^{n} \alpha_i$ *.* □

Let $\alpha = (\alpha_1 , \ldots , \alpha_n)$ be an n – tuple of non zero integers . Then we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ When $\alpha = (0 , \ldots , 0)$, note that $x^\alpha = 1$. Let $\mathcal{K}$ be an arbitrary field .

Definition (4.4.2) *: A polynomial f in* $x_1 , \ldots , x_n$ *with coefficients in* $\mathcal{K}$ *is a finite linear* *combination (with coefficients in $\mathcal{K}$) of monomials . We will write a polynomial f in the form :*

$$f(x_1 , \dots , x_n) \equiv f = \sum_{\alpha} c_\alpha \cdot x^\alpha \ , \ c_\alpha \in \mathcal{K}$$

where the sum is over a finite number of n – tuples $\alpha = (\alpha_1 , \dots , \alpha_n)$. □

The ring of polynomials in $x_1 , \dots , x_n$ and coefficients in $\mathcal{K}$ is denoted by $\mathcal{K}[x_1 , \dots , x_n]$

Definition (4.4.3) : *Given a field $\mathcal{K}$ and a positive integer n we define the n – dimensional affine space over $\mathcal{K}$ to be the set $\mathcal{K}^n = \{ (a_1 , \dots , a_n) , a_1 , \dots , a_n \in \mathcal{K} \}$* □

Definition (4.4.4) : *Let $\mathcal{K}$ be a field and let $f_1 , \dots , f_s$ be polynomials in $\mathcal{K}[x_1 , \dots , x_n]$. Then we set :*

$$\mathcal{V}(f_1 , \dots , f_s) = \{ (a_1 , \dots , a_n) \in \mathcal{K}^n : f_i(a_1 , \dots , a_n) = 0 , \forall \ 1 \le i \le s \}$$

We call $\mathcal{V}(f_1 , \dots , f_s)$ the affine variety by $f_1 , \dots , f_s$. □

Definition (4.4.5) : *A subset $\mathcal{I} \subset \mathcal{K}[x_1 , \dots , x_n]$ is an ideal , if it satisfies :*

i) $0 \in \mathcal{I}$.
ii) If $f , g \in \mathcal{I}$, then $f+g \in \mathcal{I}$.
iii) If $f \in \mathcal{I}$ and $h \in \mathcal{K}[x_1 , \dots , x_n]$, then $h \cdot f \in \mathcal{I}$. □

Definition (4.4.6) : *Let $f_1 , \dots , f_s$ be polynomials in $\mathcal{K}[x_1 , \dots , x_n]$. Then we set :*

$$\langle f_1 , \dots , f_s \rangle = \left\{ \sum_{i=1}^{s} h_i \cdot f_i : h_1 , \dots , h_s \in \mathcal{K}[x_1 , \dots , x_n] \right\}$$

□

Lemma (4.4.1) : *If $f_1 , \dots , f_s \in \mathcal{K}[x_1 , \dots , x_n]$, then $\langle f_1 , \dots , f_s \rangle$ is an ideal of $\mathcal{K}[x_1 , \dots , x_n]$.* □

Definition (4.4.7) : *We say that an ideal $\mathcal{I} \subset \mathcal{K}[x_1 , \dots , x_n]$ is finitely generated if there exist $f_1 , \dots , f_s \in \mathcal{K}[x_1 , \dots , x_n]$ such that $\mathcal{I} = \langle f_1 , \dots , f_s \rangle$ and we say that $\{f_1 , \dots , f_s\}$ is a base of $\mathcal{I}$.* □

Proposition (4.4.1) : *If $\{f_1 , \dots , f_s\}$, $\{g_1 , \dots , g_s\}$ are bases of the same ideal in $\mathcal{K}[x_1 , \dots , x_n]$ so that $\langle f_1 , \dots , f_s \rangle = \langle g_1 , \dots , g_s \rangle$ then $\mathcal{V}(f_1 , \dots , f_s) = \mathcal{V}(g_1 , \dots , g_s)$.* □

An extension of the polynomial Euclidean Division in $\mathcal{K}[x]$ can be introduced for $\mathcal{K}[x_1, \dots , x_n]$. First an ordering relation for monomials is required .

Definition (4.4.8) *: A monomial ordering over* $\mathcal{K}[x_1, \ldots, x_n]$ *, is a relation* $>$ *on* $\mathbb{N}^n$ *, or equivalently any relation on the set of monomials* x^α *,* $\alpha \in \mathbb{N}^n$ *satisfying :*

i) $>$ *is a total ordering .*

ii) *If* $\alpha > \beta$ *and* $\gamma \in \mathbb{N}^n$ *, then* $\alpha + \gamma > \beta + \gamma$ *.*

iii) $>$ *is well – ordering on* $\mathbb{N}^n$ *, or any non empty subset of* $\mathbb{N}^n$ *has a smallest element under* $>$ *.* □

Definition (4.4.9) (Lexicographic Order) *: Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and* $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ *. We say that* $\alpha >_{lex} \beta$ *if , in the vector difference* $\alpha - \beta \in \mathbb{N}^n$ *, the left most non zero entry is positive . We will write :*

$$x^\alpha >_{lex} x^\beta \text{ , if } \alpha >_{lex} \beta$$ □

Since $(1, 0, \ldots, 0) >_{lex} (0, 1, \ldots, 0) >_{lex} \cdots >_{lex} (0, 0, \ldots . 1)$ is implied that $x_1 >_{lex} \cdots >_{lex} x_n$.

Proposition (4.4.2) *: The lexicographic ordering (lex. ord.) on* $\mathbb{N}^n$ *is a monomial ordering.* □

Actually there are many other orderings (as the graded lex. ordering , reverse graded lex. ordering) which are monomial orderings . In the later we shall need to confine ourselves to the lex. ordering .

Definition (4.4.10) *: Let* $f = \sum_\alpha c_\alpha \cdot x^\alpha$ *be a non zero polynomial in* $\mathcal{K}[x_1, \ldots, x_n]$ *and let* $>$ *be a monomial order .*

i) *The multidegree of f is :* $multideg(f) = max\ \{\alpha \in \mathbb{N}^n : c_\alpha \neq 0\}$ *, (the max is taken with respect to* $>$*) .*

ii) *The leading coefficient of f is :* $LC(f) = c_{multideg(f)} \in \mathcal{K}$ *.*

iii) *The leading monomial of f is :* $LM(f) = x^{multideg(f)}$ *, (with coefficient 1) .*

iv) *The leading term of f is :* $LT(f) = LC(f) \cdot LM(f)$ *.* □

Theorem (4.4.1) (Division in $\mathcal{K}[x_1, \ldots, x_n]$***)*** *: Let* $>$ *be the lex. ord. on* $\mathbb{N}^n$ *and* $F = (f_1, \ldots, f_s)$ *an ordered s – tuple of polynomials in* $\mathcal{K}[x_1, \ldots, x_n]$ *. Then every* $f \in \mathcal{K}[x_1, \ldots, x_n]$ *can be written as :*

$$f = t_1 f_1 + \cdots + t_s f_s + r$$

where , t_i *,* $r \in \mathcal{K}[x_1, \ldots, x_n]$ *and either* $r = 0$ *, or r is a* $\mathcal{K}$ *– linear combination of monomials non of which is devisable by any of the ,* $LT(f_1), \ldots, LT(f_s)$ *. We will call r a remainder of f on division by F . Further more , if* $t_i f_i \neq 0$ *then we have :*

$$multideg(f) \geq multideg(t_i\, f_i)$$ □

Remark (4.4.1) *: The remainder and quotients (r , t_i) defined in theorem (4.5.1) are unique (modulo >) .* □

Definition (4.4.11) *: An ideal $\mathfrak{I} \subset \mathfrak{K}[x_1 , \dots , x_n]$ is a monomial ideal if there is a subset $\mathbb{A} \subset \mathbb{N}^n$ – possibly infinite – such that $\mathfrak{I}$ consists of all the polynomials which are finite sums of the form $\sum_\alpha h_a \cdot x^a$, where $\alpha \in \mathbb{A}$, $h_a \in \mathfrak{K}[x_1 , \dots , x_n]$. In this case we write $\mathfrak{I} = \langle\, x^\alpha \,;\, \alpha \in \mathbb{A} \,\rangle$.* □

Theorem (4.4.2) (Dickson's Lemma) *: A monomial ideal $\mathfrak{I} = \langle\, x^\alpha \,;\, \alpha \in \mathbb{A} \,\rangle \subset \mathfrak{K}[x_1 , \dots , x_n]$ can be written in the form $\mathfrak{I} = \langle\, x^{\alpha(1)} , \dots , x^{\alpha(s)} \,\rangle$, where $\alpha(1) , \dots , \alpha(s) \in \mathbb{A}$. In particular $\mathfrak{I}$ has a finite base .* □

Definition (4.4.12) *: Let $\mathfrak{I} \subset \mathfrak{K}[x_1 , \dots , x_n]$ be an ideal other than {0} :*
i) *We denote by $LT(\mathfrak{I})$ the set of leading terms of the elements of $\mathfrak{I}$. Thus ,*

$$LT(\mathfrak{I}) = \{\ c\ x^a : \text{there exists } f \in \mathfrak{I} \text{ with } LT(f) = c\ x^\alpha\ \}$$

ii) *We denote by $\langle\, LT(\mathfrak{I}) \,\rangle$ the ideal generated by the elements of $LT(\mathfrak{I})$.* □

Theorem (4.4.3) (Hilbert Base Theorem) *: Every ideal $\mathfrak{I} \subset \mathfrak{K}[x_1 , \dots , x_n]$ has a finite generating set . That is $\mathfrak{I} = \langle\, g_1 , \dots , g_t \,\rangle$ for some $g_1 , \dots , g_t$ in $\mathfrak{I}$.* □

Definition (4.4.13) *: Let > be the lex. ord. A finite subset $\mathfrak{G} = \{g_1 , \dots , g_t\}$ of an ideal $\mathfrak{I}$ is said to be a Groebner base , (or standard base) , if :*

$$\langle\, LT(g_1) , \dots , LT(g_t) \,\rangle = \langle\, LT(\mathfrak{I}) \,\rangle$$ □

Corollary (4.4.1) *: Let > be the lex. ord. Then every ideal $\mathfrak{I} \subset \mathfrak{K}[x_1 , \dots , x_n]$ other that {0} has a Groebner base . Furthermore , any Groebner base for an ideal $\mathfrak{I}$ is a base of $\mathfrak{I}$* □

Definition (4.4.14) *: Let $\mathfrak{I} \subset \mathfrak{K}[x_1 , \dots , x_n]$ be an ideal . We will denote by $\mathcal{V}(\mathfrak{I})$ the set :*

$$\mathcal{V}(\mathfrak{I}) = \{\ (a_1 , \dots , a_n) \in \mathfrak{K}^n : f(a_1 , \dots , a_n) = 0 \ , \forall\, f \in \mathfrak{I}\}$$

□

Proposition (4.4.3) *: $\mathcal{V}(\mathfrak{I})$ is an affine variety . In particular if $\mathfrak{I} = \langle\, f_1 , \dots , f_s \,\rangle$, then $\mathcal{V}(\mathfrak{I}) = \mathcal{V}(f_1 , \dots , f_s)$.* □

Proposition (4.4.4) *: Let $\mathcal{G} = \{g_1 , \dots , g_t\}$ be a Groebner base for an ideal $\mathcal{I} \subset \mathcal{K}[x_1 , \dots , x_n]$ and let $f \in \mathcal{I}$. Then there is a unique $r \in \mathcal{K}[x_1 , \dots , x_n]$ with the following two properties :*

i) No term of r is devisable by one of $LT(g_1) , \dots . LT(g_t)$.

ii) There is $g \in \mathcal{I}$ such that $f = g + r$. □

Corollary (4.4.2) *: Let $\mathcal{G} = \{g_1 , \dots , g_t\}$ be a Groebner base for an ideal $\mathcal{I} \subset \mathcal{K}[x_1 , \dots , x_n]$ and let $f \in \mathcal{I}$. Then $f \in \mathcal{I}$, if and only if the remainder on division of f by $\mathcal{G}$ is zero* □

Definition (4.4.15) *: A reduced Groebner base of a polynomial ideal $\mathcal{I}$ is a Groebner base for $\mathcal{I}$ such that :*

i) $LC(p) = 1$ for all $p \in \mathcal{G}$.

ii) For all $p \in \mathcal{G}$, no monomial of p lies in $\langle LT(\mathcal{G} - \{p\}) \rangle$. □

Proposition (4.4.5) *: Let $\mathcal{I} \neq \{0\}$ be a polynomial ideal . Then for the lex. monomial ord. $\mathcal{I}$ has a unique reduced Groebner base .* □

The previous results enable us to solve systems of polynomial equations by using the elimination and extension theorems .

Definition (4.4.16) *: Given $\mathcal{I} = \langle f_1 , \dots , f_s \rangle \subset \mathcal{K}[x_1 , \dots , x_n]$, the κ^{th} elimination ideal $\mathcal{I}_\kappa$ is the ideal of $\mathcal{K}[x_1 , \dots , x_n]$ defined by $\mathcal{I}_\kappa = \mathcal{I} \cap \mathcal{K}[x_{\kappa+1} , \dots , x_n]$.* □

Theorem (4.4.4) (Elimination Theorem) *: Let $\mathcal{I} \subset \mathcal{K}[x_1 , \dots , x_n]$ be an ideal and let $\mathcal{G}$ be a Groebner base with respect to the lex. ord. , where $x_1 >_{lex} \cdots >_{lex} x_n$. Then for every $0 \leq \kappa \leq n$ the set :*

$$\mathcal{G}_\kappa = \mathcal{G} \cap \mathcal{K}[x_{\kappa+1} , \dots , x_n]$$

is a Groebner base of the κ^{th} elimination ideal $\mathcal{I}_\kappa$. □

Theorem (4.4.5) (The Extension Theorem) *: Let $\mathcal{I} = \langle f_1 , \dots , f_s \rangle \subset \mathbb{C}[x_1 , \dots , x_n]$ and let $\mathcal{I}_1$ be the first elimination ideal of $\mathcal{I}$. For each $1 \leq i \leq s$, write f_i in the form :*

$$f_i = g_i(x_2 , \dots , x_n) \cdot x_1^{N_i} + \text{terms in which } x_1 \text{ has degree} < N_i$$

where $N_i \geq 0$ and $g_i \in \mathbb{C}[x_2 , \dots , x_n]$ is non zero . (We set $g_i = 0$ when $f_i = 0$) . Suppose that we have a partial solution $(a_2 , \dots , a_n) \in \mathcal{V}(\mathcal{I}_1)$. If $(a_2 , \dots , a_n) \notin \mathcal{V}(g_1 , \dots , g_s)$, there exists $a_1 \in \mathbb{C}$ such that $(a_1 , \dots , a_n) \in \mathcal{V}(\mathcal{I})$. □

Corollary (4.4.3) *: Let $\mathcal{I} = \langle f_1 , \dots , f_s \rangle \subset \mathbb{C}[x_1 , \dots , x_n]$ and let $\mathcal{I}_1$ be the first elimination ideal of $\mathcal{I}$. And assume that for some i , $1 \leq i \leq s$, f_i is of the form :*

$$f_i = c \cdot x_1^N + \text{terms in which } x_1 \text{ has degree} < N$$

where $N \geq 0$ and $c \in \mathbb{C}$ is non zero . If $\mathcal{I}_1$ is the first elimination ideal of $\mathcal{I}$ and $(a_2 , \dots , a_n) \in \mathcal{V}(\mathcal{I}_1)$ then there is $a_1 \in \mathbb{C}$ such that $(a_1 , \dots , a_n) \in \mathcal{V}(\mathcal{I})$. □

4.5. CONSTRUCTION OF THE GCDs OF A SET OF POLYNOMIALS

Now we can return to remark(4.3.2) and try to elaborate the method described over there for the construction of a gcd of a set of polynomials . Proposition(4.3.2) and remark(4.3.2) imply the following theorem . Let $\underline{p}(s) \in \mathbb{R}^m[s]$, with b.m. $P = [\ \underline{p}_0 , \underline{p}_1 , \dots , \underline{p}_\delta\ ] \in \mathbb{R}^{mx(\delta+1)}$. with *rank* $P = \rho$, $\tau = \delta + 1 - \rho$. Let $f(s) = 1 + f_1 \cdot s + \cdots + f_i \cdot s^i \in \mathbb{R}_i^0[s]$, $i = \tau , \tau - 1 , \dots , 1$ and $T_{\delta,i}(f)$ be its δ – Toeplitz representation as in (4.2.7) . Consider $T_{\delta,i}^{-1}(f)$, its elements are given by $\left(\sum_\alpha c_\alpha f^\alpha\right)_{\kappa,j}$, $\kappa = 1 , \dots , \delta + 1$, $j = \kappa , \kappa + 1 , \dots \delta + 1$.

Theorem (4.5.1) *: $f(s)$ is a gcd of $\underline{p}(s)$ if and only if there exist $f_1 , \dots , f_i$ real $f_i \neq 0$, such that the system of equations :*

$$P \cdot \underline{v} = \underline{0} \tag{4.5.1}$$

where , $\underline{v} = [\ (\sum_\alpha c_\alpha\ f^\alpha)_{1,(\delta+1)} , \dots , (\sum_\alpha c_\alpha\ f^\alpha)_{(\delta+1),(\delta+1)}\ ]^T$ and $\underline{v}$ generates an i – Toeplitz annihilating base for some $\mathcal{V}_i \subset \mathcal{N}_r\{\ P\ \}$. □

In the following we give an algorithm for the construction of the family of gcds of a set of polynomials.

Algorithm for the Construction of the GCD of a Set $\underline{p}(s) \in \mathbb{R}^m[s]$, $\deg(\underline{p}(s)) = \delta$

Let $\underline{p}(s) \in \mathbb{R}^m[s]$, $\deg(\underline{p}(s)) = \delta$, with b.m. $P = [\ \underline{p}_0 , \underline{p}_1 , \dots , \underline{p}_\delta\ ] \in \mathbb{R}^{mx(\delta+1)}$, with *rank* $P = \rho$, $\tau = \delta + 1 - \rho$. Then the degree d of the gcd is $0 \leq d \leq \tau$.

Step 1 : Set $f(s) = 1 + f_1 \cdot s + \cdots + f_d \cdot s^d \in \mathbb{R}_d^0[s]$, $f_1 , \dots , f_d$ real $f_d \neq 0$, and set $d = \tau$.

Step 2 : Set $T_{\delta,d}(f)$ the δ – Toeplitz representation of $f(s)$,

$$T_{\delta,d}(f) \triangleq \begin{bmatrix} 1 & f_1 & \cdots & f_r & 0 & \cdots & 0 \\ 0 & 1 & f_1 & \cdots & f_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & \ddots & 1 & f_1 & \cdots & f_r \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & f_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(\delta+1)x(\delta+1)}$$

Step 3 : Set $T^{-1}_{\delta,d}(f)$ the inverse of $T_{\delta,d}(f)$.

Step 4 : Set $\underline{v}$ the last column of $T^{-1}_{\delta,d}(f)$; $\underline{v} = [\ (\sum_{\alpha} c_\alpha f^\alpha)_{1,(\delta+1)}, \dots, (\sum_{\alpha} c_\alpha f^\alpha)_{(\delta+1),(\delta+1)}\ ]^T$

Step 5 : Consider the system of polynomial equations $P \cdot \underline{v} = \underline{0}$, or

$$t_1(f_1, \dots, f_d) = t_2(f_1, \dots, f_d) = \cdots = t_m(f_1, \dots, f_d) = 0 \qquad (4.5.2)$$

Step 6 : Set $\mathcal{I} = \langle t_1, t_2, \dots, t_m \rangle \subset \mathbb{R}[f_1, \dots, f_d]$.

Step 7 : Consider the lexicographic monomial order ($>_{lex}$) in $\mathbb{N}^d$, with $f_1 >_{lex} \cdots >_{lex} >_{lex} f_d$.

Step 8 : Set $\mathcal{G} = \{g_1, \dots, g_t\}$ a reduced Groebner base for the ideal $\mathcal{I} = \langle t_1, t_2, \dots, t_\mu \rangle \subset \mathbb{R}[f_1, \dots, f_d]$.

Step 9 : The solutions of (4.5.2) under the constraints $f_1, \dots, f_d$ real $f_r \neq 0$ form a variety and by propositions (4.4.1) , (4.4.3) $\mathcal{V}(t_1, t_2, \dots, t_m) = = \mathcal{V}(g_1, g_2, \dots, g_t)$.

Step 10 : According to the Elimination Theorem – theorem(4.4.4) – $\mathcal{G}_\kappa = \mathcal{G} \cap \mathbb{R}[f_{\kappa+1}, \dots, f_d]$, $\kappa = 0, \dots, d-1$, is a Groebner base for the $\kappa^{\underline{th}}$ elimination ideal $\mathcal{I}_\kappa$.

Step 11 : Set $\kappa = d-1$ in step 10 . $\mathcal{G}_{d-1}$ is a polynomial in $\mathbb{R}[f_d]$.

a) If f_d real non zero belong to $\mathcal{V}(\mathcal{G}_{d-1})$ then we apply the extension theorem – theorem(4.4.5) – to find the $\mathcal{V}(\mathcal{G}_\kappa)$, $\kappa = d-2, \dots, 0$, as long as the constrain $f_1, \dots, f_{\kappa+1}$ real holds true . If the procedure is completed successfully for all κ , then we form the matrix $W_d \in \mathbb{R}^{(\delta+1)xd}$ as in proposition(4.3.2) and test whether $\mathcal{W}_d$ is a d.a.b. for a $\mathcal{V}_d \subset \mathcal{N}_r\{P\}$, or equivalently whether $P \cdot \mathcal{W}_d = O_d$. If it is then f(s) of step 1 is a gcd of $\underline{p}(s)$.

b) If either $\mathcal{V}(\mathcal{G}_{d-1})$ is not subset of $\mathbb{R}^*$, or for some $\kappa = d-2, \dots, 0$, $\mathcal{V}(\mathcal{G}_\kappa)$ fails to comply with the constrain $f_1, \ldots, f_\kappa$ real, or $\mathcal{W}_d$ is not a d.a.b. for a $\mathcal{V}_d \subset \subset \mathcal{N}_r\{P\}$ then there exists no gcd for $\underline{p}(s)$ of degree τ. In this case we search for a gcd of degree $d = \tau - 1, \ldots, 0$ by simply setting each time $d = \tau - 1, \ldots, 0$ in step 1, (or equivalently $f_\tau = 0$, $f_{\tau-1} = 0, \ldots, f_1 = 0$ in step 8). □

Comment *: When we apply step 11 b) we need not repeat the steps 2 through 7. We can simply set $f_\tau = 0$, $f_{\tau-1} = 0, \ldots, f_1 = 0$ each time to the reduced Groebner base we have already found in step 8 and repeat the steps 9 trough 11.* □

Remark (4.5.1) *: The construction of a degree d gcd f(s) of a set of polynomials, or in other wards the construction of the $f_1, \ldots, f_d$ real $f_d \neq 0$ clearly leads to the construction of the vector $\underline{v}$ in step 5. This vector generates a d – annihilating Toeplitz base $\mathcal{W}_d$ for a $\mathcal{V}_d \subset \mathcal{N}_r\{P\}$ – proposition(4.3.2), remark(4.3.2). Then all the d – annihilating Toeplitz base $\mathcal{F}_d$ of $\mathcal{V}_d$ are characterized by the relation $F_d = W_d \cdot u$, $u \in \mathbb{R}^*$, because all the d – annihilating Toeplitz bases $\mathcal{F}_d$ of $\mathcal{V}_d$ correspond to the gcd $g(s) = f(s)\cdot u$, $u \in \mathbb{R}^*$.* □

Example (4.5.1) : Let $\underline{p}(s) = [\, s^2 - 2s + 1,\ s^3 + s^2 - s - 1,\ 2s^3 + 3s^2 - 5s \,]^T$, then the basis matrix of $\underline{p}(s)$ is :

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & -5 & 3 & 2 \end{bmatrix}$$

$\delta = 3$, $rank\ P = 2$, $\tau = \delta + 1 - rank\ P = 2$.

Step 1 : Set $f(s) = 1 + f_1 \cdot s + \cdots + f_d \cdot s^d \in \mathbb{R}_d^0[s]$, $f_1, \ldots, f_d$ real $f_d \neq 0$, and set $d = 2$.

Step 2 :

$$T_{3,2}(f) \triangleq \begin{bmatrix} 1 & f_1 & f_2 & 0 \\ 0 & 1 & f_1 & f_2 \\ 0 & 0 & 1 & f_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4x4}$$

Step 3 : $T_{3,2}^{-1}(f) \triangleq [a_{ij}]$, $i = 1, 2, 3, 4$, $j = i, \ldots, 4$ and $a_{11} = 1$, $a_{12} = -f_1$, $a_{13} = f_1^2 - f_2$, $a_{14} = -f_1^3 + 2 f_1 f_2$ and a_{ij} is the same in the $i - j$ entries.

Step 4 : $\underline{v} = [\, -f_1^3 + 2 f_1 f_2,\ f_1^2 - f_2,\ -f_1,\ 1 \,]^T$.

Step 5 : $P \cdot \underline{v} = \underline{0}$, gives the system of equations :

$$t_1(f_1\,,f_2) = f_1^3 - f_1^2 - 2\,f_1\,f_2 - f_1 + f_2 + 1 = 0$$

$$t_2(f_1\,,f_2) = -f_1^3 - 2\,f_1^2 + 2\,f_1\,f_2 - f_1 + 2\,f_2 = 0 \qquad (4.5.3)$$

$$t_3(f_1\,,f_2) = -5\,f_1^2 - 3\,f_1 + 5\,f_2 + 2 = 0$$

Step 6 : Set $\mathcal{I} = \langle\, t_1\,, t_2\,, t_3 \rangle \subset \mathbb{R}[f_1\,, f_2]$.

Step 7 : Consider the lexicographic monomial order ($>_{lex}$) in $\mathbb{N}^2$, with $f_1 >_{lex} f_2$.

Step 8 : Set $\mathcal{G} = \{g_1\,, g_2\} = \{\, f_2\,, 1 + f_1 \,\}$ a reduced Groebner base for the ideal $\mathcal{I}$.

Step 9 : The solutions of (4.5.3) under the constraints f_1 , f_2 real $f_2 \neq 0$, form a variety and by propositions(4.4.1) , (4.4.3) $\mathcal{V}(\mathcal{I}) = \mathcal{V}(t_1\,, t_2\,, t_3) = \mathcal{V}(g_1\,, g_2)$.

Step 10 : According to the Elimination Theorem(4.4.4) $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{R}[f_1\,, f_2]$, $\mathcal{G}_1 = \mathcal{G} \cap \mathbb{R}[f_2]$ are Groebner bases for the 0^{th} and 1^{st} elimination ideals $\mathcal{I}_0 = \mathcal{I} = \langle\, g_1\,, g_2 \,\rangle$, $\mathcal{I}_1 = \langle\, g_1 \rangle$.

Step 11 : $\mathcal{V}(\mathcal{I}_1) = \{0\}$ and $f_2 = 0$. Hence , part a) fails and $d = 2$ does not qualify as a degree of a gcd of $\underline{p}(s)$. Applying part b) we search for a gcd of $\underline{p}(s)$ of degree $d = 1$, or in other words set $f_2 = 0$ in step 1 and consequently in step 8 . The new Groebner base for the ideal $\mathcal{I}\langle\, t_1\,, t_2\,, t_3\,\rangle$ is $\mathcal{G} = \{g_2\}$ which imply $f_1 = -1$. The latter generates an 1 – Toeplitz annihilating base $\mathcal{W}_1$ for some $\mathcal{V}_1 \subset \mathcal{N}_r\{P\}$ since for $W_1 = [\,1\,, 1\,, 1\,, 1\,]^T$, $P \cdot W_1 = \underline{0}$ .Thus the polynomial of $1 + f_1 \cdot s = 1 - s$, qualifies as a gcd of $\underline{p}(s)$. □

4.6. PROPERTIES OF THE ELEMENTS OF $\mathcal{T}_\delta$

Let $\mathcal{T}_\delta$ be the multiplicative group of upper triangular Toeplitz matrices as it was introduced in lemma(4.2.1) and let $T_{f,\delta}$ be an element of $\mathcal{T}_\delta$ as :

$$T_{f,\delta} \triangleq \begin{bmatrix} f_0 & f_1 & \cdots & f_\kappa & f_{\kappa+1} & \cdots & f_\delta \\ 0 & f_0 & f_1 & \cdots & f_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & f_{\kappa+1} \\ \vdots & & \ddots & f_0 & f_1 & \cdots & f_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & f_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_0 \end{bmatrix} \in \mathbb{R}^{(\delta+1)x(\delta+1)} \qquad (4.6.1)$$

Let $\underline{b}^T = [\, b_0 \,, \dots, \, b_\delta \,]$ and $\underline{b}^T \cdot T_{f,\delta} = \underline{a}^T = [\, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_\delta \,]$, $\kappa = 0 \,, \dots, \delta$.

Definition (4.6.1) *: The elements $T_{f,\delta}$ of $\mathcal{T}_\delta$ for which a $\underline{b}^T = [\, b_0 \,, \dots, \, b_\delta \,]$ exists such that : $\underline{b}^T \cdot T_{f,\delta} = \underline{a}^T = [\, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_\delta \,]$, $\kappa = 0 \,, \dots, \delta - 1$ will be called δ – annihilating Toeplitz matrices or $(\delta.a.m)$.* □

Proposition (4.6.1) *: Let $\underline{b}^T = [\, b_0 \,, \dots, \, b_\delta \,]$ and $\underline{a}^T = [\, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_\delta \,]$, $\kappa = 0, \dots, \delta$. Let $T_{a,\delta}$ and $T_{b,\delta}$ be as :*

$$T_{a,\delta} \triangleq \begin{bmatrix} a_0 & a_1 & \cdots & a_\kappa & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & \ddots & a_0 & a_1 & \cdots & a_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & a_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_0 \end{bmatrix} \text{ and } T_{b,\delta} \triangleq \begin{bmatrix} b_0 & b_1 & \cdots & b_\kappa & \cdots & \cdots & b_\delta \\ 0 & b_0 & b_1 & \cdots & b_\kappa & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & b_0 & b_1 & \cdots & b_\kappa \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & b_1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_0 \end{bmatrix} \quad (4.6.2)$$

Then the matrix $T_{f,\delta} = T_{a,\delta} \cdot T_{b,\delta}^{-1} = T_{b,\delta}^{-1} \cdot T_{a,\delta}$ is a δ – Toeplitz annihilating matrix for $\underline{b}^T$ and vice versa .

Proof

(⇒) $\underline{b}^T = [\, b_0 \,, \dots, \, b_\delta \,] \cdot T_{b,\delta}^{-1} \cdot T_{a,\delta} = [\, 1 \,, 0 \,, \dots, 0 \,] \cdot T_{a,\delta} = [\, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_\delta \,] = \underline{a}^T$. Hence , $T_{f,\delta} = T_{a,\delta} \cdot T_{b,\delta}^{-1} = T_{b,\delta}^{-1} \cdot T_{a,\delta}$ is a δ.a.m.

(⇐) Let $T_{f,\delta}$ be a δ.a.m. and let $[\, b_0 \,, \dots, \, b_\delta \,] \cdot T_{f,\delta} = [\, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_\delta \,]$. This equation generates the following set of equations :

$$\begin{aligned} [\, b_0 \,, \dots, \, b_\delta \,] \cdot T_{f,\delta} &= [\, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_\delta \,] \\ &\vdots \\ [\, 0 \,, b_0 \,, \dots, \, b_{\delta-1} \,] \cdot T_{f,\delta} &= [\, 0 \,, a_0 \,, \dots, \, a_\kappa \,, 0 \,, \dots, 0_{\delta-1} \,] \\ &\vdots \\ [\, 0 \,, \dots, b_0 \,, \dots, \, b_{\delta-\kappa} \,] \cdot T_{f,\delta} &= [\, 0 \,, \dots, 0_{\delta-\kappa} \,, a_0 \,, \dots, \, a_\kappa \,] \\ &\vdots \\ [\, 0 \,, \dots, b_0 \,] \cdot T_{f,\delta} &= [\, 0 \,, \dots, 0_\delta \,, a_0 \,] \end{aligned} \quad (4.6.3)$$

The set of equations (4.6.3) can be equivalently written as $T_{b,\delta} \cdot T_{f,\delta} = T_{a,\delta}$ and hence $T_{f,\delta} = T_{a,\delta} \cdot T_{b,\delta}^{-1} = T_{b,\delta}^{-1} \cdot T_{a,\delta}$. □

Remark (4.6.1) *: Let $f(s) = a(s)/b(s)$, $a(s) = a_0 + a_1 \cdot s + \cdots + a_\kappa \cdot s^\kappa$, $b(s) = b_0 + b_1 \cdot s + \cdots + b_\delta \cdot s^\delta$ coprime . Then ,*

i) *If $\kappa \le \delta$, then for the vectors $\underline{b}^T = [\, b_0 \,, \dots , \, b_\delta \,]$ and $\underline{a}^T = [a_0 \,, \dots , \, a_\kappa \,, \, 0 \,, \dots , \, 0_\delta]$ there exists the matrix $T_{f,\delta} = T_{a,\delta} \cdot T_{b,\delta}^{-1}$, such that $\underline{b}^T \cdot T_{f,\delta} = \underline{a}^T$, where $T_{a,\delta}$, $T_{b,\delta}$ are δ – Toeplitz representations of $a(s)$, $b(s)$ respectively . The expression of $T_{f,\delta}$ as the fraction $(T_{a,\delta} \,/\, T_{b,\delta})$ is related to the expression of $f(s)$ as $a(s)/b(s)$. Hence , we can assume that $T_{f,\delta}$ is a δ – Toeplitz representation for the proper rational function $f(s)$. This representation is unique since the δ – Toeplitz representations of $a(s)$, $b(s)$ are unique .*

ii) *If $\kappa > \delta$, then we find the δ – Toeplitz representation of $f^{-1}(s)$, $T_{f^{-1},\delta} = T_{b,\delta} \cdot T_{a,\delta}^{-1}$ and set $T_{f,\delta} = T_{f^{-1},\delta}^{-1}$.*

iii) *If $a(s) = 0$, then we define as the δ – Toeplitz representation of $f(s) = 0$ the matrix $O_{\delta+1}$.* □

Let $\underline{p}(s) \in \mathbb{R}^m[s]$, with b.m. $P = [\, \underline{p}_0 \,, \underline{p}_1 \,, \dots , \underline{p}_\delta \,] \in \mathbb{R}^{m x(\delta+1)}$, $\deg(\underline{p}(s)) = \delta$. Let b(s) be a common divisor of the set of polynomials $\underline{p}(s)$ and a(s) an arbitrary polynomial with $\deg(a(s)) \le \deg(b(s))$. For all such b(s) , a(s) we take :

$$\underline{p}(s) \cdot (a(s)/b(s)) = \underline{q}(s) \text{ and } \deg(\underline{q}(s)) \le \delta \tag{4.6.4}$$

When a(s) is constant equation (4.6.4) generates the common divisors of the set of polynomials $\underline{p}(s)$. Or equivalently :

$$\underline{p}(s) \cdot (c/b(s)) = \underline{q}(s) \text{ and } \deg(\underline{q}(s)) \le \delta \tag{4.6.5}$$

If Q is the b.m. of $\underline{q}(s)$ and following the same steps as in section 4.2 equation (4.6.5) can be brought to the same form as equation (4.2.9) , namely , $[\; Q \,\vdots\, O \;] = P \cdot \widetilde{T}_{b,\delta}$, where $\widetilde{T}_{b,\delta}$ is the inverse of the δ – Toeplitz representation of b(s) . In section 4.3 we noticed that even though elements of $\mathcal{T}_\delta$ which annihilate columns of P may exist , they do not necessarily correspond to δ – Toeplitz representations of common divisors of $\underline{p}(s)$. Furthermore , those elements of $\mathcal{T}_\delta$ which correspond to δ – Toeplitz representations of common divisors of $\underline{p}(s)$ form a set $\mathcal{T}_{cd}$ which has no particular structure under the multiplication of $\mathcal{T}_\delta$. Hence , it would be interesting to try to give to that set a structure under a new operation . Let $\mathcal{T}_f$ denotes the subset of $\mathcal{T}_\delta$ which contains all the δ – Toeplitz representations of the proper rational functions (a(s)/b(s)) which satisfy equation (4.6.4) . $\mathcal{T}_f$ is a superset of $\mathcal{T}_{cd}$. In the following we shall define an operation among the elements of $\mathcal{T}_f$ and show that $\mathcal{T}_f \cup \{O_{\delta+1}\}$, ($O_{\delta+1}$ is the δ – Toeplitz representation of the zero function) , is a commutative group . Thus , $\mathcal{T}_{cd} \cup \{O_{\delta+1}\}$ is a commutative group under the new operation .

Definition (4.6.2) : *Let $T_{f,\delta}$, $T_{g,\delta}$ be two elements of $\mathcal{T}_f$ and $T_{f,\delta}$, $T_{g,\delta}$ be the*

δ – Toeplitz representations of $f(s) = (a(s)/b(s))$, $g(s) = (c(s)/d(s))$ respectively . Then we define the operation $\oplus$ over $\mathcal{T}_f$ as follows :

$$\oplus : \mathcal{T}_f \; x \; \mathcal{T}_f \rightarrow \mathcal{T}_f \text{ such that } \oplus(T_{f,\delta} \, , \, T_{g,\delta}) \triangleq T_{f,\delta} \oplus T_{g,\delta} = T_{h,\delta}$$

where , $T_{h,\delta}$ is the δ – Toeplitz representation of the proper rational function $h(s) = f(s)+ g(s)$. In the following we shall call $\oplus$ addition . □

Remark (4.6.2) *: Addition in $\mathcal{T}_f$ is well defined . Indeed , $h(s)$ is the proper rational function :*

$$h(s) = \frac{a(s)\ d(s) + c(s)\ b(s)}{b(s)\ d(s)} = \frac{e(s) + k(s)}{b'(s)\ d'(s)}$$

where $e(s)+k(s)$ and $b'(s) \cdot d'(s)$ are coprime . Then $b'(s) \cdot d'(s)$ is a common divisor of $p(s)$, $deg(e(s)+k(s)) \leq deg(b'(s)\ d'(s))$ and $h(s)$ satisfies equation (4.6.4) . Hence , the δ – Toeplitz representation of $h(s)$ belongs to $\mathcal{T}_f$ and is unique since addition in the ring of proper functions is well defined . □

Lemma (4.6.1) *: $(\mathcal{T}_f \cup \{O_{\delta+1}\} \, , \oplus)$ is a commutative group .*

Proof

Let $T_{f,\delta}$, $T_{g,\delta}$ be two elements of $\mathcal{T}_f$ and $T_{f,\delta}$, $T_{g,\delta}$ be the δ – Toeplitz representations of f(s) = (a(s)/b(s)) , g(s) = (c(s)/d(s)) respectively . Then $T_{f,\delta} \oplus T_{g,\delta} = T_{g,\delta} \oplus T_{f,\delta}$, since the functions f(s)+g(s) and g(s)+f(s) have the same δ – Toeplitz representation. Let $T_{f,\delta}$ be the δ – Toeplitz representations of f(s) = (a(s)/b(s)) , and $T_{0,\delta} = O_{\delta+1}$ be the δ – Toeplitz representations of 0(s) = 0 – the zero function . Then $T_{f,\delta} \oplus T_{0,\delta} = T_{0,\delta} \oplus T_{f,\delta} = T_{f,\delta}$, since f(s) + 0(s) = f(s) .
Let $T_{f,\delta}$, $T_{-f,\delta}$ be the δ – Toeplitz representations of f(s) = (a(s)/b(s)) , – f(s) respectively . Then $T_{f,\delta} \oplus T_{-f,\delta} = T_{-f,\delta} \oplus T_{f,\delta} = T_{0,\delta}$, since f(s) – f(s) = 0(s) .
Addition over $\mathcal{T}_f \cup \{O_{\delta+1}\}$ is associative since addition in the ring of proper rational functions is . □

Corollary (4.6.1) *: By lemma(4.6.1) $(\mathcal{T}_p \cup \{O_{\delta+1}\} \, , \oplus)$ is a commutative group .* □

Lemma (4.6.2) *: Let $\sim$ be a relation in $(\mathcal{T}_f \cup \{O_{\delta+1}\})$ x $(\mathcal{T}_f \cup \{O_{\delta+1}\})$ such that for two elements of $\mathcal{T}_f \cup \{O_{\delta+1}\}$, $T_{f,\delta}$, $T_{g,\delta}$, δ – Toeplitz representations of $f(s) = (a(s)/b(s))$, $g(s) = (c(s)/d(s))$ respectively , we have $T_{f,\delta} \sim T_{g,\delta}$, if and only if $a(s) \cdot d(s) = c(s) \cdot b(s)$. Then $\sim$ is an equivalence relation in $\mathcal{T}_f \cup \{O_{\delta+1}\}$.* □

$\mathcal{T}_f \cup \{O_{\delta+1}\}$ can be partitioned into equivalence classes as follows . Let $T_{f,\delta}$ be an element of $\mathcal{T}_f \cup \{O_{\delta+1}\}$, then we denote by $\mathcal{C}_{T_{f,\delta}}$ the set of all $T_{g,\delta}$ which are equivalent to $T_{f,\delta}$. The set of all $\mathcal{C}_{T_{f,\delta}}$ is denoted by $\mathcal{C}$.

Definition (4.6.3) *: Let $\mathcal{C}_{T_{f,\delta}}$, $\mathcal{C}_{T_{g,\delta}}$ be two elements of $\mathcal{C}$. Then an addition between the elements of $\mathcal{C}$ is defined as $\mathcal{C}_{T_{f,\delta}} + \mathcal{C}_{T_{g,\delta}} = \mathcal{C}_{T_{h,\delta}}$, where $T_{h,\delta} = T_{f,\delta} \oplus T_{g,\delta}$.* □

Remark (4.6.3) *: Addition over $\mathcal{C}$ is well defined . Let $\mathcal{C}_{T_{f,\delta}}$, $\mathcal{C}_{T_{g,\delta}}$, $\mathcal{C}_{T_{h,\delta}}$, $\mathcal{C}_{T_{k,\delta}}$ belong to $\mathcal{C}$ and $\mathcal{C}_{T_{f,\delta}} = \mathcal{C}_{T_{h,\delta}}$, $\mathcal{C}_{T_{g,\delta}} = \mathcal{C}_{T_{k,\delta}}$, with $\mathcal{C}_{T_{f,\delta}} + \mathcal{C}_{T_{g,\delta}} = \mathcal{C}_{T_{p,\delta}}$, $\mathcal{C}_{T_{h,\delta}} + \mathcal{C}_{T_{k,\delta}} = \mathcal{C}_{T_{q,\delta}}$. Then $\mathcal{C}_{T_{p,\delta}}$ corresponds to $T_{p,\delta} = T_{f,\delta} \oplus T_{g,\delta}$ and $\mathcal{C}_{T_{q,\delta}}$ corresponds to $T_{q,\delta} = T_{h,\delta} \oplus T_{k,\delta}$. But $T_{f,\delta}$, $T_{g,\delta}$ are equivalent to $T_{h,\delta}$, $T_{k,\delta}$ respectively , which implies that $T_{p,\delta}$ and $T_{q,\delta}$ are equivalent . Hence , $\mathcal{C}_{T_{p,\delta}} = \mathcal{C}_{T_{q,\delta}}$.* □

Lemma (4.6.3) *: $(\mathcal{C}\ ,+)$ is a commutative group .*

Proof

Let $\mathcal{C}_{T_{f,\delta}}$, $\mathcal{C}_{T_{g,\delta}}$ be two elements of $\mathcal{C}$. Then $\mathcal{C}_{T_{f,\delta}} + \mathcal{C}_{T_{g,\delta}} = \mathcal{C}_{T_{g,\delta}} + \mathcal{C}_{T_{f,\delta}} = \mathcal{C}_{T_{h,\delta}}$, since $T_{h,\delta} = T_{f,\delta} \oplus T_{g,\delta} = T_{g,\delta} \oplus T_{f,\delta}$.

For the $\mathcal{C}_{T_{f,\delta}}$, $\mathcal{C}_{T_{0,\delta}}$ we take that $\mathcal{C}_{T_{0,\delta}} + \mathcal{C}_{T_{f,\delta}} = \mathcal{C}_{T_{f,\delta}} + \mathcal{C}_{T_{0,\delta}} = \mathcal{C}_{T_{f,\delta}}$, since $T_{f,\delta} = T_{0,\delta} \oplus T_{f,\delta} = T_{f,\delta} \oplus T_{0,\delta}$.

For the $\mathcal{C}_{T_{f,\delta}}$, $\mathcal{C}_{T_{-f,\delta}}$ we take that $\mathcal{C}_{T_{f,\delta}} + \mathcal{C}_{T_{-f,\delta}} = \mathcal{C}_{T_{-f,\delta}} + \mathcal{C}_{T_{f,\delta}} = \mathcal{C}_{T_{0,\delta}}$, since $T_{0,\delta} = T_{f,\delta} \oplus T_{-f,\delta} = T_{-f,\delta} \oplus T_{f,\delta}$.

Addition over $\mathcal{C}$ is associative since addition over $\mathcal{T}_f \cup \{O_{\delta+1}\}$ is . □

4.7. CONCLUSIONS

An alternative characterization for the Greatest Common Divisor (GCD) , f(s) , of a set of m polynomials , $\underline{p}(s)$, of maximal degree δ has been introduced by making use of the equivalent expression of relationship $\underline{p}(s) = \underline{q}(s) \cdot f(s)$ in terms of real matrices , (basis matrices (b.m.) P , Q of $\underline{p}(s)$, $\underline{q}(s)$ respectively) , and the Toeplitz representation of f(s) . The relation between the GCD and scalar Toeplitz bases , $\mathcal{W}$, of a subspace $\mathcal{V}$ of $\mathcal{N}_r\{P\}$ has been established . The additional property , that the nonzero entries of $\mathcal{W}$ should have a certain expression involving the coefficients of the gcd f(s) and $\mathcal{V}$ has the greatest possible dimension that the latter happens has appeared in section 4.3 . This led to an algorithm for the construction of the coefficients of f(s) as a tuple taken from a certain affine variety . It has been shown that Groebner bases play an essential role in characterizing the GCD in terms of its Toeplitz representation . The present approach uses the notion of Groebner bases in an explicit manner . Although simpler methods for the computation of the GCD have already been given in the litterature , (see [Mit. 2] and the closed form solution given in [Kar. 3]) , the present method has the advantage that may be extended to matrix divisors , whereas the others have considerable difficulties . Such an extension is under investigation .

CHAPTER 5

STRUCTURAL PROPERTIES OF A MATRIX OVER A PRINCIPAL IDEAL DOMAIN

5.1. INTRODUCTION

The main aim of this chapter is to investigate further the structural properties of matrices which provide solutions to matrix equations of the type :

$$A \cdot X = B \ , \ A \in \mathcal{R}^{pxm} \ , \ B \in \mathcal{R}^{px\kappa} \ , \ X \in \mathcal{R}^{mx\kappa} \tag{5.1.1}$$

$$Y \cdot A = B \ , \ A \in \mathcal{R}^{pxm} \ , \ B \in \mathcal{R}^{\kappa xm} \ , \ Y \in \mathcal{R}^{\kappa xp} \tag{5.1.2}$$

$$A \cdot X \cdot B = C \ , \ A \in \mathcal{R}^{pxm} \ , \ B \in \mathcal{R}^{\kappa xt} \ , \ C \in \mathcal{R}^{pxt} \ , \ X \in \mathcal{R}^{mx\kappa} \tag{5.1.3}$$

$$\sum_{i=1}^{h} A_i \cdot X_i \cdot B_i = C \ , \ A_i \in \mathcal{R}^{pxm_i}, \ B_i \in \mathcal{R}^{\kappa_i xt}. \ C \in \mathcal{R}^{pxt}, \ X \in \mathcal{R}^{m_i x\kappa_i} \tag{5.1.4}$$

where the entries of the matrices are assumed over a given principal ideal domain , (PID) , $\mathcal{R}$, which in control theory problems can be either the ring of polynomials $\mathbb{R}[s]$, or proper rational functions $\mathbb{R}_{pr}(s)$, or proper and $\mathcal{P}$ stable rational functions $\mathbb{R}_{\mathcal{P}}(s)$. Notice that equation (5.1.4) is a generalization of many well know matrix equations , such as :

$$A_1 \cdot X_1 + \cdots + A_h \cdot X_h = B \ , \ A_i \in \mathcal{R}^{pxm_i}, \ B \in \mathcal{R}^{pxt}, \ X_i \in \mathcal{R}^{m_i xt} \tag{5.1.5}$$

$$Y_1 \cdot A_1 + \cdots + X_h \cdot A_h = B \ , \ A_i \in \mathcal{R}^{p_i xm}, \ B \in \mathcal{R}^{txm}, \ Y_i \in \mathcal{R}^{txp_i} \tag{5.1.6}$$

$$A \cdot X + Y \cdot B = C \ , \ A \in \mathcal{R}^{pxm}, \ B \in \mathcal{R}^{\kappa xt}, \ C \in \mathcal{R}^{pxt}, \ X \in \mathcal{R}^{mxt}, Y \in \mathcal{R}^{px\kappa} \tag{5.1.7}$$

$$X \cdot A + B \cdot Y = C \ , \ A \in \mathcal{R}^{pxm}, \ B \in \mathcal{R}^{tx\kappa}, \ C \in \mathcal{R}^{txm}, \ X \in \mathcal{R}^{txp}, Y \in \mathcal{R}^{\kappa xm} \tag{5.1.8}$$

The structural properties of a matrix over a PID , $\mathcal{R}$, are used to generate algebraic tools that will enable us , (later on in Chapter 6) , to formulate a unifying framework to deal with solvability of matrix equations over $\mathcal{R}$. The existence and characterization of families of greatest left – right common divisors , extended greatest left – right common divisors , projectors , annihilators , multiples and least multiples of a given matrix , or set of matrices , over $\mathcal{R}$ is introduced . If the known matrices in equations (5.1.1) – (5.1.8) are assumed over $\mathcal{F}$, the field of fractions of $\mathcal{R}$, then the machinery of multiples and least multiples over $\mathcal{R}$ of the rows , columns of a matrix , with entries over $\mathcal{F}$, is used in order to transform equations (5.1.1) – (5.1.8) to ones where all the matrices are over $\mathcal{R}$; thus we can apply the same algebraic approach to solve matrix equations over PIDs in the most general case , i.e. when the known matrices are assumed over $\mathcal{F}$. The relation between the algebraic tools presented in the following and the column , row $\mathcal{R}$ – modules , maximum $\mathcal{R}$ – modules of the corresponding matrix is established . In the following $\mathcal{R}$ denotes a PID , $\mathcal{F}$ is the field of fractions of $\mathcal{R}$; if $A \in \mathcal{R}^{p\times m}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p , m\}$ then we associate the following vector spaces with A :

$$\mathcal{X}^{r}_{A} = \text{row span of } \{A\} \text{ over } \mathcal{F} = \text{row space of } A \tag{5.1.9}$$

$$\mathcal{X}^{c}_{A} = \text{column span of } \{A\} \text{ over } \mathcal{F} = \text{column space of } A \tag{5.1.10}$$

$$\mathcal{N}_{r}\{A\} = \text{right null space of } A \tag{5.1.11}$$

$$\mathcal{N}_{l}\{A\} = \text{left null space of } A \tag{5.1.12}$$

We also associate the following $\mathcal{R}$ modules with A :

$$\mathcal{M}^{r}_{A} = \text{row span } \{A\} \text{ over } \mathcal{R} = \mathcal{R} \text{ row module of } A \tag{5.1.13}$$

$$\mathcal{M}^{c}_{A} = \text{column span } \{A\} \text{ over } \mathcal{R} = \mathcal{R} \text{ column module of } A \tag{5.1.14}$$

$$\hat{\mathcal{M}}^{r}_{A} = \text{the maximum } \mathcal{R} \text{ row module of } A \text{ in } \mathcal{X}^{r}_{A} \tag{5.1.15}$$

$$\hat{\mathcal{M}}^{c}_{A} = \text{the maximum } \mathcal{R} \text{ column module of } A \text{ in } \mathcal{X}^{c}_{A} \tag{5.1.16}$$

5.2. LEFT – RIGHT SQUARE DIVISORS OF A MATRIX OVER THE PID $\mathcal{R}$

We start this section with the introduction of the concept of a left , right divisor and left , right greatest common divisor of a matrix A over the PID $\mathcal{R}$. This is central to our study of structural properties of A over $\mathcal{R}$, as well as , to the characterization of related algebraic tools concerning non square divisors , projectors and annihilators over the given PID .

***Definition (5.2.1)** : Let $A \in \mathcal{R}^{p\times m}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p , m\}$. A matrix $T \in \mathcal{R}^{\rho\times\rho}$, will be called an $\mathcal{R}$ – left , right divisor , (lrd) , of A over $\mathcal{R}$, if there exist matrices $P \in \mathcal{R}^{p\times\rho}$, $Q \in \mathcal{R}^{\rho\times m}$, $rank_{\mathcal{F}}\{P\} = rank_{\mathcal{F}}\{Q\} = \rho$, such that :*

$$A = P \cdot T \cdot Q \tag{5.2.1}$$

T will be called an $\mathcal{R}$ – greatest left , right divisor , (glrd) , of A over $\mathcal{R}$ if it is a lrd of A and P , Q are left , right unimodular over $\mathcal{R}$. □

The existence of a glrd of a matrix A over $\mathcal{R}$ is established in the following result :

Proposition (5.2.1) *: Let $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$. Then there always exist matrices $P \in \mathcal{R}^{px\rho}$, $Q \in \mathcal{R}^{\rho xm}$, $T \in \mathcal{R}^{\rho x\rho}$, $rank_{\mathcal{F}}\{P\} = rank_{\mathcal{F}}\{Q\} = rank_{\mathcal{F}}\{T\} = \rho$, such that (5.2.1) holds true .*

Proof

It is well known fact , [Vid. 4] , [Ros. 1] , that when a matrix $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = =\rho \leq min\{p\ ,\ m\}$ is given , then $\mathcal{R}$ unimodular matrices U , V always exist such that A can be reduced in its Smith form over $\mathcal{R}$:

$$A = U \cdot \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \cdot V \tag{5.2.3}$$

If U , V are partitioned as :

$$U = [\ U_p^\rho\ ,\ U_p^{p-\rho}\ ]\ ,\ V = \begin{bmatrix} V_\rho^m \\ V_{m-\rho}^m \end{bmatrix} \tag{5.2.4}$$

then it is clear that the matrix $T = S_\rho$ serves as a glrd of A over $\mathcal{R}$ with $P = U_p^\rho$ and $Q = V_\rho^m$. □

Remark (5.2.1) *: If $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$, then :*

i) *If $p \geq m$ the notion of a glrd coincides with the standard notion of a greatest right divisor of A .*

ii) *If $p < m$ the notion of a glrd coincides with the standard notion of a greatest left divisor of A .* □

If $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$, then A_H^c will denote its column Hermite form .

Remark (5.2.2) *: Let $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$. If T is a glrd of A over $\mathcal{R}$ then (5.2.1) holds true and :*

i) *The rows of $T \cdot Q$ define a base for $\mathcal{M}_A^r$, the rows of Q define a base for $\hat{\mathcal{M}}_A^r$.*

ii) *The columns of $P \cdot T$ define a base for $\mathcal{M}_A^c$, the columns of P define a base for $\hat{\mathcal{M}}_A^c$* □

The above remark is helpful in characterizing the family of all glrd of a matrix $A \in \mathcal{R}^{pxm}$ over $\mathcal{R}$.

Proposition (5.2.2) *: Let $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$. If $T\ ,\ T' \in \mathcal{R}^{\rho x\rho}$ are* *two glrd of A over $\Re$ then T , T′ are equivalent over $\Re$ and we denote T E T′ .*

Proof

We can write $A = P \cdot T \cdot Q$, $A = P' \cdot T' \cdot Q'$, $P', P \in \Re^{m x \rho}$, $Q', Q \in \Re^{\rho x m}$ are left , right unimodular matrices . Remark (5.2.2) provides that $\Re$ unimodular matrices U , V exist such that :

$$P' = P \cdot U \ , \ Q' = V \cdot Q \tag{5.2.5}$$

or , using (5.2.1) for both T , T′ :

$$A = P \cdot U \cdot T' \cdot V \cdot Q = P \cdot T \cdot Q \Leftrightarrow P \cdot (U \cdot T' \cdot V - T) \cdot Q = O \tag{5.2.6}$$

Since P , Q have trivial right , left null spaces respectively , (5.2.6) implies that :

$$U \cdot T' \cdot V - T = O \Leftrightarrow U \cdot T' \cdot V = T \tag{5.2.7}$$

□

Remark (5.2.2) *: If T is a glrd of A over $\Re$, any other glrd , T′ , of A over $\Re$ is obtained by :*

$$T' = U \cdot T \cdot V \tag{5.2.8}$$

for any $\Re$ unimodular matrices U , V with compatible dimensions . It is clear that the nonzero block of the Smith form of the matrix A over $\Re$ is a glrd of A . □

5.3. NONSQUARE DIVISORS OF A MATRIX OVER THE PID $\Re$

The notion of glrd of a matrix A over $\Re$ is used next to characterize the nonsquare matrix divisors of that matrix defined in [Per. 1] :

Definition (5.3.1) *: Let $A \in \Re^{p x m}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p , m\}$. If A can be factorized as :*

$$A = L \cdot B \tag{5.3.1}$$

with , $L \in \Re^{p x q}$, $rank_{\mathfrak{F}}\{L\} = q$ and $B \in \Re^{q x m}$, then L is defined as an extended left , divisor , (eld) , of A over $\Re$. L will be called a greatest extended left divisor , (geld) , of A over $\Re$ if L is an eld of A and every other eld of A is also an eld of L . The notion of an extended right divisor , (erd) , and greatest extended right divisor , (gerd) , of A over $\Re$ is introduced in a similar manner . □

The characterization of such divisors is considered next .

Proposition (5.3.1) *: Let $A \in \mathcal{R}^{pxm}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$. Then there always exists a geld , gerd of A , L , K , respectively over $\mathcal{R}$, which has the following properties:*

i) L , K may be expressed as :

$$L = A \cdot X\ ,\ K = Y \cdot A \tag{5.3.2}$$

for some $X \in \mathcal{R}^{mx\rho}$, $Y \in \mathcal{R}^{\rho xp}$.

ii) If L , L′, K , K′ are two gelds , gerds of A over $\mathcal{R}$ respectively then they are right , left $\mathcal{R}$ equivalent and we denote $L\ E_r\ L'$, $K\ E_l\ K'$.

Proof

Let U , V be appropriate $\mathcal{R}$ unimodular matrices such that A can be expressed in its Smith form over $\mathcal{R}$:

$$A = U \cdot \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \cdot V \tag{5.3.3}$$

If U , U^{-1} , V , V^{-1} are partitioned as :

$$U = [\ U_p^\rho\ ,\ U_p^{p-\rho}\ ]\ ,\ U^{-1} = \begin{bmatrix} U_\rho^p \\ U_{p-\rho}^p \end{bmatrix},\ V = \begin{bmatrix} V_\rho^m \\ V_{m-\rho}^m \end{bmatrix},\ V^{-1} = [\ V_m^\rho\ ,\ V_m^{m-\rho}\ ] \tag{5.3.4}$$

then it is clear that the matrices $L = U_p^\rho \cdot S_\rho$, $K = S_\rho \cdot V_\rho^m$ are an eld , erd of A over $\mathcal{R}$. Furthermore if $X = V_m^\rho$, $Y = U_\rho^p$ then it is straightforward from (5.3.3) that :

$$L = A \cdot X\ ,\ K = Y \cdot A \tag{5.3.5}$$

If J′, D′ are any eld , erd of A over $\mathcal{R}$ respectively then by definition (5.3.1) is implied that :

$$A = J' \cdot B_1 = B_2 \cdot D' \tag{5.3.6}$$

with $B_1 \in \mathcal{R}^{q_1 xm}$, $B_2 \in \mathcal{R}^{pxq_2}$, $q_1 = rank_{\mathcal{F}}\{J'\}$, $q_2 = rank_{\mathcal{F}}\{D'\}$. (5.3.6) and (5.3.5) imply that L , K are a geld , gerd of A over $\mathcal{R}$.

i) (5.3.5) clearly implies (5.3.2) .

ii) Let $L \in \mathcal{R}^{pxq}$, $rank_{\mathcal{F}}\{L\} = q$, $L' \in \mathcal{R}^{pxq'}$, $rank_{\mathcal{F}}\{L\} = q'$ be two gelds of A over $\mathcal{R}$ respectively . Then by definition(5.3.1) they serve as elds of one an other respectively and thus matrices $B \in \mathcal{R}^{qxq'}$, $B' \in \mathcal{R}^{q'xq}$ exist such that :

$$L = L' \cdot B'\ ,\ L' = L \cdot B \tag{5.3.7}$$

or equivalently ,

$$L = L \cdot B \cdot B' \; , \; L' = L' \cdot B' \cdot B \qquad (5.3.8)$$

Since the gelds have trivial right null spaces (5.3.8) implies that :

$$B \cdot B' = I_q \; , \; B' \cdot B = I_{q'} \qquad (5.3.9)$$

The latter can happened if and only if $q = q'$ and B , B' are $\mathfrak{R}$ unimodular . Thus (5.3.7) implies that L E_r L' . The proof for the gerds follows along similar lines . □

From the proof of proposition (5.3.1) the link between the gelds , gerds and the gelrds of A over $\mathfrak{R}$, as well as , the corresponding decomposition of A are established . Thus we may state :

Corollary (5.3.1) *: Let $A \in \mathfrak{R}^{p x m}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p \; , \; m\}$, $T \in \mathfrak{R}^{\rho x \rho}$ be a glrd of A over $\mathfrak{R}$, and $A = P \cdot T \cdot Q$. Then a geld , gerd of A over $\mathfrak{R}$, L_l , L_r is defined by :*

$$L_l = P \cdot T \in \mathfrak{R}^{p x \rho} \; , \; L_r = T \cdot Q \in \mathfrak{R}^{\rho x m} \qquad (5.3.10)$$

respectively . Furthermore A can be factorized as :

$$A = L_l \cdot Q = P \cdot L_r \qquad (5.3.11)$$

□

Remark (5.3.1) *: Let $A \in \mathfrak{R}^{p x m}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p \; , \; m\}$, $T \in \mathfrak{R}^{\rho x \rho}$ be a glrd , $L_l \in \mathfrak{R}^{p x \rho}$; $L_r \in \mathfrak{R}^{\rho x m}$ be a geld , gerd of A respectively over $\mathfrak{R}$ and let $A = P \cdot T \cdot Q$. Remark (5.2.2) and corollary (5.3.1) imply that L_l , L_r are bases for $\mathcal{M}^c_A$, $\mathcal{M}^r_A$ respectively . If L , K denote an arbitrary eld , erd of A over $\mathfrak{R}$ respectively then by definition (5.3.1) we have :*

$$L_l = L \cdot B \; , \; L_r = C \cdot K \qquad (5.3.12)$$

and thus we can write :

$$\mathcal{M}^c_A = \mathcal{M}^c_{L_l} \subseteq \mathcal{M}^c_L \; , \; \mathcal{M}^r_A = \mathcal{M}^r_{L_r} \subseteq \mathcal{M}^r_K \qquad (5.3.13)$$

where , $\mathcal{M}^c_{L_l}$, $\mathcal{M}^c_L$, $\mathcal{M}^r_{L_r}$, $\mathcal{M}^r_K$ are the $\mathfrak{R}$ column , row modules of the matrices L_l , L , L_r , K respectively . □

From this remark is clear that the extraction of elds , erds of A over $\mathfrak{R}$ is equivalent to the creation of an ascending chain of modules , containing $\mathcal{M}^c_A$, $\mathcal{M}^r_A$; the minimal elements in these chains are $\mathcal{M}^c_A$, $\mathcal{M}^r_A$ themselves .

Remark (5.3.2) *: Proposition (5.3.1) implies that all the gelds , gerds of A over $\mathfrak{R}$ have* *exactly* $\rho = rank_{\mathcal{J}}\{A\}$ *columns , rows , whereas an eld , erd of A over* $\mathcal{R}$ *may have more .* □

5.4. NONSQUARE DIVISORS OF SETS OF MATRICES OVER THE PID $\mathcal{R}$

Having established the above results , we now proceed to define the notions of nonsquare divisors of two , or more matrices .

Definition (5.4.1) *: i) Let* $A_i \in \mathcal{R}^{p \times m_i}$, $L_l \in \mathcal{R}^{p \times \kappa}$, $i = 1 , \dots , n$ *. Then* L_l *is a common extended left divisor , (celd) , of the set of* A_i *over* $\mathcal{R}$ *if it is an eld of each* A_i *over* $\mathcal{R}$ *.* L_l *is a greatest common extended left divisor , (gceld) , of the set of* A_i *over* $\mathcal{R}$ *if it is a celd of each* A_i *over* $\mathcal{R}$ *and any other celd of all* A_i *over* $\mathcal{R}$ *is an eld of* L_l *.*
ii) Let $B_i \in \mathcal{R}^{p_i \times m}$, $L_r \in \mathcal{R}^{\kappa \times m}$, $i = 1 , \dots , n$ *. Then* L_r *is a common extended right divisor , (cerd) , of the set of* A_i *over* $\mathcal{R}$ *if it is an erd of each* B_i *over* $\mathcal{R}$ *.* L_r *is a greatest common extended right divisor , (gcerd) , of the set of* A_i *over* $\mathcal{R}$ *if it is a cerd of each* B_i *over* $\mathcal{R}$ *and any other cerd of all* B_i *over* $\mathcal{R}$ *is an erd of* L_r *.* □

The following result establishes the relation between the gcelds , gcerds of the set of A_i over $\mathcal{R}$ and the notion of gelds , gerds of the composite matrix $[A_1 , \dots , A_n]$, $[B_1^T , \dots , B_n^T]^T$ over $\mathcal{R}$.

Proposition (5.4.1) *: Let* $A_i \in \mathcal{R}^{p \times m_i}$, $B_i \in \mathcal{R}^{p_i \times m}$, $i = 1 , \dots , n$ *. The following statements hold true :*
i) $L_l \in \mathcal{R}^{p \times \kappa}$ *is a gceld of the set of* A_i *over* $\mathcal{R}$ *, if and only if it is a geld of the composite matrix :*

$$[A_1 , \dots , A_n] \tag{5.4.1}$$

over $\mathcal{R}$ *.*
ii) $L_r \in \mathcal{R}^{\kappa \times m}$ *is a gcerd of the set of* B_i *over* $\mathcal{R}$ *, if and only if it is a gerd of the composite matrix :*

$$[B_1^T , \dots , B_n^T]^T \tag{5.4.2}$$

over $\mathcal{R}$ *.*

Proof
i) (⇒) If $L_l \in \mathcal{R}^{p \times \kappa}$ is a gceld of the set of A_i over $\mathcal{R}$ then there exist matrices $C_i \in \mathcal{R}^{\kappa \times m_i}$, such that :

$$A_i = L_l \cdot C_i , \; i = 1 , \dots , n \tag{5.4.3}$$

or equivalently ,

$$[A_1 , \dots , A_n] = L_l \cdot [C_1 , \dots , C_n] \tag{5.4.4}$$

The latter implies that L_l is an eld of the composite matrix (5.4.1) . If $L \in \mathcal{R}^{pxq}$ is any eld of the matrix (5.4.1) then a matrix $\Lambda \in \mathcal{R}^{qxm}$, $(m = \sum_{i=1}^{n} m_i)$, exists such that :

$$[A_1 , \dots , A_n] = L \cdot \Lambda \tag{5.4.5}$$

If we partition the matrix Λ according to the partitioning of the matrix $[\Lambda_1 , \dots , \Lambda_n]$, then it is clear that L is an eld of each A_i over $\mathcal{R}$ and thus a celd of the set of A_i over $\mathcal{R}$. By definition (5.4.1) the latter implies that L is an eld of L_l over $\mathcal{R}$ and thus L_l is a geld of the composite matrix (5.4.1) over $\mathcal{R}$.
($\Leftarrow$) Let $L_l \in \mathcal{R}^{px\kappa}$ be a geld of the composite matrix (5.4.1) over $\mathcal{R}$. Then a matrix $D \in \mathcal{R}^{\kappa xm}$, $(m = \sum_{i=1}^{n} m_i)$, exists such that :

$$[A_1 , \dots , A_n] = L_l \cdot D \tag{5.4.6}$$

If we partition matrix D as $[D_1 , \dots , D_n]$, $D_i \in \mathcal{R}^{\kappa x m_i}$, then it is clear that L_l is a eld of each A_i over $\mathcal{R}$ and thus it is a celd of the set of A_i over $\mathcal{R}$. If $L \in \mathcal{R}^{pxq}$ is any celd of the set of A_i over $\mathcal{R}$, it also is a eld of (5.4.1) over $\mathcal{R}$ and thus L is an eld of L_l . The latter implies that L_l is a gceld of the set of A_i over $\mathcal{R}$.
ii) The proof follows along similar lines . □

The results established for the geld , gerd of a matrix over $\mathcal{R}$ may be extended for a gceld , gcerd of a set of matrices over $\mathcal{R}$.

Proposition (5.4.2) *: Let $A_i \in \mathcal{R}^{pxm_i}$, $B_i \in \mathcal{R}^{p_i xm}$, $i = 1 , \dots , n$, $A = [A_1 , \dots , A_n]$, $B = [B_1^T , \dots , B_n^T]^T$, with $rank_{\mathcal{F}}\{A\} = \rho$, $rank_{\mathcal{F}}\{B\} = \rho'$:*
i) *There exists a gceld , $L_l \in \mathcal{R}^{px\rho}$, of the set of A_i over $\mathcal{R}$, and it may be expressed as:*

$$L_l = \sum_{i=1}^{n} A_i \cdot X_i \tag{5.4.7}$$

for some matrices $X_i \in \mathcal{R}^{m_i x\rho}$. Furthermore if L_l' is any gceld of the set of A_i over $\mathcal{R}$ then L_l is $\mathcal{R}$ right equivalent with L_l' and we write $L_l \; E_r \; L_l'$.
ii) *There exists a gceld , $L_r \in \mathcal{R}^{\rho' xm}$, of the set of B_i over $\mathcal{R}$, and it may be expressed as:*

$$L_r = \sum_{i=1}^{n} Y_i \cdot B_i \tag{5.4.8}$$

for some matrices $Y_i \in \mathcal{R}^{\rho' xm_i}$. Furthermore if L_r' is any gceld of the set of A_i over $\mathcal{R}$ then L_r is $\mathcal{R}$ left equivalent with L_r' and we write $L_r \; E_l \; L_r'$.

Proof

i) Since a geld , $L_l \in \mathcal{R}^{pxp}$, of the composite matrix A over $\mathcal{R}$ exists , proposition (5.4.1) implies that $L_l \in \mathcal{R}^{pxp}$ is gceld of the set of A_i over $\mathcal{R}$. Furthermore proposition (5.3.1) has established the existence of a matrix $X \in \mathcal{R}^{mxp}$, $(m = \sum_{i=1}^{n} m_i)$, such that :

$$L_l = A \cdot X \tag{5.4.9}$$

If we partition X according to the partitioning of the matrix $[X_1^T , \dots , X_n^T]^T$, $X_i \in \mathcal{R}^{m_i x p}$, then (5.4.9) clearly implies (5.4.7) . If L_l' is any gceld of the set of A_i over $\mathcal{R}$, it is a geld of the composite matrix A over $\mathcal{R}$ as well and thus proposition (5.3.1) implies that $L_l \; E_r \; L_l'$.

ii) The proof follows along similar lines . □

The module interpretation of the geld , gerd of a matrix A over $\mathcal{R}$ can be expanded in the case of a gceld , gcerd of a set of matrices over $\mathcal{R}$.

Proposition (5.4.3) *: Let $A_i \in \mathcal{R}^{pxm_i}$, $B_i \in \mathcal{R}^{p_i xm}$, $i = 1 , \dots , n$, $A = [A_1 , \dots , A_n]$, $B=[B_1^T , \dots , B_n^T]^T$, with $rank_{\mathcal{F}}\{A\} = \rho$, $rank_{\mathcal{F}}\{B\} = \rho'$ and $L_l \in \mathcal{R}^{pxp}$, $L_r \in \mathcal{R}^{\rho' xm}$ be a gceld , gcerd of the set of A_i , B_i over $\mathcal{R}$ respectively . If $\mathcal{M}^c_{A_i}$, $\mathcal{M}^c_{L_l}$, $\mathcal{M}^r_{B_i}$, $\mathcal{M}^r_{L_r}$ denote the $\mathcal{R}$ column , row modules of A_i , L_l , B_i , L_r respectively then $\mathcal{M}^c_{L_l}$, $\mathcal{M}^r_{L_r}$ are the smallest submodules that contain each $\mathcal{M}^c_{A_i}$, $\mathcal{M}^r_{B_i}$ respectively and :*

$$\mathcal{M}^c_{L_l} = \sum_{i=1}^{n} \mathcal{M}^c_{A_i} \; , \; \mathcal{M}^r_{L_r} = \sum_{i=1}^{n} \mathcal{M}^r_{B_i} \tag{5.4.10}$$

Proof

We prove the proposition for the case of A_i , since the proof for the case of B_i follows along similar lines . It is clear that :

$$\mathcal{M}^c_{A} = \sum_{i=1}^{n} \mathcal{M}^c_{A_i} \tag{5.4.11}$$

Proposition (5.4.2) has established that L_l is a geld of A over $\mathcal{R}$ and remark (5.3.1) that

$$\mathcal{M}^c_{L_l} = \mathcal{M}^c_{A} \tag{5.4.12}$$

Thus (5.4.11) , (5.4.12) combined imply $\mathcal{M}^c_{L_l} = \sum_{i=1}^{n} \mathcal{M}^c_{A_i}$. The latter provides that $\mathcal{M}^c_{L_l}$ contains every $\mathcal{M}^c_{A_i}$. Let now $\mathcal{M}^c_K$ be an arbitrary $\mathcal{R}$ column module that contains every $\mathcal{M}^c_{A_i}$. Then if $dim\{\mathcal{M}^c_K\} = q$, matrices $K \in \mathcal{R}^{pxq}$, $rank_{\mathcal{F}}\{K\} = q$, $C_i \in \mathcal{R}^{qxm_i}$ exist such that :

$$A_i = K \cdot C_i \tag{5.4.13}$$

and thus K is a eld of each A_i over $\mathcal{R}$, or a celd of the set A_i over $\mathcal{R}$. Then K is a eld of L_l and remark (5.3.1) provides that :

$$\mathcal{M}^c_{L_l} \subseteq \mathcal{M}^c_K \tag{5.4.14}$$

which clearly implies that $\mathcal{M}^c_{L_l}$ are the smallest submodules that contain each $\mathcal{M}^c_{A_i}$. □

The module interpretation of the gcelds , gcerds of a set of matrices A_i over $\mathcal{R}$ as a base of the "minimum cover module" of all modules generated by the columns , rows of A_i will be used in the solution of matrix equations later on . With the notion of gceld , gcerd established , we proceed to define the concept of coprimeness of a set of matrices over a given PID , $\mathcal{R}$.

Definition (5.4.2) : *i) Given a set of matrices $A_i \in \mathcal{R}^{p \times m_i}$, $i = 1, \dots, n$, $A = [A_1, \dots, A_n] \in \mathcal{R}^{p \times m}$, $(m = \sum_{i=1}^{n} m_i)$, $rank_{\mathcal{F}}\{A\} = \rho$, then we say that the columns of A_i , $i = 1, \dots, n$ are $\mathcal{R}$ left coprime if the invariant factors of a gceld , (of the set of A_i over $\mathcal{R}$) , are units of $\mathcal{R}$.*
ii) Given the set of matrices $B_i \in \mathcal{R}^{p_i \times m}$, $i = 1, \dots, n$, $B = [B_1^T, \dots, B_n^T]^T \in \mathcal{R}^{p \times m}$, $(p = \sum_{i=1}^{n} p_i)$, with $rank_{\mathcal{F}}\{B\} = \rho'$, then we say that the rows of B_i , $i = 1, \dots, n$ are $\mathcal{R}$ right coprime if the invariant factors of a gcerd , (of the set of B_i over $\mathcal{R}$) , are units of $\mathcal{R}$. □

For the analysis of matrix equations over PIDs some further algebraic tools are needed . The notions of column , row projectors ; left , right annihilators and left , right inverses of a matrix $A \in \mathcal{R}^{p \times m}$ over $\mathcal{R}$ are introduced . These projectors , annihilators are shown to be generalizations of left , right inverses , and are characterized by using properties of unimodular matrices defined over the appropriate PID .

5.5. GENERALIZED COLUMN – ROW PROJECTORS OF A MATRIX OVER THE PID $\mathcal{R}$

Definition (5.5.1) : *Let $A \in \mathcal{R}^{p \times m}$, $rank_{\mathcal{F}}\{A\} = \rho \le min\{p, m\}$ and $P_l \in \mathcal{R}^{\rho \times p}$, $rank_{\mathcal{F}}\{P_l\} = \rho$, $Q_r \in \mathcal{R}^{m \times \rho}$, $rank_{\mathcal{F}}\{Q_r\} = \rho$.*
i) P_l is called an $\mathcal{R}$ column projector , ($\mathcal{R}$cp) , of A over $\mathcal{R}$, if :

$$P_l \cdot A = L_r \tag{5.5.1}$$

with L_r a gerd of A over $\mathcal{R}$.
ii) Q_r is called an $\mathcal{R}$ row projector , ($\mathcal{R}$rp) , of A over $\mathcal{R}$, if :

$$A \cdot Q_r = L_l \tag{5.5.2}$$

with L_r a gcrd of A over $\mathcal{R}$. □

Remark (5.5.1) *: By definition P_l , Q_r produces a gcrd , gcld of A over $\mathcal{R}$ and thus projects the column , row vectors of A onto the maximal $\mathcal{R}$ column , row module $\hat{\mathcal{M}}^c_A$, $\hat{\mathcal{M}}^r_A$ of A in $\mathcal{G}^c_A$, $\mathcal{G}^r_A$ respectively .* □

Proposition (5.5.1) *: Every matrix $A \in \mathcal{R}^{p \times m}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p , m\}$ has an $\mathcal{R}$cp P_l , $\mathcal{R}$rp , Q_r respectively .*

Proof

Let U , V be appropriate $\mathcal{R}$ unimodular matrices such that A can be expressed in its Smith form over $\mathcal{R}$:

$$A = U \cdot \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \cdot V \tag{5.5.3}$$

If U , U^{-1} , V , V^{-1} are partitioned as :

$$U = [\, U^\rho_p \, , \, U^{p-\rho}_p \,] \, , \; U^{-1} = \begin{bmatrix} U^p_\rho \\ U^p_{p-\rho} \end{bmatrix}, \; V = \begin{bmatrix} V^m_\rho \\ V^m_{m-\rho} \end{bmatrix}, \; V^{-1} = [\, V^\rho_m \, , \, V^{m-\rho}_m \,] \tag{5.5.4}$$

then (5.5.3) can be rewritten as :

$$A = U^\rho_p \cdot S_\rho \cdot V^m_\rho \tag{5.5.5}$$

Corollary (5.3.1) clearly implies that the matrices $L_l = U^\rho_p \cdot S_\rho$, $L_r = S_\rho \cdot V^m_\rho$ are a gcld , gcrd of A over $\mathcal{R}$ respectively . Condition (5.5.4) also implies that :

$$U^{-1} \cdot A = \begin{bmatrix} U^p_\rho \\ U^p_{p-\rho} \end{bmatrix} \cdot A = \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \cdot \begin{bmatrix} V^m_\rho \\ V^m_{m-\rho} \end{bmatrix} \Leftrightarrow U^p_\rho \cdot A = S_\rho \cdot V^m_\rho = L_r \tag{5.5.6}$$

$$A \cdot V^{-1} = A \cdot [V^\rho_m \, , \, V^{m-\rho}_m] = [U^\rho_p \, , \, U^{p-\rho}_p] \cdot \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \Leftrightarrow A \cdot V^\rho_m = U^\rho_p \cdot S_\rho = L_l \tag{5.5.7}$$

The matrices U^p_ρ , V^ρ_m are $\mathcal{R}$ right , left unimodular and (5.5.6) , (5.5.7) clearly imply that $P_l = U^p_\rho$ is an $\mathcal{R}$cp , $Q_r = V^\rho_m$ in an $\mathcal{R}$rp of A . □

Proposition (5.5.2) : Let $A \in \mathcal{R}^{p \times m}$, $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p , m\}$, and P_l , Q_r be an $\mathcal{R}$cp , $\mathcal{R}$rp of A respectively . Then P_l , Q_r are $\mathcal{R}$ right , left unimodular matrices .

Proof

We prove that P_l is an $\mathfrak{R}$ right unimodular matrix , since the proof for the case of Q_r follows along similar lines . Let us assume that P_l is not an $\mathfrak{R}$ right unimodular matrix . Then we may factorize it as :

$$P_l = Z \cdot P'_l \tag{5.5.8}$$

with , $Z \in \mathfrak{R}^{\rho x \rho}$, a non unimodular greatest left divisor of P_l , $P'_l \in \mathfrak{R}^{\rho x p}$, $rank_{\mathfrak{F}}\{P'_l\} = \rho$ On the other hand ,

$$P_l \cdot A = L_r \tag{5.5.9}$$

with L_r a gerd of A over $\mathfrak{R}$. Furthermore a matrix $B \in \mathfrak{R}^{p x \rho}$ exists , such that :

$$A = B \cdot L_r \tag{5.5.10}$$

(5.5.8) , (5.5.9) and (5.5.10) combined lead to :

$$P_l \cdot A = P_l \cdot B \cdot L_r = Z \cdot P'_l \cdot B \cdot L_r = Z \cdot W \cdot L_r \tag{5.5.11}$$

where , $W = P'_l \cdot B \in \mathfrak{R}^{\rho x \rho}$ and the matrix $Z \cdot W$ is not $\mathfrak{R}$ unimodular since Z is not . But (5.5.11) implies that the matrix $Z \cdot W \cdot L_r = L_r$ is a gcerd of A over $\mathfrak{R}$ and furthermore :

$$(Z \cdot W - I_\rho) \cdot L_r = O \tag{5.5.12}$$

But since $L_r \in \mathfrak{R}^{\rho x m}$, $rank_{\mathfrak{F}}\{L_r\} = \rho$, the left null space of L_r is trivial and thus :

$$Z \cdot W = I_\rho \tag{5.5.13}$$

which is a contradiction . □

An alternative characterization of column , row projectors of A is given in the following result .

***Proposition (5.5.3)** : Let $A \in \mathfrak{R}^{p x m}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p , m\}$, $A = P \cdot T \cdot Q$, T be a glrd of A over $\mathfrak{R}$, P , Q be bases for the maximum $\mathfrak{R}$ column , row modules , $\hat{\mathcal{M}}^c_A$, $\hat{\mathcal{M}}^r_A$ of A in $\mathfrak{S}^c_A$, $\mathfrak{S}^r_A$. Then :*

***i)** P_l is an $\mathfrak{R}$cp of A , if and only if $P_l \cdot P$ is an $\mathfrak{R}$ unimodular matrix .*

***ii)** Q_r is an $\mathfrak{R}$rp of A , if and only if $Q \cdot Q_r$ is an $\mathfrak{R}$ unimodular matrix .*

Proof

i) (⇒) If P_l is an $\mathfrak{R}$cp of A then $P_l \cdot A = L_r$ is a gerd of A over $\mathfrak{R}$. On the other hand :

$$\mathrm{L}_r = \mathrm{P}_l \cdot \mathrm{A} = \mathrm{P}_l \cdot \mathrm{P} \cdot \mathrm{T} \cdot \mathrm{Q} = \mathrm{W} \cdot \mathrm{L}_r' \tag{5.5.14}$$

where , $\mathrm{W} = \mathrm{P}_l \cdot \mathrm{P} \in \mathfrak{R}^{\rho x \rho}$, $\mathrm{L}_r' = \mathrm{T} \cdot \mathrm{Q}$ is on other gerd of A over $\mathfrak{R}$, (corollary (5.3.1)) . But since the gerds of A over $\mathfrak{R}$ are $\mathfrak{R}$ left equivalent , (proposition (5.3.1)) , (5.5.14) implies that W is an $\mathfrak{R}$ unimodular matrix .
($\Leftarrow$) If W= $\mathrm{P}_l \cdot \mathrm{P}$ is an $\mathfrak{R}$ unimodular matrix , then :

$$\mathrm{P}_l \cdot \mathrm{A} = \mathrm{P}_l \cdot \mathrm{P} \cdot \mathrm{T} \cdot \mathrm{Q} = \mathrm{W} \cdot \mathrm{L}_r' \tag{5.5.15}$$

where $\mathrm{L}_r' = \mathrm{T} \cdot \mathrm{Q}$ is a gerd of A over $\mathfrak{R}$, (corollary (5.3.1)) . But since the gerds of A over $\mathfrak{R}$ are $\mathfrak{R}$ left equivalent , (proposition (5.3.1)) , (5.5.15) implies that $\mathrm{W} \cdot \mathrm{L}_r'$ is a gerd of A over $\mathfrak{R}$ as well and thus P_l is an $\mathfrak{R}$cp of A .
ii) The proof follows along similar lines . □

Corollary (5.5.1) *: Let P_l , Q_r be a pair of $\mathfrak{R}$cp , $\mathfrak{R}$rp of A respectively , then $P_l \cdot A \cdot Q_r$ is a glrd of A over $\mathfrak{R}$, (proposition (5.2.2)) .* □

5.6. PRIME LEFT – RIGHT ANNIHILATORS OF A MATRIX OVER THE PID $\mathfrak{R}$

Definition (5.6.1) *: Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p , m\}$.*
i) *Let $p > \rho$ and $N_l \in \mathfrak{R}^{(p-\rho)xp}$, then N_l will be called an $\mathfrak{R}$ prime left annihilator , ($\mathfrak{R}$pla) , of A if N_l is an $\mathfrak{R}$ right unimodular matrix and :*

$$N_l \cdot A = O \tag{5.6.1}$$

ii) *Let $m > \rho$ and $N_r \in \mathfrak{R}^{mx(m-\rho)}$, then N_r will be called an $\mathfrak{R}$ prime right annihilator , ($\mathfrak{R}$pra) , of A if N_r is an $\mathfrak{R}$ left unimodular matrix and :*

$$A \cdot N_r = O \tag{5.6.2}$$

□

Proposition (5.6.1) *: Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p , m\}$. Then :*
i) *If $p > \rho$, A has always an $\mathfrak{R}$pla N_l . Furthermore if N_l' is any other $\mathfrak{R}$pla of A then N_l , N_l' are $\mathfrak{R}$ left equivalent , i.e. , $N_l \; E_l \; N_l'$.*
ii) *If $m > \rho$, A has always an $\mathfrak{R}$pra N_r . Furthermore if N_r' is any other $\mathfrak{R}$pra of A then N_r , N_r' are $\mathfrak{R}$ right equivalent , i.e. , $N_r \; E_r \; N_r'$.*

Proof

i) Let U , V be appropriate $\mathfrak{R}$ unimodular matrices such that A can be expressed in its

Smith form over $\mathfrak{R}$:

$$A = U \cdot \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \cdot V \tag{5.6.3}$$

If U , U^{-1} , V are partitioned as :

$$U^{-1} = \begin{bmatrix} U_\rho^p \\ U_{p-\rho}^p \end{bmatrix} , V = \begin{bmatrix} V_\rho^m \\ V_{m-\rho}^m \end{bmatrix} \tag{5.6.4}$$

Then (5.6.3) implies :

$$U_{p-\rho}^p \cdot A = O \tag{5.6.5}$$

and $N_l = U_{p-\rho}^p \in \mathfrak{R}^{(p-\rho)xp}$ is an $\mathfrak{R}$ right unimodular matrix and thus an $\mathfrak{R}$pla of A . Furthermore since $rank_{\mathfrak{F}}\{A\} = \rho < p$, the left null space of A , $\mathcal{N}_l\{A\}$, has dimension $(p-\rho)$ and thus the $\mathfrak{R}$pla N_l and any other $\mathfrak{R}$pla N_l' of A serves as a base of $\mathcal{N}_l\{A\}$. The latter implies that N_l , N_l' are $\mathfrak{R}$ left equivalent , $N_l \; E_l \; N_l'$.

ii) The proof follows along similar lines . □

The following results establish the relation between $\mathfrak{R}$cp , $\mathfrak{R}$rp and $\mathfrak{R}$pla , $\mathfrak{R}$pra of a matrix A respectively . A characterization of $\mathfrak{R}$cp , $\mathfrak{R}$rp of a matrix A via its $\mathfrak{R}$pla , $\mathfrak{R}$pra is introduced in proposition (5.6.2) .

***Corollary** (5.6.1) : Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \le min\{p , m\}$. Then :*

***i**) If $p > \rho$ there exists a pair (P_l , N_l) of an $\mathfrak{R}$cp , $\mathfrak{R}$pla of A such that the matrix :*

$$Y_l = \begin{bmatrix} P_l \\ N_l \end{bmatrix} \tag{5.6.6}$$

is $\mathfrak{R}$ unimodular .

***ii**) If $m > \rho$ there exists a pair (Q_r , N_r) of an $\mathfrak{R}$rp , $\mathfrak{R}$pra of A such that the matrix :*

$$Y_r = [\, Q_r , N_r \,] \tag{5.6.7}$$

is $\mathfrak{R}$ unimodular .

Proof

i) Let U , V be appropriate $\mathfrak{R}$ unimodular matrices such that A can be expressed in its Smith form over $\mathfrak{R}$:

$$A = U \cdot \begin{bmatrix} S_\rho & O \\ O & O \end{bmatrix} \cdot V \tag{5.6.8}$$

Then the proves of propositions (5.3.1) , (5.6.1) imply that the matrix :

$$Y_l = U^{-1} = \begin{bmatrix} U^p_\rho \\ U^p_{p-\rho} \end{bmatrix} = \begin{bmatrix} P_l \\ N_l \end{bmatrix} \tag{5.6.9}$$

satisfies (5.6.6) .

ii) Following similar arguments to those in case *i*) it can be shown that :

$$Y_r = V^{-1} = [\, V^\rho_m \,,\, V^{m-\rho}_m \,] = [\, Q_r \,,\, N_r \,] \tag{5.6.10}$$

satisfies (5.6.7) . □

Proposition (5.6.2) *: Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$, $(P_l\ ,\ N_l)$ be a pair of an $\mathfrak{R}$cp , $\mathfrak{R}$pla of A , $(Q_r\ ,\ N_r)$ be a pair of an $\mathfrak{R}$rp , $\mathfrak{R}$pra of A respectively . Then :*

i*) The general family of $\mathfrak{R}$cp of A is given by :*

α*)*

$$P'_l = U \cdot P_l + Y \cdot N_l \ ,\ if\ p > \rho \tag{5.6.11}$$

where $U \in \mathfrak{R}^{\rho x\rho}$ is an arbitrary $\mathfrak{R}$ unimodular matrix , $Y \in \mathfrak{R}^{\rho x(p-\rho)}$ is a parametric matrix .

β*)*

$$P'_l = U \cdot P_l \ ,\ if\ p = \rho \tag{5.6.12}$$

where $U \in \mathfrak{R}^{\rho x\rho}$ is an arbitrary $\mathfrak{R}$ unimodular matrix .

ii*) The general family of $\mathfrak{R}$rp of A is given by :*

α*)*

$$Q'_r = Q_r \cdot V + N_r \cdot X \ ,\ if\ m > \rho \tag{5.6.13}$$

where $V \in \mathfrak{R}^{\rho x\rho}$ is an arbitrary $\mathfrak{R}$ unimodular matrix , $X \in \mathfrak{R}^{(m-\rho)x\rho}$ is a parametric matrix .

β*)*

$$Q'_r = Q_r \cdot V \ ,\ if\ m = \rho \tag{5.6.14}$$

where $V \in \mathfrak{R}^{\rho x\rho}$ is an arbitrary $\mathfrak{R}$ unimodular matrix .

Proof

***i*) *α*)** Let $p > \rho$ and P'_l , P_l be any two $\mathfrak{R}$cps of A . Then :

$$P_l \cdot A = L_r \ ,\ P'_l \cdot A = L'_r \tag{5.6.15}$$

where , L_r , L'_r are two gerds of A over $\mathfrak{R}$. Proposition (5.3.1) has established that $L_r\ E_l\ L'_r$ and thus an $\mathfrak{R}$ unimodular matrix U exists such that :

$$P_l' \cdot A = U \cdot P_l \cdot A \tag{5.6.16}$$

or equivalently ,

$$(P_l' - U \cdot P_l) \cdot A = O \tag{5.6.17}$$

Since $p > \rho$ the left null space of A , $\mathcal{N}_l\{A\}$, has dimension $(p-\rho)$ and thus the $\mathcal{R}$pla of A , N_l , serves as base of $\mathcal{N}_l\{A\}$. Condition (5.6.17) implies that a matrix $Y \in \mathcal{R}^{\rho x(p-\rho)}$ exists such that :

$$(P_l' - U \cdot P_l) = Y \cdot N_l \Leftrightarrow P_l' = U \cdot P_l + Y \cdot N_l \tag{5.6.18}$$

β) If $p = \rho$ then all the $\mathcal{R}$pla of A , N_l , are equal to zero , i.e. $N_l = O$. Thus relation (5.6.18) becomes :

$$P_l' = U \cdot P_l \tag{5.6.19}$$

ii) The proof follows along similar lines . □

Corollary (5.6.2) *: If (P_l , N_l) is any pair of an $\mathcal{R}$cp , $\mathcal{R}$pla of A , (Q_r , N_r) is any pair of an $\mathcal{R}$rp , $\mathcal{R}$pra of A respectively . Then :*

i) *If $p > \rho$ the matrix :*

$$Y_l = \begin{bmatrix} P_l \\ N_l \end{bmatrix} \tag{5.6.20}$$

is $\mathcal{R}$ unimodular .

ii) *If $m > \rho$ the matrix :*

$$Y_r = [\, Q_r \, , \, N_r \,] \tag{5.6.21}$$

is $\mathcal{R}$ unimodular .

Proof

i) Corollary (5.6.1) has established that a pair (P_l' , N_l') of an $\mathcal{R}$cp , $\mathcal{R}$pla of A exists such that the matrix :

$$Y_l' = \begin{bmatrix} P_l' \\ N_l' \end{bmatrix} \tag{5.6.22}$$

is $\mathcal{R}$ unimodular . On the other hand matrices U , $W \in \mathcal{R}^{\rho x \rho}$ $\mathcal{R}$ unimodular , $Y \in \mathcal{R}^{\rho x(p-\rho)}$ parametric , exist such that , (proposition (5.6.2)) :

$$P_l' = U \cdot P_l + Y \cdot N_l \, , \, N_l' = W \cdot N_l \tag{5.6.23}$$

Condition (5.6.22) via (5.6.23) implies :

$$Y_l' = \begin{bmatrix} U \cdot P_l + Y \cdot N_l \\ W \cdot N_l \end{bmatrix} = \begin{bmatrix} U & Y \\ O & W \end{bmatrix} \cdot \begin{bmatrix} P_l \\ N_l \end{bmatrix} \tag{5.6.24}$$

which clearly implies (5.6.20) .
ii) The proof follows along similar lines . □

In the following the notion of left , right inverses of a matrix $A \in \mathfrak{R}^{pxm}$ over $\mathfrak{R}$ are studied . The $\mathfrak{R}$cp , $\mathfrak{R}$rp of A are generalizations of the left , right inverses over $\mathfrak{R}$.

5.7. LEFT – RIGHT INVERSES OF A MATRIX OVER THE PID $\mathfrak{R}$

Definition (5.7.1) *: Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p , m\}$ and let $A_l \in \mathfrak{R}^{mxp}$, $A_r \in \mathfrak{R}^{mxp}$. Then :*
i) A_l is called an $\mathfrak{R}$ left inverse , ($\mathfrak{R}$li) , of A if :

$$A_l \cdot A = I_m \tag{5.7.1}$$

ii) A_r is called an $\mathfrak{R}$ right inverse , ($\mathfrak{R}$ri) , of A if :

$$A \cdot A_r = I_p \tag{5.7.2}$$

□

The conditions under which an $\mathfrak{R}$li , $\mathfrak{R}$ri of a matrix A exists are examined next . We first state the following well known result :

Lemma (5.7.1) *[Per. 1] : Let $A \in \mathfrak{R}^{pxm}$ then :*
i) A left inverse $A_l \in \mathfrak{F}^{mxp}$ of A exists if and only if $rank_{\mathfrak{F}}\{A\} = m$.
ii) A right inverse $A_r \in \mathfrak{F}^{mxp}$ of A exists if and only if $rank_{\mathfrak{F}}\{A\} = p$. □

Remark (5.7.1) *: Any $\mathfrak{R}$li , $\mathfrak{R}$ri of a matrix $A \in \mathfrak{R}^{pxm}$ is also an inverse over $\mathfrak{F}$. Thus a necessary condition for the existence of $\mathfrak{R}$lis , $\mathfrak{R}$ris of A is that $rank_{\mathfrak{F}}\{A\} = m , p$ respectively .* □

Theorem (5.7.1) *: Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p , m\}$, $S_A = diag\{S_\rho , O\}$ be the Smith form of A over $\mathfrak{R}$ then :*
i) An $\mathfrak{R}$li $A_l \in \mathfrak{R}^{mxp}$ of A exists if and only if $\rho = rank_{\mathfrak{F}}\{A\} = m$ and $S_\rho^{-1} \in \mathfrak{R}^{\rho x \rho}$.

ii) An $\mathfrak{R}$ri $A_r \in \mathfrak{R}^{mxp}$ of A exists if and only if $\rho = rank_{\mathfrak{T}}\{A\} = p$ and $S_\rho^{-1} \in \mathfrak{R}^{\rho x \rho}$.

Proof

i) **(⇒)** Let $A_l \in \mathfrak{R}^{mxp}$ be an $\mathfrak{R}$li of A . Then remark (5.7.1) implies that $\rho = rank_{\mathfrak{T}}\{A\} = m$. Furthermore if U , V are appropriate $\mathfrak{R}$ unimodular matrices such that A can be expressed in its Smith form over $\mathfrak{R}$:

$$A = U \cdot \begin{bmatrix} S_m \\ O \end{bmatrix} \cdot V \tag{5.7.3}$$

then ,

$$A_l \cdot A = A_l \cdot U \cdot \begin{bmatrix} S_m \\ O \end{bmatrix} \cdot V = B \cdot \begin{bmatrix} S_m \\ O \end{bmatrix} \cdot V = I_m \tag{5.7.4}$$

where , $B = A_l \cdot U \in \mathfrak{R}^{mxp}$. If we partition B as $[\ B_m^m\ ,\ B_m^{p\text{-}m}\ ]$ then (5.7.4) is transformed to :

$$A_l \cdot A = B_m^m \cdot S_m \cdot V = I_m \tag{5.7.5}$$

It is clear that $S_m^{-1} = V \cdot B_m^m \in \mathfrak{R}^{mxm}$, or equivalently $S_\rho^{-1} \in \mathfrak{R}^{\rho x \rho}$.
(⇐) Let $\rho = rank_{\mathfrak{T}}\{A\} = m$ and $S_\rho^{-1} \in \mathfrak{R}^{\rho x \rho}$, or equivalently $S_m^{-1} \in \mathfrak{R}^{mxm}$. If U , V are appropriate $\mathfrak{R}$ unimodular matrices such that A can be expressed in its Smith form over $\mathfrak{R}$:

$$A = U \cdot \begin{bmatrix} S_m \\ O \end{bmatrix} \cdot V \tag{5.7.6}$$

Set A_l to be the matrix :

$$A_l = [\ V^{-1} \cdot S_m^{-1}\ ,\ O_m^{p\text{-}m}\ ] \cdot U^{-1} \in \mathfrak{R}^{mxp} \tag{5.7.7}$$

Then :

$$A_l \cdot A = [\ V^{-1} \cdot S_m^{-1}\ ,\ O_m^{p\text{-}m}\ ] \cdot U^{-1} \cdot U \cdot \begin{bmatrix} S_m \\ O \end{bmatrix} \cdot V = I_m \tag{5.7.8}$$

ii) The proof follows along similar lines . □

Remark (5.7.2) *: It is clear from theorem (5.7.1) that an $\mathfrak{R}$li , $\mathfrak{R}$ri of a matrix A exists, if and only if $rank_{\mathfrak{T}}\{A\} = m$, $rank_{\mathfrak{T}}\{A\} = p$ respectively and the invariant factors of A over $\mathfrak{R}$ are units of $\mathfrak{R}$.* □

The link between $\mathfrak{R}$lis , $\mathfrak{R}$ris and $\mathfrak{R}$cps , $\mathfrak{R}$rps respectively is established by the following result .

Lemma (5.7.2) : Let $A \in \mathcal{R}^{p x m}$, $rank_{\mathcal{F}}\{A\} = \rho \le min\{p , m\}$ and P_l , Q_r denote an $\mathcal{R}$cp $\mathcal{R}$rp of A respectively . Then :
i) If A has an $\mathcal{R}$li then the matrix :

$$A_l = (P_l \cdot A)^{-1} \cdot P_l \tag{5.7.9}$$

is an $\mathcal{R}$li of A as well .
ii) If A has an $\mathcal{R}$ri then the matrix :

$$A_r = Q_r \cdot (A \cdot Q_r)^{-1} \tag{5.7.9}$$

is an $\mathcal{R}$ri of A as well .

Proof

i) If A has an $\mathcal{R}$li then $rank_{\mathcal{F}}\{A\} = m$. If U , V are appropriate $\mathcal{R}$ unimodular matrices such that A can be expressed in its Smith form over $\mathcal{R}$, then :

$$A = U \cdot \begin{bmatrix} S_m \\ O \end{bmatrix} \cdot V \tag{5.7.10}$$

with $S_m^{-1} \in \mathcal{R}^{m x m}$. The proof of proposition (5.3.1) has established that the matrix $L_r = S_m \cdot V \in \mathcal{R}^{m x m}$ is a gcrd of A over $\mathcal{R}$. Furthermore :

$$P_l \cdot A = L_r' \in \mathcal{R}^{m x m} \tag{5.7.11}$$

is an other gcrd of A over $\mathcal{R}$. Proposition (5.3.1) also established that L_r , L_r' are $\mathcal{R}$ left equivalent and thus an $\mathcal{R}$ unimodular matrix W exists such that :

$$L_r' = W \cdot L_r = W \cdot S_m \cdot V \tag{5.7.12}$$

or ,

$$(L_r')^{-1} = V^{-1} \cdot S_m^{-1} \cdot W^{-1} \in \mathcal{R}^{m x m} \tag{5.7.13}$$

or by (5.7.11) ,

$$(P_l \cdot A)^{-1} = (L_r')^{-1} \in \mathcal{R}^{m x m} \tag{5.7.14}$$

which implies that the matrix $A_l = (P_l \cdot A)^{-1} \cdot P_l \in \mathcal{R}^{m x p}$. Finally ,

$$A_l = (P_l \cdot A)^{-1} \cdot P_l \cdot A = I_m \tag{5.7.15}$$

ii) The proof follows along similar lines . □

Lemma (5.7.2) suggests that the results stated for the $\mathcal{R}$cps , $\mathcal{R}$rps of a matrix A carry over to the $\mathcal{R}$lis , $\mathcal{R}$ris of that matrix , (if any) .

***Corollary (5.7.1)** : Let $A \in \mathfrak{R}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$. Then :*
i) If A has an $\mathfrak{R}$li , then the family of all $\mathfrak{R}$lis is given by :

$$A_l = (P_l \cdot A)^{-1} \cdot P_l + Y \cdot N_l \tag{5.7.16}$$

where P_l , N_l are an $\mathfrak{R}$cp , $\mathfrak{R}$pla of A respectively , $Y \in \mathfrak{R}^{mx(p-m)}$ is a parametric matrix .
ii) If A has an $\mathfrak{R}$ri , then the family of all $\mathfrak{R}$ris is given by :

$$A_r = Q_r \cdot (A \cdot Q_r)^{-1} + N_r \cdot X \tag{5.7.17}$$

where Q_r , N_r are an $\mathfrak{R}$rp , $\mathfrak{R}$pra of A respectively , $X \in \mathfrak{R}^{(m-p)xp}$ is a parametric matrix .

Proof
i) Lemma (5.7.2) implies that the matrix $A_l' = (P_l \cdot A)^{-1} \cdot P_l$ is an $\mathfrak{R}$li of A . If A_l is any other $\mathfrak{R}$li of A , then :

$$A_l' \cdot A = I_m \ ,\ A_l \cdot A = I_m \tag{5.7.18}$$

or equivalently ,

$$A_l' \cdot A - A_l \cdot A = O \Leftrightarrow (A_l' - A_l) \cdot A = O \tag{5.7.19}$$

Since $(A_l' - A_l)$ belongs to the left null space of A and N_l is a base of it , a matrix $Y \in \mathfrak{R}^{mx(p-m)}$ exists such that :

$$(A_l' - A_l) = Y \cdot N_l \Leftrightarrow A_l = (P_l \cdot A)^{-1} \cdot P_l + Y \cdot N_l \tag{5.7.20}$$

ii) The proof follows along similar lines . □

5.8. MULTIPLES AND LEAST MULTIPLES OF A MATRIX OVER THE PID $\mathfrak{R}$

In this section the ordinary concepts of multiples , (common multiples) , least multiples , (least common multiples) , is extended over $\mathfrak{R}$ for matrices with entries over $\mathfrak{F}$.

***Definition (5.8.1)** : Let $A \in \mathfrak{F}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \leq min\{p\ ,\ m\}$. Then :*
i) $M_r \in \mathfrak{R}^{pxm}$ is called a multiple of the rows of A over $\mathfrak{R}$, ($\mathfrak{R}$mr) , if a matrix $C_r \in \mathfrak{R}^{pxp}$ exists such that :

$$C_r \cdot A = M_r \tag{5.8.1}$$

M_r is called a least multiple of the rows of A over $\Re$, ($\Re$lmr) , if it is an $\Re$mr of A and for any other $\Re$mr of A , G_r , a matrix $E \in \Re^{pxp}$ exists such that $E \cdot M_r = G_r$.
ii) *$M_c \in \Re^{pxm}$ is called a multiple of the columns of A over $\Re$, ($\Re$mc) , if a matrix $C_c \in \Re^{mxm}$ exists such that :*

$$A \cdot C_c = M_c \tag{5.8.2}$$

M_c is called a least multiple of the columns of A over $\Re$, ($\Re$lmc) , if it is an $\Re$mc of A and for any other $\Re$mc of A , G_c , a matrix $E \in \Re^{mxm}$ exists such that $M_c \cdot E = G_c$. □

The following proposition establishes the existence of $\Re$mr , $\Re$mc , $\Re$lmr , $\Re$lmc of a matrix A .

Proposition (5.8.1) *: Let $A \in \mathfrak{F}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \le min\{p , m\}$ and (D , N) , (D' , N') be an $\Re$ – coprime left , right MFD of A over $\Re$ respectively . Then :*
i) *The matrix N is an $\Re$lmr of A .*
ii) *The matrix N' is an $\Re$lmc of A .*

Proof

i) Since $A = D^{-1} \cdot N$, it is clear that the matrix $N = D \cdot A$ is an $\Re$mr of A . Let M_r be any other $\Re$mr of A . Then , by definition(5.8.1) a matrix $C_r \in \Re^{pxp}$ exists such that $C_r \cdot A = M_r$ and thus :

$$[\, C_r , M_r \,] = [\, C_r , C_r \cdot A \,] = C_r \cdot [\, I_p , A \,] = C_r \cdot D^{-1} \cdot [\, D , N \,] \tag{5.8.3}$$

If $W = C_r \cdot D^{-1}$, then since [D , N] is $\Re$ – right unimodular is implied that $W \in \Re^{pxp}$. The latter implies that :

$$M_r = C_r \cdot A = C_r \cdot D^{-1} \cdot N = W \cdot N \tag{5.8.4}$$

and clearly N is an $\Re$lmr of A .
ii) The proof follows along similar lines . □

The following proposition gives a characterization of the families of $\Re$lmr , $\Re$lmc of A .

Proposition (5.8.2) *: Let $A \in \mathfrak{F}^{pxm}$, $rank_{\mathfrak{F}}\{A\} = \rho \le min\{p , m\}$ and (D , N) , (D' , N') be an $\Re$ – coprime left , right MFD of A over $\Re$ respectively . Then :*
i) If M_r is any $\Re$lmr of A then an $\Re$ – unimodular matrix U exists such that $M_r = U \cdot N$.
ii) If M_c is any $\Re$lmc of A then an $\Re$ – unimodular matrix V exists such that $M_c =$ $= N' \cdot V$.

Proof

i) Let M_r be any $\mathcal{R}$lmr of A . Proposition(5.8.1) implies that N is also an $\mathcal{R}$lmr of A and thus by definition(5.8.1) matrices W , E in $\mathcal{R}^{pxp}$ exist such that :

$$M_r = W \cdot N \ , \ N = E \cdot M_r \tag{5.8.5}$$

The latter implies that $\mathcal{N}_l\{M_r\} = \mathcal{N}_l\{ N \}$ and thus $rank_{\mathcal{F}}\{M_r\} = rank_{\mathcal{F}}\{ N \}$. Since $rank_{\mathcal{F}}\{A\} = \rho \leq min\{p \ , \ m\}$ it is implied that N can be constructed via the Smith McMillan form of A over $\mathcal{R}$ to be :

$$N = \begin{bmatrix} H \\ O \end{bmatrix} \tag{5.8.6}$$

with , $H \in \mathcal{R}^{\rho xm}$, $rank_{\mathcal{F}}\{H\} = \rho$. From the above analysis $\mathcal{R}$ – unimodular matrix K exists such that ,

$$K \cdot M_r = \begin{bmatrix} M \\ O \end{bmatrix} \tag{5.8.7}$$

with , $M \in \mathcal{R}^{\rho xm}$, $rank_{\mathcal{F}}\{M\} = \rho$. Now , if $Y = K \cdot W$, $J = E \cdot K^{-1}$ partition Y , J as :

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} , \ E = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \tag{5.8.8}$$

Conditions (5.8.5) – (5.8.8) imply $Y_3 = J_3 = O$ and :

$$M = Y_1 \cdot H \ , \ H = J_1 \cdot M \tag{5.8.9}$$

which clearly implies that the matrices Y_1 , J_1 are $\mathcal{R}$ – unimodular . Thus ,

$$W = K^{-1} \cdot \begin{bmatrix} Y_1 & Y_2 \\ O & I \end{bmatrix} \cdot \begin{bmatrix} I & O \\ O & Y_4 \end{bmatrix} \tag{5.8.10}$$

And

$$M_r = K^{-1} \cdot \begin{bmatrix} Y_1 & Y_2 \\ O & I \end{bmatrix} \cdot \begin{bmatrix} I & O \\ O & Y_4 \end{bmatrix} \cdot \begin{bmatrix} H \\ O \end{bmatrix} = K^{-1} \cdot \begin{bmatrix} Y_1 & Y_2 \\ O & I \end{bmatrix} \cdot \begin{bmatrix} H \\ O \end{bmatrix} = U \cdot N \tag{5.8.11}$$

with U $\mathcal{R}$ – unimodular .

ii) The proof follows along similar lines . □

In the following we study the concepts of common left , right multiples , least common left , right multiples of a set of matrices .

Definition (5.8.2) : i) *Let $A_i \in \mathcal{R}^{pxm_i}$, $M \in \mathcal{R}^{pxm}$, $i = 1 , \dots , n$, $m = \sum_i m_i$. Then M*

is a common left multiple of the set of A_i , ($\mathcal{R}$clm) , over $\mathcal{R}$, if it is an $\mathcal{R}$mr of the composite matrix $[A_1 , \dots , A_n]$. M is a least common left multiple , ($\mathcal{R}$lclm) , of the set of A_i over $\mathcal{R}$, if it is an $\mathcal{R}$lmr of the composite matrix $[A_1 , \dots , A_n]$.
ii) Let $B_i \in \mathcal{R}^{p_i x m}$, $\Lambda \in \mathcal{R}^{p x m}$, $i = 1 , \dots , n$, $p = \sum_i p_i$. Then Λ is a common right multiple of the set of B_i , ($\mathcal{R}$crm) , over $\mathcal{R}$, if it is an $\mathcal{R}$mc of the composite matrix $[B_1^T , \dots , B_n^T]^T$. Λ is a least common right multiple , ($\mathcal{R}$lcrm) , of the set of B_i over $\mathcal{R}$, if it is an $\mathcal{R}$lmc of the composite matrix $[B_1^T , \dots , B_n^T]^T$. □

The above definition is different from what one would have expected ; this is due to the fact that our analysis is oriented on the use of multiples , over $\mathcal{R}$, of a matrix in the transformation of matrix equations defined over $\mathcal{F}$ to ones with known matrices defined over $\mathcal{R}$. This will become clear in chapter 6 where these issues are studied .

5.9. CONCLUSIONS

In Chapter 5 we have investigated structural properties of matrices A over a PID , $\mathcal{R}$. The matrices have been assumed to have entries over $\mathcal{R}$, apart from the case of multiples , least multiples where the matrices A have entries over the field of fractions of $\mathcal{R}$, $\mathcal{F}$. These properties have been used to generate algebraic tools that will enable us to formulate a unifying framework to deal with solvability of matrix equations over $\mathcal{R}$. The existence and characterization of families of greatest left – right divisors , extended greatest left – right divisors , projectors , annihilators , left – right inverses , multiples and least multiples over $\mathcal{R}$ of the matrices A has been introduced . The relation between these algebraic tools and the column , row $\mathcal{R}$ – modules , maximum $\mathcal{R}$ – modules of the matrices under investigation has been established .

CHAPTER 6

SOLVABILITY OF MATRIX EQUATIONS OVER A PRINCIPAL IDEAL DOMAIN

6.1. INTRODUCTION

The formulation and solvability of many control synthesis problems via the algebraic framework of what is termed as the matrix fraction description , (MFD) , approach , can be associated with the study of certain matrix equations over the ring of interest $\mathfrak{R}$; depending on the nature of the problem in question this ring can be either $\mathbb{R}[s]$, or $\mathbb{R}_{pr}(s)$, or $\mathbb{R}_{\mathfrak{P}}(s)$, and certainly a principal ideal domain , (PID) . Our aim in this chapter is to try to develop a unifying algebraic approach for solving matrix equations related to control synthesis problems , (such as stabilization problems , model matching, disturbance decoupling , noninteracting control and the regulator problem) , by making use of the structural properties of the given matrices over the PID of interest . The matrix equations we deal with are of the type :

$$Z \cdot X = E \ , \ Z \in \mathfrak{F}^{p x m} \ , \ E \in \mathfrak{F}^{p x \kappa} \ , \ X \in \mathfrak{R}^{m x \kappa} \tag{6.1.1}$$

$$Y \cdot Z = E \ , \ Z \in \mathfrak{F}^{p x m} \ , \ E \in \mathfrak{F}^{\kappa x m} \ , \ Y \in \mathfrak{R}^{\kappa x p} \tag{6.1.2}$$

$$Z \cdot X \cdot E = H \ , \ Z \in \mathfrak{F}^{p x m} \ , \ E \in \mathfrak{F}^{\kappa x t} \ , \ H \in \mathfrak{F}^{p x t} \ , \ X \in \mathfrak{R}^{m x \kappa} \tag{6.1.3}$$

$$\sum_{i=1}^{h} Z_i \cdot X_i \cdot E_i = H \ , \ Z_i \in \mathfrak{F}^{p x m_i}, \ E_i \in \mathfrak{F}^{\kappa_i x t}, \ H \in \mathfrak{F}^{p x t}, \ X_i \in \mathfrak{R}^{m_i x \kappa_i} \tag{6.1.4}$$

where the entries of the given matrices are supposed to be over the field of fractions , $\mathfrak{F}$ of a given PID , $\mathfrak{R}$, . Notice that equation (6.1.4) is a generalization of many well know matrix equations , such as :

$$Z_1 \cdot X_1 + \cdots + Z_h \cdot X_h = E \ , \ Z_i \in \mathfrak{F}^{p x m_i}, \ E \in \mathfrak{F}^{p x t}, \ X_i \in \mathfrak{R}^{m_i x t} \tag{6.1.5}$$

$$Y_1 \cdot Z_1 + \cdots + X_h \cdot Z_h = E \ , \ Z_i \in \mathfrak{F}^{p_i x m}, \ E \in \mathfrak{F}^{t x m}, \ Y_i \in \mathfrak{R}^{t x p_i} \tag{6.1.6}$$

$$Z \cdot X + Y \cdot E = H \ , \ Z \in \mathfrak{F}^{p x m}, \ E \in \mathfrak{F}^{\kappa x t}, \ H \in \mathfrak{F}^{p x t}, \ X \in \mathfrak{R}^{m x t}, Y \in \mathfrak{R}^{p x \kappa} \tag{6.1.7}$$

$$X \cdot Z + E \cdot Y = H \ , \ Z \in \mathfrak{F}^{p x m}, \ E \in \mathfrak{F}^{l x \kappa}, \ H \in \mathfrak{F}^{l x m}, \ X \in \mathfrak{R}^{l x p}, Y \in \mathfrak{R}^{\kappa x m} \tag{6.1.8}$$

Matrix equations of this type have been discussed in the literature , [Rot. 1] , [Kuc. 2] , [Emr. 2] , [Zac. 1] , [Per. 1] , [Var. 5] , [Ozg. 1] and references therein . Each of the matrix equations in (6.1.1) – (6.1.4) are studied separately and solvability conditions as well as parametrization of solutions are given in terms of greatest left – right divisors , column , row projectors and annihilators over $\mathfrak{R}$ of the known matrices . The machinery that has been developed in chapter 5 can be used on equations (6.1.1) – (6.1.8) if they are transformed to equivalent ones over $\mathcal{R}$, via the concepts of multiples of the rows , columns and common left , right multiples over $\mathcal{R}$, of the known matrices . In the following if A is a matrix we shall denote with $\mathcal{M}^c_A$, $\mathcal{M}^r_A$, the $\mathcal{R}$ column , row span modules of A respectively , $\rho_A = rank\{A\} = rank_{\mathcal{F}}\{A\}$ as well as the dimension of the finitely generated free modules $\mathcal{M}^c_A$, $\mathcal{M}^r_A$; we shall denote by $\mathcal{N}_l\{A\}$, $\mathcal{N}_r\{A\}$ the left , right null space of the matrix A .

6.2. STUDY OF THE MATRIX EQUATION Z·X = E , (Y·Z = E) , OVER THE PID $\mathcal{R}$

The matrix equations (6.1.1) , (6.1.2) are central in the formulation of many control synthesis problems like : the exact model and stable exact model matching problems , stabilization problems like the centralized and decentralized stabilization of linear unstable systems , where the equations in question are met in the matrix Diophantine type , $(D\cdot X + N\cdot Y = I \ , \ X\cdot D + Y\cdot N = I)$. Notice that (6.1.1) and (6.1.2) are dual and thus all the arguments and results stated and proved for (6.1.1) have their duals holding true for (6.1.2) , so we shall only prove results for (6.1.1) . In the following we consider the matrix equation :

$$Z\cdot X = E \ , \ Z\in\mathcal{F}^{p\times m} \ , \ E\in\mathcal{F}^{p\times\kappa} \ , \ X\in\mathcal{R}^{m\times\kappa}$$
$$Y\cdot Z = E \ , \ Z\in\mathcal{F}^{p\times m} \ , \ E\in\mathcal{F}^{\kappa\times m} \ , \ Y\in\mathcal{R}^{\kappa\times p}$$

where $\mathcal{R}$ is a given PID , $\mathcal{F}$ is the field of fractions of $\mathcal{R}$. If (D , N) , (D′ , N′) denote an $\mathcal{R}$ – coprime left , right MFD of the matrices $M = [\ Z \ , \ E\ ]$, $M' = [\ Z^T \ , \ E^T\ ]^T$ respectively , $(D\cdot M = [A \ , B] = N \ , \ M'\cdot D' = [A^T, B^T]^T = N')$, then N , N′ are an $\mathcal{R}$lmr, $\mathcal{R}$lmc of M , M′ respectively and the above equations can be equivalently transformed to :

$$A\cdot X = B \ , \ A\in\mathcal{R}^{p\times m} \ , \ B\in\mathcal{R}^{p\times\kappa} \ , \ X\in\mathcal{R}^{m\times\kappa} \qquad (6.2.1)$$
$$Y\cdot A = B \ , \ A\in\mathcal{R}^{p\times m} \ , \ B\in\mathcal{R}^{\kappa\times m} \ , \ Y\in\mathcal{R}^{\kappa\times p} \qquad (6.2.2)$$

Thus in the following we can implement the algebraic tools developed in chapter 5 to achieve solvability and characterization of solutions of (6.1.1) , (6.1.2) via (6.2.1) , (6.2.2) .

Theorem (6.2.1) : *i) The equation (6.2.1) has a solution over $\mathcal{R}$ if and only if :*

$$\mathcal{M}^c_B \subseteq \mathcal{M}^c_A \Leftrightarrow \rho_B \le \rho_A \qquad (6.2.3)$$

ii) The equation (6.2.2) has a solution over $\mathfrak{R}$ if and only if :

$$\mathcal{M}^r_B \subseteq \mathcal{M}^r_A \Leftrightarrow \rho_B \leq \rho_A \qquad (6.2.4)$$

Proof

i) ($\Rightarrow$) Let equation (6.2.1) has a solution X over $\mathfrak{R}$. Then each column of B , $\underline{b}_i$, $i =$ $= 1$, ... , κ , can be expressed as :

$$\underline{b}_i = \sum_{j=1}^{m} x_{ji} \cdot \underline{a}_j \ , \ i = 1 \ , \ldots , \ \kappa \qquad (6.2.5)$$

where x_{ji} belongs to $\mathfrak{R}$. Thus $\underline{b}_i \in \mathcal{M}^c_A$, $i = 1$, ... , κ and finally (6.2.3) holds true .
($\Leftarrow$) Let (6.2.3) hold true . Then $\underline{b}_i \in \mathcal{M}^c_A$, $i = 1$, ... , κ and there exist $x_{ji} \in \mathfrak{R}$, $j = 1$, ... , m such that :

$$\underline{b}_i = \sum_{j=1}^{m} x_{ji} \cdot \underline{a}_j \ , \ i = 1 \ , \ldots , \ \kappa \qquad (6.2.6)$$

If $X \in \mathfrak{R}^{m x \kappa}$ is set to be the matrix $[x_{ji}]$, $j = 1, \ldots , m$, $i = 1$, ... , κ , (6.2.6) implies that :

$$A \cdot X = B \qquad (6.2.7)$$

ii) The proof follows along similar lines . □

The module inclusion properties (6.2.3) , (6.2.4) will be the base of our analysis . In the following conditions for the characterization of these properties will be derived . In the previous chapter , (chapter 5) , we defined the notion of non square matrix divisors over a PID $\mathfrak{R}$. The following result due to Pernebo , [Per. 1] , defines solutions of (6.2.1) , (6.2.2) over $\mathfrak{R}$ by using the concept of non square divisors .

Theorem (6.2.2) [Per. 1] : i) Equation (6.2.1) has a solution over $\mathfrak{R}$, if and only if a geld of A over $\mathfrak{R}$ is an eld of B over $\mathfrak{R}$ as well .
ii) Equation (6.2.2) has a solution over $\mathfrak{R}$, if and only if a gerd of A over $\mathfrak{R}$ is an erd of B over $\mathfrak{R}$ as well .

Proof

i) ($\Rightarrow$) Let (6.2.1) have a solution $X \in \mathfrak{R}^{m x \kappa}$ and L_l be a geld of A over $\mathfrak{R}$. Then we can write :

$$A = L_l \cdot A_0 \qquad (6.2.8)$$

where , $A_0 \in \mathfrak{R}^{\rho_A x m}$. Then :

$$A \cdot X = L_l \cdot A_0 \cdot X = L_l \cdot W = B \qquad (6.2.9)$$

where , $W \in \mathfrak{R}^{\rho_A x \kappa}$ and it is clear that L_l is a eld of B over $\mathfrak{R}$.

($\Leftarrow$) Let L_l be a geld of A over $\mathcal{R}$ and assume that it is an eld of B over $\mathcal{R}$ as well . Then :

$$B = L_l \cdot B_0 \tag{6.2.10}$$

where , $B_0 \in \mathcal{R}^{\rho_A x \kappa}$. Proposition (5.3.1) has established that a matrix $K \in \mathcal{R}^{m x \rho_A}$ exists such that :

$$L_l = A \cdot K \tag{6.2.11}$$

Thus (6.2.10) , (6.2.11) imply :

$$B = L_l \cdot B_0 = A \cdot K \cdot B_0 \tag{6.2.12}$$

If we set $X \in \mathcal{R}^{m x \kappa}$ to be the matrix $K \cdot B_0$, then X is a solution of (6.2.1) over $\mathcal{R}$.
ii) The proof follows along similar lines . □

Remark (6.2.1) *: Notice that the latter result is almost identical to the former . In fact from the analysis in chapter5 the gelds , gerds of A over* $\mathcal{R}$ *serve as bases for the* $\mathcal{M}^c_A$, $\mathcal{M}^r_A$ *and thus if* L_l , L_r *is a pair of a geld , gerd of A over* $\mathcal{R}$ *respectively , from theorems (6.2.1) , (6.2.2) is implied that :*

$$\mathcal{M}^c_B \subseteq \mathcal{M}^c_A = \mathcal{M}^c_{L_l} \; , \; \mathcal{M}^r_B \subseteq \mathcal{M}^r_A = \mathcal{M}^r_{L_r} \tag{6.2.13}$$

□

Now if a solution of (6.2.1) , (6.2.2) over $\mathcal{R}$ exists , it is important to determine if the family of solutions over $\mathcal{R}$ can be generated . The following corollary provides a characterization of the family of solutions of (6.2.1) , (6.2.2) over $\mathcal{R}$, whenever such a solution exists .

Corollary (6.2.1) *: **i**) If X_0 is a particular solution of equation (6.2.1) over* $\mathcal{R}$ *then the family of solutions of (6.2.1) over* $\mathcal{R}$ *is characterized by the following properties :*
α) If $\mathcal{N}_r\{A\} = \{\underline{0}\}$, then X_0 is uniquely defined .
β) If $\mathcal{N}_r\{A\} \neq \{\underline{0}\}$, and N_r is an $\mathcal{R}$*pra of A , then the family of solutions of (6.2.1) over* $\mathcal{R}$ *is given by :*

$$X = X_0 + N_r \cdot K \; , \; K \in \mathcal{R}^{(m-\rho_A) x \kappa} \text{parametric} \tag{6.2.14}$$

***ii**) If Y_0 is a particular solution of equation (6.2.2) over* $\mathcal{R}$ *then the family of solutions of (6.2.2) over* $\mathcal{R}$ *is characterized by the following properties :*
α) If $\mathcal{N}_l\{A\} = \{\underline{0}\}$, then Y_0 is uniquely defined .
β) If $\mathcal{N}_l\{A\} \neq \{\underline{0}\}$, and N_l is an $\mathcal{R}$*pla of A , then the family of solutions of (6.2.2) over* $\mathcal{R}$ *is given by :*

$$Y = Y_0 + C \cdot N_l \; , \; C \in \mathcal{R}^{\kappa x (p-\rho_A)} \text{parametric} \tag{6.2.15}$$

Proof

***i*) α)** Let us suppose that an other solution , X , of (6.2.1) over $\mathfrak{R}$ exists . Then :

$$A \cdot X_0 = B \ , \ A \cdot X = B \Leftrightarrow A \cdot (X_0 - X) = O \qquad (6.2.16)$$

But since A has trivial right null space (6.2.16) implies that $(X_0 - X) = O$ and thus X_0 is uniquely defined .

β) If X is any solution of (6.2.1) then as in case α) (6.2.16) holds true . Since the right null space of A is not trivial then N_r serves as a base of $\mathcal{N}_r\{A\}$ and a matrix $K \in \mathfrak{R}^{(m-\rho_A)x\kappa}$ exists such that :

$$(X_0 - X) = N_r \cdot K \qquad (6.2.17)$$

which clearly implies (6.2.14) .

***ii*)** The proof follows along similar lines . □

We now state a further result on the characterization of solutions of (6.2.1) , (6.2.2) over $\mathfrak{R}$.

***Corollary (6.2.2)** : i) Let Q_r be an $\mathfrak{R}$rp of A and assume that a geld , L_l , of A over $\mathfrak{R}$ is an eld of B over $\mathfrak{R}$, i.e. $B = L_l \cdot B_0$. Then equation (6.2.1) has a solution of the type:*

$$X_0 = Q_r \cdot W \qquad (6.2.18)$$

where , $W = V \cdot B_0$, $V \in \mathfrak{R}^{\rho_A x \rho_A}$, is an $\mathfrak{R}$ unimodular . The characterization of solutions of (6.2.1) over $\mathfrak{R}$ is given by corollary (6.2.1) .

***ii)** Let P_l be an $\mathfrak{R}$cp of A and assume that a gerd , L_l , of A over $\mathfrak{R}$ is an eld of B over $\mathfrak{R}$, i.e. $B = B_0 \cdot L_r$. Then equation (6.2.1) has a solution of the type :*

$$Y_0 = W \cdot P_l \qquad (6.2.19)$$

where , $W = B_0 \cdot U$, $U \in \mathfrak{R}^{\rho_A x \rho_A}$, is an $\mathfrak{R}$ unimodular . The characterization of solutions of (6.2.1) over $\mathfrak{R}$ is given by corollary (6.2.1) .

Proof

***i*)** Since the geld of A over $\mathfrak{R}$, L_l , is an eld of B over $\mathfrak{R}$ theorem (6.2.2) implies that (6.2.1) has a solution , X_0 , over $\mathfrak{R}$. Furthermore :

$$A \cdot Q_r = L_l' \qquad (6.2.20)$$

with , L_l' being an other geld of A over $\mathfrak{R}$. Since L_l' , L_l are $\mathfrak{R}$ right equivalent , (proposition (5.3.1)) , an $\mathfrak{R}$ unimodular matrix $V \in \mathfrak{R}^{\rho_A x \rho_A}$ exists such that :

$$L_l = L_l' \cdot V \tag{6.2.21}$$

(6.2.20) and (6.2.21) combined together provide :

$$B = L_l \cdot B_0 = L_l' \cdot V \cdot B_0 = A \cdot Q_r \cdot V \cdot B_0 = A \cdot Q_r \cdot W \tag{6.2.22}$$

which clearly implies (6.2.18) .
ii) The proof follows along similar lines . □

The results so far have established conditions under which the matrix equations (6.2.1) , (6.2.2) are solvable over $\mathfrak{R}$. However these conditions are not readily verifiable , i.e. the decomposition of B in (6.2.1) , (6.2.2) as a product of two matrices one of which is a geld , gerd of A respectively can not be easily determined and thus simpler solvability conditions are sought .

Theorem (6.2.3) [Vid. 4] : i) *The equation (6.2.1) has a solution over $\mathfrak{R}$ if the matrices [A , B] and [A , O] are $\mathfrak{R}$ right equivalent .*
ii) *The equation (6.2.2) has a solution over $\mathfrak{R}$ if the matrices $[A^T , B^T]^T$ and $[A^T , O]^T$ are $\mathfrak{R}$ left equivalent .*

Proof
i) **(⇒)** Let a solution X of (6.2.1) over $\mathfrak{R}$ exists . Then :

$$[A , B] = [A , A \cdot X] = [A , O] \cdot \begin{bmatrix} I_m & X \\ O & I_\kappa \end{bmatrix} \tag{6.2.23}$$

which clearly implies that the matrices [A , B] and [A , O] are $\mathfrak{R}$ right equivalent .
(⇐) Let the matrices [A , B] and [A , O] be $\mathfrak{R}$ right equivalent . Then an $\mathfrak{R}$ unimodular matrix $U \in \mathfrak{R}^{(m+\kappa)x(m+\kappa)}$ such that :

$$[A , B] = [A , O] \cdot U = [A , O] \cdot \begin{bmatrix} U_m & U_m^\kappa \\ U_\kappa^m & U_\kappa \end{bmatrix} = [A \cdot U_m , A \cdot U_m^\kappa] \tag{6.2.24}$$

which clearly implies that the matrix $X = U_m^\kappa \in \mathfrak{R}^{m x \kappa}$ is a solution of equation (6.2.1) over $\mathfrak{R}$.

ii) The proof follows along similar lines . □

***Corollary (6.2.3)** : i) Let A_H^c be the column Hermite form of A . Equation (6.2.1) has a solution over $\mathfrak{R}$, if and only if the column Hermite form of $[A , B]$ is $[A_H^c , O]$.*
ii) Let A_H^r be the row Hermite form of A . Equation (6.2.2) has a solution over $\mathfrak{R}$, if and only if the row Hermite form of $[A^T , B^T]^T$ is $[(A_H^r)^T , O]^T$. □

The latter result may be used for the derivation of a more practical way of checking the solvability of equations (6.2.1) , (6.2.2) over $\mathfrak{R}$. Attention is now focused on a more direct approach to solvability of equations (6.2.1) , (6.2.2) over $\mathfrak{R}$, involving the machinery developed in chapter 5 .

***Definition (6.2.1)** : Let $A \in \mathfrak{R}^{pxm}$, $\rho_A = rank_{\mathfrak{F}}\{A\}$. Then :*
i) If $\rho_A = p < m$, then A will be called left regular .
ii) If $\rho_A = m < p$, then A will be called right regular .
iii) If $\rho_A = m = p$, then A will be called regular .
iv) If $\rho_A < min\{p , m\}$, then A will be called irregular . □

***Remark (6.2.2)** : i) If $A \in \mathfrak{R}^{pxm}$ is left regular , then it is a gerd of itself over $\mathfrak{R}$. As a result A can be factorized , (corollary (5.3.1)) , as :*

$$A = T \cdot Q \tag{6.2.25}$$

where , $T \in \mathfrak{R}^{pxp}$ is a glrd of A over $\mathfrak{R}$, $Q \in \mathfrak{R}^{pxm}$, is an $\mathfrak{R}$ right unimodular matrix with an $\mathfrak{R}$ri .
ii) If $A \in \mathfrak{R}^{pxm}$ is right regular , then it is a geld of itself over $\mathfrak{R}$. As a result A can be factorized , (corollary (5.3.1)) , as :

$$A = P \cdot T \tag{6.2.26}$$

where , $T \in \mathfrak{R}^{pxp}$ is a glrd of A over $\mathfrak{R}$, $P \in \mathfrak{R}^{pxm}$, is an $\mathfrak{R}$ left unimodular matrix with an $\mathfrak{R}$li . □

In the following we shall denote by (P_l , N_l) a pair of an $\mathfrak{R}$cp , $\mathfrak{R}$pla of the given matrix A in (6.2.1) , or (6.2.2) ; $L_r = P_l \cdot A$, to be a gerd of A over $\mathfrak{R}$ associated with P_l ; Y_l to be the $\mathfrak{R}$ unimodular matrix $[P_l^T , N_l^T]^T$. We shall also denote (Q_r , N_r) a pair of an $\mathfrak{R}$rp , $\mathfrak{R}$pra of the given matrix A in (6.2.1) , or (6.2.2) ; $L_l = A \cdot Q_r$, to be a geld of A over $\mathfrak{R}$ associated with Q_r ; Y_r to be the $\mathfrak{R}$ unimodular matrix $[Q_r , N_r]$.

***Proposition (6.2.1)** : i) α) Assume that the given matrix A in (6.2.1) is right regular .*

Then (6.2.1) has a solution over $\mathfrak{R}$ *, if and only if :*

$$N_l \cdot B = O \;,\; L_r^{-1} \cdot P_l \cdot B \in \mathfrak{R}^{m x \kappa} \qquad (6.2.27)$$

Furthermore if a solution , X_0 *, of (6.2.1) over* $\mathfrak{R}$ *exists , then it is unique and given by:*

$$X_0 = L_r^{-1} \cdot P_l \cdot B \in \mathfrak{R}^{m x \kappa} \qquad (6.2.28)$$

β) *Assume that the given matrix A in (6.2.1) is left regular . Then (6.2.1) has a solution over* $\mathfrak{R}$ *if and only if :*

$$L_l^{-1} \cdot B \in \mathfrak{R}^{p x \kappa} \qquad (6.2.29)$$

Furthermore if (6.2.29) holds true then the matrix :

$$X_0 = Y_r \cdot \begin{bmatrix} L_l^{-1} \cdot B \\ W \end{bmatrix} \in \mathfrak{R}^{m x \kappa} \;,\; W \text{ arbitrary matrix} \qquad (6.2.30)$$

qualifies as a solution of (6.2.1) over $\mathfrak{R}$ *and the family of solutions of (6.2.1) over* $\mathfrak{R}$ *is given by :*

$$X = X_0 + N_r \cdot K \;,\; K \in \mathfrak{R}^{(m-p) x \kappa} \text{parametric} \qquad (6.2.31)$$

ii) α) *Assume that the given matrix A in (6.2.2) is left regular . Then (6.2.2) has a solution over* $\mathfrak{R}$ *if and only if :*

$$B \cdot N_r = O \;,\; B \cdot Q_r \cdot L_l^{-1} \in \mathfrak{R}^{\kappa x p} \qquad (6.2.32)$$

Furthermore if a solution , Y_0 *, of (6.2.2) over* $\mathfrak{R}$ *exists , then it is unique and given by:*

$$Y_0 = B \cdot Q_r \cdot L_l^{-1} \in \mathfrak{R}^{\kappa x p} \qquad (6.2.33)$$

β) *Assume that the given matrix A in (6.2.2) is right regular . Then (6.2.2) has a solution over* $\mathfrak{R}$ *if and only if :*

$$B \cdot L_r^{-1} \in \mathfrak{R}^{\kappa x m} \qquad (6.2.34)$$

Furthermore if (6.2.34) holds true then the matrix :

$$Y_0 = [\, B \cdot L_r^{-1} \;,\; W \,] \cdot Y_l \in \mathfrak{R}^{\kappa x p} \qquad (6.2.35)$$

qualifies as a solution of (6.2.2) over $\mathfrak{R}$ *and the family of solutions of (6.2.2) over* $\mathfrak{R}$ *is* *given by :*

$$Y = Y_0 + C\cdot N_l \ , \ C\in \mathfrak{R}^{\kappa x(p-m)} \ \textit{parametric} \qquad (6.2.36)$$

Proof

i) α) Assume that the matrix A in equation (6.2.1) is right regular . Then :

($\Rightarrow$) Let (6.2.1) have a solution $X_0 \in \mathfrak{R}^{mx\kappa}$ over $\mathfrak{R}$. Then the equivalent equation :

$$Y_l\cdot A\cdot X = Y_l\cdot B \qquad (6.2.37)$$

has X_0 as a solution over $\mathfrak{R}$ as well . If we perform the multiplications , (6.2.37) can be rewritten as :

$$\begin{bmatrix} L_r \\ O \end{bmatrix}\cdot X_0 = \begin{bmatrix} P_l\cdot B \\ N_l\cdot B \end{bmatrix} \qquad (6.2.38)$$

which clearly implies :

$$N_l\cdot B = O \ , \ L_r^{-1}\cdot P_l\cdot B \in \mathfrak{R}^{mx\kappa} \qquad (6.2.39)$$

should hold true . (6.2.38) also implies that $X_0 = L_r^{-1}\cdot P_l\cdot B \in \mathfrak{R}^{mx\kappa}$ and it is a unique solution of (6.2.1) over $\mathfrak{R}$ since the right null space of A is trivial , (A is right regular) .

($\Leftarrow$) Assume that the following holds true :

$$N_l\cdot B = O \ , \ L_r^{-1}\cdot P_l\cdot B \in \mathfrak{R}^{mx\kappa} \qquad (6.2.40)$$

then ,

$$P_l\cdot A = L_r \Leftrightarrow L_r^{-1}\cdot P_l\cdot A\cdot = I_m \overset{\mathcal{N}_r\{A\}=\underline{0}}{\Leftrightarrow} A\cdot L_r^{-1}\cdot P_l\cdot A = A \qquad (6.2.41)$$

The latter clearly implies that the rows of the matrix $(A\cdot L_r^{-1}\cdot P_l - I_p)$ belong to the left null space of A . Since N_l is a base for $\mathcal{N}_l\{A\}$ a matrix $D \in \mathfrak{F}^{px(p-m)}$ exists such that :

$$(A\cdot L_r^{-1}\cdot P_l - I_p) = D\cdot N_l \Leftrightarrow A\cdot L_r^{-1}\cdot P_l = I_p + D\cdot N_l \qquad (6.2.42)$$

Set now $X_0 \in \mathfrak{R}^{mx\kappa}$ to be the matrix $L_r^{-1}\cdot P_l\cdot B$. Then by (6.2.40) , (6.2.42) the following holds true :

$$A\cdot X_0 = A\cdot L_r^{-1}\cdot P_l\cdot B = (I_p + D\cdot N_l)\cdot B = B \qquad (6.2.43)$$

Thus (6.2.1) has a solution over $\mathfrak{R}$, $X_0 = L_r^{-1}\cdot P_l\cdot B \in \mathfrak{R}^{mx\kappa}$ is such a solution and it is unique since the right null space of A is trivial , (A is right regular) .

β) Assume that the given matrix A in equation (6.2.1) is left regular . Then :

($\Rightarrow$) Let (6.2.1) have a solution $X_0 \in \mathfrak{R}^{mx\kappa}$ over $\mathfrak{R}$. Then X_0 is a solution over $\mathfrak{R}$ of the equivalent equation :

$$A \cdot Y_r \cdot Y_r^{-1} \cdot X = B \Leftrightarrow [L_l\ ,\ O] \cdot G = B \tag{6.2.44}$$

where , $G = Y_r^{-1} \cdot X$. Let $G_0 = Y_r^{-1} \cdot X_0 \in \mathfrak{R}^{m x \kappa}$, then G_0 satisfies (6.2.44) and partition G_0 as :

$$G_0 = \begin{bmatrix} G_{01} \\ G_{02} \end{bmatrix},\ G_{01} \in \mathfrak{R}^{p x \kappa},\ G_{02} \in \mathfrak{R}^{(m-p) x \kappa} \tag{6.2.45}$$

Finally (6.2.44) , (6.2.45) implies that the following relation should hold :

$$L_l \cdot G_{01} = B \Leftrightarrow L_l^{-1} \cdot B \in \mathfrak{R}^{p x \kappa} \tag{6.2.46}$$

($\Leftarrow$) Assume that the following holds true :

$$L_l^{-1} \cdot B \in \mathfrak{R}^{p x \kappa} \tag{6.2.47}$$

Then set $X_0 \in \mathfrak{R}^{m x \kappa}$ to be the matrix :

$$X_0 = Y_r \cdot \begin{bmatrix} L_l^{-1} \cdot B \\ W \end{bmatrix} \in \mathfrak{R}^{m x \kappa}\ ,\ \text{W arbitrary matrix} \tag{6.2.48}$$

Then it is trivial to verify that X_0 is a solution of (6.2.1) over $\mathfrak{R}$, i.e. :

$$A \cdot X_0 = A \cdot Y_r \cdot \begin{bmatrix} L_l^{-1} \cdot B \\ W \end{bmatrix} = [\ L_l\ ,\ O\ ] \cdot \begin{bmatrix} L_l^{-1} \cdot B \\ W \end{bmatrix} = B \tag{6.2.49}$$

If (6.2.29) holds true and since the right null space of A is not trivial corollary (6.2.1) implies that the family of solutions of (6.2.1) over $\mathfrak{R}$ is given by :

$$X = X_0 + N_r \cdot K\ ,\ K \in \mathfrak{R}^{(m-p) x \kappa} \text{parametric} \tag{6.2.50}$$

ii) The proof follows along similar lines . □

***Remark (6.2.3)** : If the given matrix A in (6.2.1) , (6.2.2) is regular then it is clear that these equations have a solution over* $\mathfrak{R}$ *, if and only if the matrices* $A^{-1} \cdot B \in \mathfrak{R}^{p x p}$, $B \cdot A^{-1} \in \mathfrak{R}^{p x p}$ *respectively . If the latter holds true then equations (6.2.1) , (6.2.2) have the unique solutions over* $\mathfrak{R}$, $X = A^{-1} \cdot B$, $Y = B \cdot A^{-1}$. □

So far we have studied a more practical approach for solvability and characterization of solutions (6.2.1) , (6.2.2) when A is either left regular , right regular or regular . In the following we do so in the more general case when A is irregular .

***Proposition (6.2.2)** : Let $A \in \mathcal{R}^{p\times m}$, $\rho_A = rank_{\mathcal{F}}\{A\} < min\{p , m\}$. Then :*

i) *Equation (6.2.1) has a solution over* $\mathcal{R}$ *, if and only if :*

α)

$$N_l \cdot B = O \qquad (6.2.51)$$

and equation ,

$$L_r \cdot X = P_l \cdot B \qquad (6.2.52)$$

is solvable over $\mathcal{R}$ *. Or equivalently ,*

β) *Equation ,*

$$L_l \cdot W = B \qquad (6.2.53)$$

has a solution over $\mathcal{R}$ *, where* $W = [I_{\rho_A} , O] \cdot Y_r^{-1} \cdot X$ *.*

c) *The family of solutions of (6.2.1) over* $\mathcal{R}$ *is given by the family of solutions of equation (6.2.52) , or equivalently ,*

d) *The family of solutions of (6.2.1) over* $\mathcal{R}$ *is given by :*

$$X = Y_r \cdot \begin{bmatrix} W \\ R \end{bmatrix} \qquad (6.2.54)$$

where W is an arbitrary solution of (6.2.53) over $\mathcal{R}$ *, R is an arbitrary parametric matrix over* $\mathcal{R}$ *.*

ii) *Equation (6.2.2) has a solution over* $\mathcal{R}$ *if and only if :*

α)

$$B \cdot N_r = O \qquad (6.2.55)$$

and equation ,

$$Y \cdot L_l = B \cdot Q_r \qquad (6.2.56)$$

is solvable over $\mathcal{R}$ *. Or equivalently ,*

β) *Equation ,*

$$W \cdot L_r = B \qquad (6.2.57)$$

has a solution over $\mathcal{R}$ *, where* $W = Y \cdot Y_l^{-1} \cdot [I_{\rho_A}^T , O]^T$ *.*

c) *The family of solutions of (6.2.2) over* $\mathcal{R}$ *is given by the family of solutions of equation (6.2.56) , or equivalently ,*

d) *The family of solutions of (6.2.2) over* $\mathcal{R}$ *is given by :*

$$Y = [W , R] \cdot Y_l \qquad (6.2.58)$$

where W is an arbitrary solution of (6.2.57) over $\mathcal{R}$ *, R is an arbitrary parametric matrix over* $\mathcal{R}$ *.*

Proof

i) α) ($\Rightarrow$) If (6.2.1) has a solution , X_0 , over $\mathfrak{R}$ then so does the equivalent equation :

$$Y_l \cdot A \cdot X = Y_l \cdot B \tag{6.2.59}$$

Thus ,

$$Y_l \cdot A \cdot X_0 = Y_l \cdot B \Leftrightarrow \begin{bmatrix} L_r \\ O \end{bmatrix} \cdot X_0 = \begin{bmatrix} P_l \cdot B \\ N_l \cdot B \end{bmatrix} \tag{6.2.60}$$

which clearly implies that (6.2.51) holds true and equation (6.2.52) is solvable over $\mathfrak{R}$.
($\Leftarrow$) Let (6.2.51) hold true and equation (6.2.52) be solvable over $\mathfrak{R}$. Then a matrix X_0 over $\mathfrak{R}$ exists such that :

$$\begin{bmatrix} L_r \\ O \end{bmatrix} \cdot X_0 = \begin{bmatrix} P_l \cdot B \\ O \end{bmatrix} = \begin{bmatrix} P_l \cdot B \\ N_l \cdot B \end{bmatrix} \Leftrightarrow Y_l \cdot A \cdot X_0 = Y_l \cdot B \tag{6.2.61}$$

which implies that (6.2.1) is solvable over $\mathfrak{R}$.
β) ($\Rightarrow$) If (6.2.1) has a solution , X_0 , over $\mathfrak{R}$ then so does the equivalent equation :

$$A \cdot Y_r \cdot Y_r^{-1} \cdot X = B \tag{6.2.62}$$

Thus if K is set to be the matrix $Y_r^{-1} \cdot X$, $K_0 = Y_r^{-1} \cdot X_0 \in \mathfrak{R}^{m x \kappa}$, (6.2.62) implies that

$$[\, L_l \, , \, O \,] \cdot K_0 = B \Leftrightarrow L_l \cdot W_0 = B \tag{6.2.63}$$

which implies that equation (6.2.53) is solvable over $\mathfrak{R}$.
($\Leftarrow$) Let equation (6.2.53) have a solution , W_0 , over $\mathfrak{R}$. Then set X_0 to be the matrix :

$$X_0 = Y_r \cdot \begin{bmatrix} W_0 \\ R \end{bmatrix} \in \mathfrak{R}^{m x \kappa} \tag{6.2.64}$$

R an arbitrary parametric matrix over $\mathfrak{R}$. Then ,

$$A \cdot X_0 = A \cdot Y_r \cdot \begin{bmatrix} W_0 \\ R \end{bmatrix} = [\, L_l \, , \, O \,] \cdot \begin{bmatrix} W_0 \\ R \end{bmatrix} = B \tag{6.2.65}$$

which implies that (6.2.1) is solvable over $\mathfrak{R}$.
c) , *d*) The proves are straightforward from the analysis in α) , β) .
ii) The proof follows along similar lines . □

Remark (6.2.3) : *Proposition (6.2.2) implies that the solvability of (6.2.1) , (6.2.2) over $\mathfrak{R}$ can be reduced to the solvability of equations of the same type but with left , or right regular matrices instead of an irregular A . (6.2.52) , (6.2.53) , (6.2.56) , (6.2.57) are solved in the way introduced by proposition (6.2.1) .* □

We conclude this section by examining the solvability over $\mathfrak{R}$ of the more general matrix Diophantine equations (6.1.5) , (6.1.6) . If (D , N) , (D′ , N′) denote an $\mathfrak{R}$ – coprime left , right MFD of the matrices $M = [\, Z_1 , \dots , Z_h , E \,]$, $M' = [Z_1^T , \dots , Z_h^T$ $E^T \,]^T$ respectively , ($D \cdot M = [A_1 , \dots , A_h , B] = N$, $M' \cdot D' = [A_1^T , \dots , A_h^T , B^T]^T = N'$) , then N , N′ are an $\mathfrak{R}$lmr , $\mathfrak{R}$lmc of M , M′ respectively and (6.1.5) , (6.1.6) equations can be equivalently transformed to :

$$A_1 \cdot X_1 + \cdots + A_h \cdot X_h = B \ , \ Z_i \in \mathfrak{R}^{p x m_i}, \ B \in \mathfrak{R}^{p x t}, \ X_i \in \mathfrak{R}^{m_i x t} \tag{6.2.66}$$

$$Y_1 \cdot A_1 + \cdots + X_h \cdot A_h = B \ , \ A_i \in \mathfrak{R}^{p_i x m}, \ B \in \mathfrak{R}^{t x m}, \ Y_i \in \mathfrak{R}^{t x p_i} \tag{6.2.67}$$

The results introduced in proposition (6.2.2) are used in the following analysis :

Proposition (6.2.3) : ***i***) *Equation (6.2.66) is solvable over $\mathfrak{R}$, if and only if equation :*

$$A \cdot X = B \Leftrightarrow [A_1 , \dots , A_h] \cdot [X_1^T , \dots , X_h^T]^T = B \tag{6.2.68}$$

is solvable over $\mathfrak{R}$. The family of solutions of (6.2.66) over $\mathfrak{R}$ is the family of solutions of (6.2.68) over $\mathfrak{R}$.
ii) *Equation (6.2.67) is solvable over $\mathfrak{R}$, if and only if equation :*

$$Y \cdot A = B \Leftrightarrow [Y_1 , \dots , Y_h] \cdot [A_1^T , \dots , A_h^T]^T = B \tag{6.2.69}$$

is solvable over $\mathfrak{R}$. The family of solutions of (6.2.67) over $\mathfrak{R}$ is the family of solutions of (6.2.69) over $\mathfrak{R}$. □

6.3. STUDY OF THE MATRIX EQUATION Z·X·E = H OVER THE PID $\mathfrak{R}$

The matrix equation (6.1.3) is central in the formulation and solvability of control synthesis problems such as the disturbance decoupling and noninteracting control problems with or without the internal stability requirement for the feedback system . This equation is also important in the study of solvability of (6.1.4) over $\mathcal{R}$. In the following we consider equation :

$$Z \cdot X \cdot E = H \ , \ Z \in \mathcal{F}^{p x m} \ , \ E \in \mathcal{F}^{\kappa x t} \ , \ H \in \mathcal{F}^{p x t} \ , \ X \in \mathcal{R}^{m x \kappa}$$

If (D , N) , (D$'$, B) denote an $\mathcal{R}$ – coprime left , right MFD of the matrices M = [Z , H] , E respectively , ($D \cdot M = [A \ , \ \Gamma] = N$, $E \cdot D' = B$) , then N , B are an $\mathcal{R}$lmr, $\mathcal{R}$lmc of M , E respectively and equation (6.1.3) can be equivalently transformed to :

$$A \cdot X \cdot B = C \ , \ A \in \mathcal{R}^{p x m} \ , \ B \in \mathcal{R}^{\kappa x t} \ , \ C \in \mathcal{R}^{p x t} \ , \ X \in \mathcal{R}^{m x \kappa} \tag{6.3.1}$$

Thus we can implement the algebraic tools developed in chapter 5 to achieve solvability and characterization of solutions of (6.1.3) via (6.2.1) . In the following we associate the matrices A , B in (6.3.1) with the well known algebraic machinery established in chapter 5 . Let (P_l^a , N_l^a) , (Q_r^a , N_r^a) denote two pairs of an ($\mathcal{R}$cp , $\mathcal{R}$pla) , ($\mathcal{R}$rp , $\mathcal{R}$pra) of A respectively ; (P_l^b , N_l^b) , (Q_r^b , N_r^b) denote two pairs of an ($\mathcal{R}$cp , $\mathcal{R}$pla) , ($\mathcal{R}$rp , $\mathcal{R}$pra) of B respectively . Also let :

$$\begin{cases} Y_l^a = [\ (P_l^a)^T \ , \ (N_l^a)^T \ ]^T \ , \ Y_r^a = [\ Q_r^a \ , \ N_r^a \ ] \\ Y_l^b = [\ (P_l^b)^T \ , \ (N_l^b)^T \ ]^T \ , \ Y_r^b = [\ Q_r^b \ , \ N_r^b \ ] \end{cases}$$

be the unimodular matrices associated with the pairs of ($\mathcal{R}$cps , $\mathcal{R}$plas) , ($\mathcal{R}$rps , $\mathcal{R}$pras) of A , B respectively . If A , B are represented as :

$$A = P_a \cdot T_a \cdot Q_a \ , \ B = P_b \cdot T_b \cdot Q_b$$

where T_a , T_b are a glrd of A , B over $\mathcal{R}$ respectively , then we denote by :

$$L_r^a = T_a \cdot Q_a = P_l^a \cdot A \ , \ L_l^a = P_a \cdot T_a = A \cdot Q_r^a$$

a pair of a (gerd , geld) of A over $\mathcal{R}$;

$$L_r^b = T_b \cdot Q_b = P_l^b \cdot B \ , \ L_l^b = P_b \cdot T_b = B \cdot Q_r^b$$

a pair of a (gerd , geld) of B over $\mathcal{R}$.

Proposition (6.3.1) *: i) If the matrix A in equation (6.3.1) is left regular then (6.3.1) has a solution over* $\Re$ *, if and only if the equation :*

$$Y \cdot B = (L_l^a)^{-1} \cdot C \tag{6.3.2}$$

is solvable over $\Re$ *. If (6.3.1) is solvable over* $\Re$ *then the family of solutions of (6.3.1) over* $\Re$ *is given by :*

$$X = Y_r^a \cdot \begin{bmatrix} Y \\ R \end{bmatrix} \in \Re^{m x \kappa} \tag{6.3.3}$$

where , Y is an arbitrary solution of (6.3.2) over $\Re$ *and R is an arbitrary parametric matrix over* $\Re$ *.*

ii) *If the matrix A in (6.3.1) is right regular then (6.3.1) has a solution over* $\Re$ *, if and only if :*

$$N_l^a \cdot C = O \tag{6.3.4}$$

and the equation :

$$X \cdot B = (L_r^a)^{-1} \cdot P_l^a \cdot C \tag{6.3.5}$$

is solvable over $\Re$ *. If (6.3.1) is solvable over* $\Re$ *, then the family of solutions of (6.3.1) over* $\Re$ *is given by the family of solutions of (6.3.5) over* $\Re$ *.*

iii) *If the matrix B in (6.3.1) is left regular then (6.3.1) has a solution over* $\Re$ *, if and only if :*

$$C \cdot N_r^b = O \tag{6.3.6}$$

and the equation :

$$A \cdot X = C \cdot Q_r^b \cdot (L_l^b)^{-1} \tag{6.3.7}$$

is solvable over $\Re$ *. If (6.3.1) is solvable over* $\Re$ *, then the family of solutions of (6.3.1) over* $\Re$ *is given by the family of solutions of (6.3.7) over* $\Re$ *.*

iv) *If the matrix B in equation (6.3.1) is right regular then (6.3.1) has a solution over* $\Re$ *if and only if the equation :*

$$A \cdot Y = C \cdot (L_r^b)^{-1} \tag{6.3.8}$$

is solvable over $\Re$ *. If (6.3.1) is solvable over* $\Re$ *then the family of solutions of (6.3.1) over* $\Re$ *is given by :*

$$X = [\ Y\ ,\ R\ ] \cdot Y_l^b \in \Re^{m x \kappa} \tag{6.3.9}$$

where , Y is an arbitrary solution of (6.3.8) over $\Re$ *and R is an arbitrary parametric matrix over* $\Re$ *.*

Proof

i) (⇒) Let equation (6.3.1) have a solution , X_0 , over $\mathfrak{R}$, then :

$$A \cdot X_0 \cdot B = A \cdot Y_r^a \cdot (Y_r^a)^{-1} \cdot X_0 \cdot B = [L_l^a \ , O] \cdot W_0 \cdot B = C \tag{6.3.10}$$

where , $W_0 = (Y_r^a)^{-1} \cdot X_0 = [Y_0^T \ , R_0^T]^T \in \mathfrak{R}^{m x \kappa}$. Since A is left regular , then $L_l^a \in \mathfrak{R}^{p x p}$ and $rank_{\mathfrak{F}}\{L_l^a\} = p$. Thus (6.3.10) implies that :

$$L_l^a \cdot Y_0 \cdot B = C \Leftrightarrow Y_0 \cdot B = (L_l^a)^{-1} \cdot C \tag{6.3.11}$$

and is clear that equation (6.3.2) is solvable over $\mathfrak{R}$.

(⇐) Assume that (6.3.2) has a solution , Y_0 , over $\mathfrak{R}$. Then set $X_0 \in \mathfrak{R}^{m x \kappa}$ to the matrix :

$$X_0 = Y_r^a \cdot W_0 = Y_r^a \cdot [Y_0^T \ , R_0^T]^T \tag{6.3.12}$$

where , R_0 is a parametric matrix over $\mathfrak{R}$. Then ,

$$A \cdot X_0 \cdot B = [L_l^a \ , O] \cdot W_0 \cdot B = L_l^a \cdot Y_0 \cdot B = C \tag{6.3.13}$$

which implies that (6.3.1) is solvable over $\mathfrak{R}$. Furthermore , if (6.3.1) is solvable over $\mathfrak{R}$ and X is any solution of it over $\mathfrak{R}$ then (6.3.10) implies that there always exists a corresponding matrix $W = (Y_r^a)^{-1} \cdot X = [Y^T \ , R^T]^T \in \mathfrak{R}^{m x \kappa}$, with Y a solution of (6.3.2) and R a matrix over $\mathfrak{R}$ respectively and thus (6.3.3) holds true for all the solutions of (6.3.1) over $\mathfrak{R}$.

ii) (⇒) Let (6.3.1) have a solution , X_0 , over $\mathfrak{R}$. Then X_0 is a solution over $\mathfrak{R}$ of the equivalent equation :

$$Y_l^a \cdot A \cdot X \cdot B = Y_l^a \cdot C \tag{6.3.14}$$

Thus ,

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot B = \begin{bmatrix} P_l^a \cdot C \\ N_l^a \cdot C \end{bmatrix} \tag{6.3.15}$$

which clearly implies that :

$$N_l^a \cdot C = O \tag{6.3.16}$$

and since A is right regular , $L_r^a \in \mathfrak{R}^{m x m}$ and $rank_{\mathfrak{F}}\{L_r^a\} = m$, and the equation :

$$X_0 \cdot B = (L_r^a)^{-1} \cdot P_l^a \cdot C \tag{6.3.17}$$

is solvable over $\mathfrak{R}$.

(⇐) Assume that (6.3.4) holds true and equation (6.3.5) is solvable over $\mathfrak{R}$. Then it is obvious that if X_0 is a solution of (6.3.5) then ,

$$\begin{cases} X_0 \cdot B = (L_r^a)^{-1} \cdot P_l^a \cdot C \Leftrightarrow L_r^a \cdot X_0 \cdot B = P_l^a \cdot C & (6.3.18) \\ N_l^a \cdot C = O & (6.3.19) \end{cases}$$

or equivalently ,

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot B = \begin{bmatrix} P_l^a \cdot C \\ N_l^a \cdot C \end{bmatrix} \Leftrightarrow Y_l^a \cdot A \cdot X_0 \cdot B = Y_l^a \cdot C \qquad (6.3.20)$$

which clearly implies that equation (6.3.1) is solvable over $\mathfrak{R}$. Furthermore it is obvious from the above analysis that the family of solutions of (6.3.1) over $\mathfrak{R}$ is given by the family of solutions of (6.3.5) over $\mathfrak{R}$.
***iii*) , *iv*)** The proof follows along similar lines . □

Remark (6.3.1) : ***i*)** *If the matrix A in (6.3.1) is regular and (D', N') is an $\mathfrak{R}$ – coprime right MFD of $M' = [B^T, (A^{-1} \cdot C)^T]^T$, $(M' \cdot D' = [\Delta^T, \Theta^T]^T = N')$, then N' is an $\mathfrak{R}$lmc of M' and it is obvious that (6.3.1) is solvable over $\mathfrak{R}$, if and only if the equation :*

$$X \cdot \Delta = \Theta \qquad (6.3.21)$$

is solvable over $\mathfrak{R}$. Furthermore , the family of solutions of (6.3.1) over $\mathfrak{R}$ is given by the family of solutions of (6.3.21) over $\mathfrak{R}$.
***ii*)** *If the matrix B in (6.3.1) is regular and (D , N) is an $\mathfrak{R}$ – coprime left MFD of M = = $[A , C \cdot B^{-1}]$, $(D \cdot M = [\Delta , \Theta] = N)$, then N is an $\mathfrak{R}$lmr of M and it is obvious that (6.3.1) is solvable over $\mathfrak{R}$ if and only if the equation :*

$$\Delta \cdot X = \Theta \qquad (6.3.22)$$

is solvable over $\mathfrak{R}$. Furthermore , the family of solutions of (6.3.1) over $\mathfrak{R}$ is given by the family of solutions of (6.3.22) over $\mathfrak{R}$. □

Remark (6.3.2) : *Proposition (6.3.1) and remark (6.3.1) suggest that if either of the matrices A , B appearing in (6.3.1) are left , right regular , or regular the solvability of (6.3.1) over $\mathfrak{R}$ can be reduced to the solvability over $\mathfrak{R}$ of matrix equations of the type (6.2.1) , (6.2.2) .* □

The next result deals with the existence and characterization of solutions of (6.3.1) over $\mathfrak{R}$ when both A , B are irregular matrices .

***Proposition** (6.3.3) : The following statements are equivalent . Equation (6.3.1) is solvable over* $\mathfrak{R}$ *, if and only if :*

i)

$$N_l^a \cdot C = O \tag{6.3.23}$$

and the equation :

$$L_r^a \cdot X \cdot B = P_l^a \cdot C \tag{6.3.24}$$

is solvable over $\mathfrak{R}$ *. The family of solutions over* $\mathfrak{R}$ *of (6.3.1) is given by the family of solutions over* $\mathfrak{R}$ *of (6.3.24) .*

ii)

$$L_l^a \cdot Y \cdot B = C \tag{6.3.25}$$

is solvable over $\mathfrak{R}$ *. If (6.3.1) is solvable over* $\mathfrak{R}$ *then the family of solutions of (6.3.1) over* $\mathfrak{R}$ *is given by :*

$$X = Y_r^a \cdot \begin{bmatrix} Y \\ R \end{bmatrix} \in \mathfrak{R}^{m x \kappa} \tag{6.3.26}$$

where Y is any solution of (6.3.25) over $\mathfrak{R}$ *, R is a parametric matrix over* $\mathfrak{R}$ *.*

iii)

$$C \cdot N_r^b = O \tag{6.3.27}$$

and the equation :

$$A \cdot X \cdot L_l^b = C \cdot Q_r^b \cdot \tag{6.3.28}$$

is solvable over $\mathfrak{R}$ *. If (6.3.1) is solvable over* $\mathfrak{R}$ *, then the family of solutions of (6.3.1) over* $\mathfrak{R}$ *is given by the family of solutions of (6.3.28) over* $\mathfrak{R}$ *.*

iv)

$$A \cdot Y \cdot L_r^b = C \tag{6.3.29}$$

is solvable over $\mathfrak{R}$ *. If (6.3.1) is solvable over* $\mathfrak{R}$ *then the family of solutions of (6.3.1) over* $\mathfrak{R}$ *is given by :*

$$X = [\, Y \,,\, R \,] \cdot Y_l^b \in \mathfrak{R}^{m x \kappa} \tag{6.3.30}$$

where , Y is an arbitrary solution of (6.3.29) over $\mathfrak{R}$ *and R is an arbitrary parametric matrix over* $\mathfrak{R}$ *.*

Proof

i) (⇒) Let (6.3.1) have a solution , X_0 , over $\mathfrak{R}$. Then X_0 is a solution over $\mathfrak{R}$ of the equivalent equation :

$$Y_l^a \cdot A \cdot X \cdot B = Y_l^a \cdot C \tag{6.3.31}$$

Thus ,

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot B = \begin{bmatrix} P_l^a \cdot C \\ N_l^a \cdot C \end{bmatrix} \tag{6.3.32}$$

which clearly implies that :

$$N_l^a \cdot C = O \tag{6.3.33}$$

and the equation :

$$L_r^a \cdot X_0 \cdot B = P_l^a \cdot C \tag{6.3.34}$$

is solvable over $\mathfrak{R}$.

(⇐) Assume that (6.3.23) holds true and equation (6.3.24) is solvable over $\mathfrak{R}$. Then it is obvious that if X_0 is a solution of (6.3.24) then ,

$$\begin{cases} L_r^a \cdot X_0 \cdot B = P_l^a \cdot C & (6.3.35) \\ N_l^a \cdot C = O & (6.3.36) \end{cases}$$

or equivalently ,

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot B = \begin{bmatrix} P_l^a \cdot C \\ N_l^a \cdot C \end{bmatrix} \Leftrightarrow Y_l^a \cdot A \cdot X_0 \cdot B = Y_l^a \cdot C \tag{6.3.37}$$

which clearly implies that equation (6.3.1) is solvable over $\mathfrak{R}$. Furthermore it is obvious from the above analysis that the family of solutions of (6.3.1) over $\mathfrak{R}$ is given by the family of solutions of (6.3.24) over $\mathfrak{R}$.

ii) **(⇒)** **Let equation (6.3.1) have a solution , X_0 , over $\mathfrak{R}$, then :**

$$A \cdot X_0 \cdot B = A \cdot Y_r^a \cdot (Y_r^a)^{-1} \cdot X_0 \cdot B = [L_l^a \ , O] \cdot W_0 \cdot B = C \tag{6.3.38}$$

where , $W_0 = (Y_r^a)^{-1} \cdot X_0 = [Y_0^T \ , R_0^T]^T \in \mathfrak{R}^{m x \kappa}$. Thus (6.3.38) implies that :

$$L_l^a \cdot Y_0 \cdot B = C \tag{6.3.39}$$

and is clear that equation (6.3.25) is solvable over $\mathfrak{R}$.

(⇐) Assume that (6.3.25) has a solution , Y_0 , over $\mathfrak{R}$. Then , set $X_0 \in \mathfrak{R}^{m x \kappa}$ to the matrix :

$$X_0 = Y_r^a \cdot W_0 = Y_r^a \cdot [Y_0^T \ , R_0^T]^T \tag{6.3.40}$$

where , R_0 is a parametric matrix over $\mathfrak{R}$. Then ,

$$A \cdot X_0 \cdot B = [L_l^a \ , O] \cdot W_0 \cdot B = L_l^a \cdot Y_0 \cdot B = C \tag{6.3.41}$$

which implies that (6.3.1) is solvable over $\mathfrak{R}$. Furthermore , if (6.3.1) is solvable over $\mathfrak{R}$ and X is any solution of it over $\mathfrak{R}$, then (6.3.38) implies that there always exists a corresponding matrix $W = (Y_r^a)^{-1} \cdot X = [Y^T , R^T]^T \in \mathfrak{R}^{m x \kappa}$, with Y a solution of (6.3.25) and R a matrix over $\mathfrak{R}$ respectively and thus (6.3.26) holds true for all the solutions of (6.3.1) over $\mathfrak{R}$.

iii) , ***iv)*** The proof follows along similar lines .

The equivalence between *i)* , *ii)* , *iii)* , *vi)* is obvious since all of them result to solvability of (6.3.1) over $\mathfrak{R}$ and vice versa . □

So far our approach to solvability of (6.3.1) over $\mathfrak{R}$ has been entirely based on the reduction of (6.3.1) to matrix equations of the type (6.2.1) , (6.2.2) . In the following we present a more direct approach for the study of (6.3.1) over $\mathfrak{R}$, avoiding the intermediate equations (6.2.1) , (6.2.2) . This method will be proved useful later on when we shall study equation (6.1.4) over $\mathfrak{R}$.

Proposition (6.3.3) : ***i)*** *Equation :*

$$A \cdot X \cdot B = O \qquad (6.3.43)$$

is solvable over $\mathfrak{R}$, (has a nontrivial solution over $\mathfrak{R}$) , if and only if :

$$\rho_A = rank_{\mathfrak{F}}\{A\} < m \text{ , or } \rho_B = rank_{\mathfrak{F}}\{B\} < \kappa \text{ , or both hold true} \qquad (6.3.44)$$

Furthermore the family of all solutions of (6.3.43) over $\mathfrak{R}$ is given by :

$$X = Y_r^a \cdot \begin{bmatrix} O & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \cdot Y_l^b \qquad (6.3.45)$$

where , $Y_2 \in \mathfrak{R}^{\rho_A x(\kappa-\rho_B)}$, $Y_3 \in \mathfrak{R}^{(m-\rho_A)x\rho_B}$, $Y_4 \in \mathfrak{R}^{(m-\rho_A)x(\kappa-\rho_B)}$ are arbitrary parametric matrices .

ii) *Equation (6.3.1) is solvable over $\mathfrak{R}$, if and only if :*

$$\begin{cases} P_l^a \cdot C \cdot N_r^b = O \text{ , } N_l^a \cdot C \cdot Q_r^b = O \text{ , } N_l^a \cdot C \cdot N_r^b = O & (6.3.46) \\ T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \in \mathfrak{R}^{\rho_A x \rho_B} & (6.3.47) \end{cases}$$

If a solution of (6.3.1) over $\mathfrak{R}$ exists then the matrix :

$$X_0 = (Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1} \qquad (6.3.48)$$

with $(Q_a^r)^{-1}$, $(P_b^l)^{-1}$ an $\mathfrak{R}ri$, $\mathfrak{R}li$ of Q_a, P_b respectively, is a solution of (6.3.1) over $\mathfrak{R}$ and the family of solutions of (6.3.1) over $\mathfrak{R}$ is given by :

$$X = X_0 + Y_r^a \cdot \begin{bmatrix} O & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \cdot Y_l^b \tag{6.3.49}$$

where , $Y_2 \in \mathfrak{R}^{\rho_A x(\kappa-\rho_B)}$, $Y_3 \in \mathfrak{R}^{(m-\rho_A)x\rho_B}$, $Y_4 \in \mathfrak{R}^{(m-\rho_A)x(\kappa-\rho_B)}$ are arbitrary parametric matrices .

Proof

i) (⇒) Let equation (6.3.43) have a non trivial solution , X_0 , over $\mathfrak{R}$. Then :

$$Y_l^a \cdot A \cdot X_0 \cdot B \cdot Y_r^b = \begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot [\, L_l^b \, , O \,] = \begin{bmatrix} L_r^a \cdot X_0 \cdot L_l^b & O \\ O & O \end{bmatrix} = O \tag{6.3.50}$$

which clearly implies that equation :

$$L_r^a \cdot X \cdot L_l^b = O \; , \; L_r^a \in \mathfrak{R}^{\rho_A x m} , \; L_l^b \in \mathfrak{R}^{\kappa x \rho_B} \tag{6.3.51}$$

has a nontrivial solution , X_0 , over $\mathfrak{R}$. Since $\rho_A = rank_{\mathfrak{F}}\{L_r^a\}$ and $\rho_B = rank_{\mathfrak{F}}\{L_l^b\}$, if both $\rho_A = rank_{\mathfrak{F}}\{A\} = m$ and $\rho_B = rank_{\mathfrak{F}}\{B\} = \kappa$, then L_r^a , L_l^b would be invertible and (6.3.51) would have only trivial solutions , ($X_0 = O$) , something that contradicts the truth . Thus (6.3.51) implies that (6.3.44) holds true .

(⇐) Assume that $\rho_A = rank_{\mathfrak{F}}\{A\} < m$, or $\rho_B = rank_{\mathfrak{F}}\{B\} < \kappa$, or both relations hold true true . Then set $X_0 \in \mathfrak{R}^{m x \kappa}$ to be the matrix :

$$X_0 = Y_r^a \cdot \begin{bmatrix} O & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \cdot Y_l^b \in \mathfrak{R}^{m x \kappa} \tag{6.3.52}$$

for some nontrivial matrices $Y_2 \in \mathfrak{R}^{\rho_A x(\kappa-\rho_B)}$, $Y_3 \in \mathfrak{R}^{(m-\rho_A)x\rho_B}$, $Y_4 \in \mathfrak{R}^{(m-\rho_A)x(\kappa-\rho_B)}$, (such matrices exist because (6.3.44) holds true) . Then :

$$A \cdot X_0 \cdot B = [\, L_l^a \, , O \,] \cdot X_0 \cdot \begin{bmatrix} L_r^b \\ O \end{bmatrix} = [\, L_l^a \, , O \,] \cdot \begin{bmatrix} O & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \cdot \begin{bmatrix} L_r^b \\ O \end{bmatrix} = O \tag{6.3.53}$$

and the latter clearly implies that equation (6.3.43) has a nontrivial solution , X_0 , over $\mathfrak{R}$. Furthermore , if X is any nontrivial solution of (6.3.43) over $\mathfrak{R}$ then :

$$A\cdot Y_r^a\cdot(Y_r^a)^{-1}\cdot X\cdot(Y_l^b)^{-1}\cdot Y_l^b\cdot B = [\ L_l^a\ ,\ O\ ]\ \cdot Y\cdot\begin{bmatrix} L_r^b \\ O \end{bmatrix} = O \qquad (6.3.54)$$

where , $Y = (Y_r^a)^{-1}\cdot X\cdot(Y_l^b)^{-1} \in \mathfrak{R}^{m x \kappa}$. If the matrix Y is partitioned as :

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \qquad (6.3.55)$$

then (6.3.54) implies :

$$A\cdot X\cdot B = [\ L_l^a\ ,\ O\ ]\ \cdot\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\cdot\begin{bmatrix} L_r^b \\ O \end{bmatrix} = L_l^a\cdot Y_1\cdot L_r^b = O \qquad (6.3.56)$$

Since the right null space of L_l^a , and the left null space of L_r^b are trivial (6.3.56) implies that $Y_1 = O$ and thus the family of solutions of (6.3.43) over $\mathfrak{R}$ is given by :

$$X = Y_r^a\cdot\begin{bmatrix} O & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\cdot Y_l^b \qquad (6.3.57)$$

ii) **(⇒)** Let (6.3.1) have a solution , X_0 , over $\mathfrak{R}$. Then :

$$A\cdot X_0\cdot B = C \Leftrightarrow Y_l^a\cdot A\cdot X_0\cdot B\cdot Y_r^b = Y_l^a\cdot C\cdot Y_r^b \qquad (6.3.58)$$

or ,

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix}\cdot X_0\cdot[\ L_l^b\ ,\ O\ ] = \begin{bmatrix} L_r^a\cdot X_0\cdot L_l^b & O \\ O & O \end{bmatrix} = \begin{bmatrix} P_l^a\cdot C\cdot Q_r^b & P_l^a\cdot C\cdot N_r^b \\ N_l^a\cdot C\cdot Q_r^b & N_l^a\cdot C\cdot N_r^b \end{bmatrix} \qquad (6.3.50)$$

the latter implies that the following relations should hold true :

$$P_l^a\cdot C\cdot N_r^b = O\ ,\ N_l^a\cdot C\cdot Q_r^b = O\ ,\ N_l^a\cdot C\cdot N_r^b = O \qquad (6.3.60)$$

$$L_r^a\cdot X_0\cdot L_l^b = P_l^a\cdot C\cdot Q_r^b \qquad (6.3.61)$$

Since , $L_r^a = T_a\cdot Q_a$, $L_l^b = P_b\cdot T_b$, (6.3.61) implies that the matrix :

$$T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} = Q_a \cdot X_0 \cdot P_b \in \mathfrak{R}^{\rho_A x \rho_B} \tag{6.3.62}$$

(6.3.60) and (6.3.62) imply that (6.3.46) and (6.3.47) should hold .
($\Leftarrow$) Let (6.3.46) and (6.3.47) hold true , then set $X_0 \in \mathfrak{R}^{m x \kappa}$ to be the matrix :

$$X_0 = (Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1} \tag{6.3.63}$$

where , $Q_a \cdot (Q_a^r)^{-1} = I_{\rho_A}$, $(P_b^l)^{-1} \cdot P_b = I_{\rho_B}$, $(Q_a^r)^{-1}$, $(P_b^l)^{-1}$ are defined over $\mathfrak{R}$. Then , since we have supposed $A = P_a \cdot T_a \cdot Q_a$, $B = P_b \cdot T_b \cdot Q_b$, the following must hold true :

$$A \cdot X_0 \cdot B = (P_a \cdot T_a \cdot Q_a) \cdot \{(Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1}\} \cdot (P_b \cdot T_b \cdot Q_b) \tag{6.3.64}$$

or,

$$A \cdot X_0 \cdot B = P_a \cdot P_l^a \cdot C \cdot Q_r^b \cdot Q_b \tag{6.3.65}$$

We also observe that :

$$\begin{cases} L_r^a = P_l^a \cdot A \Leftrightarrow P_a \cdot L_r^a = P_a \cdot P_l^a \cdot A = A \Leftrightarrow \{P_a \cdot P_l^a - I_p\} \cdot A = O & (6.3.66) \\ L_l^b = B \cdot Q_r^b \Leftrightarrow L_l^b \cdot Q_b = B \cdot Q_r^b \cdot Q_b = B \Leftrightarrow B \cdot \{Q_r^b \cdot Q_b - I_t\} = O & (6.3.67) \end{cases}$$

Since N_l^a , N_r^b are bases of the left , right null spaces of A , B respectively , then matrices E_1 , E_2 over $\mathfrak{R}$ exist such that :

$$\begin{cases} P_a \cdot P_l^a = I_p + E_1 \cdot N_l^a & (6.3.68) \\ Q_r^b \cdot Q_b = I_t + N_r^b \cdot E_2 & (6.3.69) \end{cases}$$

Furthermore , if we make use of (6.3.46) , (6.3.68) , (6.3.69)

$$P_a \cdot P_l^a \cdot C \cdot Q_r^b \cdot Q_b = (I_p + E_1 \cdot N_l^a) \cdot C \cdot Q_r^b \cdot Q_b = C \cdot Q_r^b \cdot Q_b = C + C \cdot N_r^b \cdot E_2 \tag{6.3.70}$$

$$P_a \cdot P_l^a \cdot C \cdot Q_r^b \cdot Q_b = P_a \cdot P_l^a \cdot C \cdot (I_t + N_r^b \cdot E_2) = P_a \cdot P_l^a \cdot C = C + E_1 \cdot N_l^a \cdot C \tag{6.3.71}$$

$$P_a \cdot P_l^a \cdot C \cdot Q_r^b \cdot Q_b = (I_p + E_1 \cdot N_l^a) \cdot C \cdot (I_t + N_r^b \cdot E_2) = C + E_1 \cdot N_l^a \cdot C + C \cdot N_r^b \cdot E_2 \tag{6.3.72}$$

(6.3.70) , (6.3.71) , (6.3.72) imply that :

$$E_1 \cdot N_l^a \cdot C = C \cdot N_r^b \cdot E_2 = O \tag{6.3.73}$$

Hence , (6.3.65) via (6.3.72) , (6.3.73) gives :

$$A \cdot X_0 \cdot B = P_a \cdot P_l^a \cdot C \cdot Q_r^b \cdot Q_b = C \tag{6.3.74}$$

and thus (6.3.1) is solvable over $\mathcal{R}$ and $X_0 \in \mathcal{R}^{m x \kappa}$ in (6.3.63) is a solution of (6.3.1) over $\mathcal{R}$. Now if X is any solution of (6.3.1) over $\mathcal{R}$ then X_0 in (6.3.48) is also a solution of (6.3.1) over $\mathcal{R}$ and thus :

$$A \cdot X \cdot B = A \cdot X_0 \cdot B \Leftrightarrow A \cdot (X - X_0) \cdot B = O \tag{6.3.75}$$

which clearly implies that the family of solutions of (6.3.1) is given by :

$$X = X_0 + Y_r^a \cdot \begin{bmatrix} O & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \cdot Y_l^b \tag{6.3.76}$$

where , $Y_2 \in \mathcal{R}^{\rho_A x(\kappa - \rho_B)}$, $Y_3 \in \mathcal{R}^{(m-\rho_A) x \rho_B}$, $Y_4 \in \mathcal{R}^{(m-\rho_A)x(\kappa-\rho_B)}$ are arbitrary parametric matrices . □

6.4. STUDY OF THE MATRIX EQUATION $\sum_{i=1}^{h} Z_i \cdot X_i \cdot E_i = H$ OVER THE PID $\mathcal{R}$

The matrix equation (6.1.4) is a generalization of the matrix equations :

$$Z \cdot X + Y \cdot E = H \ , \ Z \in \mathcal{F}^{p x m}, \ E \in \mathcal{F}^{\kappa x t}, \ H \in \mathcal{F}^{p x t}, \ X \in \mathcal{R}^{m x t}, Y \in \mathcal{R}^{p x \kappa} \tag{6.4.1}$$

$$X \cdot Z + E \cdot Y = H \ , \ Z \in \mathcal{F}^{p x m}, \ E \in \mathcal{F}^{l x \kappa}, \ H \in \mathcal{F}^{l x m}, \ X \in \mathcal{R}^{l x p}, Y \in \mathcal{R}^{\kappa x m} \tag{6.4.2}$$

that arise from control synthesis problems , such as the regulator problem with measurement feedback and noninteracting control . In the following we consider the matrix equation :

$$\sum_{i=1}^{h} Z_i \cdot X_i \cdot E_i = H \ , \ Z_i \in \mathcal{F}^{p x m_i}, \ E_i \in \mathcal{F}^{\kappa_i x t}, \ H \in \mathcal{F}^{p x t}, \ X_i \in \mathcal{R}^{m_i x \kappa_i}$$

with , $\sum_{i=1}^{n} m_i = m$, $\sum_{i=1}^{n} \kappa_i = \kappa$. If (D , N) , (D′ , N′) denote an $\mathcal{R}$ – coprime left , right MFD of the matrices $M = [\, Z_1 , \dots , Z_h , H \,]$, $M' = [\, E_1^T , \dots , E_h^T \,]^T$ respectively , ($D \cdot M = [A_1 , \dots , A_h , \Gamma] = N$, $M' \cdot D' = [\, B_1^T , \dots , B_h^T \,]^T = N'$) , then N , N′ are an $\mathcal{R}$lmr , $\mathcal{R}$lmc of M , M′ respectively and equation (6.1.4) can be equivalently transformed to :

$$\sum_{i=1}^{h} A_i \cdot X_i \cdot B_i = C \ , \ A_i \in \mathcal{R}^{p x m_i}, \ B_i \in \mathcal{R}^{\kappa_i x t}, \ C \in \mathcal{R}^{p x t}, \ X_i \in \mathcal{R}^{m_i x \kappa_i} \tag{6.4.3}$$

or ,

$$[\, A_1 \,, \ldots , A_n \,] \cdot \begin{bmatrix} X_1 & & O \\ & \ddots & \\ O & & X_n \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} = A \cdot X \cdot B = C \qquad (6.4.4)$$

with , $A = [A_1 \,, \ldots , A_n] \in \mathfrak{F}^{p \times m}$, $B = [B_1^T \,, \ldots , B_n^T\,]^T \in \mathfrak{F}^{\kappa \times t}$, $X = diag\{X_1 \,, \ldots , X_n\}$.

Remark (6.4.1) *: Equation (6.4.4) clearly implies that solvability of (6.4.3) over* $\mathfrak{R}$ *can be reduced to the search of special type solutions , (block diagonal) , of the matrix equation* $A \cdot Y \cdot B = C$ *, with* $A = [A_1 \,, \ldots , A_n]$ *,* $B = [B_1^T \,, \ldots , B_n^T\,]^T$ *.* □

In the following we associate the matrices A , B in (6.4.4) with the well known algebraic machinery established in chapter 5 . Let $(P_l^a \,, N_l^a)$, $(Q_r^a \,, N_r^a)$ denote two pairs of an ($\mathfrak{R}$cp , $\mathfrak{R}$pla) , ($\mathfrak{R}$rp , $\mathfrak{R}$pra) of A respectively ; $(P_l^b \,, N_l^b)$, $(Q_r^b \,, N_r^b)$ denote two pairs of an ($\mathfrak{R}$cp , $\mathfrak{R}$pla) , ($\mathfrak{R}$rp , $\mathfrak{R}$pra) of B respectively . Also let :

$$\begin{cases} Y_l^a = [\, (P_l^a)^T \,, (N_l^a)^T \,]^T \,, \; Y_r^a = [\, Q_r^a \,, N_r^a \,] \\ Y_l^b = [\, (P_l^b)^T \,, (N_l^b)^T \,]^T \,, \; Y_r^b = [\, Q_r^b \,, N_r^b \,] \end{cases}$$

be the unimodular matrices associated with the pairs of ($\mathfrak{R}$cps , $\mathfrak{R}$plas) , ($\mathfrak{R}$rps , $\mathfrak{R}$pras) of A , B respectively . If A , B are represented as :

$$A = P_a \cdot T_a \cdot Q_a \,, \; B = P_b \cdot T_b \cdot Q_b$$

where T_a , T_b are a glrd of A , B over $\mathfrak{R}$ respectively , then we denote by :

$$L_r^a = T_a \cdot Q_a = P_l^a \cdot A \,, \; L_l^a = P_a \cdot T_a = A \cdot Q_r^a$$

a pair of a (gerd , geld) of A over $\mathfrak{R}$;

$$L_r^b = T_b \cdot Q_b = P_l^b \cdot B \,, \; L_l^b = P_b \cdot T_b = B \cdot Q_r^b$$

a pair of a (gerd , geld) of B over $\mathfrak{R}$. Furthermore , if $\rho_A = rank_{\mathfrak{F}}\{A\}$, $\rho_B = rank_{\mathfrak{F}}\{B\}$ then :

Proposition (6.4.1) ***: i) If*** $\rho_A = rank_{\mathfrak{F}}\{A\} = m$ *,* $\rho_B = rank_{\mathfrak{F}}\{B\} = \kappa$ ***, then equation (6.4.4) is solvable over*** $\mathfrak{R}$ ***, if and only if :***

$$P_l^a \cdot C \cdot N_r^b = O \,, \; N_l^a \cdot C \cdot Q_r^b = O \,, \; N_l^a \cdot C \cdot N_r^b = O \qquad (6.4.6)$$

$$T_a^{-1}\cdot P_l^a\cdot C\cdot Q_r^b\cdot T_b^{-1}\in\mathfrak{R}^{m x \kappa} \qquad (6.4.7)$$

And the matrix :

$$X_0 = (Q_a^r)^{-1}\cdot T_a^{-1}\cdot P_l^a\cdot C\cdot Q_r^b\cdot T_b^{-1}\cdot (P_b^l)^{-1} = diag\{X_1^0\ ,\ \dots\ ,\ X_n^0\}\in\mathfrak{R}^{m x \kappa} \qquad (6.4.8)$$

where , $(Q_a^r)^{-1}$, $(P_b^l)^{-1}$ are an $\mathfrak{R}ri$, $\mathfrak{R}li$ of Q_a , P_b respectively .

ii) ***α)*** *If either $\rho_A = rank_{\mathfrak{F}}\{A\} < m$, or $\rho_B = rank_{\mathfrak{F}}\{B\} < \kappa$, or both relations hold , then equation (6.4.4) is solvable over $\mathfrak{R}$, if and only if :*

$$\begin{cases} P_l^a\cdot C\cdot N_r^b = O\ ,\ N_l^a\cdot C\cdot Q_r^b = O\ ,\ N_l^a\cdot C\cdot N_r^b = O & (6.4.8) \\ T_a^{-1}\cdot P_l^a\cdot C\cdot Q_r^b\cdot T_b^{-1}\in\mathfrak{R}^{\rho_A x \rho_B} & (6.4.9)\end{cases}$$

And the equation :

$$Q_a\cdot Y\cdot P_b = T_a^{-1}\cdot P_l^a\cdot C\cdot Q_r^b\cdot T_b^{-1} \qquad (6.4.10)$$

has a solution $X_0 = diag\{X_1^0\ ,\ \dots\ ,\ X_n^0\}\in\mathfrak{R}^{m x \kappa}$.

β) *A sufficient condition for equation (6.4.4) to be solvable over $\mathfrak{R}$ is that the matrix :*

$$X_0 = (Q_a^r)^{-1}\cdot T_a^{-1}\cdot P_l^a\cdot C\cdot Q_r^b\cdot T_b^{-1}\cdot (P_b^l)^{-1} = diag\{X_1^0\ ,\ \dots\ ,\ X_n^0\}\in\mathfrak{R}^{m x \kappa} \qquad (6.4.11)$$

where , $Q_a\cdot(Q_a^r)^{-1} = I_{\rho_A}$, $(P_b^l)^{-1}\cdot P_b = I_{\rho_B}$, $((Q_a^r)^{-1}$, $(P_b^l)^{-1}$ are an $\mathfrak{R}ri$, $\mathfrak{R}li$ of Q_a , P_b respectively) .

Proof

i) Assume that $\rho_A = rank_{\mathfrak{F}}\{A\} = m$, $\rho_B = rank_{\mathfrak{F}}\{B\} = \kappa$, then proposition (6.3.3) implies that the homogeneous equation :

$$A\cdot Y\cdot B = O \qquad (6.4.12)$$

with , $A = [A_1\ ,\ \dots\ ,\ A_n]\in\mathfrak{R}^{p x m}$, $B = [B_1^T\ ,\ \dots\ ,\ B_n^T\ ]^T\in\mathfrak{R}^{\kappa x t}$, has only trivial solutions over $\mathfrak{R}$.

(⇒) Let equation (6.4.4) have a solution , $X_0 = diag\{X_1^0\ ,\ \dots\ ,\ X_n^0\}$, over $\mathfrak{R}$, then X_0 is a solution over $\mathfrak{R}$ of equation :

$$A\cdot Y\cdot B = C \qquad (6.4.13)$$

with , $A = [A_1\ ,\ \dots\ ,\ A_n]\in\mathfrak{F}^{p x m}$, $B = [B_1^T\ ,\ \dots\ ,\ B_n^T\ ]^T\in\mathfrak{F}^{\kappa x t}$ and proposition (6.3.3) implies that :

$$P_l^a\cdot C\cdot N_r^b = O\ ,\ N_l^a\cdot C\cdot Q_r^b = O\ ,\ N_l^a\cdot C\cdot N_r^b = O \qquad (6.4.14)$$

$$T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \in \mathfrak{R}^{m x \kappa} \tag{6.4.15}$$

as well as the matrix :

$$Y_0 = (Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1} \in \mathfrak{R}^{m x \kappa} \tag{6.4.16}$$

with , $Q_a \cdot (Q_a^r)^{-1} = I_m$, $(P_b^l)^{-1} \cdot P_b = I_\kappa$, ($(Q_a^r)^{-1}$, $(P_b^l)^{-1}$ are an $\mathfrak{R}$ri , $\mathfrak{R}$li of Q_a , P_b respectively) , is a solution of (6.4.13) over $\mathfrak{R}$. But since the homogeneous equation (6.4.12) has only trivial solutions over $\mathfrak{R}$ proposition (6.3.3) implies that (6.4.13) has a unique solution Y_0 over $\mathfrak{R}$ and so does equation (6.4.4) . Thus ,

$$X_0 = Y_0 = (Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1} = diag\{X_1^0 , \dots , X_n^0\} \in \mathfrak{R}^{m x \kappa} \tag{6.4.17}$$

(6.4.14) , (6.4.15) and (6.4.16) imply (6.4.6) , (6.4.7) and (6.4.8) .

($\Leftarrow$) Assume that :

$$\begin{cases} P_l^a \cdot C \cdot N_r^b = O \ , \ N_l^a \cdot C \cdot Q_r^b = O \ , \ N_l^a \cdot C \cdot N_r^b = O & (6.4.18) \\ T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \in \mathfrak{R}^{m x \kappa} & (6.4.19) \end{cases}$$

and the matrix :

$$X_0 = (Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1} = diag\{X_1^0 , \dots , X_n^0\} \in \mathfrak{R}^{m x \kappa} \tag{6.4.16}$$

Then proposition (6.3.3) implies that X_0 is the unique solution over $\mathfrak{R}$ of equation :

$$A \cdot Y \cdot B = C \tag{6.4.17}$$

with , $A = [A_1 , \dots , A_n] \in \mathfrak{F}^{p x m}$, $B = [B_1^T , \dots , B_n^T]^T \in \mathfrak{F}^{\kappa x t}$. The latter implies that (6.4.4) is solvable over $\mathfrak{R}$.

ii) α) Let $\rho_A = rank_{\mathfrak{F}}\{A\} < m$, or $\rho_B = rank_{\mathfrak{F}}\{B\} < \kappa$, or both relations hold , then :

($\Rightarrow$) Let equation (6.4.4) have a solution , $X_0 = diag\{X_1^0 , \dots , X_n^0\}$, over $\mathfrak{R}$, then X_0 is a solution over $\mathfrak{R}$ of equation $A \cdot Y \cdot B = C$, with , $A = [A_1 , \dots , A_n] \in \mathfrak{R}^{p x m}$, $B = [B_1^T , \dots , B_n^T]^T \in \mathfrak{R}^{\kappa x t}$. Thus :

$$A \cdot X_0 \cdot B = C \Leftrightarrow Y_l^a \cdot A \cdot X_0 \cdot B \cdot Y_r^b = Y_l^a \cdot C \cdot Y_r^b \tag{6.4.18}$$

or ,

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot [\, L_l^b \ , O \,] = \begin{bmatrix} L_r^a \cdot X_0 \cdot L_l^b & O \\ O & O \end{bmatrix} = \begin{bmatrix} P_l^a \cdot C \cdot Q_r^b & P_l^a \cdot C \cdot N_r^b \\ N_l^a \cdot C \cdot Q_r^b & N_l^a \cdot C \cdot N_r^b \end{bmatrix} \tag{6.4.19}$$

the latter implies that the following relations should hold true :

$$P_l^a \cdot C \cdot N_r^b = O \ , \ N_l^a \cdot C \cdot Q_r^b = O \ , \ N_l^a \cdot C \cdot N_r^b = O \tag{6.4.20}$$

$$L_r^a \cdot X_0 \cdot L_l^b = P_l^a \cdot C \cdot Q_r^b \tag{6.4.21}$$

Since , $L_r^a = T_a \cdot Q_a$, $L_l^b = P_b \cdot T_b$, (6.4.21) implies that the matrix :

$$T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} = Q_a \cdot X_0 \cdot P_b \in \mathcal{R}^{\rho_A x \rho_B} \tag{6.4.22}$$

Relations (6.4.20) , (6.4.22) result to the truth of (6.4.8) , (6.4.9) , (6.4.10) .
($\Leftarrow$) Assume that (6.4.8) , (6.4.9) , (6.4.10) hold true . Then :

$$\begin{bmatrix} L_r^a \\ O \end{bmatrix} \cdot X_0 \cdot [\ L_l^b \ , O\ ] = \begin{bmatrix} L_r^a \cdot X_0 \cdot L_l^b & O \\ O & O \end{bmatrix} \tag{6.4.23}$$

Since , $L_r^a = T_a \cdot Q_a$, $L_l^b = P_b \cdot T_b$, (6.4.8) , (6.4.9) , (6.4.10) , (6.4.23) imply that :

$$\begin{bmatrix} L_r^a \cdot X_0 \cdot L_l^b & O \\ O & O \end{bmatrix} = \begin{bmatrix} P_l^a \cdot C \cdot Q_r^b & P_l^a \cdot C \cdot N_r^b \\ N_l^a \cdot C \cdot Q_r^b & N_l^a \cdot C \cdot N_r^b \end{bmatrix} \tag{6.4.24}$$

or equivalently ,

$$Y_l^a \cdot A \cdot X_0 \cdot B \cdot Y_r^b = Y_l^a \cdot C \cdot Y_r^b \Leftrightarrow A \cdot X_0 \cdot B = C \tag{6.4.25}$$

which clearly implies the solvability of (6.4.4) over $\mathcal{R}$.
β) The proof follows along the same arguments employed in the proof of the sufficient in part *ii*) of proposition (6.3.3) . □

6.5. EXAMPLES

In this appendix we present examples of solving matrix equations by making use of the method introduced in the previous sections .

Example (6.5.1) : Investigate the solvability , over the ring of polynomials , $\mathbb{R}[s]$, of the matrix equation :

$$Z \cdot X = E \tag{6.5.1}$$

where ,

$$Z = \begin{bmatrix} \frac{s^2+1}{s^3+4\,s^2+4\,s} & 0 & 1 \\ 1 & \frac{1}{s^3+4\,s^2+4\,s} & s \end{bmatrix}, \; E = \begin{bmatrix} \frac{1}{s^3+4\,s^2+4\,s} & 0 \\ 0 & \frac{1}{s^3+4\,s^2+4\,s} \end{bmatrix} \tag{6.5.2}$$

An $\mathbb{R}[s]$ – coprime left MFD of the matrx [Z , E] is given by the pair (D , N) with :

$$D = \begin{bmatrix} s^3+4\,s^2+4\,s & 0 \\ 0 & s^3+4\,s^2+4\,s \end{bmatrix}, \; N = \begin{bmatrix} s^2+1 & 0 & s^3+4\,s^2+4\,s & 1 & 0 \\ s^3+4\,s^2+4\,s & 1 & s^4+4\,s^3+4\,s^2 & 0 & 1 \end{bmatrix} \tag{6.5.3}$$

Thus N is an $\mathbb{R}[s]$lmr of [Z , E] and equation (6.5.1) can be transformed to :

$$A \cdot X = B \tag{6.5.4}$$

with ,

$$A = \begin{bmatrix} s^2+1 & 0 & s^3+4\,s^2+4\,s \\ s^3+4\,s^2+4\,s & 1 & s^4+4\,s^3+4\,s^2 \end{bmatrix}, \; B = I_2 \tag{6.5.5}$$

A is clearly an $\mathbb{R}[s]$ – right unimodular matrix and thus a left regular matrix , i.e. $\rho_A = = rank_{\mathbb{R}(s)}\{A\} = 2$; Proposition (6.2.1) , (part *i*) , *β*)) , implies that (6.5.4) is solvable over $\mathbb{R}[s]$, if and only if :

$$(L_l^a)^{-1} \cdot B \in \mathbb{R}^{2x2}[s] \tag{6.5.6}$$

for an arbitrary geld , L_l^a , of A over $\mathbb{R}[s]$. Since A is an $\mathbb{R}[s]$ – right unimodular matrix a geld of it is given by $L_l^a = I_2$ and (6.5.6) holds true . Clearly a solution X_0 of (6.5.4) over $\mathbb{R}[s]$ is given by an $\mathbb{R}[s]$ri of A and the family of solutions over $\mathbb{R}[s]$ of (6.5.4) and thus of (6.5.1) is given by :

$$X = X_0 + N_r \cdot K \tag{6.5.7}$$

where , N_r is an $\mathbb{R}[s]$pra of A and $K \in \mathbb{R}^{2x2}[s]$ is an arbitrary parameter and X_0 is given by :

$$X_0 = \begin{bmatrix} \frac{3\,s^2+16\,s+25}{25} & \frac{-(12\,s^4+73\,s^3+148\,s^2+100\,s)}{25} & \frac{-3\,s-36}{25} \\ 0 & 1 & 0 \end{bmatrix}^T$$

□

Example (6.5.2) : Investigate the solvability , over the ring of polynomials , $\mathbb{R}[s]$, of the matrix equation :

$$Z \cdot X \cdot E = H \tag{6.5.8}$$

where ,

$$Z = \begin{bmatrix} \frac{s+1}{2\,s+1} & 0 & 1 \\ 1 & \frac{1}{2\,s+1} & s \end{bmatrix}, E = \begin{bmatrix} \frac{1}{s+1} & 1 \\ 0 & \frac{1}{s} \end{bmatrix}, H = \begin{bmatrix} \frac{1}{2\,s+1} & 0 \\ 0 & \frac{1}{2\,s+1} \end{bmatrix} \tag{6.5.9}$$

An $\mathbb{R}[s]$ – coprime left MFD of the matrix [Z , H] is given by the pair (D , N) with :

$$D = \begin{bmatrix} 2\,s+1 & 0 \\ 0 & 2\,s+1 \end{bmatrix}, N = \begin{bmatrix} s+1 & 0 & 2\,s+1 & 1 & 0 \\ 2\,s+1 & 1 & 2\,s^2+s & 0 & 1 \end{bmatrix} \tag{6.5.10}$$

Whereas , an $\mathbb{R}[s]$ – coprime right MFD of the matrix E is given by the pair (D′ , N′) with :

$$D' = \begin{bmatrix} -(s^3+s^2) & 1-s^2 \\ s^2+s & s \end{bmatrix}, N' = \begin{bmatrix} s & 1 \\ s+1 & 1 \end{bmatrix} \tag{6.5.11}$$

Thus N , N′ are $\mathbb{R}[s]$lmr , $\mathbb{R}[s]$lmc of [Z , H] , E respectively and equation (6.5.8) can be transformed to :

$$A \cdot X \cdot B = C \tag{6.5.12}$$

where ,

$$A = \begin{bmatrix} s+1 & 0 & 2\,s+1 \\ 2\,s+1 & 1 & 2\,s^2+s \end{bmatrix}, B = \begin{bmatrix} s & 1 \\ s+1 & 1 \end{bmatrix}, C = \begin{bmatrix} -(s^3+s^2) & 1-s^2 \\ s^2+s & s \end{bmatrix} \tag{6.5.13}$$

Proposition (6.3.3) implies that equation (6.5.12) is solvable over $\mathbb{R}[s]$ if and only if :

$$P_l^a \cdot C \cdot N_r^b = O \ , \ N_l^a \cdot C \cdot Q_r^b = O \ , \ N_l^a \cdot C \cdot N_r^b = O \tag{6.5.14}$$

$$T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \in \mathbb{R}^{2x2}[s] \tag{6.5.15}$$

Since A is an $\mathbb{R}[s]$ – right unimodular and thus left regular , B is an $\mathbb{R}[s]$ – unimodular and thus regular , $N_l^a = O$, $N_r^b = O$ respectively and thus (6.5.14) holds . On the other hand , a pair of glrds of A , B over $\mathbb{R}[s]$ respectively , is given by $T_a = I_2$, T_b= B and a pair of $\mathbb{R}[s]$cp , $\mathbb{R}[s]$rp of A , B respectively are given by $P_l^a = I_2$, $Q_r^b = I_2$. The latter implies that (6.5.15) holds true . A solution of (6.5.12) X_0 over $\mathbb{R}[s]$ is given by (6.3.48) :

$$X_0 = (Q_a^r)^{-1} \cdot T_a^{-1} \cdot P_l^a \cdot C \cdot Q_r^b \cdot T_b^{-1} \cdot (P_b^l)^{-1}$$

where ,

$$(Q_a^r)^{-1} = \begin{bmatrix} 2 & 2s^2-3s-2 & -1 \\ 0 & 1 & 0 \end{bmatrix}^T , \ (P_b^l)^{-1} = I_2 \tag{6.5.16}$$

Then ,

$$X_0 = \begin{bmatrix} 2s+2 & 2s^3-s^2-5s-2 & -s-1 \\ -2s^2-2s & -2s^4+s^3+5s^2+3s & s^2+s \end{bmatrix}^T \tag{6.5.17}$$

The family of solutions over $\mathbb{R}[s]$ of (6.5.12) and thus (6.5.8) is given by :

$$X = X_0 + Y_r^a \cdot \begin{bmatrix} O \\ Y_3 \end{bmatrix} \tag{6.5.18}$$

where , $Y_3 \in \mathbb{R}^{1x2}[s]$ is arbitrary parameter and ,

$$Y_r^a = \begin{bmatrix} 2 & 2s^2-3s-2 & -1 \\ 0 & 1 & 0 \\ -2s-1 & -2s^3+s^2+3s+1 & s+1 \end{bmatrix}^T$$

□

6.6. CONCLUSIONS

In Chapter 6 we have tackled the very important issue of formulating a unifying approach for solving the matrix equations (6.1.1) – (6.1.4) over the PID of interest , $\mathfrak{R}$. In our attempt to do so we use the results have been derived in Chapter 5 . The given matrices Z , E , Z_i . E_i , H , in (6.1.1) – (6.1.4) have been considered over the field of fractions , $\mathfrak{F}$, of $\mathfrak{R}$. whereas the unknown matrices X , Y , X_i are required to be over $\mathfrak{R}$. The set of equations (6.1.1) – (6.1.4) has been transformed via the implementation of the concept of multiples , least multiples over $\mathfrak{R}$ of the rows , columns of a matrix , to an equivalent one with known matrices A , B , A_i , B_i , C over $\mathfrak{R}$. Conditions for the existence as well as parametrization of solutions of the equations in question have been provided in terms of greatest left – right divisors , greatest extended left – right divisors , projectors , annihilators , and right , left inverses of the given matrices as well as parametric matrices over $\mathfrak{R}$. Further investigation in the derivation and characterization of solutions of equation (6.1.4) over $\mathfrak{R}$ is needed .

CHAPTER 7

THE MINIMAL DESIGN PROBLEM AND RELATED ISSUES FOR DISCRETE TIME LINEAR SYSTEMS

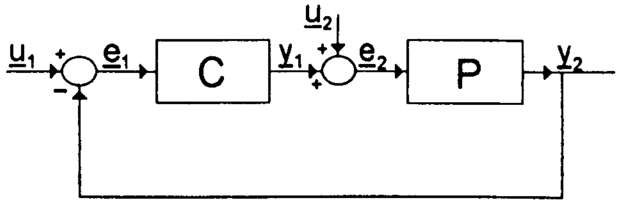
7.1. INTRODUCTION

The problem of stabilizing unstable linear control systems , has motivated the representation of the plant and controller , involved in the system configuration , as fractions of matrices with entries in special rings of interest . This representation describes the property of stability in an algebraic sense , [Vid. 1] – [Vid. 4] , [Fra. 1] , [Fra. 2] , [Des. 1] , [Sae. 1] , [Sae. 2] , [Var. 3] . The basic control schemes comprised by a precompensator , (or feedback compensator) , and unity output feedback , which are used to stabilize unstable plants , always lead to the study of a Matrix Diophantine Equation (MDE) over the ring of interest . The problem of finding solutions of MDE corresponding to controllers with minimum number of poles is referred to as the Minimal Design Problem (MDP) . In the following we consider the MDP as it arises from the study of Total Finite Settling Time Stabilization (TFSTS) , for MIMO discrete time , linear , time invariant , systems , [Kar. 1] , [Mil. 1] . TFSTS requires all the internal and external variables , (signals) , of the system to settle to a new steady – state after finite time from the application of a step change to its input and for every initial condition. The TFSTS comprises the dead – beat response problem , i.e. the forcing of the state or output vector from any initial state to the origin in minimum time , [Ber. 1] , [Ise. 1] , [Kal. 1] , [Kuo 1] , [Kuc. 1] – [Kuc. 8] , [Vid. 4] . The study of controllers which are defined by solutions of Polynomial MDE (PMDE) with minimum number of poles refers to the definition of the Extended McMillan Degree (EMD) of a rational matrix via its Polynomial Matrix Fractional Description (PMFD) .

After an initial introduction and formulation of the problem in section 7.2 , parametrization issues for such stabilizing controllers are examined in section 7.3 . The importance of characterizing solutions of the PMDE is established . If the plant and controller are represented by a left (right) MFD , right (left) MFD , when the number of inputs are greater than , (less than) , or equal the number of outputs , respectively ; we prove that the solutions of a PMDE – with an arbitrary unimodular matrix on its right half side – which correspond to column , (row) , reduced matrices form a nonempty , dense but neither open , nor closed subset of the its family of solutions . Bearing in mind that the EMD , δ^* , of a controller defined by a column , (row) , reduced PMFD is equal to the sum of column , (row) , degrees of that PMFD , the latter result implies that the sum of minimum column degrees that occur in the set of solutions of a PMDE is more likely to serve as an upper bound rather than be equal to δ^* . The approach employed for the parametrization of least column , (row) , degrees solutions is based on the expression of the PMDE via its Toeplitz matrix representation. This approach leads to a very simple algorithm involving only the computation of right (left) null spaces of real matrices . Employing the exterior products of the rows , columns , (columns rows) , of its matrices , the PMDE can be reduced to a vector matrix equation the characterization of least column degree solutions of which yields of a lower bound for δ^* . Additional issues , such as , the PI controller design problem and fixed controllability index stabilizing controllers are studied as well .

7.2. STATEMENT OF THE PROBLEM

Consider the standard feedback configuration associated with a discrete time system in the d – representation , [Mil. 1] :



where ,

$$P = N \cdot D^{-1} = \tilde{D}^{-1} \cdot \tilde{N} \in \mathbb{R}^{mxl}[d] \tag{7.2.1}$$
$$C = N_c \cdot D_c^{-1} = \tilde{D}_c^{-1} \cdot \tilde{N}_c \in \mathbb{R}^{lxm}[d] \tag{7.2.2}$$

We assume that both plant and controller are represented by the coprime MFDs . The solution of TFSTS problem , [Kar. 1] , [Mil. 1] is reduced to a solution of the :

$$\tilde{D}\,D_c + \tilde{N}\,N_c = I_m \text{ , or } [\,\tilde{D},\ \tilde{N}\,]\begin{bmatrix} D_c \\ N_c \end{bmatrix} = I_m \text{ , if } l \geq m \tag{7.2.3}$$

or equivalently ,

$$\tilde{D}_c\,D_c + \tilde{N}_c\,N_c = I_l \text{ , or } [\,\tilde{D}_c\,,\ \tilde{N}_c\,]\begin{bmatrix} D \\ N \end{bmatrix} = I_l \text{ , if } l < m \tag{7.2.4}$$

In the following , we shall represent both plant and controller in terms of composite matrices as :

$$T_p^l(d) \triangleq [\,\tilde{D}\,,\,\tilde{N}\,] \in \mathbb{R}^{mx(m+l)}[d] \tag{7.2.5.a}$$

$$T_p^r(d) \triangleq \begin{bmatrix} D \\ N \end{bmatrix} \in \mathbb{R}^{(m+l)xl}[d] \tag{7.2.5.b}$$

$$T_c^l(d) \triangleq [\,\tilde{D}_c\,,\ \tilde{N}_c\,] \in \mathbb{R}^{lx(m+l)}[d] \tag{7.2.6.a}$$

$$T_c^r(d) \triangleq \begin{bmatrix} D_c \\ N_c \end{bmatrix} \in \mathbb{R}^{(m+l)xm}[d] \tag{7.2.6.b}$$

If ν , μ are the observability , controllability indices of the plant respectively [Kai. 1] , then we may express $T_p^l(d)$, $T_p^r(d)$ as :

$$T_p^l(d) = [\ \widetilde{D}_0\ ,\ \widetilde{N}_0] + d\ [\ \widetilde{D}_1\ ,\ \widetilde{N}_1] + \dots + d^\nu\ [\ \widetilde{D}_\nu\ ,\ \widetilde{N}_\nu] =$$

$$= \widetilde{T}_0 + d\ \widetilde{T}_1 + \dots + d^\nu\ \widetilde{T}_\nu \in \mathbb{R}^{m x (m+l)}[d] \qquad (7.2.7.a)$$

$$T_p^r(d) = \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} + d \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} + \dots + d^\mu \begin{bmatrix} D_\mu \\ N_\mu \end{bmatrix} =$$

$$= T_0 + d\ T_1 + \dots + d^\mu\ T_\mu \in \mathbb{R}^{(m+l) x l}[d] \qquad (7.2.7.b)$$

Similarly , if p , τ are the controllability , observability indices of the controller respectively , then we may express $T_c^l(d)$, $T_c^r(d)$ as:

$$T_c^l(d) = [\ \widetilde{D}_{c0}\ ,\ \widetilde{N}_{c0}] + d\ [\ \widetilde{D}_{c1}\ ,\ \widetilde{N}_{c1}] + \dots + d^\tau\ [\ \widetilde{D}_{c\tau}\ ,\ \widetilde{N}_{c\tau}] =$$

$$= \widetilde{T}_{c0} + d\ \widetilde{T}_{c1} + \dots + d^\tau\ \widetilde{T}_{c\tau} \in \mathbb{R}^{l x (m+l)}[d] \qquad (7.2.8.a)$$

$$T_c^r(d) = \begin{bmatrix} D_{c0} \\ N_{c0} \end{bmatrix} + d \begin{bmatrix} D_{c1} \\ N_{c1} \end{bmatrix} + \dots + d^p \begin{bmatrix} D_{cp} \\ N_{cp} \end{bmatrix} =$$

$$= T_{c0} + d\ T_{c1} + \dots + d^p\ T_{cp} \in \mathbb{R}^{(m+l) x m}[d] \qquad (7.2.8.b)$$

In the following we shall consider the formulation of the problem based on equation (7.2.3) – similar analysis may be used for equation (7.2.4) . From this equation the following problems are put forward :

Problem (i) : (Fixed controllability , (observability) , solutions) .
Given the plant P , determine the necessary and sufficient conditions such that the Diophantine equation (7.2.3) , (or (7.2.4)) , has a solution for given controllability (observability) index controller . If a solution exists then parametrize the whole family of such solutions . □

Problem (ii) : (McMillan degree characterization , parametrization) .
Among the family of given controllability (observability) index solutions , determine those with a given McMillan degree ; investigate the parametrization of the family of given

McMillan degree solutions . □

***Problem (iii)** : (Minimal design problem) .*
Define the minimal controllability , (observability) , indices solutions of the Diophantine equation and define the condition characterizing the minimal McMillan degree amongst the whole family . □

An integral part of the above study is the investigation of the following subproblem :

***Problem (iv)** : (Parameter space , characterization of McMillan degree) .*
Derive the relationship or characterization of McMillan degree in terms of the properties of the matrix coefficients of the polynomial matrix $T^l(d)$, or $T^r(d)$. □

The above problems have been studied in [Kar. 1] , [Mil. 1] for the SISO case . In the following we do so for the more general case of MIMO plants . Our approach to the parametrization of minimum McMillan , (or more general Extended McMillan) , degree controllers , defined by solutions of polynomial Diophantine equations , concentrates more on the investigation of topological properties of certain types of solutions of the matrix Diophantine equations in question . The general issues of controller parametrization and McMillan degree characterization are examined first .

7.3. PARAMETRIZATION OF CONTROLLERS AND RELATED ISSUES

Throughout this study we shall concentrate on the (7.2.3) form of the Diophantine equation which will be referred to as right Diophantine equation , since the controller is represented by a right MFD ; similarly equation (7.2.4) will be referred to as left Diophantine equation . The study of fixed complexity solutions of the Diophantine equations is intimately related to the different ways we can characterize the controller complexity and thus parametrize the composite $T_c^r(d)$ matrix . In the following we examine two alternative types of parametrization of the $T_c^r(d)$. These are :

i) The Forney dynamic index parametrization .

ii) The set of Forney dynamic indices parametrization .

Those two fundamental parametrizations of the $T_c^r(d)$ matrix , are considered first and then we link these parametrizations to the McMillan degree . The first parametrization is the one defined by (7.2.8.b) , where p is the controllability index of the controller and it is expressed as :

$$T_c^r(d) = \begin{bmatrix} D_{c0} \\ N_{c0} \end{bmatrix} + d \begin{bmatrix} D_{c1} \\ N_{c1} \end{bmatrix} + \ldots + d^p \begin{bmatrix} D_{cp} \\ N_{cp} \end{bmatrix} =$$

$$= [\ T_{c0}\ ,\ \ T_{c1}\ ,\ \ldots\ ,\ T_{cp}\ ] \begin{bmatrix} I_m \\ d\ I_m \\ \vdots \\ d^p\ I_m \end{bmatrix} =$$

$$= T_c^r(d)\ S_{m,p}(d) \tag{7.3.1}$$

The above parametrization is defined by the indices $(l\ ,\ m\ ,\ p)$ completely and the matrix $T_c^r(d)$, which has dimensions $(m+l)\ x\ m\ (p+1)$. Given that $T_c^r(d)$ describes an MFD we must have that :

$$|\ D_{c0} + d\ D_{c1} + \ldots + d^p\ D_{cp}\ |\ \neq 0 \tag{7.3.2.a}$$

or equivalently ,

$$|\ [\ D_{c0}\ \ D_{c1}\ \ \cdots\ \ D_{cp}\ ]\ S_{m,p}(d)\ | \neq 0 \tag{7.3.2.b}$$

We may summarize as :

Remark (7.3.1) *: The Forney dynamic index parametrization defined by (7.3.1) corresponds to an MFD , if and only if condition (7.3.2) is satisfied . The MFD is causal if* $|\ D_{c0}\ | \neq 0$ *, and clearly the latter condition also guarantees the existence of an MFD .* □

The above parametrization will be referred to as a right $-(l\ ,\ m\ ,\ p)$ parametrization and its characteristic is that we fix the maximal Forney index p of the space $colsp_{\mathbb{R}(s)}\{T_c^r(d)\}$; this representation does not specify the Forney dynamical order of the latter space , but it just gives an upper bound for it .

Remark (7.3.2) *: If* δ *is the Forney dynamical order of* $\mathfrak{X}_{cr} \triangleq colsp_{\mathbb{R}(s)}\{\ T_c^r(d)\ \}$ *, then for the family of* $X_{cr} \in \mathfrak{X}_{cr}$ *defined by the right* $-(l\ ,\ m\ ,\ p)$ *parametrizations we have that :*

$$\delta \leq m \cdot p$$ □

Clearly , the right $-(l\ ,\ m\ ,\ p)$ parametrization is rather simple , but the Forney order of the resulting matrix is not apparent from the parametrization . An alternative parametrization that avoids the above problem is considered next . Let $\{\ r\ \} = \{\ r_1\ ,\ r_2, \ldots, r_m\ \}$ denote the degrees of the columns of the matrix $T_c^r(d)$. We may write :

$$T_c^r(d) = [\ \underline{t}_1(d)\ ,\ \underline{t}_2(d)\ ,\ \ldots\ ,\ \underline{t}_m(d)\ ] = [\ T'_{r_1} \vdots \cdots \vdots\ T'_{r_m}\ ] \cdot \begin{bmatrix} \underline{e}_{r_1}(d) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \underline{e}_{r_m}(d) \end{bmatrix} =$$

$$= (T'_c)^r\ S'_{m,\{r\}}(d) \tag{7.3.3.a}$$

where ,

$$\underline{t}_i(d) = \underline{t}_{i0} + d\,\underline{t}_{i1} + \cdots + d^{r_i}\,\underline{t}_{ir_i} =$$

$$= [\,\underline{t}_{i0} \;\; \underline{t}_{i1} \;\cdots\; \underline{t}_{ir_i}] \cdot \begin{bmatrix} 1 \\ d \\ \vdots \\ d^{r_i} \end{bmatrix} =$$

$$= T'_{r_i} \cdot \underline{e}_{r_i}(d) \tag{7.3.3.b}$$

Note that , $(T'_c)^r \in \mathbb{R}^{(l+m)x\omega}$, $\omega = \sum_{i=1}^{m} r_i + m$, and if we partition $(T'_c)^r$ as :

$$(T'_c)^r = \begin{bmatrix} (T'_{c_1})^r \\ \cdots\cdots \\ (T'_{c_2})^r \end{bmatrix} ,\; (T'_{c_1})^r \in \mathbb{R}^{mx\omega} \text{ and } (T'_{c_2})^r \in \mathbb{R}^{lx\omega} \tag{7.3.3.c}$$

then the above form corresponds to a right MFD if :

$$|\, T'^{r}_{c_1} \cdot S'_{m,\{r\}}(d) \,| \neq 0 \tag{7.3.4}$$

If $\underline{d}_{i0}$ denotes the first column of the i-*th* column block in $(T'_{c_1})^r$, then the condition for causality of the corresponding MFD representation is :

$$|\,(T'_{c_1})^r \cdot S'_{m,\{r\}}(0)\,| = |\,[\,\underline{d}_{10} , \underline{d}_{20} , \ldots , \underline{d}_{m0}\,]\,| \neq 0 \tag{7.3.5}$$

The above parametrization will be called a right $-\{\, l , m , \{r\} \,\}$ parametrization , where $\{\, r \,\} = \{\, r_1 , r_2 , \ldots , r_m \,\}$. Note that in the case where $T^r_c(d)$ is coprime and column reduced then the set $\{\, r \,\} = \{\, r_i , i \in \underline{m}\}$ are the Forney indices of $\mathcal{K}_{cr}$.

Remark (7.3.3) *: If δ is the Forney dynamical order of $\mathcal{K}_{cr}$, then the family of $\mathcal{K}_{cr}$ defined by the right$-\{l , m , \{r\}\}$ parametrizations , defines a right MFD if condition (7.3.4) is satisfied ; the representation is causal if (7.3.5) holds true . Further more , for all parametrizations $\delta \leq \sum_{i=1}^{m} r_i$.* □

Remark (7.3.4) *: The set of solutions of equation (7.2.3) consists of $\mathbb{R}[d]-$left unimodular matrices $[\, D_c^T , N_c^T \,]^T$. Indeed , since $[\,\tilde{D} , \tilde{N}\,]$ is an $\mathbb{R}[d]-$right unimodular matrix , it never loses rank over $\mathbb{C}$, hence if $[\, D_c^T , N_c^T \,]^T$ does lose rank on $d_0 \in \mathbb{C}$ then $C_m([\, D_c^T(d_0) , N_c^T(d_0) \,]^T) = \underline{0}$ and :*

$$C_m([\,\tilde{D}(d_0) , \tilde{N}(d_0)]) \cdot C_m([\, D_c^T(d_0) , N_c^T(d_0) \,]^T) = 0$$

which not true , since $C_m([\,\tilde{D}(d_0)\,,\ \tilde{N}(d_0)]) \cdot C_m([\,D_c^T(d_0)\,,\ N_c^T(d_0)\,]^T) = 1$. □

***Remark** (7.3.5) [Vid. 4] : Almost all the solutions of (7.2.3) correspond to a coprime MFD , i.e. the set of solutions of (7.2.3) with* $|D_c(d)| \neq 0$ *is open and dense in the set of all solutions of (7.2.3) .* □

Equation (7.2.3) appears in stabilization problems in the more general form :

$$\tilde{D}\,D_c + \tilde{N}\,N_c = U_m \text{ , or } [\,\tilde{D}\,,\ \tilde{N}\,]\begin{bmatrix} D_c \\ N_c \end{bmatrix} = U_m \text{ , or } T_p^l(d)\cdot T_c^r(d) = U_m \qquad (7.3.6)$$

where , U_m is a unimodular matrix ; in other words controllers with coprime MFD representations (D_c , N_c) are required such that the result of the matrix Diophantine equation $\tilde{D}\,D_c + \tilde{N}\,N_c$ is a unimodular matrix U_m . It is well known , [Che. 1] , [Kuc. 2], that if (D_c , N_c) is a coprime MFD representation of a controller and satisfies equation (7.3.6) , then the column degrees of $[\,D_c^T\,,\ N_c^T\,]^T$ serve as the controllability indices , if and only if $[\,D_c^T\,,\ N_c^T\,]^T$ is column reduced . In the case of proper controllers the column degrees of D_c^T serve as the controllability indices , if and only if D_c^T is column reduced . In the above cases the complexity of the stabilizing controllers is equal to the sum of the column degrees of $[\,D_c^T\,,\ N_c^T\,]^T$, (when the controller in non proper) , D_c^T , (when the controller is proper) . More generally , when the stabilizing controllers are not proper we have the following definition :

***Definition** (7.3.1) [Ros. 1] : Let* $C \in \mathbb{R}^{l \times m}(d)$ *be a stabilizing controller not necessarily proper . Then the Extended McMillan degree (EMD)* δ_M^* *of C is defined as the total number of finite and infinite poles of C .* □

***Lemma** (7.3.1) [Var. 5] : Let* $C \in \mathbb{R}^{l \times m}(d)$ *be a stabilizing controller not necessarily proper and* $(A_1\,,\ B_1)$, $(A_2\,,\ B_2)$ *be any pair of coprime right , left MFDs of C respectively . Then the EMD of C ,* $\delta_M^*(C)$ *, is :*

$$\delta_M^*(D_c^T) = deg[C_m(T_A^l(d))] = deg[C_l(T_A^r(d))] \qquad (7.3.7)$$

where , $T_A^r(d) = [\,A_1^T\,,\ B_1^T\,]^T$, $T_A^l(d) = [\,A_2\,,\ B_2\,]$. □

The previous analysis motivates the study of the property of column reduceness among the solutions of (7.3.6) . In the following we shall assume that U_m is a unimodular matrix .

***Remark* (7.3.6)** *: Equation (7.3.6) has always a solution $T^r_c(d)$, since equation (7.2.3) has always a solution , due to the fact that the matrix $T^l_p(d)$ corresponds to a coprime left MFD of the plant . If $T^r_c(d)_0$ is a solution of (7.3.6) then the family of solutions of (7.3.6) is :*

$$\mathcal{T} = \{ T^r_c(d) : T^r_c(d) = T^r_c(d)_0 + V \cdot R \} \tag{7.3.8}$$

where , V is the matrix formed by a base $\mathcal{V}$ of $\mathcal{N}_r\{T^l_p(d)\}$, and R is an arbitrary polynomial matrix . □

Let $\mathcal{T}_{cr}$ denotes the family of solutions of (7.3.6) which are column reduced and let $\mathcal{R}$ denotes the family :

$$\mathcal{R} = \{ R \in \mathbb{R}^{l x m}[d] : T^r_c(d) \in \mathcal{T}_{cr} , T^r_c(d) = T^r_c(d)_0 + V \cdot R \} \tag{7.3.9}$$

Furthermore choose V to correspond to a minimal polynomial base $\mathcal{V}$ of $\mathcal{N}_r\{T^l_p(d)\}$. Then V is a column reduced matrix , i.e. its highest column order coefficient matrix $[V]^c_h$ has full column rank . Finally , $\mathbb{R}^{l x m}[d]$ becomes a metric space if it is endowed with the following metric :

***Definition* (7.3.2)** *: Define ϱ_M to be a matrix metric over the space $\mathbb{R}^{l x m}[d]$ such that :*

$$\varrho_M : \mathbb{R}^{l x m}[d] \; x \; \mathbb{R}^{l x m}[d] \rightarrow \mathbb{R}_+ \cup \{0\} \tag{7.3.10}$$

and for all matrices $A = [a_{ij}]$, $B = [b_{ij}]$, of $\mathbb{R}^{l x m}[d]$,

$$\varrho_M (A , B) = \max_{\forall i,j} \{ || a_{ij} - b_{ij} ||_p \} \tag{7.3.11}$$

where $||\cdot||_p$ is any of the classical polynomial norms . (Such a norm for example can be:

$$||\cdot||_p : \mathbb{R}[d] \rightarrow \mathbb{R}_+ \cup \{0\} : || r ||_p \equiv || r(d) ||_p = \sum_{i=0}^{n} | r_i | \tag{7.3.12}$$

where , $r(d) = r_0 + r_1 \cdot d + \cdots + r_n \cdot d^n$) . □

***Remark* (7.3.7)** *: It is straightforward to prove that ϱ_M is a metric over $\mathbb{R}^{l x m}[d]$. Further more ϱ_M defines convergence of matrix sequences over $\mathbb{R}^{l x m}[d]$ in the following natural manner : If $P_n = [p^n_{ij}]$, $Q = [q_{ij}]$ are a matrix sequence , a matrix over $\mathbb{R}^{l x m}[d]$ respectively then :*

$$\lim_{n \to \infty} P_n = Q \Leftrightarrow \lim_{n \to \infty} p^n_{i,j} = q_{i,j} , \forall i,j \tag{7.3.13}$$

□

If $\bar{\mathcal{R}}$ denotes the closure of $\mathcal{R}$ and $\mathcal{T}$ is defined in (7.3.8) , we can proceed with the statement and proof of the following result concerning the property of column reduceness of the solutions of (7.3.6) for an arbitrary unimodular matrix U_m .

***Theorem (7.3.1)** : The set of column reduced solutions of (7.3.6) is dense in $\mathcal{T}$, or :*

$$\bar{\mathcal{R}} = \mathbb{R}^{l x m}[d] \tag{7.3.14}$$

Proof

It is obvious that $\bar{\mathcal{R}} \subset \mathbb{R}^{lxm}[d]$. Thus we must show that $\mathbb{R}^{lxm}[d] \subset \bar{\mathcal{R}}$. Let $R \in \mathbb{R}^{lxm}[d]$ and $T^r_c(d)_R = T^r_c(d)_0 + V \cdot R$. Then:

i) If $T^r_c(d)_R \in \mathcal{T}_{cr}$, then R belongs to $\mathcal{R}$ and thus $\mathbb{R}^{lxm}[d] \subset \bar{\mathcal{R}}$.

ii) If $T^r_c(d)_R \notin \mathcal{T}_{cr}$, then in order to show that R belongs to $\bar{\mathcal{R}}$ we must find a sequence R_n of elements of $\mathcal{R}$ such that $R_n \to R$. The latter can be achieved as follows . Write $T^r_c(d)_R$ as :

$$T^r_c(d)_R = [\, \underline{t}_1(d) , \underline{t}_2(d) , \dots , \underline{t}_m(d) \,] \tag{7.3.15}$$

and let $\mu_1 , \mu_2 , \dots , \mu_m$ be its column degrees . Write V as :

$$V = [\, \underline{v}_1(d) , \underline{v}_2(d) , \dots , \underline{v}_l(d) \,] \tag{7.3.16}$$

and let $\nu_1 , \nu_2 , \dots , \nu_l$ be its column degrees . Bearing in mind that V corresponds to a minimal polynomial base $\mathcal{V}$ of $\mathcal{N}_r\{T^l_p(d)\}$ and $l \geq m$, then V is a column reduced matrix $rank\{[V]^c_h\} = l$. Let V_m be the matrix formed by the first m columns of V . Then it is clear that $rank\{[V_m]^c_h\} = m$ and V_m is a column reduced matrix . Now consider the sequence $R_n \in \mathbb{R}^{lxm}[d]$ such that :

$$R_n = R + C_n \tag{7.3.17}$$

where , $C_n \in \mathbb{R}^{lxm}[d]$, $C_n = [c^n_{i,j}]$ and :

$$c^n_{i,j} = \begin{cases} 0 , \text{ when } i > m , j = 1 , \dots , m \\ 0 , \text{ when } i \leq m , i \neq j , j = 1 , \dots , m \\ \frac{1}{n} \cdot d^{\kappa_j} , \text{ when } i \leq m , i = j , j = 1 , \dots , m \end{cases} \tag{7.3.18}$$

$$\kappa_j = \begin{cases} \mu_j - \nu_j + 1 , \text{ when } \mu_j \geq \nu_j , j = 1 , \dots , m \\ \\ 0 , \text{ when } \mu_j < \nu_j , j = 1 , \dots , m \end{cases} \tag{7.3.19}$$

Clearly $\lim_{n \to \infty} C_n = O$, which implies that $\lim_{n \to \infty} R_n = \lim_{n \to \infty} (R + C_n) = R$. It remains to prove that R_n belongs to $\mathfrak{R}$. Consider :

$$T_c^r(d)_{R_n} = T_c^r(d)_0 + V\cdot R_n = T_c^r(d)_0 + V\cdot(R + C_n) =$$

$$= (T_c^r(d)_0 + V\cdot R) + V\cdot C_n = T_c^r(d)_R + V\cdot C_n$$

By (7.3.15) , (7.3.16) we take that :

$$T_c^r(d)_{R_n} = [\ \underline{t}_1(d)\ ,\underline{t}_2(d)\ ,\ldots,\underline{t}_m(d)\ ] + [\ \underline{v}_1(d)\ ,\underline{v}_2(d)\ ,\ldots,\underline{v}_l(d)\ ]\cdot C_n$$

by (7.3.18) , (7.3.19) we take that :

$$T_c^r(d)_{R_n} = [\ \underline{t}_1(d) + \underline{v}_1(d)\cdot c_{11}^n\ ,\ \ldots,\underline{t}_m(d) + \underline{v}_m(d)\cdot c_{mm}^n\ ] \qquad (7.3.20)$$

By (7.3.20) it is clear that $[\ T_c^r(d)_{R_n}\ ]_h^c = (1/n)\cdot[\ V_m\ ]_h^c$ and $rank\{\ [\ T_c^r(d)_{R_n}\ ]_h^c\ \} = m$, which implies that $T_c^r(d)_{R_n}$ are column reduced $\forall\ n\in\mathbb{N}$. The latter implies that the sequence R_n belongs to $\mathfrak{R}$ and finally $\mathbb{R}^{l x m}[d] \subset \overline{\mathfrak{R}}$. □

***Remark (7.3.8)** : It is obvious that theorem (7.3.1) is invariant of the selection of the unimodular matrix U_m on the right hand side of (7.3.6) . Furthermore , from the proof of theorem (7.3.1) it is implied that the set of column reduced solutions of (7.3.6) is non empty .* □

Although the set of column reduced solutions of a matrix Diophantine equation , such as (7.3.6) , is a dense subset of its set of solutions , it is not open . In other words the solutions of (7.3.6) are not generically column reduced . This result is derived by the following approach :

***Definition (7.3.2)** : Consider equation (7.3.6) for the two arbitrary $\mathbb{R}[d]$ – unimodular matrices U_1 , U_2 on its right hand side . Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ denote the corresponding sets of solutions and let f be the function defined as :*

$$f\ :\mathfrak{F}_1 \rightarrow \mathfrak{F}_2\ ,\ \forall\ X\in\mathfrak{F}_1\ ,\ f(X) = X\cdot U_1^{-1}\cdot U_2 = X\cdot G \qquad (7.3.21)$$

and f is well defined . □

***Remark (7.3.9)** : In the following we consider the matrix metric ϱ_M of definition(7.3.2) expanded over the cartesian product $\mathbb{R}^{(m+l)xm}[d]\ x\ \mathbb{R}^{(m+l)xm}[d]$.* □

Proposition (7.3.1) : *The function f defined in definition(7.3.2) is a homeomorphism .*

Proof

α) f is a bijection . Indeed let X_1 , X_2 be two arbitrary elements of $\mathcal{F}_1$, such that $X_1 \neq X_2$. Then $f(X_1) = X_1 \cdot U_1^{-1} \cdot U_2$ and $f(X_2) = X_2 \cdot U_1^{-1} \cdot U_2$. Thus $f(X_1) \neq f(X_2)$.

β) f is a surjection . Indeed let Y be an arbitrary element of $\mathcal{F}_2$, then the matrix X defined by $X = Y \cdot U_2^{-1} \cdot U_1$, is an element of $\mathcal{F}_1$, since $T_p^l(d) \cdot Y \cdot U_2^{-1} \cdot U_1 = U_2 \cdot U_2^{-1} \cdot U_1$, and $f(X) = Y$.

γ) f is continuous . We shall prove that for every X , Y in $\mathcal{F}_1$ a positive real number ω exists such that :

$$\varrho_M (f(X) , f(Y)) \leq \omega \cdot \varrho_M (X , Y) \tag{7.3.22}$$

Indeed , if $X = [x_{ij}]$, $Y = [y_{ij}]$ are any elements in $\mathcal{F}_1$, $G = [g_{ij}]$ is the matrix of (7.3.21) then :

$$\varrho_M (f(X) , f(Y)) = \varrho_M (X \cdot G , Y \cdot G) = \max_{\forall i,j} \{ \| a_{ij} - b_{ij} \|_p \} \tag{7.3.23}$$

where , $A = [a_{ij}] = X \cdot G$, $B = [b_{ij}] = Y \cdot G$. Since :

$$a_{ij} = \sum_{\kappa=1}^{m} x_{i\kappa} \cdot g_{\kappa j} , \; b_{ij} = \sum_{\kappa=1}^{m} y_{i\kappa} \cdot g_{\kappa j} , \; \forall j = 1 , \dots , m , \forall i = 1 , \dots , l \tag{7.3.24}$$

Then , $\forall j = 1 , \dots , m , \forall i = 1 , \dots , l$

$$\| a_{ij} - b_{ij} \|_p = \left\| \sum_{\kappa=1}^{m} x_{i\kappa} \cdot g_{\kappa j} - \sum_{\kappa=1}^{m} y_{i\kappa} \cdot g_{\kappa j} \right\|_p = \left\| \sum_{\kappa=1}^{m} (x_{i\kappa} - y_{i\kappa}) \cdot g_{\kappa j} \right\|_p \tag{7.3.25}$$

or , $\forall j = 1 , \dots , m , \forall i = 1 , \dots , l$

$$\| a_{ij} - b_{ij} \|_p \leq \sum_{\kappa=1}^{m} \| (x_{i\kappa} - y_{i\kappa}) \cdot g_{\kappa j} \|_p \leq \sum_{\kappa=1}^{m} \| x_{i\kappa} - y_{i\kappa} \|_p \cdot \| g_{\kappa j} \|_p \tag{7.3.26}$$

(since , for the classical polynomial norms , like for example the coefficient norm demostrated in (7.3.12) , $\| p \cdot q \| \leq \| p \| \cdot \| q \|$) . If $\lambda = \max_{\forall \kappa,j} \{ \| g_{\kappa j} \|_p \}$, then (7.3.26) implies that $\forall j = 1 , \dots , m , \forall i = 1 , \dots , l$:

$$\| a_{ij} - b_{ij} \|_p \leq \lambda \cdot \sum_{\kappa=1}^{m} \| x_{i\kappa} - y_{i\kappa} \|_p \leq \lambda \cdot m \cdot \max_{\forall i,\kappa} \{ \| x_{i\kappa} - y_{i\kappa} \|_p \}$$

and finally , $\forall j = 1 , \dots , m , \forall i = 1 , \dots , l , \kappa = 1 , \dots , m$

$$\varrho_M (f(X) , f(Y)) = \max_{\forall i,j} \{ \| a_{ij} - b_{ij} \|_p \} \leq \lambda \cdot m \cdot \max_{\forall i,\kappa} \{ \| x_{i\kappa} - y_{i\kappa} \|_p \} =$$

$$= \omega \cdot \varrho_M (X , Y) \tag{7.3.27}$$

with $\omega = \lambda \cdot m$ a real positive number . Thus (7.3.23) holds true and f is a uniformly continuous and hence continuous function .

δ) f^{-1} is continuous . The proof follows similar arguments as in the case of γ) .

Considering α) , β) , γ) , δ) together it is implied that f is a homeomorphism . □

***Remark (7.3.10)** : Let $\mathfrak{F}_{cr}$ denote the set of column reduced solutions of (7.3.6) for an $\mathbb{R}[d]$ – unimodular matrix U . Then a unimodular matrix V exists such that the set $\mathfrak{F}_{cr} \cdot V$ contains no column reduced matrices . Such a matrix V , for example , is given by :*

$$V = \begin{bmatrix} 1 & d & 0 & \cdots & \cdots & 0 \\ 1 & 1+d & 0 & \cdots & \cdots & 0 \\ 1 & d & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 1 & d & 0 & & \ddots & 0 \\ 1 & d & 0 & \cdots & \cdots & 1 \end{bmatrix} \qquad (7.3.28)$$

□

***Proposition (7.3.2)** : There exists no unimodular matrix U , such that the set of column reduced solutions , $\mathfrak{F}_U^{cr}$, of (7.3.6) is either open , or closed .*

Proof

Let U be a unimodular matrix such that $\mathfrak{F}_U^{cr}$ is open . Set $W = U \cdot V$, where V is the unimodular matrix defined in remark(7.3.10) . If $\mathfrak{F}_W$, $\mathfrak{F}_W^{cr}$ denote the set of solutions , column reduced solutions of (7.3.6) respectively , for W , theorem(7.3.1) implies that $\mathfrak{F}_W^{cr}$ is a dense subset of $\mathfrak{F}_W$. If $\mathfrak{F}_U$ is the set of solutions of (7.3.6) , for U , definition(7.3.2) and proposition(7.3.1) provide us with a homeomorphism f defined between $\mathfrak{F}_U$, $\mathfrak{F}_W$ as :

$$f(X) = X \cdot U^{-1} \cdot W = X \cdot V \ , \ \forall \ X \in \mathfrak{F}_U \qquad (7.3.29)$$

Consider now the set $f(\mathfrak{F}_U^{cr}) = \mathfrak{F}_U^{cr} \cdot V$. Remark(7.3.10) clearly implies that $f(\mathfrak{F}_U^{cr})$ contains no column reduced matrices . On the other hand we shall prove that $f(\mathfrak{F}_U^{cr})$ is an open dense subset of $\mathfrak{F}_W$. Since $\mathfrak{F}_U^{cr}$ is open and f is a homeomorphism it is implied that $f(\mathfrak{F}_U^{cr})$ is open ; whereas if Y is an arbitrary element of $\mathfrak{F}_W$ then $f^{-1}(Y) = X$ belongs to $\mathfrak{F}_U$ and a sequence of elements of $\mathfrak{F}_U^{cr}$, X_n , exists , (theorem(7.3.1)) , such that $X_n \to X$. The latter implies that $Y_n = f(X_n)$ is a sequence of elements of $f(\mathfrak{F}_U^{cr})$ and :

$$\lim_{n \to \infty} Y_n = \lim_{n \to \infty} f(X_n) = \lim_{n \to \infty} (X_n \cdot V) = X \cdot V = f(X) = Y \qquad (7.3.30)$$

Thus $f(\mathfrak{F}_U^{cr})$ is an open dense subset of $\mathfrak{F}_W$. Finally the complement of $f(\mathfrak{F}_U^{cr})$, $f(\mathfrak{F}_U^{cr})^c$, is a closed subset of $\mathcal{F}_W$ and :

$$\mathcal{F}_W^{cr} \subset f(\mathcal{F}_U^{cr})^c \tag{7.3.31}$$

but ,

$$\mathcal{F}_W = \overline{\mathcal{F}_W^{cr}} \subset \overline{f(\mathcal{F}_U^{cr})^c} = f(\mathcal{F}_U^{cr})^c \subset \mathcal{F}_W. \tag{7.3.32}$$

or equivalently ,

$$\mathcal{F}_W = f(\mathcal{F}_U^{cr})^c \tag{7.3.33}$$

The latter implies that $f(\mathcal{F}_U^{cr})$ is an empty set , something that contradicts the truth , since :

$$\overline{f(\mathcal{F}_U^{cr})} = \mathcal{F}_W \tag{7.3.34}$$

Thus our initial assumption that $\mathcal{F}_U^{cr}$ is open is wrong . If on the other hand $\mathcal{F}_U^{cr}$ is closed then :

$$f(\mathcal{F}_U^{cr}) = f(\overline{\mathcal{F}_U^{cr}}) = f(\mathcal{F}_U) = f(\mathcal{F}_W) \tag{7.3.35}$$

which implies that $\mathcal{F}_W$ contains no column reduced solutions , since $f(\mathcal{F}_U^{cr})$ does not , something that contradicts remark(7.3.8) □

7.4. FIXED INDEX SOLUTIONS OF THE MATRIX DIOPHANTINE EQUATION

Let us consider the plant described by a left – coprime MFD as in (7.2.1) and assume that $T_p^l(d)$ is column reduced and that ν is the observability index i.e. we can write :

$$T_p^l(d) = [\,\widetilde{D}_0 , \widetilde{N}_0] + d\,[\,\widetilde{D}_1 , \widetilde{N}_1] + \ldots + d^\nu\,[\,\widetilde{D}_\nu , \widetilde{N}_\nu] =$$

$$= \widetilde{T}_0 + d\,\widetilde{T}_1 + \ldots + d^\nu\,\widetilde{T}_\nu \in \mathbf{R}^{mx(m+l)}[d] \tag{7.4.1}$$

We also assume that the controller is represented by the composite matrix associated with a right – MFD i.e. :

$$T_c^r(d) = \begin{bmatrix} D_{c0} \\ N_{c0} \end{bmatrix} + d \begin{bmatrix} D_{c1} \\ N_{c1} \end{bmatrix} + \ldots + d^p \begin{bmatrix} D_{cp} \\ N_{cp} \end{bmatrix} =$$

$$= T_{c0} + d\,T_{c1} + \ldots + d^p\,T_{cp} \in \mathbf{R}^{(m+l)xm}[d] \tag{7.4.2}$$

where we fix the index p , (maximum of the indices (column degrees) of the columns of $T_c^r(d)$) . It is not difficult to see that the condition :

$$T_p^l(d) \cdot T_c^r(d) = I_m \tag{7.4.3}$$

implies the following set of conditions :

$$\begin{aligned}
&\widetilde{T}_0 \cdot T_{c0} = I_m \\
&\widetilde{T}_1 \cdot T_{c0} + \widetilde{T}_0 \cdot T_{c1} = O \\
&\widetilde{T}_2 \cdot T_{c0} + \widetilde{T}_1 \cdot T_{c1} + \widetilde{T}_0 \cdot T_{c2} = O \\
&\qquad\vdots \\
&\qquad\vdots \\
&\widetilde{T}_\nu \cdot T_{cp} = O
\end{aligned} \tag{7.4.4}$$

which in matrix form may be written as :

$$\begin{bmatrix} \widetilde{T}_0 & O & \cdots & \cdots & O \\ \widetilde{T}_1 & \widetilde{T}_0 & \ddots & & \vdots \\ \vdots & \widetilde{T}_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ \widetilde{T}_\nu & \vdots & \ddots & \ddots & \widetilde{T}_0 \\ O & \widetilde{T}_\nu & \ddots & \ddots & \widetilde{T}_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ O & \cdots & \cdots & O & \widetilde{T}_\nu \end{bmatrix} \cdot \begin{bmatrix} T_{c0} \\ T_{c1} \\ \vdots \\ \vdots \\ T_{cp} \end{bmatrix} = \begin{bmatrix} I_m \\ O \\ O \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ O \\ O \end{bmatrix} \tag{7.4.5}$$

Condition (7.4.5) is equivalent to the right Diophantine equation (7.4.3) and the fixed controllability index solutions of (7.4.3) are investigated as solutions of (7.4.5) . We shall denote by :

$$Q_p \triangleq \begin{bmatrix} \widetilde{T}_0 & O & \cdots & \cdots & O \\ \widetilde{T}_1 & \widetilde{T}_0 & \ddots & & \vdots \\ \vdots & \widetilde{T}_1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ \widetilde{T}_\nu & \vdots & \ddots & \ddots & \widetilde{T}_0 \\ O & \widetilde{T}_\nu & \ddots & \ddots & \widetilde{T}_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ O & \cdots & \cdots & O & \widetilde{T}_\nu \end{bmatrix} \tag{7.4.6}$$

the r – Toeplitz matrix defined by $T_p^l(d)$. Equation (7.4.5) is equivalent to :

$$\begin{bmatrix} \tilde{T}_0 & O & \cdots & \cdots & O & -I_m \\ \tilde{T}_1 & \tilde{T}_0 & \ddots & & \vdots & O \\ \vdots & \tilde{T}_1 & \ddots & \ddots & \vdots & O \\ \vdots & \vdots & \ddots & \ddots & O & \vdots \\ \tilde{T}_\nu & \vdots & \ddots & \ddots & \tilde{T}_0 & \vdots \\ O & \tilde{T}_\nu & \ddots & \ddots & \tilde{T}_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & O \\ O & \cdots & \cdots & O & \tilde{T}_\nu & O \end{bmatrix} \cdot \begin{bmatrix} T_{c0} \\ T_{c1} \\ \vdots \\ \vdots \\ T_{cp} \\ I_m \end{bmatrix} = \begin{bmatrix} O \\ O \\ O \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ O \\ O \end{bmatrix} \tag{7.4.7}$$

and the following notation will be employed :

$$S_p \triangleq \begin{bmatrix} \tilde{T}_0 & O & \cdots & \cdots & O & -I_m \\ \tilde{T}_1 & \tilde{T}_0 & \ddots & & \vdots & O \\ \vdots & \tilde{T}_1 & \ddots & \ddots & \vdots & O \\ \vdots & \vdots & \ddots & \ddots & O & \vdots \\ \tilde{T}_\nu & \vdots & \ddots & \ddots & \tilde{T}_0 & \vdots \\ O & \tilde{T}_\nu & \ddots & \ddots & \tilde{T}_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & O \\ O & \cdots & \cdots & O & \tilde{T}_\nu & O \end{bmatrix}, \quad T_p \triangleq \begin{bmatrix} T_{c0} \\ T_{c1} \\ \vdots \\ \vdots \\ T_{cp} \\ I_m \end{bmatrix} \tag{7.4.8}$$

Proposition (7.4.1) *: The least possible controllability index of $T_c^r(d)$ is equal to the first index p of S_p , for which (7.4.7) has a solution of the type T_p as in (7.4.8) , corresponding to a column reduced MFD .* □

In the following we present necessary and sufficient conditions such that a fixed p be the least Forney index among the Forney indices of the column space of $T_c^r(d)$, in either cases of $T_c^r(d)$ corresponding to a causal or non causal controller .

Proposition (7.4.2) *: A necessary condition for (7.4.7) to have a solution of the type T_p is that :*

$$rank\ S_p \leq m \cdot p \tag{7.4.9}$$

Proof

Since the solutions of (7.4.7) of type T_p are full column rank matrices the right null space of S_p , which we denote as $\mathcal{N}_r\{ S_p \}$ must have dimension greater than or equal to m , i.e.,

$$dim\mathcal{N}_r\{ S_p \} \geq m$$

or ,

$$m(p+1) - rank\ \mathrm{S}_p \geq m$$

or,

$$rank\ \mathrm{S}_p \leq m\ p$$ □

This condition is invariant of the selection of matrices in (7.4.7) and hence in (7.2.3) . Let $\mathrm{W} = [\ \underline{\mathrm{w}}_1\ , \dots , \underline{\mathrm{w}}_j\ ]$ be a base for the $\mathcal{N}_r\{\ \mathrm{S}_p\ \}$, with $j \geq m$. If we partition W according to the partition of T_p in (7.4.8) then it follows that :

$$\mathrm{W} = \begin{bmatrix} \mathrm{W}_0 \\ \mathrm{W}_1 \\ \vdots \\ \vdots \\ \mathrm{W}_{p+1} \end{bmatrix} \tag{7.4.10}$$

If Q is a matrix in $\mathbb{R}^{mxj}$ then by $\mathrm{C}_m(\mathrm{Q})$ we denote the $1x\binom{j}{m}$ matrix consisting of the mxm minors of Q taken in lexicographical order . The conditions stated in the following propositions are invariant of the selection of the base W . The solutions of (7.4.7) does not necessarily correspond to a causal MFD .

***Proposition (7.4.3)** : A necessary and sufficient condition for solvability of (7.4.7) for a given p is that :*

$$C_m(W_{p+1}) \neq \underline{0}^{\mathrm{T}} \tag{7.4.11}$$

Proof :

(⇒) Let W be a base for the $\mathcal{N}_r\{\ \mathrm{S}_p\ \}$ and partitioned as in (7.4.10) . Suppose that $\mathrm{C}_m(\mathrm{W}_{p+1}) = \underline{0}^{\mathrm{T}}$ and let B be an other base of $\mathcal{N}_r\{\ \mathrm{S}_p\ \}$. Then , if we partition B as we did with W we have :

$$\mathrm{C}_m(\mathrm{B}_{p+1}) = \underline{0}^{\mathrm{T}}$$

Indeed , there exists an $\mathbb{R}^{jxj}$ unimodular matrix U such that : $\mathrm{B} = \mathrm{W} \cdot \mathrm{U}$, or , $\mathrm{B}_{p+1} = \mathrm{W}_{p+1} \cdot \mathrm{U}$. So , $\mathrm{C}_m(\mathrm{B}_{p+1}) = \mathrm{C}_m(\mathrm{W}_{p+1}) \cdot \mathrm{C}_m(\mathrm{U}) = \underline{0}^{\mathrm{T}}$. If (7.4.7) has a solution T_p , then it can be written as :

$$\mathrm{T}_p = \mathrm{W} \cdot \mathrm{P}$$

where , P is an $\mathbb{R}^{jxm}$ parametric matrix , or

$$\begin{bmatrix} \mathrm{T}_{c0} \\ \mathrm{T}_{c1} \\ \vdots \\ \vdots \\ \mathrm{T}_{cp} \\ \mathrm{I}_m \end{bmatrix} = \begin{bmatrix} \mathrm{W}_0 \\ \mathrm{W}_1 \\ \vdots \\ \vdots \\ \mathrm{W}_{p+1} \end{bmatrix} \cdot \mathrm{P} \tag{7.4.12}$$

Equation (7.4.12) then implies that :

$$I_m = W_{p+1} \cdot P \text{ , or } C_m(W_{p+1}) \cdot C_m(P) = 1 \tag{7.4.13}$$

which does not hold true , since $C_m(W_{p+1}) = \underline{0}^T$. Hence , there exists no solution for (7.4.7) if for an arbitrary base of $\mathcal{N}_r\{ S_p \}$, W , we have $C_m(W_{p+1}) = \underline{0}^T$.

($\Leftarrow$) Let W be an arbitrary base for the $\mathcal{N}_r\{ S_p \}$ and partitioned as in (7.4.10) . Suppose that $C_m(W_{p+1}) \neq \underline{0}^T$. We can write :

$$W_{p+1} = [\underline{w}_1^{p+1} , \dots , \underline{w}_j^{p+1}]$$

That means that we can select at least one m – tuple of columns of W_{p+1} with the following property :

$$| \underline{w}_{i_1}^{p+1}, \dots , \underline{w}_{i_m}^{p+1} | \neq 0 \tag{7.4.14}$$

We select now the columns i_1 , ... , i_m from the base W and form the following matrix :

$$\begin{bmatrix} \underline{w}_{i_1}^0 & \underline{w}_{i_2}^0 & \cdots & \cdots & \underline{w}_{i_m}^0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^1 & \underline{w}_{i_2}^1 & \cdots & \cdots & \underline{w}_{i_m}^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^{p+1} & \underline{w}_{i_2}^{p+1} & \cdots & \cdots & \underline{w}_{i_m}^{p+1} \end{bmatrix} \tag{7.4.15}$$

where , $\underline{w}_{i_t}^{\kappa}$ is a column vector in $\mathbb{R}^{(m+l)x1}$, when $\kappa = 0$, ... , p and in $\mathbb{R}^{mx1}$, when $\kappa =$ $= p + 1$. If T denotes the matrix (7.4.15) then we have :

$$T \equiv \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ \vdots \\ T_p \\ T_{p+1} \end{bmatrix} = \begin{bmatrix} \underline{w}_{i_1}^0 & \underline{w}_{i_2}^0 & \cdots & \cdots & \underline{w}_{i_m}^0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^1 & \underline{w}_{i_2}^1 & \cdots & \cdots & \underline{w}_{i_m}^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^{p+1} & \underline{w}_{i_2}^{p+1} & \cdots & \cdots & \underline{w}_{i_m}^{p+1} \end{bmatrix} \tag{7.4.16}$$

Where , by (7.4.14) the matrix $T_{p+1} = [\, \underline{w}_{i_1}^{p+1}, \; \ldots \; , \underline{w}_{i_m}^{p+1} \,]$ has the property :

$$| T_{p+1} | = | \underline{w}_{i_1}^{p+1}, \; \ldots \; , \underline{w}_{i_m}^{p+1} | \neq 0 \tag{7.4.17}$$

Now we can create the matrix $T_p = T \cdot T_{p+1}^{-1}$ and T_p is a solution of (7.4.7) . Indeed :

$$S_p \cdot T_p = S_p \cdot T \cdot T_{p+1}^{-1} = \mathbf{O}$$

since $S_p \cdot T = \mathbf{O}$, (each column of T is a column of the base W of $\mathcal{N}_r\{ S_p \}$) , and :

$$T_p = \begin{bmatrix} T_{c0} \\ T_{c1} \\ \vdots \\ \vdots \\ T_{cp} \\ I_m \end{bmatrix} = \begin{bmatrix} T_0 \cdot T_{p+1}^{-1} \\ T_1 \cdot T_{p+1}^{-1} \\ \vdots \\ \vdots \\ T_p \cdot T_{p+1}^{-1} \\ T_{p+1} \cdot T_{p+1}^{-1} \end{bmatrix}$$

□

By remark(7.2.1) we require $| D_{c0} | \neq 0$, so the MFD $C = N_c \cdot D_c^{-1}$ will be causal . If we write equation (7.4.7) as :

$$\begin{bmatrix} \tilde{D}_0 & \tilde{N}_0 & O & O & \cdots & \cdots & O & O & : & -I_m \\ \tilde{D}_1 & \tilde{N}_1 & \tilde{D}_0 & \tilde{N}_0 & \ddots & & \vdots & \vdots & : & O \\ \vdots & \vdots & \tilde{D}_1 & \tilde{N}_1 & \ddots & \ddots & \vdots & \vdots & : & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & O & O & : & \vdots \\ \tilde{D}_\nu & \tilde{N}_\nu & \vdots & \vdots & & \ddots & \tilde{D}_0 & \tilde{N}_0 & : & \vdots \\ O & O & \tilde{D}_\nu & \tilde{N}_\nu & & & \tilde{D}_1 & \tilde{N}_1 & : & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & : & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & : & O \\ O & O & \cdots & \cdots & O & O & \tilde{D}_\nu & \tilde{N}_\nu & : & O \end{bmatrix} \cdot \begin{bmatrix} D_{c0} \\ N_{c0} \\ D_{c1} \\ N_{c1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ I_m \end{bmatrix} = \underline{O} \tag{7.4.18}$$

Then we define :

$$\tilde{S}_p \triangleq \begin{bmatrix} \tilde{D}_0 & \tilde{N}_0 & O & O & \cdots & \cdots & O & O & : & -I_m \\ \tilde{D}_1 & \tilde{N}_1 & \tilde{D}_0 & \tilde{N}_0 & \ddots & & \vdots & \vdots & : & O \\ \vdots & \vdots & \tilde{D}_1 & \tilde{N}_1 & \ddots & \ddots & \vdots & \vdots & : & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & O & O & : & \vdots \\ \tilde{D}_\nu & \tilde{N}_\nu & \vdots & \vdots & & \ddots & \tilde{D}_0 & \tilde{N}_0 & : & \vdots \\ O & O & \tilde{D}_\nu & \tilde{N}_\nu & & & \tilde{D}_1 & \tilde{N}_1 & : & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & : & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & : & O \\ O & O & \cdots & \cdots & O & O & \tilde{D}_\nu & \tilde{N}_\nu & : & O \end{bmatrix} , \; \tilde{T}_p \triangleq \begin{bmatrix} D_{c0} \\ N_{c0} \\ D_{c1} \\ N_{c1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ I_m \end{bmatrix} \tag{7.4.19}$$

Consider now a base of $\mathcal{N}_r\{ \widetilde{S}_p \}$, W , and partition it according to the partition of $\widetilde{T}_p$ in (7.4.19) , that is :

$$W = \begin{bmatrix} W_{D_{c0}} \\ W_{N_{c0}} \\ \vdots \\ \vdots \\ W_{I_m} \end{bmatrix} \tag{7.4.20}$$

If we compare the partition of $\mathcal{W}$ in (7.4.10) with the one in (7.4.20) we clearly have :

$$W_1 = \begin{bmatrix} W_{D_{c0}} \\ W_{N_{c0}} \end{bmatrix} , W_2 = \begin{bmatrix} W_{D_{c1}} \\ W_{N_{c1}} \end{bmatrix} , \dots , W_{p+1} = \begin{bmatrix} W_{I_m} \end{bmatrix} \tag{7.4.21}$$

Where , $W_{D_{c\kappa}}$ belongs to $\mathbb{R}^{mxj}$, $W_{N_{c\kappa}}$ belongs to $\mathbb{R}^{lxj}$, $\kappa = 0 , 1 , \dots , p$ and W_{I_m} belongs to $\mathbb{R}^{mxj}$.

Proposition (7.4.4) *: A necessary and sufficient condition for the existence of a solution $\widetilde{T}_p$ of (7.4.18) with $| D_{c0} | \neq 0$, is that both $C_m\left(W_{D_{c0}}\right)$ and $C_m\left(W_{I_m}\right)$ are non zero vectors , for an arbitrary base , W , of $\mathcal{N}_r\{ \widetilde{S}_p \}$.*

Proof

In proposition(7.4.2) we have shown that the necessary and sufficient condition for the existence of a solution T_p of (7.4.7) and hence of a solution $\widetilde{T}_p$, (without the constraint $|D_{c0}| \neq 0$) , of (7.4.18) is that for an arbitrary base W of $\mathcal{N}_r\{ S_p \} \equiv \mathcal{N}_r\{ \widetilde{S}_p \}$, partitioned as in (7.4.10) :

$$C_m(W_{p+1}) \neq \underline{0}^T$$

If the base $\mathcal{W}$ is partitioned as in (7.4.20) then by (7.4.21) we take :

$$C_m(W_{p+1}) = C_m\left(W_{I_m}\right) \neq \underline{0}^T \tag{7.4.22}$$

So , while (7.4.22) holds true , it is enough to examine condition $C_m\left(W_{D_{c0}}\right) \neq \underline{0}^T$.

($\Rightarrow$) If (7.4.18) has a solution :

$$\widetilde{T}_p \triangleq \begin{bmatrix} D_{c0} \\ N_{c0} \\ D_{c1} \\ N_{c1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ I_m \end{bmatrix}$$

with $| D_{c0} | \neq 0$, then $\widetilde{S}_p \cdot \widetilde{T}_p = \underline{O}$ and $\widetilde{T}_p$ is a full column rank matrix ; that means , that $\widetilde{T}_p$ can be completed , if necessary , to give a base B of $\mathcal{N}_r\{ \widetilde{S}_p \}$. If B is partitioned as in (7.4.20) and because of its construction :

$$C_m\left(B_{D_{c0}}\right) \neq \underline{0}^T \tag{7.4.23}$$

in other wards , if we take the $C_m\left(B_{D_{c0}}\right)$ then , at least the minor formed by the columns of D_{c0} is non zero . Any other base of $\mathcal{N}_r\{ \widetilde{S}_p \}$, let say , W is expressed as :

$$W = B \cdot U$$

where U is an $\mathbb{R}^{jxj}$ unimodular matrix . Then by (7.4.20) we have :

$$W_{D_{c0}} = B_{D_{c0}} \cdot U \tag{7.4.24}$$

By (7.4.23) we conclude that $B_{D_{c0}}$ is a full row rank matrix , ($W_{D_{c0}}$, $B_{D_{c0}} \in \mathbb{R}^{mxj}$, $j \geq m$) , and because U is unimodular $W_{D_{c0}}$ must be a full row rank matrix as well . Hence :

$$C_m\left(W_{D_{c0}}\right) \neq \underline{0}^T \tag{7.4.25}$$

So , (7.4.22) and (7.4.25) hold simultaneously .

($\Leftarrow$) Consider now an arbitrary base of $\mathcal{N}_r\{ \widetilde{S}_p \}$, W , partitioned as in (7.4.20) and :

$$C_m\left(W_{D_{c0}}\right) \text{ and } C_m\left(W_{I_m}\right) \neq \underline{0}^T \tag{7.4.26}$$

Because $W_{D_{c0}}$, W_{I_m} are in $\mathbb{R}^{mxj}$ and $j \geq m$ is implied that the elements of $C_m\left(W_{D_{c0}}\right)$ and $C_m\left(W_{I_m}\right)$ are the mxm minors formed by the $(i_1 , \dots , i_m)$ columns of $W_{D_{c0}}$ and W_{I_m} respectively taken in lexicographical order . By $(1 , (i_1 , \dots , i_m))$ we denote the element of $C_m(W_i)$, $i = D_{c0}$, I_m , formed by the minor of the $(i_1 , \dots , i_m)$ columns of

W_i . If the $C_m(W_{D_{c0}})$ and $C_m(W_{I_m})$ have at least one non zero element at the same position $(1\ ,\ (i_1\ ,\ \dots\ ,\ i_m))$ then by selecting the $(i_1\ ,\ \dots\ ,\ i_m)$ columns of W , preserving its partition (7.4.20) , we form a matrix :

$$\begin{bmatrix} \underline{w}_{i_1}^{D_{c0}} & \underline{w}_{i_2}^{D_{c0}} & \cdots & \cdots & \underline{w}_{i_m}^{D_{c0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^{N_{c0}} & \underline{w}_{i_2}^{N_{c0}} & \cdots & \cdots & \underline{w}_{i_m}^{N_{c0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^{I_m} & \underline{w}_{i_2}^{I_m} & \cdots & \cdots & \underline{w}_{i_m}^{I_m} \end{bmatrix} \tag{7.4.27}$$

where , $\underline{w}_{i_t}^{D_{c\kappa}}$ is a column vector in $\mathbb{R}^{m x 1}$, $\underline{w}_{i_t}^{N_{c\kappa}}$ is a column vector in $\mathbb{R}^{l x 1}$, $\kappa = 0\ ,\ \dots\ ,$ p and $\underline{w}_{i_t}^{I_m}$ is a column vector in $\mathbb{R}^{m x 1}$. If $\widetilde{T}$ denotes the matrix (7.4.26) then we have :

$$\widetilde{T} \equiv \begin{bmatrix} T_{D_{c0}} \\ T_{N_{c0}} \\ \vdots \\ \vdots \\ \vdots \\ T_{I_m} \end{bmatrix} \equiv \begin{bmatrix} \underline{w}_{i_1}^{D_{c0}} & \underline{w}_{i_2}^{D_{c0}} & \cdots & \cdots & \underline{w}_{i_m}^{D_{c0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^{N_{c0}} & \underline{w}_{i_2}^{N_{c0}} & \cdots & \cdots & \underline{w}_{i_m}^{N_{c0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{w}_{i_1}^{I_m} & \underline{w}_{i_2}^{I_m} & \cdots & \cdots & \underline{w}_{i_m}^{I_m} \end{bmatrix} \tag{7.4.28}$$

and

$$|\, T_{D_{c0}} \,| \equiv |\ \underline{w}_{i_1}^{D_{c0}}\ \underline{w}_{i_2}^{D_{c0}} \cdots \underline{w}_{i_m}^{D_{c0}}\ | \neq 0 \tag{7.4.29}$$

$$|\, T_{I_m} \,| \equiv |\ \underline{w}_{i_1}^{I_m}\ \underline{w}_{i_2}^{I_m} \cdots \underline{w}_{i_m}^{I_m}\ | \neq 0 \tag{7.4.30}$$

By (7.4.28) , (7.4.29) the matrix $T_{I_m}^{-1}$ exists and we can set :

$$\widetilde{T}_p = \begin{bmatrix} D_{c0} \\ N_{c0} \\ \vdots \\ \vdots \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} T_{D_{c0}} \cdot T_{I_m}^{-1} \\ T_{N_{c0}} \cdot T_{I_m}^{-1} \\ \vdots \\ \vdots \\ \vdots \\ T_{I_m} \cdot T_{I_m}^{-1} \end{bmatrix}$$

and hence by (7.4.27) it is implied that :

$$\widetilde{S}_p \cdot \widetilde{T}_p = \widetilde{S}_p \cdot \widetilde{T} \cdot T^{-1}_{I_m} = O \tag{7.4.31}$$

and $|\ D_{c0}\ | = |\ T_{D_{c0}} \cdot T^{-1}_{I_m}\ | = |\ T_{D_{c0}}\ | \cdot |\ T^{-1}_{I_m}\ | \neq 0$. So , $\widetilde{T}_p$ is a solution for (7.4.18) . If the $C_m\left(W_{D_{c0}}\right)$ and $C_m\left(W_{I_m}\right)$ have none of their non zero elements at the same position then we can create an other base of $\mathcal{N}_r\{\ \widetilde{S}_p\ \}$, B , such that , the $C_m\left(B_{D_{c0}}\right)$ and $C_m\left(B_{I_m}\right)$ have at least one non zero element at the same position . Let the (1 , (i_1 , … , i_m)) element of $C_m\left(W_{D_{c0}}\right)$ and the (1 , (t_1 , … , t_m)) element of $C_m\left(W_{I_m}\right)$ be non zero with $(i_1 , \dots , i_m) \neq (t_1 , \dots , t_m)$. Create now the base of $\mathcal{N}_r\{\ \widetilde{S}_p\ \}$, B , as :

$$B = W \cdot U \tag{7.4.32}$$

where U is an $\mathbb{R}^{jxj}$ unimodular matrix which multiplies the i_κ column of W by the real constant c_κ and adds the resultant column to the t_κ column of W when $i_\kappa \neq t_\kappa$, $\kappa = 1$, 2 , … , m . For this new base B , if partitioned as in (7.4.20) , we shall prove that the (1 , (t_1 , … , t_m)) element of $C_m\left(B_{D_{c0}}\right)$ and $C_m\left(B_{I_m}\right)$ are non zero for some appropriate selection of c_κ . By $C_m\left(B_{D_{c0}}\right)_{t_1}^{t_m}$ and $C_m\left(B_{I_m}\right)_{t_1}^{t_m}$ is denoted the (1 , (t_1 , … , t_m)) element of $C_m\left(B_{D_{c0}}\right)$ and $C_m\left(B_{I_m}\right)$ respectively . Before we continue with the proof some additional notation is introduced .

Consider the sets $P_1 = \{\ i_1 , t_1\ \}$, $P_2 = \{\ i_2 , t_2\}$, … , $P_\rho = \{\ i_\rho , t_\rho\ \}$, with $\rho \in \mathbb{N}^*$, $\rho \leq m$. If Γ denotes the set :

$$\Gamma = \{\ \gamma^\rho\ \} = \{\gamma_1 , \gamma_2 , \dots , \gamma_\rho\ \} \in P_1\ x\ P_2\ x \cdots x\ P_\rho \tag{7.4.33}$$

It is clear that the cardinal of Γ is 2^ρ . Suppose now that $i_\kappa \neq t_\kappa$ for κ_1 , κ_2 , … , κ_ρ , whereas $i_\kappa = t_\kappa$ for $\kappa \in \{\ 1 , 2 , \dots , m\ \} - \{\ \kappa_1 , \kappa_2 , \dots , \kappa_\rho\ \}$; then without lose of generality it can be assumed that:

$$i_\kappa \neq t_\kappa\ ,\ \kappa = 1 , 2 , \dots , \rho \text{ and } i_\kappa = t_\kappa\ ,\ \kappa = \rho + 1 , \rho + 2 , \dots , m \tag{7.4.34}$$

by interchanging the i_{κ_1} , i_{κ_2} , … , i_{κ_ρ} columns of W with the i_κ for $\kappa \in \{1 , 2 , \dots , m\} - \{\ \kappa_1 , \kappa_2 , \dots , \kappa_\rho\ \}$ respectively . Now set :

$$d_\kappa = \gamma_\kappa - t_\kappa = \begin{cases} d_\kappa = 0\ ,\ \text{when } \gamma_\kappa = t_\kappa \\ \\ d_\kappa \neq 0\ ,\ \text{when } \gamma_\kappa = i_\kappa \end{cases} ,\ \kappa = 1 , 2 , \dots , \rho$$

and $q_\kappa = (1/(i_\kappa - t_\kappa))$, $q_\kappa \neq 0$, $\kappa = 1 , 2 , \dots , \rho$. According to the procedure for the construction of B we take that :

$$C_m\left(B_{D_{c0}}\right)_{t_1}^{t_m} = \left\| \left[\left(c_1 \cdot \underline{w}_{i_1}^{D_{c0}} + \underline{w}_{t_1}^{D_{c0}}\right), \cdots, \left(c_\rho \cdot \underline{w}_{i_\rho}^{D_{c0}} + \underline{w}_{t_\rho}^{D_{c0}}\right), \underline{w}_{t_{\rho+1}}^{D_{c0}}, \dots, \underline{w}_{t_m}^{D_{c0}} \right] \right\| =$$

$$= \sum_{\{\gamma^\rho\}} \left(c_1^{d_1}\right)^{q_1} \cdot \left(c_2^{d_2}\right)^{q_2} \cdot \cdots \cdot \left(c_\rho^{d_\rho}\right)^{q_\rho} \cdot \mid \underline{w}_{\gamma_1}^{D_{c0}}, \cdots, \underline{w}_{\gamma_\rho}^{D_{c0}}, \underline{w}_{t_{\rho+1}}^{D_{c0}}, \dots, \underline{w}_{t_m}^{D_{c0}} \mid \tag{7.4.35}$$

where , $\left(c_\kappa^{d_\kappa}\right)^{q_\kappa} = c_\kappa$, when $\gamma_\kappa = i_\kappa$ and $\left(c_\kappa^{d_\kappa}\right)^{q_\kappa} = 1$, when $\gamma_\kappa = t_\kappa$, $\kappa = 1 , 2 , \dots , \rho$. For $\{ \gamma^\rho \} = \{i_1 , i_2 , \dots , i_\rho \}$, (7.4.35) becomes :

$$C_m\left(B_{D_{c0}}\right)_{t_1}^{t_m} = c_1 \cdot c_2 \cdot \cdots \cdot c_\rho \cdot \mid \underline{w}_{i_1}^{D_{c0}}, \cdots, \underline{w}_{i_\rho}^{D_{c0}}, \underline{w}_{t_{\rho+1}}^{D_{c0}}, \dots, \underline{w}_{t_m}^{D_{c0}} \mid +$$

$$+ \sum_{\{\gamma^\rho\}^*} \left(c_1^{d_1}\right)^{q_1} \cdot \left(c_2^{d_2}\right)^{q_2} \cdot \cdots \cdot \left(c_\rho^{d_\rho}\right)^{q_\rho} \cdot \mid \underline{w}_{\gamma_1}^{D_{c0}}, \cdots, \underline{w}_{\gamma_\rho}^{D_{c0}}, \underline{w}_{t_{\rho+1}}^{D_{c0}}, \dots, \underline{w}_{t_m}^{D_{c0}} \mid \tag{7.4.36}$$

where , $\{ \gamma^\rho \}^* = \Gamma - \{i_1 , i_2 , \dots , i_\rho \}$. By (7.4.34) , (7.4.36) becomes :

$$C_m\left(B_{D_{c0}}\right)_{t_1}^{t_m} = c_1 \cdot c_2 \cdot \cdots \cdot c_\rho \cdot \mid \underline{w}_{i_1}^{D_{c0}}, \cdots, \underline{w}_{i_\rho}^{D_{c0}}, \underline{w}_{i_{\rho+1}}^{D_{c0}}, \dots, \underline{w}_{i_m}^{D_{c0}} \mid +$$

$$+ \sum_{\{\gamma^\rho\}^*} \left(c_1^{d_1}\right)^{q_1} \cdot \left(c_2^{d_2}\right)^{q_2} \cdot \cdots \cdot \left(c_\rho^{d_\rho}\right)^{q_\rho} \cdot \mid \underline{w}_{\gamma_1}^{D_{c0}}, \cdots, \underline{w}_{\gamma_\rho}^{D_{c0}}, \underline{w}_{t_{\rho+1}}^{D_{c0}}, \dots, \underline{w}_{t_m}^{D_{c0}} \mid \tag{7.4.37}$$

Similarly we have :

$$C_m\left(B_{I_m}\right)_{t_1}^{t_m} = \left\| \left[\left(c_1 \cdot \underline{w}_{i_1}^{I_m} + \underline{w}_{t_1}^{I_m}\right), \cdots, \left(c_\rho \cdot \underline{w}_{i_\rho}^{I_m} + \underline{w}_{t_\rho}^{I_m}\right), \underline{w}_{t_{\rho+1}}^{I_m}, \dots, \underline{w}_{t_m}^{I_m} \right] \right\| =$$

$$= \sum_{\{\gamma^\rho\}} \left(c_1^{d_1}\right)^{q_1} \cdot \left(c_2^{d_2}\right)^{q_2} \cdot \cdots \cdot \left(c_\rho^{d_\rho}\right)^{q_\rho} \cdot \mid \underline{w}_{\gamma_1}^{I_m}, \cdots, \underline{w}_{\gamma_\rho}^{I_m}, \underline{w}_{t_{\rho+1}}^{I_m}, \dots, \underline{w}_{t_m}^{I_m} \mid \tag{7.4.38}$$

where , $\left(c_\kappa^{d_\kappa}\right)^{q_\kappa} = c_\kappa$, when $\gamma_\kappa = i_\kappa$ and $\left(c_\kappa^{d_\kappa}\right)^{q_\kappa} = 1$, when $\gamma_\kappa = t_\kappa$, $\kappa = 1 , 2 , \dots , \rho$. For $\{ \gamma^\rho \} = \{t_1 , t_2 , \dots , t_\rho \}$, (7.4.38) becomes :

$$C_m\left(B_{I_m}\right)_{t_1}^{t_m} = \mid \underline{w}_{t_1}^{I_m}, \cdots, \underline{w}_{t_\rho}^{I_m}, \underline{w}_{t_{\rho+1}}^{I_m}, \dots, \underline{w}_{t_m}^{I_m} \mid +$$

$$+ \sum_{\{\gamma^\rho\}^*} \left(c_1^{d_1}\right)^{q_1} \cdot \left(c_2^{d_2}\right)^{q_2} \cdot \cdots \cdot \left(c_\rho^{d_\rho}\right)^{q_\rho} \cdot \mid \underline{w}_{\gamma_1}^{I_m}, \cdots, \underline{w}_{\gamma_\rho}^{I_m}, \underline{w}_{t_{\rho+1}}^{I_m}, \dots, \underline{w}_{t_m}^{I_m} \mid \tag{7.4.39}$$

where , $\{ \gamma^{\rho} \}^{*} = \Gamma - \{t_1 , t_2 , \dots , t_{\rho} \}$. Using the hypothesis , that is :

$$| \underline{w}_{i_1}^{D_{c0}} , \cdots , \underline{w}_{i_{\rho}}^{D_{c0}} , \underline{w}_{i_{\rho+1}}^{D_{c0}} , \dots , \underline{w}_{i_m}^{D_{c0}} | \neq 0$$

$$| \underline{w}_{t_1}^{I_m} , \cdots , \underline{w}_{t_{\rho}}^{I_m} , \underline{w}_{t_{\rho+1}}^{I_m} , \dots , \underline{w}_{t_m}^{I_m} | \neq 0$$

we take that for appropriate selection of $c_1 , c_2 , \dots , c_{\rho}$, (7.4.37) and (7.4.39) , namely , $C_m\left(B_{D_{c0}}\right)_{t_1}^{t_m}$ and $C_m\left(B_{I_m}\right)_{t_1}^{t_m}$ are non zero . Thus the $(1 , (t_1 , \dots , t_m))$ element of $C_m\left(B_{D_{c0}}\right)$ and $C_m\left(B_{I_m}\right)$ is non zero . Now we construct a solution of (7.4.18) following the steps (7.4.27) through (7.4.31) for the new base $\mathfrak{B}$. □

Remark (7.4.1) *: For a parametrization of the solutions $\widetilde{T}_p$ of (7.4.18) we argue as follows . The existence of a solution of (7.4.18) requires the existence of a base W of $\mathcal{N}_r\{ \widetilde{S}_p \}$, for which , under the partition (7.4.20) , both $C_m\left(W_{D_{c0}}\right)$ and $C_m\left(W_{I_m}\right)$ are non zero vectors ; it is clear that the existence of such a base leads to the conclusion that for all the bases B of $\mathcal{N}_r\{ \widetilde{S}_p \}$, under the partition (7.4.20) , both $C_m\left(B_{D_{c0}}\right)$ and $C_m\left(B_{I_m}\right)$ are non zero vectors , (by simply generalizing the steps in (⇒) of proposition(7.4.4)) . A solution $\widetilde{T}_p$ of (7.4.18) is a full column rank matrix , hence, it is a base for a subspace of $\mathcal{N}_r\{ \widetilde{S}_p \}$, let say , $\mathfrak{T}$ with dimension m . So , each solution $\widetilde{T}_p$ can be completed to be a base for $\mathcal{N}_r\{ \widetilde{S}_p \}$. Thus , all solutions of (7.4.18), if any , can be obtained by extracting from the bases B of $\mathcal{N}_r\{ \widetilde{S}_p \}$ their $\{i_1 , i_2 , \dots , i_m\}$ columns for which $C_m\left(B_{D_{c0}}\right)$ and $C_m\left(B_{I_m}\right)$ have a non zero element at the $(1 , \{i_1 , i_2 , \dots , i_m \})$ position . All the bases B can be obtained by simply multiplying one of them with an arbitrary $\mathbb{R}^{j \times j}$ unimodular matrix U . So , first an arbitrary base B of $\mathcal{N}_r\{\widetilde{S}_p\}$ is examined for the truth of the conditions introduced in proposition(7.4.3) , then a solution of (7.4.18) can be constructed , (as in (⇐) of proposition(7.4.3)) . For the parametrization of the solutions of (7.4.18) , we multiply this base B by an arbitrary $\mathbb{R}^{j \times j}$ unimodular matrix U and each time the parameters of U take a value , a new solution can be found by repeating the steps of the (⇐) part of the proof of proposition(7.4.3) for the new base $B \cdot U$.* □

7.5. FIXED COMPLEXITY SOLUTIONS – PI CONTROLLERS

In the following we consider the PI controller problem , where the complexity of the controller is fixed and equal to m . Let $P \in \mathbb{R}^{m \times l}(d)$ denote the plant and $C = C_0 + C_1 \cdot (1/(1-d)) \in \mathbb{R}^{l \times m}(d)$ denote a PI controller with $C_1 \in \mathbb{R}^{l \times m}$ full column rank matrix when $l \geq m$, full row rank matrix when $l < m$. Then the plant and controller may be represented by $\mathbb{R}[d]$ – coprime MFDs as :

$$P = N \cdot D^{-1} = \widetilde{D}^{-1} \cdot \widetilde{N} \quad \in \mathbb{R}^{mxl}[d] \tag{7.5.1}$$

$$C = N_c \cdot D_c^{-1} = \widetilde{D}_c^{-1} \cdot \widetilde{N}_c \in \mathbb{R}^{lxm}[d] \tag{7.5.2}$$

where ,

$$\begin{aligned} C_0 &= A_2 \cdot A_0^{-1}, \text{ when } l \geq m \\ D_c &= (1-d)\, A_0 \text{ , with } A_0 \in \mathbb{R}^{mxm} \text{ and } |\, A_0 \,| \neq 0 \\ N_c &= A_1 + (1-d)\, A_2 \text{ , with } A_1 = C_1 \cdot A_0 \text{ , } A_2 \in \mathbb{R}^{lxm} \end{aligned}$$

$$\tag{7.5.2.a}$$

$$\begin{aligned} C_0 &= \widetilde{A}_0^{-1} \cdot \widetilde{A}_2 \text{ , when } l < m \\ \widetilde{D}_c &= (1-d)\, \widetilde{A}_0 \text{ , with } \widetilde{A}_0 \in \mathbb{R}^{lxl} \text{ and } |\, \widetilde{A}_0 \,| \neq 0 \\ \widetilde{N}_c &= \widetilde{A}_1 + (1-d)\, \widetilde{A}_2 \text{ , with } \widetilde{A}_1 = \widetilde{A}_0 \cdot C_1 \text{ , } \widetilde{A}_2 \in \mathbb{R}^{lxm} \end{aligned}$$

In the following we consider (7.5.2.a) under the transformation $w = (1-d)$. The stabilization problem for the plant P with the PI controller C , (as in (7.5.2) , (7.5.2.a)) leads us to examine the following problem .

***Problem** : Given a plant as in (7.5.1) find all the possible controllers C , (as in (7.5.2) , (7.5.2.a)) such that the following Diophantine equations are satisfied :*

$$\widetilde{D} D_c + \widetilde{N} N_c = I_m \text{ , or } [\, \widetilde{D},\ \widetilde{N}\,] \begin{bmatrix} D_c \\ N_c \end{bmatrix} = I_m \text{ , when } l \geq m \tag{7.5.3}$$

or ,

$$\widetilde{D}_c\, D_c + \widetilde{N}_c\, N_c = I_l \text{ , or } [\widetilde{D}_c \text{ , } \widetilde{N}_c] \begin{bmatrix} D \\ N \end{bmatrix} = I_l \text{ , when } l < m \tag{7.5.4}$$

□

In the following , we shall represent both plant and controller in terms of composite matrices as :

$$T_p^l(w) \triangleq [\, \widetilde{D} \text{ , } \widetilde{N}\,] \in \mathbb{R}^{mx(m+l)}[w] \tag{7.5.5.a}$$

$$T_c^r(w) \triangleq \begin{bmatrix} D_c \\ N_c \end{bmatrix} \in \mathbb{R}^{(m+l)xm}[w] \tag{7.5.5.b}$$

Furthermore we consider only equation (7.5.3) , since all the results for (7.5.3) apply to equation (7.5.4) as well in their dual form .

***Remark (7.5.1)** : Equation (7.5.3) suggests that the matrix $[\widetilde{D},\ \widetilde{N}] \in \mathbb{R}^{mx(m+l)}[w]$ is right unimodular and $[D_c^T \text{ , } N_c^T]^T \in \mathbb{R}^{(m+l)xm}[w]$ is leftt unimodular . So , rank $[D_c^T \text{ , } N_c^T]^T$ must be equal to m for all the $w \in \mathbb{C}$. For $w = 0$ we take rank $[O \text{ , } A_1^T]^T = m$, which implies that $l \geq m$. Similar arguments for equation (7.5.4) imply that $l < m$.* □

By (7.5.5.a) and (7.5.5.b) we take :

$$T_p^l(w) = [\,\tilde{D}_0\,,\,\tilde{N}_0] + w\,[\,\tilde{D}_1\,,\,\tilde{N}_1] + \ldots + w^\nu\,[\,\tilde{D}_\nu\,,\,\tilde{N}_\nu] =$$

$$= \tilde{T}_0 + w\,\tilde{T}_1 + \ldots + w^\nu\,\tilde{T}_\nu \in \mathbb{R}^{mx(m+l)}[w] \qquad (7.5.6.a)$$

$$T_c^r(d) = \begin{bmatrix} D_{c0} \\ N_{c0} \end{bmatrix} + w \begin{bmatrix} D_{c1} \\ N_{c1} \end{bmatrix} = T_{c0} + w\,T_{c1} \in \mathbb{R}^{(m+l)xm}[w] \qquad (7.5.6.b)$$

where , ν is the observability index of the plant P . By (7.5.2.a) , (7.5.6.b) becomes :

$$T_c^r(w) = \begin{bmatrix} O \\ A_1 \end{bmatrix} + w \begin{bmatrix} A_0 \\ A_2 \end{bmatrix} = T_{c0} + w\,T_{c1} \in \mathbb{R}^{(m+l)xm}[w] \qquad (7.5.6.b)$$

Then equation (7.5.3) gives :

$$T_p^l(w) \cdot T_c^r(w) = I_m \qquad (7.5.7)$$

which implies the following set of conditions :

$$\begin{aligned} &\tilde{T}_0 \cdot T_{c0} = I_m \\ &\tilde{T}_1 \cdot T_{c0} + \tilde{T}_0 \cdot T_{c1} = O \\ &\qquad\vdots \\ &\tilde{T}_\nu \cdot T_{c1} = O \end{aligned} \qquad (7.5.8)$$

which in matrix form may be written as :

$$\begin{bmatrix} \tilde{T}_0 & O \\ \tilde{T}_1 & \tilde{T}_0 \\ \vdots & \tilde{T}_1 \\ \vdots & \vdots \\ \tilde{T}_\nu & \vdots \\ O & \tilde{T}_\nu \end{bmatrix} \cdot \begin{bmatrix} T_{c0} \\ T_{c1} \end{bmatrix} = \begin{bmatrix} I_m \\ O \\ O \\ \vdots \\ \vdots \\ O \end{bmatrix} \qquad (7.5.9)$$

or equivalently ,

$$\begin{bmatrix} \tilde{D}_0 & \tilde{N}_0 & O & O \\ \tilde{D}_1 & \tilde{N}_1 & \tilde{D}_0 & \tilde{N}_0 \\ \vdots & \vdots & \tilde{D}_1 & \tilde{N}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_\nu & \tilde{N}_\nu & \vdots & \vdots \\ O & O & \tilde{D}_\nu & \tilde{N}_\nu \end{bmatrix} \cdot \begin{bmatrix} O \\ A_1 \\ A_0 \\ A_2 \end{bmatrix} = \begin{bmatrix} I_m \\ O \\ O \\ \vdots \\ \vdots \\ O \end{bmatrix} \qquad (7.5.10)$$

or equivalently ,

$$\begin{bmatrix} \tilde{D}_0 & \tilde{N}_0 & O & O & \vdots & -I_m \\ \tilde{D}_1 & \tilde{N}_1 & \tilde{D}_0 & \tilde{N}_0 & \vdots & O \\ \vdots & \vdots & \tilde{D}_1 & \tilde{N}_1 & \vdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_\nu & \tilde{N}_\nu & \vdots & \vdots & \vdots & \vdots \\ O & O & \tilde{D}_\nu & \tilde{N}_\nu & \vdots & O \end{bmatrix} \cdot \begin{bmatrix} O \\ A_1 \\ A_0 \\ A_2 \\ I_m \end{bmatrix} = \begin{bmatrix} O \\ O \\ O \\ \vdots \\ \vdots \\ O \end{bmatrix} \tag{7.5.11}$$

Let $M \in \mathbb{R}^{(\nu+2)mx(3m+2l)}$, $X \in \mathbb{R}^{(3m+2l)xm}$ denote the matrices :

$$M = \begin{bmatrix} \tilde{D}_0 & \tilde{N}_0 & O & O & \vdots & -I_m \\ \tilde{D}_1 & \tilde{N}_1 & \tilde{D}_0 & \tilde{N}_0 & \vdots & O \\ \vdots & \vdots & \tilde{D}_1 & \tilde{N}_1 & \vdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_\nu & \tilde{N}_\nu & \vdots & \vdots & \vdots & \vdots \\ O & O & \tilde{D}_\nu & \tilde{N}_\nu & \vdots & O \end{bmatrix} , \tilde{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} O \\ A_1 \\ A_0 \\ A_2 \\ I_m \end{bmatrix} \tag{7.5.12}$$

with $X_1 = O \in \mathbb{R}^{mxm}$, $X_2 \in \mathbb{R}^{lxm}$ full column rank , $X_3 \in \mathbb{R}^{mxm}$ and $|X_3| \neq 0$, $X_4 \in \mathbb{R}^{lxm}$, $X_5 = I_m \in \in \mathbb{R}^{mxm}$ and $|X_5| \neq 0$, whereas $M' \in \mathbb{R}^{(\nu+2)mx2(m+l)}$ denotes the matrix :

$$M' = \begin{bmatrix} \tilde{N}_0 & O & O & \vdots & -I_m \\ \tilde{N}_1 & \tilde{D}_0 & \tilde{N}_0 & \vdots & O \\ \vdots & \tilde{D}_1 & \tilde{N}_1 & \vdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{N}_\nu & \vdots & \vdots & \vdots & \vdots \\ O & \tilde{D}_\nu & \tilde{N}_\nu & \vdots & O \end{bmatrix} \tag{7.5.13}$$

Equation (7.5.3) has been transformed to the form (7.5.11) , or , by using the notation (7.5.12) , to the form:

$$M \cdot \tilde{X} = \mathbf{O} \tag{7.5.14}$$

Hence , it suffices to solve equation (7.5.14) under the constraints $X_1 = O$, X_2 full column rank , $|X_3| \neq 0$, $|X_5| \neq 0$, (not necessarily I_m) , and set $A_0 = X_3 \cdot X_5^{-1}$, $A_1 = X_2 \cdot X_5^{-1}$, $A_2 = X_4 \cdot X_5^{-1}$. In the following $\mathcal{N}_r\{ M \}$ denotes the right null space of M , $\mathcal{N}_r^0\{ M \}$ denotes the subspace of $\mathcal{N}_r\{ M \}$, the vectors of which have their first m rows zero , $\mathcal{N}_r\{ M' \}$ denotes the right null space of M' . If we consider the matrices :

$$\widetilde{X}^0 = \begin{bmatrix} O \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} \text{ and } X = \begin{bmatrix} X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

then it is straightforward to show htat :

$$M \cdot \widetilde{X}^0 = O \Leftrightarrow M' \cdot X = O \tag{7.5.15}$$

Relation (7.5.15) implies the existence of an isomorphism ϕ between the vector spaces $\mathcal{N}_r^0\{\ M\ \}$ and $\mathcal{N}_r\{\ M'\ \}$, namely :

$$\phi : \mathcal{N}_r^0\{\ M\ \} \to \mathcal{N}_r\{\ M'\ \}\ ,\ \phi\left(\widetilde{X}^0\right) := X \tag{7.5.16}$$

Hence the vector spaces $\mathcal{N}_r^0\{\ M\ \}$ and $\mathcal{N}_r\{\ M'\ \}$ are isomorphic and have the same dimension . Now we can proceed with the solution of (7.5.14) under the constraints mentioned there . The matrices $\widetilde{X}^0$ which satisfy (7.5.14) are formed by m linearly independent vectors of $\mathcal{N}_r\{\ M\ \}$ and thus the first condition is derived from this fact .

***Proposition** (7.5.1) : A necessary condition for the existence of a solution of (7.5.14) is that :*

$$rank\ M \leq 2\ (m+l) \tag{7.5.17}$$

Proof

Since the dimension of $\mathcal{N}_r\{\ M\ \}$ must be greater than or equal m we take :

$$dim\ \mathcal{N}_r\{\ M\ \} = (3\ m + 2\ l) - rank\ M\ \geq m \Leftrightarrow rank\ M \leq 2\ (m+l) \qquad \square$$

More precisely , considering the constraints of equation (7.5.14) , we see that the vectors of the solutions $\widetilde{X}^0$ belong to the subspace of $\mathcal{N}_r\{\ M\ \}$, $\mathcal{N}_r^0\{\ M\ \}$. Thus , a necessary condition for the existence of a solution of (7.5.14) is :

***Proposition** (7.5.2) : A necessary condition for the existence of a solution of (7.5.14) is that :*

$$rank\ M' \leq (m + 2\ l) \tag{7.5.18}$$

Proof

For the existence of m linearly independent vectors of $\mathcal{N}_r^0\{M\}$ defining a solution of (7.5.14) it is necessary that $dim\ \mathcal{N}_r^0\{M\} \geq m$, or by (7.5.16) , $dim\ \mathcal{N}_r^0\{M\} = dim\ \mathcal{N}_r\{M'\} \geq m$, or :

$$dim\ \mathcal{N}_r\{ M' \} = 2\ (m + l) - rank\ M' \geq m \Leftrightarrow rank\ M' \leq (m + 2\ l) \qquad \square$$

***Remark (7.5.2)** : By the construction of M′ we see that :*

$$rank\ M' \leq (m + 2\ l) \Rightarrow rank\ M \leq 2\ (m + l) \qquad \square$$

***Remark (7.5.3)** : The condition (7.5.18) is sufficient for the existence of a solution* $\widetilde{X}^0$ *of (7.5.14) without the constraints* X_2 *full column rank ,* $|X_3| \neq 0$ *,* $|X_5| \neq 0$ *. Indeed let rank* $M' \leq (m + 2\ l)$ *, then dim* $\mathcal{N}_r\{ M' \} = 2\ (m + l) - rank\ M' \geq m \Leftrightarrow$ *by (7.5.16) ,*

$$dim\ \mathcal{N}_r^0\{ M \} = dim\ \mathcal{N}_r\{ M' \} \geq m \qquad \square$$

In order to find solutions of (7.5.14) satisfying the rest of the constraints , X_2 full column rank , $| X_3 | \neq 0$, $| X_5 | \neq 0$, we proceed as follows . Because of the isomorphism (7.5.16) it suffices to find a solution X of the equation $M' \cdot X = \mathbf{O}$ with :

$$X = \begin{bmatrix} X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

and X satisfies the constraints X_2 full column rank , $| X_3 | \neq 0$, $| X_5 | \neq 0$. Then a solution $\widetilde{X}$ of (7.5.14) is :

$$\widetilde{X}^0 = \phi^{-1}(X)$$

or equivalently ,

$$\widetilde{X}^0 = \begin{bmatrix} O \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

Consider now the vector space $\mathcal{N}_r\{ M' \}$. The X for which $M' \cdot X = \mathbf{O}$, consist of m linearly independent vectors of $\mathcal{N}_r\{ M' \}$. Condition (7.5.18) is necessary and sufficient for the existence of such an X , (without necessarily satisfying the constraints X_2 full column rank , $| X_3 | \neq 0$, $| X_5 | \neq 0$) . Let W be a base of $\mathcal{N}_r\{ M' \}$ with $j \geq m$ columns and partitioned according to the partition of X , that is :

$$W = \begin{bmatrix} W_2 \\ W_3 \\ W_4 \\ W_5 \end{bmatrix} \qquad (7.5.19)$$

where , $W_2 \in \mathbb{R}^{lxj}$, $W_3 \in \mathbb{R}^{mxj}$, $W_4 \in \mathbb{R}^{lxj}$, $W_5 \in \mathbb{R}^{mxj}$. If $C_m(W_i)$ denotes the $1x\binom{j}{m}$ real matrix with elements the mxm minors of W_i taken in lexicographical order the following conditions are invariant of the selection of the base W .

***Proposition (7.5.3)** : A necessary and sufficient condition for the existence of a solution X of $M' \cdot X = O$ satisfying the constraints X_2 full column rank , $|X_3| \neq 0$, $|X_5| \neq 0$ is that $C_m(W_2)$ is a non zero matrix , $C_m(W_3)$ and $C_m(W_5)$ are non zero vectors for an arbitrary base W of $\mathcal{N}_r\{ M' \}$.*

Proof

The proof is similar to the one of proposition(7.4.3) if $\mathcal{N}_r\{ M' \}$ replaces $\mathcal{N}_r\{ \widetilde{S}_p \}$. □

***Remark (7.5.4)** : Summarizing the above analysis , in order to construct solution of (7.5.14) , we construct a solution X of $M' \cdot X = O$, if such a solution exists , and then $\widetilde{X}^0 = \phi^{-1}(X)$ is a solution of (7.4.14) .* □

7.6. MINIMAL COMPLEXITY SOLUTIONS

Consider again equation (7.2.3) . Our next task is to try to find a minimal complexity solution for it . In order to do so we have to find the least possible column degrees of solutions $[D_c^T , N_c^T]^T$ of (7.3.2) for all the unimodular matrices U on its right hand side. Then the least complexity of solutions of (7.2.3) will be the sum of these degrees . In the following we give a simple algorithm for the evaluation of the least column degrees of solutions of (7.3.2) which serves as an upper bound for the least complexity . A low bound will be introduced in section 7.7 . Using the notation (7.2.7.a) , (7.2.8.a) , (7.3.3.a) , (7.3.3.b) we may write (7.2.3) as :

$$T_p^l(d) \cdot T_c^r(d) = I_m \tag{7.6.1}$$

$$T_p^l(d) = [\widetilde{D}_0 , \widetilde{N}_0] + d [\widetilde{D}_1 , \widetilde{N}_1] + \dots + d^\nu [\widetilde{D}_\nu , \widetilde{N}_\nu] =$$

$$= \widetilde{T}_0 + d\, \widetilde{T}_1 + \dots + d^\nu\, \widetilde{T}_\nu \in \mathbf{R}^{mx(m+l)}[d] \tag{7.6.2}$$

$$T_c^r(d) = \begin{bmatrix} D_c \\ N_c \end{bmatrix} = [\underline{t}_1(d) , \underline{t}_2(d) , \dots , \underline{t}_m(d)] \tag{7.6.3.a}$$

$$\underline{t}_i(d) = \underline{t}_{i0} + d\, \underline{t}_{i1} + \dots + d^{r_i}\, \underline{t}_{ir_i} \tag{7.6.3.b}$$

and by (7.6.3.a) , equation (7.5.1) becomes :

$$[T_p^l(d) \cdot \underline{t}_1(d)\ ,\ T_p^l(d) \cdot \underline{t}_2(d)\ ,\ \ldots\ ,\ T_p^l(d) \cdot \underline{t}_m(d)\ ] = I_m \tag{7.6.4}$$

(7.6.4) implies the following set of equations :

$$T_p^l(d) \cdot \underline{t}_i(d) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i^{\underline{th}} \text{ row}\ ,\ i = 1\ ,\ 2\ ,\ \ldots\ ,\ m \tag{7.6.5}$$

Remark (7.6.3) *: A least possible column degrees solution of (7.6.1) , $T_c^r(d)$, consists of least degree solutions $\underline{t}_i(d)$, $(i = 1\ ,\ 2\ ,\ \ldots\ ,\ m)$, of (7.6.5) . Hence , in order to find a least column degrees solution of (7.6.1) it suffices to find least degree solutions $\underline{t}_i(d)$ of the set of equations (7.6.5) . In the following we show how such solutions can be obtained* □

Consider an arbitrary equation from the set (7.6.5) :

$$T_p^l(d) \cdot \underline{t}_i(d) = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i^{\underline{th}} \text{ row} \tag{7.6.6}$$

Then by (7.6.2) , (7.6.3.b) and (7.6.6) the following set of conditions is derived :

$$\begin{aligned} &\widetilde{T}_0 \cdot \underline{t}_{i0} = \underline{0} \\ &\vdots \\ &\widetilde{T}_1 \cdot \underline{t}_{i0} + \widetilde{T}_0 \cdot \underline{t}_{i1} = \underline{0} \\ &\vdots \\ &\widetilde{T}_{i-1} \cdot \underline{t}_{i0} + \cdots + \widetilde{T}_{i-r_i} \cdot \underline{t}_{ir_i} = 1 \\ &\vdots \\ &\widetilde{T}_\nu \cdot \underline{t}_{ir_i} = \underline{0} \end{aligned} \tag{7.6.7}$$

which in matrix form yields :

$$i^{\underline{th}} \text{ row} \rightarrow \begin{bmatrix} \widetilde{T}_0 & O & \cdots & \cdots & O & \underline{0} \\ \widetilde{T}_1 & \widetilde{T}_0 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & -1 \\ \vdots & \ddots & \ddots & \ddots & O & \vdots \\ \widetilde{T}_\nu & \ddots & \ddots & \ddots & \widetilde{T}_0 & \underline{0} \\ O & \widetilde{T}_\nu & \ddots & \ddots & \widetilde{T}_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \underline{0} \\ O & \cdots & \cdots & O & \widetilde{T}_\nu & \underline{0} \end{bmatrix} \cdot \begin{bmatrix} \underline{t}_{i0} \\ \underline{t}_{i1} \\ \vdots \\ \vdots \\ \underline{t}_{ir_i} \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \underline{0} \\ \underline{0} \end{bmatrix} \tag{7.6.8}$$

and we denote by T'_{r_i} the coefficient matrix :

$$T'_{r_i} = \begin{bmatrix} \widetilde{T}_0 & O & \cdots & \cdots & O & \underline{0} \\ \widetilde{T}_1 & \widetilde{T}_0 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & -1 \\ \vdots & \ddots & \ddots & \ddots & O & \vdots \\ \widetilde{T}_\nu & \ddots & \ddots & \ddots & \widetilde{T}_0 & \underline{0} \\ O & \widetilde{T}_\nu & \ddots & \ddots & \widetilde{T}_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \underline{0} \\ O & \cdots & \cdots & O & \widetilde{T}_\nu & \underline{0} \end{bmatrix} \tag{7.6.9}$$

In order to find the solutions of (7.6.6) it suffices to solve equation (7.6.8) or , equation :

$$T'_{r_i} \cdot T_i = \underline{0} \tag{7.6.10}$$

with ,

$$T_i \triangleq \begin{bmatrix} \underline{t}_0 \\ \underline{t}_1 \\ \vdots \\ \vdots \\ \underline{t}_{r_i} \\ t_{r_i+1} \end{bmatrix} \tag{7.6.11}$$

and $\underline{t}_\kappa \in \mathbb{R}^{(m+l)x1}$ when $\kappa = 0 , 1 , \dots , r_i$ and t_{r_i+1} is a non zero real number . Then a solution of (7.6.8) is the vector $T_i \cdot (1/t_{r_i+1})$. Let $\mathcal{N}_r\{ T'_{r_i} \}$ denotes the right null space of T'_{r_i} and W_i an arbitrary base of it , with $j \geq 1$ columns . Partition W_i according to (7.6.11) , i.e. ,

$$W_i = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_{r_i} \\ W_{r_i+1} \end{bmatrix} \tag{7.6.12}$$

with , $W_\kappa \in \mathbb{R}^{(m+l)xj}$ when $\kappa = 0 , 1 , \dots , r_i$, $W_{r_i+1} \in \mathbf{R}^{1xj}$.

***Remark (7.6.4)** : The least degree solutions of (7.6.6) are those with r_i least possible ;*

thus such solutions correspond to solutions of (7.6.8) with least number of row blocks r_i+1 *, or equivalently to solutions* T_i *of (7.6.10) with the same number of row blocks and* $t_{r_i+1} \neq 0$ *. In the following we give conditions for the construction of such solutions of (7.6.10) . The conditions are invariant of the selection of the base* W_i *of* $\mathcal{N}_r\{ T'_{r_i} \}$ *.* □

Proposition (7.6.1) *: A necessary and sufficient condition for* r_i *to be the least degree of solutions of (7.6.6) is that* T'_{r_i} *is the first element of the sequence* T'_κ *,* $\kappa = 0 , \dots ,$ *for which* $W_{r_i+1} \neq (0 , \dots , 0)$ *, for an arbitrary base* W_i *of* $\mathcal{N}_r\{ T'_{r_i} \}$ *, partitioned as in (7.6.12) .*

Proof

(⇒) Let $\underline{t}_i(d) = \underline{t}_{i0} + d\, \underline{t}_{i1} + \cdots + d^{r_i}\, \underline{t}_{ir_i}$ be a solution of (7.6.6) with r_i the least degree among the solutions of (7.6.6) . Then the matrix :

$$\begin{bmatrix} \underline{t}_{i0} \\ \underline{t}_{i1} \\ \vdots \\ \vdots \\ \underline{t}_{ir_i} \\ 1 \end{bmatrix}$$

is a solution of equation (7.6.8) and if we set $\underline{t}_0 = \underline{t}_{i0}$, $\underline{t}_1 = \underline{t}_{i1}$, … , $\underline{t}_{r_i} = \underline{t}_{ir_i}$, $t_{r_i+1} = 1 \neq 0$, the matrix :

$$T_i \triangleq \begin{bmatrix} \underline{t}_0 \\ \underline{t}_1 \\ \vdots \\ \vdots \\ \underline{t}_{r_i} \\ t_{r_i+1} \end{bmatrix}$$

is a solution of (7.6.10) and it can be completed with $j-1$ vectors of $\mathcal{N}_r\{ T'_{r_i} \}$ to be a base W_i . If W_i is partitioned as in (7.6.12) we take that $W_{r_i+1} = (\dots , t_{r_i+1} , \dots) \neq \underline{0}^T$ and T'_{r_i} is the first element of the sequence T'_κ , $\kappa = 0 , \dots ,$ for which this holds true .

(⇐) Let T'_{r_i} be the first element of the sequence T'_κ , $\kappa = 0 , \dots ,$ for which $W_{r_i+1} \neq \underline{0}^T$, for an arbitrary base W_i of $\mathcal{N}_r\{ T'_{r_i} \}$ partitioned as in (7.6.11) . Let w_{r_i+1} be a non zero element of W_{r_i+1}, then we select the column of W_i which corresponds to w_{r_i+1} and form the matrix :

$$T_i \triangleq \begin{bmatrix} \underline{t}_0 \\ \underline{t}_1 \\ \vdots \\ \vdots \\ \underline{t}_{r_i} \\ t_{r_i+1} \end{bmatrix} = \begin{bmatrix} \underline{w}_0 \\ \underline{w}_1 \\ \vdots \\ \vdots \\ \underline{w}_{r_i} \\ w_{r_i+1} \end{bmatrix}$$

which is a solution for equation (7.6.10) . Consequently the matrix $T_i \cdot (1/t_{r_i+1})$ is a solution of equation (7.6.8) and hence $\underline{t}_i(d) = (\underline{t}_0 + d\, \underline{t}_1 + \cdots + d^{r_i}\, \underline{t}_{r_i}) \cdot (1/t_{r_i+1}) = \underline{t}_{i0} + d\, \underline{t}_{i1} + \cdots + d^{r_i}\, \underline{t}_{ir_i}$ is a solution of (7.6.6) with degree r_i least among the degrees of the family of solutions of (7.6.6) . □

***Remark (7.6.5)** : If $W_{r_i+1} \neq \underline{0}^T$, for a base W_i of $\mathcal{N}_r\{ T'_{r_i} \}$, then $B_{r_i+1} \neq \underline{0}^T$, for all the bases B_i of $\mathcal{N}_r\{ T'_{r_i} \}$ partitioned as in (7.6.11) . Indeed for any base of $\mathcal{N}_r\{ T'_{r_i} \}$, $\mathfrak{B}_i$, there exists an $\mathbb{R}^{j \times j}$ unimodular matrix U such that $B_i = B_i \cdot U$ and hence $B_{r_i+1} = W_{r_i+1} \cdot U$, which implies that $B_{r_i+1} \cdot U^{-1} = W_{r_i+1} \neq \underline{0}^T$. Hence , $B_{r_i+1} \neq \underline{0}^T$* □

***Proposition (7.6.2)** : The family of solutions $\underline{t}_i(d)$ of (7.6.6) , $i = 1 , \ldots , m$ with least column degree r_i is given by :*

$$\mathcal{T}_{r_i} = \left\{ \underline{t}_i(d) = \left[\begin{matrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{matrix} \;\vdots\; \begin{matrix} d & & O \\ & \ddots & \\ O & & d \end{matrix} \;\vdots\; \cdots \;\vdots\; \begin{matrix} d^{r_i} & & O \\ & \ddots & \\ O & & d^{r_i} \end{matrix} \;\vdots\; \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right] \cdot W_i \cdot \underline{\lambda}_i \;,\; \underline{\lambda}_i \text{ satisfy } W_{r_i+1} \cdot \underline{\lambda}_i = 1 \right\} \tag{7.6.13}$$

where , W_i is an arbitrarily chosen base of $\mathcal{N}_r\{T'_{r_i}\}$, partitioned as in (7.6.12) and r_i the first index of the sequence T'_κ , $\kappa = 0 , \ldots$ for which $W_{r_i+1} \neq \underline{0}^T$.

Proof

(⇒) Let $\underline{t}_i(d) = \underline{t}_{i0} + d\, \underline{t}_{i1} + \cdots + d^{r_i}\, \underline{t}_{ir_i}$ be a least column degree solution of (7.6.6) for an $i \in \{ 1 , 2 , \ldots , m \}$. Then the matrix C_{r_i} :

$$C_{r_i} = \begin{bmatrix} \underline{t}_{i0} \\ \underline{t}_{i1} \\ \vdots \\ \vdots \\ \underline{t}_{ir_i} \\ 1 \end{bmatrix} \tag{7.6.14}$$

formed by the coefficients of $\underline{t}_i(d)$ satisfies equation (7.6.8) , or equivalently :

$$T'_{r_i} \cdot C_{r_i} = \underline{0}^T \tag{7.6.15}$$

with T'_{r_i} the matrix defined in (7.6.9) . C_{r_i} can be completed to a base W_i of $\mathcal{N}_r\{T'_{r_i}\}$. Partition W_i as in (7.6.12) , then a vector $\underline{\lambda}_i = [\ 0 \cdots 1 \cdots 0\ ]^T$ exists such that :

$$W_i \cdot \underline{\lambda}_i = C_{r_i} \tag{7.6.16}$$

and thus ,

$$W_{r_i+1} \cdot \underline{\lambda}_i = 1 \tag{7.6.17}$$

Since r_i is a least degree (7.6.17) implies that r_i is the least possible for which $W_{r_i+1} \neq \underline{0}^T$. It is obvious that :

$$\underline{t}_i(d) = \underline{t}_{i0} + d\,\underline{t}_{i1} + \cdots + d^{r_i}\,\underline{t}_{ir_i} = \left[\begin{array}{cc:cc:c:cc:c} 1 & O & d & O & \cdots & d^{r_i} & O & 0 \\ & \ddots & & \ddots & & & \ddots & \vdots \\ O & 1 & O & d & & O & d^{r_i} & 0 \end{array}\right] \cdot W_i \cdot \underline{\lambda}_i \tag{7.6.18}$$

($\Leftarrow$) Let ,

$$\underline{t}_i(d) = \left[\begin{array}{cc:cc:c:cc:c} 1 & O & d & O & \cdots & d^{r_i} & O & 0 \\ & \ddots & & \ddots & & & \ddots & \vdots \\ O & 1 & O & d & & O & d^{r_i} & 0 \end{array}\right] \cdot W_i \cdot \underline{\lambda}_i \tag{7.6.19}$$

where , $\underline{\lambda}_i$ satisfies $W_{r_i+1} \cdot \underline{\lambda}_i = 1$, W_{r_i+1} is the last row of an arbitrarily chosen base W_i of $\mathcal{N}_r\{T'_{r_i}\}$ partitioned as in (7.6.12) and r_i the least possible for which $W_{r_i+1} \neq \underline{0}^T$. Set the matrix C_{r_i} to be :

$$C_{r_i} = \begin{bmatrix} \underline{t}_{i0} \\ \underline{t}_{i1} \\ \vdots \\ \vdots \\ \underline{t}_{ir_i} \\ 1 \end{bmatrix} = W_i \cdot \underline{\lambda}_i \tag{7.6.20}$$

and

$$T'_{r_i} \cdot C_{r_i} = \underline{0}^T \tag{7.6.21}$$

with r_i least possible . Then ,

$$\underline{t}_i(d) = \underline{t}_{i0} + d\,\underline{t}_{i1} + \cdots + d^{r_i}\,\underline{t}_{ir_i} = \left[\begin{array}{cc:cc:c:cc:c} 1 & O & d & O & \cdots & d^{r_i} & O & 0 \\ & \ddots & & \ddots & & & \ddots & \vdots \\ O & 1 & O & d & & O & d^{r_i} & 0 \end{array}\right] \cdot C_{r_i} \tag{7.6.20}$$

(7.6.7) and (7.6.8) imply that $\underline{t}_i(d)$ is a least degree solution of (7.6.6) . □

In the following we present an algorithm for the construction of $\mathcal{T}_{r_i}$ with r_i minimum .

ALGORITHM FOR THE CONSTRUCTION OF $\mathcal{T}_{r_i}$ WITH r_i MINIMUM

Step 1 : Write $T_p^l(d) = \widetilde{T}_0 + d\,\widetilde{T}_1 + \dots + d^{\nu}\,\widetilde{T}_{\nu}$.

Step 2 : Create the sequence of real matrices :

$$S_0 = \begin{bmatrix} \widetilde{T}_0 \\ \vdots \\ \widetilde{T}_{\nu} \end{bmatrix} ,\ S_n = \begin{bmatrix} S_{n-1} & O \\ O & S_0 \end{bmatrix} ,\ n = 1 , \dots ,\ T_n' = \left[\begin{array}{c|c} S_n & \begin{matrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \leftarrow i^{\underline{th}} \text{ row}$$

Step 3 : Find the first n , $(n = 0 , 1 , \dots)$, for which the matrix W formed by an arbitrarily chosen base of $\mathcal{N}_r\{T_n'\}$, has its last row nonzero .

Step 4 : Set $r_i = n$, $W_i = W$, partition W_i as in (7.6.12) and set :

$$\mathcal{T}_{r_i} = \left\{ \underline{t}_i(d) = \left[\begin{array}{cc|cc|c|cc|c} 1 & O & d & O & \cdots & d^{r_i} & O & 0 \\ & \ddots & & \ddots & & & \ddots & \vdots \\ O & 1 & O & d & \cdots & O & d^{r_i} & 0 \end{array}\right] \cdot W_i \cdot \underline{\lambda}_i \ ,\ \underline{\lambda}_i \text{ satisfy } W_{r_i+1} \cdot \underline{\lambda}_i = 1 \right\}$$

□

Corollary (7.6.1) *: Applying steps 2 – 4 of the above algorithm for $i = 1, \dots , m$ we take that the family of least column degrees solutions of (7.6.1) is given by :*

$$\mathcal{T}_{lcd} = \{\ T_c^r(d) : T_c^r(d) = [\,\underline{t}_1(d) , \underline{t}_2(d) , \dots , \underline{t}_m(d)\,] \text{ and } \underline{t}_i(d) \text{ are taken from } \mathcal{T}_{r_i}\ \}$$

□

If the set of least column degrees $\{\ r_i\ ,\ i \in \underline{m}\ \}$ of the solutions of equation (7.6.1) is constructed then the least complexity will be $\delta \leq \sum_{i=1}^{m} r_i$. A lower bound for δ is constructed in the next section .

7.7. MINIMAL EXTENDED MCMILLAN DEGREE SOLUTIONS

In this section a lower bound for the minimal extended McMillan degree of the solutions of equation (7.7.1) is introduced . The analysis is based on the minimal solution of the scalar , (polynomial) , Diophantine equation that applies in the case of SISO discrete time ssystems , [Kar. 1] , [Mil. 1] . Consider the equation :

$$T_p^l(d) \cdot T_c^r(d) = I_m \tag{7.7.1}$$

where ,

$$\mathrm{T}_p^l(\mathrm{d}) \triangleq [\, \widetilde{\mathrm{D}} \;,\; \widetilde{\mathrm{N}}\,] \in \mathbb{R}^{mx(m+l)}[\mathrm{d}] \;,\; \mathrm{T}_c^r(\mathrm{d}) \triangleq [\, \mathrm{D}_c^{\mathrm{T}} \;,\; \mathrm{N}_c^{\mathrm{T}}\,]^{\mathrm{T}} \in \mathbb{R}^{(m+l)xm}[\mathrm{d}]$$

Then the following equation can be derived using the Binet – Cauchy theorem [Gan. 1] :

$$\mathrm{C}_m\big(\mathrm{T}_p^l(\mathrm{d})\big) \cdot \mathrm{C}_m\big(\mathrm{T}_c^r(\mathrm{d})\big) = 1 \tag{7.7.2}$$

or ,

$$\mathrm{C}_m\big(\mathrm{T}_p^l(\mathrm{d})\big) = \widetilde{\mathrm{M}}_0 + \mathrm{d}\,\widetilde{\mathrm{M}}_1 + \ldots + \mathrm{d}^n\,\widetilde{\mathrm{M}}_n \in \mathbb{R}^{1xt}[\mathrm{d}] \;,\; t = \binom{m+l}{m} \tag{7.7.3.a}$$

$$\mathrm{C}_m\big(\mathrm{T}_c^r(\mathrm{d})\big) = \mathrm{M}_{c0} + \mathrm{d}\,\mathrm{M}_{c1} + \ldots + \mathrm{d}^a\,\mathrm{M}_{ca} \in \mathbb{R}^{tx1}[\mathrm{d}] \;,\; t = \binom{m+l}{m} \tag{7.7.3.b}$$

Equation (7.7.1) implies the following set of conditions :

$$\begin{aligned}
&\widetilde{\mathrm{M}}_0 \cdot \mathrm{M}_{c0} = 1 \\
&\widetilde{\mathrm{M}}_1 \cdot \mathrm{M}_{c0} + \widetilde{\mathrm{M}}_0 \cdot \mathrm{M}_{c1} = 0 \\
&\widetilde{\mathrm{M}}_2 \cdot \mathrm{M}_{c0} + \widetilde{\mathrm{M}}_1 \cdot \mathrm{M}_{c1} + \widetilde{\mathrm{M}}_0 \cdot \mathrm{M}_{c2} = 0 \\
&\qquad \vdots \\
&\widetilde{\mathrm{M}}_n \cdot \mathrm{M}_{ca} = 0
\end{aligned} \tag{7.7.4}$$

which in matrix form may be written as :

$$\begin{bmatrix}
\widetilde{\mathrm{M}}_0 & \mathrm{O} & \cdots & \cdots & \mathrm{O} \\
\widetilde{\mathrm{M}}_1 & \widetilde{\mathrm{M}}_0 & \ddots & & \vdots \\
\vdots & \widetilde{\mathrm{M}}_1 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \mathrm{O} \\
\widetilde{\mathrm{M}}_n & \vdots & \ddots & \ddots & \widetilde{\mathrm{M}}_0 \\
\mathrm{O} & \widetilde{\mathrm{M}}_n & \ddots & \ddots & \widetilde{\mathrm{M}}_1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
\mathrm{O} & \cdots & \cdots & \mathrm{O} & \widetilde{\mathrm{M}}_n
\end{bmatrix} \cdot \begin{bmatrix} \mathrm{M}_{c0} \\ \mathrm{M}_{c1} \\ \vdots \\ \vdots \\ \mathrm{M}_{ca} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{7.7.5}$$

Equation (7.7.5) is equivalent to :

$$\begin{bmatrix} \widetilde{M}_0 & O & \cdots & \cdots & O & -1 \\ \widetilde{M}_1 & \widetilde{M}_0 & \ddots & & \vdots & 0 \\ \vdots & \widetilde{M}_1 & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & O & \vdots \\ \widetilde{M}_n & \vdots & \ddots & \ddots & \widetilde{M}_0 & \vdots \\ O & \widetilde{M}_n & \ddots & \ddots & \widetilde{M}_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & 0 \\ O & \cdots & \cdots & O & \widetilde{M}_n & 0 \end{bmatrix} \cdot \begin{bmatrix} M_{c0} \\ M_{c1} \\ \vdots \\ \vdots \\ M_{ca} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (7.7.6)$$

We shall denote by :

$$S_a \triangleq \begin{bmatrix} \widetilde{M}_0 & O & \cdots & \cdots & O & -1 \\ \widetilde{M}_1 & \widetilde{M}_0 & \ddots & & \vdots & 0 \\ \vdots & \widetilde{M}_1 & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & O & \vdots \\ \widetilde{M}_n & \vdots & \ddots & \ddots & \widetilde{M}_0 & \vdots \\ O & \widetilde{M}_n & \ddots & \ddots & \widetilde{M}_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & 0 \\ O & \cdots & \cdots & O & \widetilde{M}_n & 0 \end{bmatrix}, \quad T_a \triangleq \begin{bmatrix} M_{c0} \\ M_{c1} \\ \vdots \\ \vdots \\ M_{ca} \\ 1 \end{bmatrix}$$

Let $\mathcal{N}_r\{ S_a \}$ denotes the right null space of S_a and W be an arbitrary base of $\mathcal{N}_r\{ S_a \}$, with $j \geq 1$ columns . Partition W as in T_a , namely :

$$W = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_a \\ W_{a+1} \end{bmatrix} \quad (7.7.8)$$

where , $W_\kappa \in \mathbb{R}^{txj}$ when $\kappa = 0 , 1 , \ldots , \alpha$, $W_{a+1} \in \mathbb{R}^{1xj}$.

***Remark (7.7.1)** : Because of its construction equation (7.7.2) produces solutions the least degree of which is a lower bound for the minimal extended McMillan degree of the solutions of equation (7.7.1) . In the following we shall construct this lower bound , which is invariant of the selection of base W of $\mathcal{N}_r\{ S_a \}$.* □

***Proposition (7.7.1)** : A necessary and sufficient condition for α to be the least degree of* *the solutions of (7.7.2) is that S_a is the first element of the sequence S_κ , $\kappa = 0$, ... , for which $W_{a+1} \neq (0\ ,\ \dots\ ,\ 0)$, for an arbitrary base W of $\mathcal{N}_r\{\ S_a\ \}$ partitioned as in (7.7.8).*

Proof

($\Rightarrow$) Let $C_m\left(T_c^r(d)\right) = M_{c0} + d\ M_{c1} + \dots + d^a\ M_{ca} \in \mathbb{R}^{t \times 1}[d]$, $t = \binom{m+l}{m}$ be a solution of (7.7.2) with α the least degree among the solutions of (7.7.2) . Then the matrix :

$$T_a \triangleq \begin{bmatrix} M_{c0} \\ M_{c1} \\ \vdots \\ \vdots \\ M_{ca} \\ 1 \end{bmatrix}$$

is a solution of equation (7.7.6) and be completed with $j-1$ vectors of $\mathcal{N}_r\{\ S_a\ \}$ to be a base W . If W is partitioned as in (7.7.8) we take that $W_{a+1} = (\ \dots\ ,\ 1\ ,\ \dots\ ) \neq \underline{0}^T$ and S_a is the first element of the sequence S_κ , $\kappa = 0$, ... , for which this holds true .

($\Leftarrow$) Let S_a be the first element of the sequence S_κ , $\kappa = 0$, ... , for which $W_{a+1} \neq \underline{0}^T$, for an arbitrary base W of $\mathcal{N}_r\{\ S_a\ \}$ partitioned as in (7.7.8) . Let w_{a+1} be a non zero element of W_{a+1}, then we select the column of W which corresponds to w_{a+1} and form the matrix:

$$T'_a = \begin{bmatrix} \underline{w}_0 \\ \underline{w}_1 \\ \vdots \\ \vdots \\ \underline{w}_a \\ w_{a+1} \end{bmatrix}$$

which is a solution for equation $S_a \cdot T'_a = \underline{0}$. Consequently the matrix $T_a = T'_a \cdot (1/w_{a+1})$ is a solution of equation (7.7.6) and hence :

$$C_m\left(T_c^r(d)\right) = (\ \underline{w}_0 + d\ \underline{w}_1 + \cdots + d^a\ \underline{w}_a\ ) \cdot (1/w_{a+1}) =$$

$$= M_{c0} + d\ M_{c1} + \dots + d^a\ M_{ca}$$

is a solution of (7.7.2) with degree a least among the degrees of the family of solutions of

(7.7.2) . □

Remark (7.7.2) *: When the upper and lower bounds of the minimal extended Mc Millan degree coincide then , the minimal extended McMillan degree is equal to*

$$\alpha = \delta^* = \sum_{i=1}^{m} r_i \ , \ \{ r_i \ , i \in \underline{m} \} \qquad (7.7.9)$$

the set of least column degrees of the solutions of equation (7.7.1) . □

7.8 EXAMPLES

In this section we present examples for sections 7.4 , 7.5 , 7.6 , 7.7 respectively . We start with an example about the construction of a least possible maximum column degree solution , (the maximum of column degrees of $T_c^r(d)$ is minimum among the maximum of column degrees of solutions) , of equation (7.4.3) .

Example (7.8.1) : Let $P = \widetilde{D}^{-1} \widetilde{N}$ be an MFD representation for the plant P , with :

$$\widetilde{D} = \begin{bmatrix} d^2+1 & 1 \\ 0 & d+1 \end{bmatrix} \in \mathbb{R}^{2x2}[d] \ , \ \widetilde{N} = \begin{bmatrix} 1 & d & 0 \\ 1 & 0 & 2 \end{bmatrix} \in \mathbb{R}^{2x3}[d]$$

and :

$$\left[\widetilde{D} \ , \widetilde{N} \right] = \begin{bmatrix} d^2+1 & 1 & 1 & d & 0 \\ 0 & d+1 & 1 & 0 & 2 \end{bmatrix}$$

or equivalently :

$$\left[\widetilde{D} \ , \widetilde{N} \right] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} d + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} d^2$$

$$\left[\widetilde{D} \ , \widetilde{N} \right] = \widetilde{T}_0 + \widetilde{T}_1 d + \widetilde{T}_2 d^2$$

Following the method described in proposition(7.4.4) , we take :

$$S_0 = \begin{bmatrix} \widetilde{T}_0 & \vdots & -I_2 \\ \widetilde{T}_1 & \vdots & O_2 \\ \widetilde{T}_2 & \vdots & O_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A base of $\mathcal{N}_r\{ S_0 \}$ is the matrix :

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.5 & 1 & 0 \end{bmatrix}^T$$

And if W is partitioned as in (7.4.20) then $C_2(W_{D_{c0}}) = 0$. whereas $C_2(W_{I_m}) = 1$, gives a solution for the equation (7.4.3) , which does not correspond to an MFD representation of the controller , since :

$$D_{c0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence , we have to examine S_1 , which is

$$S_1 = \begin{bmatrix} \tilde{T}_0 \; O_5 & : & -I_2 \\ \tilde{T}_1 \; \tilde{T}_0 & : & O_2 \\ \tilde{T}_2 \; \tilde{T}_1 & : & O_2 \\ O_5 \; \tilde{T}_2 & : & O_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a base W of $\mathcal{N}_r\{ S_0 \}$ is the matrix :

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

Because $C_2(W_{D_{c0}}) \neq (0 , \dots , 0)$ and $C_2(W_{I_m}) \neq (0 , \dots , 0)$ if we add the fourth column of W to the second and select the first two resulting columns we form the matrix :

$$\tilde{T}_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^T$$

then ,

$$\begin{bmatrix} D_{c0} \\ N_{c0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0.5 \end{bmatrix}^T$$

$$\begin{bmatrix} D_{c1} \\ N_{c1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

Hence , a solution with least maximum column index among the maximum column indices of solutions of (7.4.3) is :

$$\begin{bmatrix} D_{c} \\ N_{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -d & 0 \\ 0 & -2 & 2 & 0 & d+0.5 \end{bmatrix}^T$$

and it corresponds to a causal MFD representation of a controller . The latter implies that the least controllability index of the stabilizing controllers for the plant P is either 1 , or 0 . □

The second example concerns the PI controllers . We shall use the method introduced in section 7.5 for the PI controller problem .

Example (7.8.2) : Let $P = \tilde{D}^{-1}\tilde{N}$ be an MFD representation for the plant P , with :

$$\tilde{D} = \begin{bmatrix} d^2+1 & 1 \\ 0 & d+1 \end{bmatrix} \in \mathbb{R}^{2x2}[d] \ , \ \tilde{N} = \begin{bmatrix} 1 & d & 0 \\ 1 & 0 & 2 \end{bmatrix} \in \mathbb{R}^{2x3}[d]$$

$$\left[\tilde{D}\ ,\tilde{N}\right] = \begin{bmatrix} d^2+1 & 1 & 1 & d & 0 \\ 0 & d+1 & 1 & 0 & 2 \end{bmatrix}$$

or equivalently :

$$\left[\tilde{D}\ ,\tilde{N}\right] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} d + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} d^2$$

$$\left[\tilde{D}\ ,\tilde{N}\right] = \tilde{T}_0 + \tilde{T}_1\, d + \tilde{T}_2\, d^2$$

Using the study of PI stabilizing controllers problem of section 7.5 we take :

$$M = \begin{bmatrix} \tilde{T}_0 & O_5 & : & -I_2 \\ \tilde{T}_1 & \tilde{T}_0 & : & O_2 \\ \tilde{T}_2 & \tilde{T}_1 & : & O_2 \\ O_5 & \tilde{T}_2 & : & O_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and as in (7.5.12) :

$$M' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Condition (7.5.18) holds true since $rank\ M' = 7 \leq (m + 2\,l) = 8$. A base of W of $\mathcal{N}_r\{M'\}$ is the matrix:

$$W = \begin{bmatrix} 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

If we partition W as in (7.5.19) is clear that the conditions of proposition(7.5.3) do not hold true , since :

$$C_2(W_2) \neq O_2 \,,\ C_2(W_3) = (0 \,, \dots , 0) \text{ and } C_2(W_5) \neq (0 \,, \dots , 0)$$

So , there is no solution of (7.5.7) and hence of (7.5.3) which corresponds to a PI controller. □

The next example concerns the minimal complexity of solutions of the equation (7.6.1) as has been introduced in section 7.6 .

Example (7.8.3) : Let $P = \tilde{D}^{-1}\,\tilde{N}$ be an MFD representation for the plant P , with :

$$\widetilde{D} = \begin{bmatrix} d+1 & 1 \\ 2 & d \end{bmatrix} \in \mathbb{R}^{2x2}[d] \ , \ \widetilde{N} = \begin{bmatrix} d^2 & 1 \\ 0 & d^2+1 \end{bmatrix} \in \mathbb{R}^{2x2}[d]$$

and :

$$\left[\, \widetilde{D} \ , \widetilde{N} \,\right] = \begin{bmatrix} d+1 & 1 & d^2 & 1 \\ 2 & d & 0 & d^2+1 \end{bmatrix}$$

or equivalently :

$$\left[\, \widetilde{D} \ , \widetilde{N} \,\right] = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} d + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} d^2$$

$$\left[\, \widetilde{D}, \widetilde{N} \,\right] = \widetilde{T}_0 + \widetilde{T}_1\, d + \widetilde{T}_2\, d^2$$

Setting $i = 1$ in (7.6.5) and using proposition(7.6.1) , for $\kappa = 0$ we take :

$$T_0' = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

rank $T_0' \doteq 5$, so $\mathcal{N}_r\{\, T_0' \,\} = \{\, \underline{0} \,\}$. For $\kappa = 1$, we take :

$$T_1' = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a base W of $\mathcal{N}_r\{\, T_1' \,\}$ is given by :

$$W = \frac{1}{5} \cdot [\, 1 \ \ 6 \ \ 3 \ -2 \ \ -3 \ \ 2 \ \ 0 \ \ 0 \ \ 5 \,]^T$$

by proposition(7.6.1) we take that the least column degree for the first column of the solutions of (7.6.1) is 1 and such a column , satisfying (7.6.5) for $\kappa = 1$ is given by :

$$\underline{t}_1(s) = \frac{1}{5} \cdot [\ 1-3d \ \ 6+2d \ \ 3 \ \ -2\ ]^T$$

Setting $i = 2$ in (7.6.5) , we follow the same steps as above and have , for $\kappa = 0$:

$$T_0' = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

rank $T_0' = 5$, so $\mathcal{N}_r\{\ T_0'\ \} = \{\ \underline{0}\ \}$. For $\kappa = 1$, we take :

$$T_1' = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a base W of $\mathcal{N}_r\{\ T_1'\ \}$ is given by :

$$W = \frac{1}{5} \cdot [\ 1 \ \ -4 \ \ -2 \ \ 3 \ \ 2 \ \ -3 \ \ 0 \ \ 0 \ \ 5\ ]^T$$

by proposition(7.6.1) we take that the least column degree for the first column of the solutions of (7.6.1) is 1 and such a column , satisfying (7.6.5) for $\kappa = 1$ is given by :

$$\underline{t}_2(s) = \frac{1}{5} \cdot [\ 1+2d \ \ -4-3d \ \ -2 \ \ 3\ ]^T$$

Hence , a least complexity solution for (7.6.1) is the matrix :

$$T_c^r(s) = [\ \underline{t}_1(d)\ ,\ \underline{t}_2(d)\ ]$$

and an upper bound for the least complexity of solutions of (7.6.1) is 2 . We observe that for d = 0 ,

$$D_c(0) = \begin{bmatrix} (1/5) & (1/5) \\ (6/5) & (-4/5) \end{bmatrix}$$

$|\, D_c(0) \,| = -(2/5) \neq 0$, so the solution corresponds to a causal MFD representation of the controller and 2 serves as an upper bound for the minimal extended McMillan degree of (7.6.1) . □

The last example concerns the construction of a lower bound for the minimal extended McMillan degree of stabilizing controllers the coprime MFD representations of which are taken as solutions of (7.6.1) for the plant P as in example (7.9.3) .

Example (7.8.4) : Consider the plant of example (7.9.3) . It is already known that 2 is an upper bound for the minimal McMillan degree of the solutions of equation (7.6.1) . We construct $C_2\left([\,\tilde{D}\,,\tilde{N}\,]\right)$, which is :

$$C_2\left([\,\tilde{D}\,,\tilde{N}\,]\right) = [\, -2+d+d^2 \,,\, -2d^2 \,,\, -1+d+d^2+d^3 \,,\, -d^3 \,,\, 1-d+d^2 \,,\, d^2+d^4 \,]$$

or,

$$C_2\left([\,\tilde{D}\,,\tilde{N}\,]\right) = [-2\;\;0\;\;-1\;\;0\;\;1\;\;0\,] + [\,1\;\;0\;\;1\;\;0\;\;-1\;\;0\,]\,d + [\,1\;\;-2\;\;1\;\;0\;\;1\;\;1\,]\,d^2 +$$

$$+ [\,0\;\;0\;\;1\;\;-1\;\;0\;\;0\,]\,d^3 + [\,0\;\;0\;\;0\;\;0\;\;0\;\;1\,]\,d^4$$

Applying proposition(7.7.1) for $\alpha = 0$ we take :

$$S_0 = \begin{bmatrix} \tilde{M}_0 & \vdots & -1 \\ \tilde{M}_1 & \vdots & 0 \\ \tilde{M}_2 & \vdots & 0 \\ \tilde{M}_3 & \vdots & 0 \\ \tilde{M}_4 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Making use of proposition(7.7.1) a base of $\mathcal{N}_r\{\, S_0 \,\}$ is given by :

$$W = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}^T$$

Then a least degree solution for equation (7.7.2) is derived by :

$$T_0 = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}^T$$

or by (7.7.2) , (7.7.3.b) , (7.7.6) :

$$C_2\left(T_c^r(s) \right) = M_{c0}^T = \left[-1 \ 0 \ 1 \ 1 \ 0 \ 0 \right]^T$$

Hence , the least degree is $\alpha = 0$ and the minimal extended McMillan degree δ^* is bounded as : $0 \le \delta^* \le 2$. □

7.9. CONCLUSIONS

In Chapter 7 the standard Polynomial Matrix Diophantine equation A X+B Y = U (7.9.1) , (with (A , B) , (X , Y) coprime polynomial MFDs , U a unimodular matrix) , arising from many stabilization problems , like the Total Finite Time Settling Stabilization (TFSTS) of discrete – time linear systems , has been studied . Solutions of (7.9.1) , (for U = I) , satisfying various constraints like minimal controllability index , least complexity , fixed complexity – PI controllers , minimal McMillan degree were studied . The expression of [A , B] , $[X^T , Y^T]^T$ by composite matrices leads to the transformation of the Diophantine equation to an equivalent one employing Toeplitz matrix representation of the product $[A , B] \cdot [X^T , Y^T]^T = I$.

Some topological properties of solutions of (7.9.1) such that , the set of column reduced solutions is dense but not open or closed subset of the set of solutions , were introduced in section 7.3. A characterization of the least column degrees solutions of (7.9.1) , (for U = I) , as well as the least column degree solutions of equation $C_m([A , B]) \cdot C_m([X^T , Y^T]^T) = 1$ are examined in the light of the expression of the PMDE as a set of products of the Toeplitz matrix representation of the left (right) MFD of the plant by the matrix vector representation of each column (row) of the right (left) MFD of the controller . This approach leads to a very simple algorithm involving only the computation of right (left) null spaces of real matrices . The above has served as an upper and lower bound for the minimum extended McMillan degree of the stabilizing controllers . The construction of the set of least column degrees that occur among the family of sets of least column degrees of solutions of (7.9.1) for all unimodular matrices U is still under investigation . As an additional issue to the investigation of fixed complexity solutions of (7.9.1) , (for U = I) , necessary and sufficient conditions for the existence of a PI stabilizing controller for a discrete – time time invariant linear system were given in section 7.5 .

CHAPTER 8

THE GENERAL DECENTRALIZED STABILIZATION PROBLEM: PARAMETRIZATION ISSUES

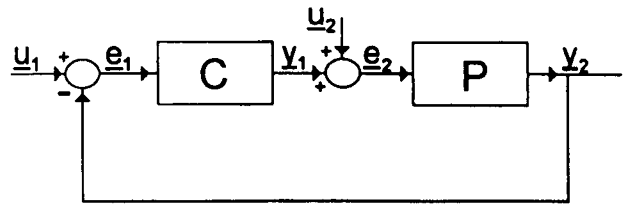
8.1. INTRODUCTION

Restrictions on the feedback compensator structure are often encountered in large scale systems . These systems have several local control stations ; each local compensator observes only the corresponding local outputs . Such decentralized control of systems results in a block diagonal compensator matrix structure , [San. 1] , [Des. 2] , [Wan. 1] . Achieving stabilization of an unstable system by using a decentralized compensator and unity feedback scheme defines the decentralized stabilization problem (DSP) . Wang and Davison , [Wan. 1] and Corfmat and Morse , [Cor. 1] , [Cor. 2] have introduced synthesis methods for the design of stabilizing decentralized compensators . It has been derived that a necessary and sufficient condition for the existence of local control laws with dynamic compensation to stabilize a given system is that the system has no "fixed modes" , [Wan. 1] . Further study of the problem has been done in [And. 1] , [And. 2] , [Kar. 2] , [Özg. 1] , [Güc. 1] , [Kar. 3] , [Vid. 3] . An algebraic approach to the problem based on the factorization of the plant and compensator into coprime matrix fraction descriptions (MFDs) , over the ring of proper and $\mathcal{P}$ - stable functions $\mathbb{R}_{\mathcal{P}}(s)$, has been derived by Gündes and Desoer , [Gün. 1] and a procedure for the design of a stabilizing decentralized compensator is given . A parametrization of all stabilizing block diagonal compensators is introduced there , in terms of parameters which however are not fully described . An other attempt has been made in [Özg. 1] , where the parametrization refers to two block decentralized controllers and the family of parameters is described generically . Our aim in this chapter is to study alternative means of parametrization for the solutions of DSP and try to provide closed form descriptions of the families of parameters in some cases .

In section 8.2 we give a statement of the problem and present the framework of our approach to it . If (D , N) denotes an $\mathbb{R}_{\mathcal{P}}(s)$ - coprime left MFD of the plant , T_i are the matrices formed from the p_i , m_i columns of the partitioning of D , N according to the number of local inputs - outputs respectively , then the parametrization of solutions of the DSP is derived from the set of left unimodular solutions , X_i , of the set of equations $T_i \cdot X_i = U_i$, $i = 1 , \dots , \kappa$, for which $[\ U_1 , \dots , U_\kappa\ ]$ is unimodular . In our study we show that the above parametrization requires the existence of a constructive method that enables us to generate the family of all unimodular matrices of a given dimension , as well as the families of left , (right) unimodular matrices which complete given left , (right) , unimodular matrices to square unimodular ones . Such methods are examined in section 8.3 . The main result of this chapter is introduced in section 8.4 , where an alternative parametrization of solutions of the DSP is established . The parameters are expressed in terms of upper , lower triangular matrices which must satisfy certain constraints . These constraints introduce a necessary and sufficient criterion that enables us to identify the admissible parameters . Although , in the general case , the family of qualifying parameters is not described in closed form there are particular cases when this is possible . These cases are based on the structure , [Vid. 4] , of the Smith forms of the T_i when the latter are generic . Then a closed form description of the family of parameters of the parametrization problem is given in section 8.5 . Many times in the following , especially when we refer to a block partitioning of a matrix , we shall denote by A_m , A^n , A_m^n a square matrix mxm , a matrix with n columns , a matrix with m rows and n columns , respectively . We shall also use this notation for convenience when we want to emphasize on the dimensions of a matrix .

8.2. STATEMENT OF THE PROBLEM – PRELIMINARY RESULTS

Consider the standard feedback configuration associated with a linear time invariant well posed system :



where , $P \in \mathbb{R}_{pr}^{pxm}(s)$ is the transfer function of the plant , $C \in \mathbb{R}^{mxp}(s)$ is the transfer function of the controller . Assume that P is $\mathfrak{P}$ – stabilizable , $\mathfrak{P}$ – detectable , with $\mathfrak{P}^c$ the area of stability .

***Decentralized Stabilization Problem (DSP)** : The decentralized stabilization problem is to determine necessary and sufficient conditions under which a decentralized (block diagonal) stabilizing controller may be defined such that the closed loop system is stable.*□

If $\mathfrak{P} = \mathbb{C}_+ \cup \{\infty\}$ and $\mathbb{R}_{\mathfrak{P}}(s)$ denotes the ring of proper and $\mathfrak{P}$ – stable functions , then an $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime MFD of the plant P is defined by $P = D^{-1} \cdot N$, where $D \in \mathbb{R}_{\mathfrak{P}}^{pxp}(s)$, $N \in \mathbb{R}_{\mathfrak{P}}^{pxm}(s)$ and (D , N) is an $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime pair . Let $C = \text{diag}\{C_1 , \dots , C_\kappa\} = N_c \cdot D_c^{-1}$ be an $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime MFD of the decentralized controller , where , $C_i = N_i \cdot D_i^{-1} \in \mathbb{R}_{\mathfrak{P}}^{m_i x p_i}(s)$, $(i = 1 , 2 , \dots , \kappa , \sum_{i=1}^{\kappa} m_i = m , \sum_{i=1}^{\kappa} p_i = p)$, are $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime MFDs of C_i and $N_c = \text{diag}\{N_1 , \dots , N_\kappa\}$ and $D_c = \text{diag}\{D_1 , \dots , D_\kappa\}$. It is known , [Vid. 4] , that the controller internally stabilizes the feedback system , if and only if there exists some $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular matrix U such that :

$$D D_c + N N_c = U \tag{8.2.1}$$

Partitioning D , N in terms of columns , (8.2.1) is expressed as :

$$[\, D^{p_1}, D^{p_2}, \dots , D^{p_\kappa} \,] \cdot \begin{bmatrix} D_1 & & & O \\ & D_2 & & \\ & & \ddots & \\ O & & & D_\kappa \end{bmatrix} + [\, N^{m_1}, N^{m_2}, \dots , N^{m_\kappa} \,] \cdot \begin{bmatrix} N_1 & & & O \\ & N_2 & & \\ & & \ddots & \\ O & & & N_\kappa \end{bmatrix} =$$

$$= [\, U_1 \,, U_2 \,, \dots , U_\kappa \,] \tag{8.2.2}$$

Or equivalently ,

$$[\, D^{p_i} \vdots N^{m_i} \,] \cdot \begin{bmatrix} D_i \\ N_i \end{bmatrix} = U_i \,,\, i = 1 \,, 2 \,, \dots , \kappa \tag{8.2.3}$$

where , $T_i = [\, D^{p_i} \vdots N^{m_i} \,] \in \mathbb{R}_{\mathfrak{P}}^{p x (p_i + m_i)}(s)$ are matrices defined by the plant and $X_i =$ $= [\, D_i^T \,,\, N_i^T \,]^T \in \mathbb{R}_{\mathfrak{P}}^{(p_i + m_i) x p_i}(s)$ characterize the p_i input , m_i output local controllers . The U_i are arbitrary matrices of $\mathbb{R}_{\mathfrak{P}}^{p x p_i}(s)$, with the additional property that $U \triangleq [\, U_1 \,, U_2 \,, \dots , U_\kappa \,]$ is $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular . The latter condition implies that U_i are left unimodular in $\mathbb{R}_{\mathfrak{P}}^{p x p_i}(s)$.

Remark (8.2.1) *: The solvability of DSP is equivalent to the determination of necessary and sufficient solvability conditions for the set of equations (8.2.3) , with the additional constraint that* $[\, U_1 \,, U_2 \,, \dots , U_\kappa \,]$ *is* $\mathbb{R}_{\mathfrak{P}}(s)$ *– unimodular .* □

Definition (8.2.1) *: The plant P is said to have a "decentralized fixed eigenvalue" , ("fixed mode") , at* $s_0 \in \mathfrak{P}$ *with respect to decentralized controllers C if and only if* s_0 *is a pole of the closed loop system transfer function determined for all C .* □

Theorem (8.2.1) [Wan. 1] , [Gün. 1] *: A necessary and sufficient solvability condition for the DSP is that the plant P has no "decentralized fixed eigenvalues" , ("fixed modes") .* □

Corollary (8.2.1) [Gün. 1] *: Theorem (8.2.1) implies that a necessary and sufficient solvability condition for the DSP is that the matrices* T_i *in (8.2.3) have at least* p_i *unit invariant factors .* □

Although conditions for the existence of a solution of the DSP are known , [Wan. 1] , [Gün. 1] , the parametrization of all DSP solutions in closed form has remained an open issue so far . Our aim is to study this problem and give closed form parametrization in special cases . The latter is possible for generic matrices T_i in (8.2.3) , i.e. matrices the Smith forms of which satisfy the conditions of the following lemma :

***Lemma (8.2.1) [Vid. 4]** : Let m , $n \in \mathbb{N}$. Then :*
i) If $m < n$ the set $S_A = \{A \in \mathbb{R}_{\mathcal{P}}^{m x n}(s) : A$ is equivalent to $[\, I_m \,\vdots\, O^{n-m} \,]\}$ is an open dense subset of $\mathbb{R}_{\mathcal{P}}^{m x n}(s)$.
ii) If $m > n$ the set $S_A = \{A \in \mathbb{R}_{\mathcal{P}}^{m x n}(s) : A$ is equivalent to $[\, I_n \,\vdots\, O^{m-n} \,]^T\}$ is an open dense subset of $\mathbb{R}_{\mathcal{P}}^{m x n}(s)$.
iii) If $m = n$ the set $S_A = \{A \in \mathbb{R}_{\mathcal{P}}^{m x m}(s) : A$ is equivalent to $diag\{I_{m-1} , |\, A \,|\}\}$ is an open dense subset of $\mathbb{R}_{\mathcal{P}}^{m x m}(s)$. □

A problem that is intimately related to the parametrization of solutions of the DSP is the characterization of the family of unimodular matrices of a given dimension , as well as the completion of a left or right unimodular matrix to a square unimodular . The need to derive the above characterizations arise from the necessary and sufficient parametrization constraints that the family of parameters , of the DSP , should satisfy . These issues are considered next .

8.3. CHARACTERIZATION OF UNIMODULAR MATRICES AND RELATED ISSUES

Let K be a Euclidean domain , $A_i^j \in K^{i x j}$ and $C_j(A_i^j)$ be the $j^{\underline{th}}$ order compound matrix of A_i^j . Also let $Q_{l,t}$ be the sequence of lexicographically ordered l – tuples from the set $\{1 , \ldots , t\}$, $\gamma = (i_1 , \ldots , i_{l+1}) \in Q_{l+1,t}$ and $Q_{l,l+1}^{\gamma}$ be the subset of $Q_{l,t}$ with elements the lexicographically ordered l – tuples from γ . If $\mu_\gamma[i_\kappa] = (i_1 , \ldots , i_{\kappa-1} , i_{\kappa+1} , \ldots , i_{l+1}) \in Q_{l,l+1}^{\gamma}$ and $\tau = \binom{t}{l+1}$ then :

***Definition (8.3.1) [Kar. 4]** : If $\underline{a}$ is a vector over K with coordinates given by the set $\{a_\omega, \omega \in Q_{l,t}\}$ then :*
i) The vector $\underline{a}$ is said to be decomposable over $K^{t x l}$, if a matrix $A \in K^{t x l}$ exists such that $C_l(A_t^l) = \underline{a}$.
ii) The Grasmann matrix of $\underline{a}$ is defined by , $\Phi_t^l(\underline{a}) = [\phi_{\gamma,j}]$ for all $\gamma \in Q_{l+1,t}$, $j = 1, \ldots , t$ and :

$$\phi_{\gamma,j} = \begin{cases} 0 , \text{ if } j \notin \gamma \\ sign(i_\kappa , \mu_\gamma[i_\kappa]) \cdot a_{\mu_\gamma[i_\kappa]} , \text{ if } j = i_\kappa \in \gamma \end{cases}$$

Clearly $\Phi_t^l(\underline{a})$ has dimensions $\tau x t$. □

***Lemma (8.3.1) [Kar. 4]** : Let $\underline{a}_m \in K^{m x 1}$. Then there always exist matrices $A_m^{m-1} \in K^{m x (m-1)}$ such that $C_{m-1}\left(A_m^{m-1}\right) = \underline{a}_m$; that is $\underline{a}_m$ is always decomposable to a* *matrix A_m^{m-1} . The matrices A_m^{m-1} are determined by the right null space of the Grassman matrix of $\underline{a}_m$.* □

Let $\mathcal{U}$ denotes the family of all unimodular matrices U_m of $K^{m x m}$, $\underline{u}_i = [\ u_j^i\ ] \in K^{i x 1}$, $i = m , \ldots , 2$ be arbitrary coprime vectors , u_1 be a unit of K ; then we state and prove the following results .

***Theorem (8.3.1)** : A characterization of the elements of $\mathcal{U}$ is given by :*

$$U_m = [\ \underline{u}_m \vdots A_m^{m-1} \cdot \underline{u}_{m-1} \vdots A_m^{m-1} \cdot A_{m-1}^{m-2} \cdot \underline{u}_{m-2} \vdots \cdots\cdots\cdots \vdots A_m^{m-1} \cdot A_{m-1}^{m-2} \cdot \ \ldots\ \cdot A_2^1 \cdot u_1\ ] \quad (8.3.1)$$

where , $A_i^{i-1} \in K^{i x (i-1)}$ are the decompositions of the vectors $\underline{a}_i = [\ a_j^i\ ] \in K^{i x 1}$, ($\underline{a}_i = C_{i-1}\left(A_i^{i-1}\right)$, lemma (8.3.1)) , for which the following relation holds true :

$$\sum_{j=1}^{i} (-1)^{j+1} \cdot a_{i-j+1}^i \cdot u_j^i = 1 \quad (8.3.2)$$

Proof

First we shall prove that a matrix of the form (8.3.1) is unimodular and then that an arbitrary unimodular matrix can be written as in (8.3.1) .

(⇒) Let U_m be a matrix of the type (8.3.1) . Then (8.3.1) can be viewed as :

$$U_m = [\ \underline{u}_m \vdots A_m^{m-1} \cdot U_{m-1}\ ] , \ldots , U_i = [\ \underline{u}_i \vdots A_i^{i-1} \cdot U_{i-1}\ ] , \ldots , U_2 = [\ \underline{u}_2 \vdots A_2^1 \cdot u_1\ ] ,$$
$$i = m-1 , \ldots , 3 \quad (8.3.3)$$

Consider the matrix $U_i = [\ \underline{u}_i \vdots A_i^{i-1} \cdot U_{i-1}\ ]$, then by the assumptions of the theorem , $\underline{u}_i$ is a coprime vector and A_i^{i-1} such that the vector $\underline{a}_i = [\ a_j^i\ ] = C_i\left(A_i^{i-1}\right)$ satisfies (8.3.2) . The latter implies that :

$$|\ U_i\ | = \sum_{j=1}^{i} (-1)^{j+1} \cdot a_{i-j+1}^i \cdot u_j^i \cdot |\ U_{i-1}\ | = |\ U_{i-1}\ | \quad (8.3.4)$$

$\forall\ i = m , m-1 , \ldots , 2$, (8.3.4) implies that : $|\ U_m\ | = |\ U_{m-1}\ | = \cdots = |\ U_2\ | = u_1$, which by assumption is a unit and thus U_m is unimodular .

(⇐) Let U_i be a unimodular matrix over $K^{i x i}$, $i = m , \ldots , 2$. Then U_i can be expressed as $U_i = [\ \underline{u}_i \vdots B_i^{i-1}\ ]$, with $\underline{u}_i$ a coprime vector and :

$$|\ U_i\ | = \sum_{j=1}^{i} (-1)^{j+1} \cdot b_{i-j+1}^i \cdot u_j^i = u \quad (8.3.5)$$

where , $\underline{b}_i = [\ b_1^i\ b_2^i \ldots b_i^i\ ]^T = C_{i-1}\left(B_i^{i-1}\right)$, u a unit of K . If U_{i-1} denotes a unimodular matrix with $|\ U_{i-1}\ | = u$, (such a matrix always exists) , then by lemma (8.3.1) the matrix $A_i^{i-1} = B_i^{i-1} \cdot U_{i-1}^{-1}$ is a decomposition of $\underline{a}_i = \underline{b}_i \cdot u^{-1}$ and (8.3.5) implies that :

$$\sum_{j=1}^{i} (-1)^{j+1} \cdot \left(b_{i-j+1}^{i} \cdot u^{-1}\right) \cdot u_j^i = \sum_{j=1}^{i} (-1)^{j+1} \cdot a_{i-j+1}^{i} \cdot u_j^i = 1 \qquad (8.3.6)$$

Thus $U_i = [\ \underline{u}_i : A_i^{i-1} \cdot U_{i-1}\ ]$, $\forall\ i = m, \ldots, 2$ and finally

$$U_m = [\ \underline{u}_m : A_m^{m-1} \cdot \underline{u}_{m-1} : A_m^{m-1} \cdot A_{m-1}^{m-2} \cdot \underline{u}_{m-2} : \cdots\cdots\cdots : A_m^{m-1} \cdot A_{m-1}^{m-2} \cdot \ \ldots\ \cdot A_2^1 \cdot u_1\ ]$$

where , $\underline{u}_i = [\ u_j^i\ ] \in K^{ix1}$, $i = m, \ldots, 2$ are arbitrary coprime vectors , u_1 is a unit of K, $A_i^{i-1} \in K^{ix(i-1)}$ are the decompositions of the vectors $\underline{a}_i = [\ a_j^i\ ] \in K^{ix1}$, ($\underline{a}_i = C_{i-1}(A_i^{i-1})$, for which relation (8.3.2) holds true . □

Theorem (8.3.1) states that all unimodular matrices of given dimension m are expressed as in (8.3.1) and vice versa . Furthermore , this result provides a method for constructing all unimodular matrices with given dimension m . Throughout the rest of this section we deal with the problem of characterizing all left , (right) , unimodular matrices which complete a given left , (right) , unimodular matrix to a square unimodular . These two cases are dual and thus we deal only with left unimodular matrices . Let $U_1 \in K^{mx\kappa_1}$, $m>\kappa_1$, be a left unimodular matrix , $\mathfrak{F}$ the family of all left unimodular matrices $F \in K^{mx(m-\kappa_1)}$, such that the matrix :

$$U = [\ U_1 : F\ ] \qquad (8.3.7)$$

If V_1 is a unimodular matrix for which $V_1 \cdot U_1 = [\ I_{\kappa_1} : O\ ]^T$, then $V_1^{-1} = [\ U_1 : F_0\ ]$, $F_0 \in \mathfrak{F}$ and :

***Proposition (8.3.1)** : The elements of $\mathfrak{F}$ are given by :*

$$\mathfrak{F} = \{\ F \in K^{mx(m-\kappa_1)} : F = (\ U_1 \cdot R + F_0 \cdot L\ )\ ,\ R \in K^{\kappa_1 x(m-\kappa_1)}\ ,\ \textit{arbitrary parametric matrix}\ ,\ L \in K^{(m-\kappa_1)x(m-\kappa_1)}\ ,\ \textit{arbitrary unimodular matrix}\ \} \qquad (8.3.8)$$

Proof

($\Rightarrow$) Let F be a left unimodular matrix such that $U = [\ U_1 : F\ ]$ is unimodular . On the other hand $V_1^{-1} = [\ U_1 : F_0\ ]$ is unimodular and the product :

$$V_1 \cdot U = \begin{bmatrix} I_{\kappa_1} & \vdots & \\ O & \vdots & V_1 \cdot F \\ O & \vdots & \end{bmatrix} \qquad (8.3.9)$$

is a unimodular matrix . Thus $V_1 \cdot F = [\ R^T : L^T\ ]^T$, with $R \in K^{\kappa_1 x(m-\kappa_1)}$, $L \in K^{(m-\kappa_1)x(m-\kappa_1)}$ unimodular . Consequently :

$$F = V_1^{-1} \cdot [\ R^T : L^T\ ]^T = [\ U_1 : F_0\ ] \cdot [\ R^T : L^T\ ]^T = (U_1 \cdot R + F_0 \cdot L) \in \mathcal{F}$$

($\Leftarrow$) Let $F \in \mathcal{F}$. Then $F = (U_1 \cdot R + F_0 \cdot L)$ for some $R \in K^{\kappa_1 x(m-\kappa_1)}$, $L \in K^{(m-\kappa_1)x(m-\kappa_1)}$ unimodular and the matrix :

$$U = [\ U_1 : F\ ] = [\ U_1 : F_0\ ] \cdot \begin{bmatrix} I_{\kappa_1} & R \\ O & L \end{bmatrix} = V_1^{-1} \cdot \begin{bmatrix} I_{\kappa_1} & R \\ O & L \end{bmatrix} \qquad (8.3.10)$$

is clearly unimodular . □

8.4. PARAMETRIZATION ISSUES FOR THE DSP

In this section a parametrization method for the solutions of the DSP is studied . All solutions of DSP are defined in terms of the left unimodular matrices X_i which satisfy the set of equations (8.2.3) , with $U \triangleq [\ U_1\ ,\ U_2\ ,\ \dots\ ,\ U_\kappa\ ]$ unimodular . Let $\rho_i = rank\{T_i\}$, S_i denote the Smith form of T_i over $\mathcal{P}$; U_l^i , U_r^i denote the $\mathbb{R}_{\mathcal{P}}^{pxp}(s)$, $\mathbb{R}_{\mathcal{P}}^{(p_i+m_i)x(p_i+m_i)}(s)$ unimodular matrices respectively for which $T_i = U_l^i \cdot S_i \cdot U_r^i$. Corollary (8.2.1) implies that the DSP has a solution if and only if S_i can be partitioned as :

$$S_i = \begin{bmatrix} I_{p_i} & \vdots & O & \vdots & O \\ \cdots & & \cdots & & \cdots \\ O & \vdots & S_{\rho_i - p_i} & \vdots & O \\ \cdots & & \cdots & & \cdots \\ O & \vdots & O & \vdots & O \end{bmatrix} \qquad (8.4.1)$$

If $\sum_{i=1}^{\kappa} p_i = p$, $\sum_{i=1}^{\kappa} m_i = m$ then denote by M_i the matrix :

$$M_i = \begin{bmatrix} O & \vdots & I_{p_i} & \vdots & O \\ \cdots & & \cdots & & \cdots \\ I_{\tau_{i-1}} & \vdots & O & \vdots & O \\ \cdots & & \cdots & & \cdots \\ O & \vdots & O & \vdots & I_{p-\tau_i} \end{bmatrix} ,\ \tau_i = \sum_{j=1}^{i} p_j \qquad (8.4.2)$$

Suppose that the DSP has a solution . Then :

***Theorem (8.4.1)** : All the solutions X_i of the set of equations (8.2.3) are parametrized as:*

$$X_i = (U_r^i)^{-1} \cdot Z_i^{-1} \cdot \begin{bmatrix} I_{p_i} \\ O \end{bmatrix} \qquad (8.4.3)$$

where ,

$$Z_i^{-1} = \begin{bmatrix} Z_{\rho_i} & O \\ Z^{\rho_i}_{(p_i+m_i)-\rho_i} & Z_{(p_i+m_i)-\rho_i} \end{bmatrix} \in \mathbb{R}_{\mathfrak{P}}^{(p_i+m_i)x(p_i+m_i)}(s) \qquad (8.4.4)$$

are unimodular with the additional property that there exist unimodular matrices $K_i \in \mathbb{R}_{\mathfrak{P}}^{pxp}(s)$, $L_i \in \mathbb{R}_{\mathfrak{P}}^{(p-p_i)x(p-p_i)}(s)$ *such that the following conditions hold true :*

$$i)\ K_i = \begin{bmatrix} K_{\rho_i} & K_{\rho_i}^{p-\rho_i} \\ O & K_{p-\rho_i} \end{bmatrix} \qquad (8.4.5.i)$$

$$ii)\ K_i \cdot S_i = S_i \cdot Z_i^{-1} \overset{(8.4.1)}{\Leftrightarrow} K_{\rho_i} \cdot S_{\rho_i} = S_{\rho_i} \cdot Z_{\rho_i} \qquad (8.4.5.ii)$$

$$iii)\ U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} = U_l^2 \cdot K_2 \cdot \begin{bmatrix} I_{p_2} & O \\ O & L_2 \end{bmatrix} \cdot M_2 = \cdots = U_l^{\kappa} \cdot K_{\kappa} \cdot \begin{bmatrix} I_{p_\kappa} & O \\ O & L_\kappa \end{bmatrix} \cdot M_\kappa \qquad (8.4.5.iii)$$

Proof

First we shall show that for an arbitrary set of solutions X_i of (8.2.3) conditions (8.4.3) , (8.4.4) , (8.4.5*i* , *ii* , *iii*) hold true . Then that a set of matrices X_i which satisfy conditions (8.4.3) , (8.4.4) , (8.4.5*i* , *ii* , *iii*) qualifies as a set of solutions of (8.2.3) .

(⇒) Let X_i be an arbitrary set of solutions of (8.2.3) . Then :

$$[\, T_1 \cdot X_1 \vdots T_2 \cdot X_2 \vdots \cdots\cdots\cdots \vdots T_\kappa \cdot X_\kappa \,] = U \qquad (8.4.6)$$

with U unimodular ,

$$[\, U^{-1} \cdot T_1 \cdot X_1 \vdots U^{-1} \cdot T_2 \cdot X_2 \vdots \cdots\cdots\cdots \vdots U^{-1} \cdot T_\kappa \cdot X_\kappa \,] = I_p \qquad (8.4.7)$$

or ,

$$U^{-1} \cdot T_i \cdot X_i = \begin{bmatrix} O \\ I_{p_i} \\ O \end{bmatrix} \leftarrow i^{\underline{th}} \text{ block}\ , \ \forall\, i = 1\ , \ldots , \kappa \qquad (8.4.8)$$

Finally ,

$$M_i \cdot U^{-1} \cdot T_i \cdot X_i = [\, I_{p_i} \vdots O \,]^T\ , \ \forall\, i = 1\ , \ldots , \kappa \qquad (8.4.9)$$

with M_i as in (8.4.2) , X_i are left unimodular matrices . Using the results of section 8.3 , left unimodular matrices A_i exist such that :

$$Y_i = [\, X_i \vdots A_i \,] \in \mathbf{R}_{\mathfrak{P}}^{(p_i+m_i)x(p_i+m_i)}(s) - \text{unimodular} \qquad (8.4.10)$$

Applying (8.4.10) to (8.4.9) we take :

$$M_i \cdot U^{-1} \cdot T_i \cdot Y_i = \begin{bmatrix} I_{p_i} & \vdots & \Omega_{p_i}^{m_i} \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & \Omega_{p-p_i}^{m_i} \end{bmatrix}, \forall\, i = 1 , \dots , \kappa \tag{8.4.11}$$

Multiplying (8.4.11) on the right by an appropriate unimodular matrix , we can eliminate $\Omega_{p_i}^{m_i}$ on the right hand side . Indeed :

$$M_i \cdot U^{-1} \cdot T_i \cdot Y_i \cdot \begin{bmatrix} I_{p_i} & \vdots & -\Omega_{p_i}^{m_i} \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & I_{m_i} \end{bmatrix} = \begin{bmatrix} I_{p_i} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & \Omega_{p-p_i}^{m_i} \end{bmatrix}, \forall\, i = 1 , \dots , \kappa \tag{8.4.12}$$

Set L_i , R_i the $\mathbb{R}_{\mathcal{P}}^{(p-p_i)x(p-p_i)}(s)$, $\mathbb{R}_{\mathcal{P}}^{m_i x m_i}(s)$ unimodular matrices respectively for which :

$$L_i \cdot \Omega_{p-p_i}^{m_i} \cdot R_i = S_i' \ , \ \forall\, i = 1 , \dots , \kappa \tag{8.4.13}$$

where , S_i' is the Smith form of $\Omega_{p-p_i}^{m_i}$ over $\mathcal{P}$. (8.4.12) , (8.4.13) imply $\forall\, i = 1 , \dots , \kappa$:

$$\begin{bmatrix} I_{p_i} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & L_i \end{bmatrix} \cdot M_i \cdot U^{-1} \cdot T_i \cdot Y_i \cdot \begin{bmatrix} I_{p_i} & \vdots & -\Omega_{p_i}^{m_i} \cdot R_i \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ O & \vdots & R_i \end{bmatrix} = \begin{bmatrix} I_{p_i} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & S_i' \end{bmatrix} \tag{8.4.14}$$

On the other hand :

$$(U_l^i)^{-1} \cdot T_i \cdot (U_r^i)^{-1} = S_i \ , \ \forall\, i = 1 , \dots , \kappa \tag{8.4.15}$$

Now $\forall\, i = 1 , \dots , \kappa$ set :

$$W_i = \begin{bmatrix} I_{p_i} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & L_i \end{bmatrix} \cdot M_i \cdot U^{-1} \ , \ Q_i = Y_i \cdot \begin{bmatrix} I_{p_i} & \vdots & -\Omega_{p_i}^{m_i} \cdot R_i \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ O & \vdots & R_i \end{bmatrix} \tag{8.4.16}$$

Combining (8.4.14) , (8.4.15) , (8.4.16) together it is implied that :

$$S_i = \begin{bmatrix} I_{p_i} & \vdots & O & \vdots & O \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ O & \vdots & S_{\rho_i - p_i} & \vdots & O \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ O & \vdots & O & \vdots & O \end{bmatrix} = \begin{bmatrix} I_{p_i} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & S_i' \end{bmatrix}, \forall\, i = 1 , \dots , \kappa \tag{8.4.17}$$

and $$(U_l^i)^{-1}\cdot W_i^{-1}\cdot S_i\cdot Q_i^{-1}\cdot (U_r^i)^{-1} = S_i \ , \ \forall\ i = 1\ , \dots , \kappa$$

or , $$(U_l^i)^{-1}\cdot W_i^{-1}\cdot S_i = S_i\cdot U_r^i\cdot Q_i \ , \ \forall\ i = 1\ , \dots , \kappa \qquad (8.4.18)$$

Set : $$K_i = (U_l^i)^{-1}\cdot W_i^{-1} \ , \ Z_i^{-1} = U_r^i\cdot Q_i \ , \ \forall\ i = 1\ , \dots , \kappa \qquad (8.4.19)$$

K_i , Z_i^{-1} are unimodular and satisfy (8.4.18) . If the operations in (8.4.18) are carried out the result implies :

$$\begin{bmatrix} K_{\rho_i}\cdot S_{\rho_i} & O \\ K^{\rho_i}_{p-\rho_i}\cdot S_{\rho_i} & O \end{bmatrix} = \begin{bmatrix} S_{\rho_i}\cdot Z_{\rho_i} & S_{\rho_i}\cdot Z_{\rho_i}^{(p_i+m_i)-\rho_i} \\ O & O \end{bmatrix} , \ \forall\ i = 1\ , \dots , \kappa$$

or , $$K_{\rho_i}\cdot S_{\rho_i} = S_{\rho_i}\cdot Z_{\rho_i} \ , \ K^{\rho_i}_{p-\rho_i}\cdot S_{\rho_i} = O \ , \ S_{\rho_i}\cdot Z_{\rho_i}^{(p_i+m_i)-\rho_i} = O \ , \ \forall\ i = 1\ , \dots , \kappa \qquad (8.4.20)$$

(8.4.18) , (8.4.19) , (8.4.20) imply that :

$$Y_i = [\ X_i : A_i\ ] = (U_r^i)^{-1}\cdot Z_i^{-1}\cdot \begin{bmatrix} I_{p_i} & \vdots & \Omega_{p_i}^{m_i} \\ \cdots\ \cdots & \vdots & \cdots\ \cdots \\ O & \vdots & R_i^{-1} \end{bmatrix}$$

or ,

$$X_i = (U_r^i)^{-1}\cdot Z_i^{-1}\cdot \begin{bmatrix} I_{p_i} \\ O \end{bmatrix} , \ \forall\ i = 1\ , \dots , \kappa \qquad (8.4.21)$$

with , Z_i^{-1} as in (8.4.4) . Furthermore ,

$$U_l^i\cdot K_i\cdot \begin{bmatrix} I_{p_i} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots \\ O & \vdots & L_i \end{bmatrix}\cdot M_i = U \ , \ \forall\ i = 1\ , \dots , \kappa$$

which finally implies :

$$U_l^1\cdot K_1\cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} = U_l^2\cdot K_2\cdot \begin{bmatrix} I_{p_2} & O \\ O & L_2 \end{bmatrix}\cdot M_2 = \cdots = U_l^\kappa\cdot K_\kappa\cdot \begin{bmatrix} I_{p_\kappa} & O \\ O & L_\kappa \end{bmatrix}\cdot M_\kappa \qquad (8.4.22)$$

(8.4.18) , (8.4.19) , (8.4.20) , (8.4.21) , (8.4.22) imply (8.4.3) , (8.4.4) , (8.4.5*i* , *ii* , *iii*) .
(⇐) Let a set of left unimodular matrices X_i satisfy (8.4.3) , (8.4.4) , (8.4.5*i* , *ii* , *iii*) . Then :

$$T_i\cdot X_i = U_l^i\cdot S_i\cdot U_r^i\cdot (U_r^i)^{-1}\cdot Z_i^{-1}\cdot \begin{bmatrix} I_{p_i} \\ O \end{bmatrix} , \ \forall\ i = 1\ , \dots , \kappa$$

or ,

$$T_i \cdot X_i = U_l^i \cdot S_i \cdot Z_i^{-1} \cdot \begin{bmatrix} I_{p_i} \\ O \end{bmatrix}, \ \forall\ i = 1, \ldots, \kappa \qquad (8.4.23)$$

(8.4.1) , (8.4.5ii) imply that :

$$T_i \cdot X_i = U_l^i \cdot K_i \cdot S_i \cdot \begin{bmatrix} I_{p_i} \\ O \end{bmatrix} = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} \\ O \end{bmatrix}, \ \forall\ i = 1, \ldots, \kappa \qquad (8.4.24)$$

Now partition L_1 in (8.4.5iii) as follows :

$$L_1 = [\, L_1^{p_2} \vdots L_1^{p_3} \vdots \cdots\cdots \vdots L_1^{p_\kappa} \,] \in \mathbb{R}_{\mathcal{P}}^{(p-p_1)x(p-p_1)}(s) \qquad (8.4.25)$$

(8.4.5iii) implies that :

$$U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} & O \\ O & L_i \end{bmatrix} \cdot M_i, \ \forall\ i = 2, \ldots, \kappa \qquad (8.4.26)$$

or ,

$$U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} \cdot M_i^{-1} = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} & O \\ O & L_i \end{bmatrix}, \ \forall\ i = 2, \ldots, \kappa \qquad (8.4.27)$$

or ,

$$U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} \cdot \begin{bmatrix} O & \vdots & I_{\tau_{i-1}} & \vdots & O \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ I_{p_i} & \vdots & O & \vdots & O \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ O & \vdots & O & \vdots & I_{p-\tau_i} \end{bmatrix} = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} & O \\ O & L_i \end{bmatrix}, \ \forall\ i = 2, \ldots, \kappa \qquad (8.4.28)$$

with τ_i defined in (8.4.2) . Finally , (8.4.25) , (8.4.28) imply that $\forall\ i = 2, \ldots, \kappa$:

$$U_l^1 \cdot K_1 \cdot \begin{bmatrix} O & \vdots & I_{p_1} & \vdots & O \\ \cdots\cdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \\ L_1^{p_i} & \vdots & O & \vdots & L_1^{p-(p_1+p_i)} \end{bmatrix} = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} & O \\ O & L_i \end{bmatrix} \qquad (8.4.29)$$

or ,

$$U_l^1 \cdot K_1 \cdot \begin{bmatrix} O \\ \cdots \\ L_1^{p_i} \end{bmatrix} = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} \\ \cdots \\ O \end{bmatrix}, \ \forall\ i = 2, \ldots, \kappa \qquad (8.4.30)$$

Applying (8.4.30) to (8.4.24) is implied that :

$$T_1 \cdot X_1 = U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_i} \\ \cdots \\ O \end{bmatrix} \tag{8.4.31}$$

$$T_i \cdot X_i = U_l^i \cdot K_i \cdot \begin{bmatrix} I_{p_i} \\ \cdots \\ O \end{bmatrix} = U_l^1 \cdot K_1 \cdot \begin{bmatrix} O \\ \cdots \\ L_1^{p_i} \end{bmatrix}, \forall\, i = 2, \ldots, \kappa \tag{8.4.32}$$

The set of equations (8.2.3) is then satisfied by the left unimodular matrices X_i defined in (8.4.3) and by (8.4.25) , (8.4.31) , (8.4.32) the matrix :

$$U = [\, T_1 \cdot X_1 \vdots T_2 \cdot X_2 \vdots \cdots\cdots\cdots \vdots T_\kappa \cdot X_\kappa \,] =$$

$$= U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_i} & O & O & \cdots & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & L_1^{p_2} & L_1^{p_3} & \cdots & L_1^{p_\kappa} \end{bmatrix} = U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} \tag{8.4.33}$$

is unimodular . □

Despite the fact that the parametrization method of theorem (8.4.1) is not in closed form – since the set of parameters which satisfy conditions (8.4.4) , (8.4.5*i* , *ii* , *iii*) is not fully characterized – there are cases in which closed form parametrization can be achieved . The first of such cases is described next . Let T_i be the matrices defined by (8.2.3) which satisfy the condition $p \le (p_i + m_i)$. Then $\rho_i = p$ and generically T_i is equivalent to the matrix $[\, I_p \vdots O^{(p_i + m_i) - p} \,]$, (lemma (8.2.1)) . Then (8.4.4) , (8.4.5*i* *ii*) , imply that $K_i = Z_{\rho_i} = K_{\rho_i}$. Under the above assumptions the parametrization of X_i in (8.2.3) is formulated using (8.4.5.*i* , *ii* , *iii*) :

Parametrization of solutions of DSP when $p \le (p_i + m_i)$

Step 1 : For all the arbitrary unimodular matrices K_1 , L_1 , L_2 , $Z_{(p_1+m_1)-\rho_1}$ and arbitrary parametric $Z_{(p_1+m_1)-\rho_1}^{\rho_1}$, define :
α) K_2 to be the unimodular matrix :

$$K_2 = (U_l^2)^{-1} \cdot U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} \cdot (M_2)^{-1} \cdot \begin{bmatrix} I_{p_2} & O \\ O & L_2^{-1} \end{bmatrix} \tag{8.4.34}$$

***β*) Z_1^{-1} to be unimodular matrix as in (8.4.4) for $i = 1$, $Z_{\rho_1} = K_1$.**

Step 2 : For all the matrices of *α*) in step 1 and all arbitrary unimodular matrices L_3 ,

$Z_{(p_2+m_2)\text{-}\rho_2}$ and arbitrary parametric $Z^{\rho_2}_{(p_2+m_2)\text{-}\rho_2}$, define :

α) K_3 to be the unimodular matrix :

$$K_3 = (U_l^3)^{-1}\cdot U_l^2\cdot K_2\cdot\begin{bmatrix} I_{p_2} & O \\ O & L_2\end{bmatrix}\cdot M_2\cdot(M_3)^{-1}\cdot\begin{bmatrix} I_{p_3} & O \\ O & L_3^{-1}\end{bmatrix} \tag{8.4.34}$$

β) Z_2^{-1} to be unimodular matrix as in (8.4.4) for $i = 2$, $Z_{\rho_2} = K_2$.

Following similar arguments and after finite number of steps the process terminates with steps $\kappa - 1$, κ :

Step $\kappa - 1$: For all the matrices of step $\kappa - 2$ and all arbitrary unimodular matrices L_κ , $Z_{(p_{\kappa-1}+m_{\kappa-1})\text{-}\rho_{\kappa-1}}$ and arbitrary parametric $Z^{\rho_{\kappa-1}}_{(p_{\kappa-1}+m_{\kappa-1})\text{-}\rho_{\kappa-1}}$, define :

α) K_κ to be the unimodular matrix :

$$K_\kappa = (U_l^\kappa)^{-1}\cdot U_l^{\kappa-1}\cdot K_{\kappa-1}\cdot\begin{bmatrix} I_{p_{\kappa-1}} & O \\ O & L_{\kappa-1}\end{bmatrix}\cdot M_{\kappa-1}\cdot(M_\kappa)^{-1}\cdot\begin{bmatrix} I_{p_\kappa} & O \\ O & L_\kappa^{-1}\end{bmatrix} \tag{8.4.39}$$

β) $Z_{\kappa-1}^{-1}$ to be unimodular matrix as in (8.4.4) for $i = \kappa - 1$, $Z_{\rho_{\kappa-1}} = K_{\kappa-1}$.

Step κ : For all the matrices K_κ of *α*) in step $\kappa - 1$, all arbitrary unimodular matrices $Z_{(p_\kappa+m_\kappa)\text{-}\rho_\kappa}$ and arbitrary parametric $Z^{\rho_\kappa}_{(p_\kappa+m_\kappa)\text{-}\rho_\kappa}$, define : Z_κ^{-1} to be unimodular matrix as in (8.4.4) for $i = \kappa$, $Z_{\rho_\kappa} = K_\kappa$. □

By inspection of theorem (8.4.1) it follows that the set of parametric matrices Z_i^{-1} which parametrize the set of solutions X_i of (8.2.3) is generated by the above algorithm and vice versa . A more practical way to view the parametrization described above follows next :

Proposition (8.4.1) *: All the solutions X_i of the set of equations (8.2.3) are parametrized as:*

$$\begin{bmatrix} X_1 & & O \\ & X_2 & \\ & & \ddots \\ O & & X_\kappa\end{bmatrix} = [\, V_1 \vdots \cdots \vdots V_\kappa \,]\cdot\begin{bmatrix} U & & O \\ & U & \\ & & \ddots \\ O & & U\end{bmatrix}\cdot E + [\, G_1 \vdots \cdots \vdots G_\kappa \,]\cdot\begin{bmatrix} P_1 & & O \\ & P_2 & \\ & & \ddots \\ O & & P_\kappa\end{bmatrix} \tag{8.4.40}$$

where , V_i , G_i belong to $\mathbb{R}_{\mathfrak{P}}^{(p_i+m_i)\times p}(s)$, $\mathbb{R}_{\mathfrak{P}}^{(p_i+m_i)\times(p_i+m_i-p)}(s)$ respectively and

$$[\, V_i : G_i \,] = (U_r^i)^{-1} \cdot \begin{bmatrix} (U_l^i)^{-1} & O \\ O & I_{p_i + m_i - p} \end{bmatrix} \tag{8.4.41}$$

U is an arbitrary pxp unimodular matrix , P_i is an arbitrary $(p_i + m_i - p)xp_i$ parametric matrix and if I_p is partitioned as $[\, I^{p_1} : I^{p_2} : \cdots : I^{p_\kappa} \,]$ then E is defined as $E = diag\, \{ E_1 \;\; E_2 \,, \ldots , E_\kappa \} \in \mathbb{R}^{(\kappa \cdot p)xp}$, where :

$$E_i = \begin{bmatrix} O \\ I_p^{p_i} \\ O \end{bmatrix} \leftarrow i^{\underline{th}} \; block \in \mathbb{R}^{pxp_i} \tag{8.4.42}$$

Proof

($\Rightarrow$) Let X_i be a set of matrices as in (8.4.40) . Then if the columns of U are partitioned according to the partitioning of p , namely , $U = [\, U^{p_1} : U^{p_2} : \cdots : U^{p_\kappa} \,]$, and the operations in (8.4.40) carried out , the X_i are formulated as :

$$X_i = [\, V_i : G_i \,] \cdot \begin{bmatrix} U^{p_i} \\ P_i \end{bmatrix} = (U_r^i)^{-1} \cdot \begin{bmatrix} (U_l^i)^{-1} & O \\ O & I_{p_i + m_i - p} \end{bmatrix} \cdot \begin{bmatrix} U^{p_i} \\ P_i \end{bmatrix} \tag{8.4.43}$$

(8.4.43) implies that :

$$T_i \cdot X_i = U_l^i \cdot [\, I_p : O^{p_i + m_i - p} \,] \cdot U_r^i \cdot (U_r^i)^{-1} \cdot \begin{bmatrix} (U_l^i)^{-1} & O \\ O & I_{p_i + m_i - p} \end{bmatrix} \cdot \begin{bmatrix} U^{p_i} \\ P_i \end{bmatrix} = U^{p_i} \tag{8.4.44}$$

By (8.4.44) it is clear that the matrix :

$$[\, T_1 \cdot X_1 : T_2 \cdot X_2 : \cdots\cdots\cdots : T_\kappa \cdot X_\kappa \,] = [\, U^{p_1} : U^{p_2} : \cdots : U^{p_\kappa} \,] = U \tag{8.4.45}$$

is unimodular and thus X_i qualify for a solution of (8.2.3) .

($\Leftarrow$) Let a set of matrices X_i satisfy (8.3.2) . Then there exists a unimodular matrix U such that :

$$[\, T_1 \cdot X_1 : T_2 \cdot X_2 : \cdots\cdots\cdots : T_\kappa \cdot X_\kappa \,] = U = [\, U^{p_1} : U^{p_2} : \cdots : U^{p_\kappa} \,] \tag{8.4.46}$$

or equivalently ,

$$T_i \cdot X_i = U_l^i \cdot [\, I_p : O^{p_i + m_i - p} \,] \cdot U_r^i \cdot X_i = U^{p_i}$$

or,

$$[\, I_p : O^{p_i + m_i - p} \,] \cdot \begin{bmatrix} U_l^i & O \\ O & I_{p_i + m_i - p} \end{bmatrix} \cdot U_r^i \cdot X_i = U^{p_i} \tag{8.4.47}$$

Set :

$$Y_i = \begin{bmatrix} U_l^i & O \\ O & I_{p_i + m_i - p} \end{bmatrix} \cdot U_r^i \cdot X_i \tag{8.4.48}$$

Partition Y_i as :

$$Y_i = \begin{bmatrix} Y_p^{p_i} \\ Y_{p_i + m_i - p}^{p_i} \end{bmatrix} \tag{8.4.49}$$

(8.4.47) , (8.4.48) (8.4.49) combined imply that $Y_p^{p_i} = U^{p_i}$. Thus , by (8.4.48)

$$X_i = (U_r^i)^{-1} \cdot \begin{bmatrix} (U_l^i)^{-1} & O \\ O & I_{p_i + m_i - p} \end{bmatrix} \cdot \begin{bmatrix} U^{p_i} \\ Y_{p_i + m_i - p}^{p_i} \end{bmatrix} = [\, V_i : G_i \,] \cdot \begin{bmatrix} U^{p_i} \\ P_i \end{bmatrix} \tag{8.4.50}$$

It is clear that X_i can be arranged as indicated in (8.4.40) . □

Next we consider the parametrization problem for the case of two block diagonal controller ($\kappa = 2$) ; the generic and some non generic cases are examined .

8.5. TWO BLOCKS DECENTRALIZED STABILIZING CONTROLLERS PARAMETRIZATION ISSUES

Assume that the stabilizing controller has two blocks . Then the parametrization of the solutions of the DSP reduces to the parametrization of the solutions of the following two equations :

$$[\, D^{p_i} : N^{m_i} \,] \cdot \begin{bmatrix} D_i \\ N_i \end{bmatrix} = U_i \ , \ i = 1 \ , 2 \tag{8.5.1}$$

where , $T_i = [\, D_p^{p_i} : N_p^{m_i} \,] \in \mathbb{R}_{\mathfrak{P}}^{px(p_i + m_i)}(s)$ are matrices defined by the plant and $X_i = [\, D_i^T \ , N_i^T \,]^T \in \mathbb{R}_{\mathfrak{P}}^{(p_i + m_i)xp_i}(s)$ characterize the p_i input , m_i output local controllers . The U_i are arbitrary matrices of $\mathbb{R}_{\mathfrak{P}}^{pxp_i}(s)$, with the additional property that $U \triangleq [\, U_1 \ , U_2 \,]$ is $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular .

CASE 1 : Assume that none of the T_i is square and their Smith form is given by :

$$S_i = \begin{bmatrix} I_{\rho_i} & O \\ O & O \end{bmatrix} \tag{8.5.2}$$

where , $1 \le \rho_i \le min\ \{\ p\ ,\ (p_i + m_i)\ \}$. It is clear that when ρ_i is either p , or $(p_i + m_i)$ then we have the generic case for the T_i , whereas when $\rho_i < min\ \{\ p\ ,\ (p_i + m_i)\ \}$ we have a non generic case for the T_i . Theorem (8.4.1) appropriately adjusted to suit the above mentioned assumptions provides a parametrization for the solutions of (8.5.1) , namely :

Theorem (8.5.1) *: All the solutions X_i of the set of equations (8.5.1) are parametrized as:*

$$X_i = (U_r^i)^{-1} \cdot Z_i^{-1} \cdot \begin{bmatrix} I_{\rho_i} \\ O \end{bmatrix} \tag{8.5.3}$$

where ,

$$Z_i^{-1} = \begin{bmatrix} Z_{\rho_i} & O \\ Z^{\rho_i}_{(p_i+m_i)-\rho_i} & Z_{(p_i+m_i)-\rho_i} \end{bmatrix} \in \mathbb{R}_{\mathfrak{P}}^{(p_i+m_i)x(p_i+m_i)}(s) \tag{8.5.4}$$

are unimodular with the additional property that there exist unimodular matrices $K_i \in \mathbb{R}_{\mathfrak{P}}^{pxp}(s)$, $L_i \in \mathbb{R}_{\mathfrak{P}}^{(p-p_i)x(p-p_i)}(s)$ such that the following conditions hold true :

$$\text{i)}\ K_i = \begin{bmatrix} K_{\rho_i} & K_{\rho_i}^{p-\rho_i} \\ O & K_{p-\rho_i} \end{bmatrix} \tag{8.5.5.i}$$

$$\text{ii)}\ K_i \cdot S_i = S_i \cdot Z_i^{-1} \overset{(8.5.2)}{\Leftrightarrow} K_{\rho_i} \cdot I_{\rho_i} = I_{\rho_i} \cdot Z_{\rho_i} \Leftrightarrow K_{\rho_i} = Z_{\rho_i} \tag{8.5.5.ii}$$

$$\text{iii)}\ U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} = U_l^2 \cdot K_2 \cdot \begin{bmatrix} O & I_{p_2} \\ L_2 & O \end{bmatrix} \tag{8.5.5.iii}$$

□

Remark (8.5.1) ***: The parametrization described in theorem (8.5.1) is in closed form if and only if the set of parameters which satisfy (8.5.4) , (8.5.5i , ii , iii) can be fully generated . Inspection of conditions (8.5.4) , (8.5.5i , ii , iii) implies that it suffices to fully generate the family of matrices K_i which satisfy (8.5.5i , iii) , since all the Z_i^{-1} which satisfy (8.5.4) , (8.5.5.ii) can be generated by setting $Z_{\rho_i} = K_{\rho_i}$, $Z^{\rho_i}_{(p_i+m_i)-\rho_i}$, an arbitrary parametric matrix , $Z_{(p_i+m_i)-\rho_i}$ an arbitrary unimodular matrix .*** □

In the following we study the closed form parametrization of the matrices K_i which satisfy (8.5.5*i* , *iii*) . Condition (8.5.5.*iii*) can be equivalently transformed to :

$$(K_2)^{-1} \cdot (U_l^2)^{-1} \cdot U_l^1 \cdot K_1 = \begin{bmatrix} O & I_{p_2} \\ L_2 & O \end{bmatrix} \cdot \begin{bmatrix} I_{p_1} & O \\ O & (L_1)^{-1} \end{bmatrix}$$

or,

$$(K_2)^{-1} \cdot G \cdot K_1 = \begin{bmatrix} O & (L_1)^{-1} \\ L_2 & O \end{bmatrix} = L \qquad (8.5.6)$$

Note that $(K_2)^{-1}$ has upper triangular structure as in (8.5.5.*i*) .

Definition (8.5.1) *: Define* $\mathfrak{F}$ *the set of all pairs* (K_1 , K_2) *such that (8.5.5.i) , (8.5.5.iii) , (or (8.5.6)) , hold true . Define the relation ,* $\sim$ *, between the elements of* $\mathfrak{F}$ *as* $(K_1 , K_2) \sim (H_1 , H_2) \Leftrightarrow \exists\, L$ *: (8.5.6) holds true for the pairs* (K_1 , K_2) *,* (H_1 , H_2) *and the same* L *.* □

The above defines an equivalence relation and partitions $\mathfrak{F}$ to a family of equivalence classes $C_{(K_1,K_2)}$. It is clear that the matrix L characterizes the equivalence classes . If L is changed then a new equivalence class is determined . Thus the parametrization of the matrices K_i which satisfy (8.5.5.*i*) , (8.5.5.*iii*) is equivalent to the description of a process which generates all the elements of $\mathfrak{F}/\sim$. This task involves the following two steps : Let the pair (K_1 , K_2) be an element of $\mathfrak{F}$. The first step is to determine representatives for all the equivalence classes in $\mathfrak{F}/\sim$, in terms of (K_1 , K_2) . The second step is to parametrize the elements of an arbitrary equivalence class in terms of its representative determined in step 1 . It is clear that this process parametrizes all the elements of $\mathfrak{F}/\sim$ and thus the set $\mathfrak{F}$ in closed form .

STEP 1 : Generation of representatives for the elements of $\mathfrak{F}/\sim$

Let (K_1 , K_2) be an element of $\mathfrak{F}$, $C_{(K_1,K_2)}$ be the equivalence class with representative (K_1 , K_2) , then a matrix L exists such that (8.5.6) holds true . Let B_1 , B_2 be the *pxp* unimodular matrices :

$$B_1 = \begin{bmatrix} M_1^{-1} \cdot L_1 & O \\ O & I_{p_1} \end{bmatrix} , B_2 = \begin{bmatrix} L_2^{-1} \cdot M_2 & O \\ O & I_{p_2} \end{bmatrix} \qquad (8.5.7)$$

where , M_i are arbitrary $(p-p_i)x(p-p_i)$ unimodular matrices , L_i are defined by L in (8.5.6) . A process for generating representatives for the elements of $\mathcal{F}/\sim$ in terms of (K_1, K_2) is described by the following result .

***Proposition (8.5.1)** : A representative of an arbitrary equivalence class in $\mathcal{F}/\sim$ is expressed in terms of (K_1 , K_2) as :*

$$P_1 = K_1 \cdot B_2 \ , \ P_2 = K_2 \cdot B_1^{-1} \tag{8.5.8}$$

with , B_1 , B_2 as in (8.5.7) .

Proof

Let B_1 , B_2 be two unimodular matrices defined as in (8.5.7) . Set $P_1 = K_1 \cdot B_2$, $P_2 = K_2 \cdot B_1^{-1}$. Then :

$$P_1 = \begin{bmatrix} K_{\rho_1} & K_{\rho_1}^{p-\rho_1} \\ O & K_{p-\rho_1} \end{bmatrix} \cdot \left[\begin{array}{cc:c} L_2^{-1}\cdot M_2 & & \\ & I_{\rho_1 - p_1} & \\ \hdashline & & I_{p-\rho_1} \end{array}\right] = \begin{bmatrix} P_{\rho_1} & P_{\rho_1}^{p-\rho_1} \\ O & P_{p-\rho_1} \end{bmatrix} \tag{8.5.9}$$

$$P_2 = \begin{bmatrix} K_{\rho_2} & K_{\rho_2}^{p-\rho_2} \\ O & K_{p-\rho_2} \end{bmatrix} \cdot \left[\begin{array}{cc:c} L_1^{-1}\cdot M_1 & & \\ & I_{\rho_2 - p_2} & \\ \hdashline & & I_{p-\rho_2} \end{array}\right] = \begin{bmatrix} P_{\rho_2} & P_{\rho_2}^{p-\rho_2} \\ O & P_{p-\rho_2} \end{bmatrix} \tag{8.5.10}$$

(8.5.9) , (8.5.10) imply that (P_1 , P_2) satisfy (8.5.5.*i*) . Furthermore ,

$$(P_2)^{-1}\cdot G\cdot P_1 = B_1\cdot (K_2)^{-1}\cdot G\cdot K_1\cdot B_2 = B_1\cdot \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix}\cdot B_2 = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} = M \tag{8.5.11}$$

For (P_1 , P_2) , (8.5.5.*iii*) , or equivalently (8.5.6) holds true for L = M . Thus (P_1 , P_2) can be viewed as a representative of an equivalence class $C_{(P_1,P_2)}$ with elements all the pairs (F_1 , F_2) for which (8.5.5.*i*) and $F_2^{-1}\cdot G\cdot F_1 = M$ hold true . Since the matrices M_i are arbitrarily selected , the matrix M which characterizes $C_{(P_1,P_2)}$ is arbitrary and thus $C_{(P_1,P_2)}$ is arbitrary . □

When a representative of an equivalence class is known then , the parametrization of its elements is required .

STEP 2 : Parametrization of the elements of $C_{(P_1,P_2)}$ in terms of (P_1 , P_2)

Consider the arbitrary equivalence class $C_{(P_1,P_2)}$ characterized by the unimodular matrix M :

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} = P_2^{-1} \cdot G \cdot P_1 \tag{8.5.12}$$

and M_1 , M_2 have dimensions $p_2 x p_2$, $p_1 x p_1$ respectively . Since we have assumed that the DSP has a solution , corollary (8.2.1) implies that $\rho_i \geq p_i$. Partition M , M^{-1} as,

$$M = \begin{bmatrix} M_{\rho_2}^{\rho_1} & M_{\rho_2}^{p-\rho_1} \\ M_{p-\rho_2}^{\rho_1} & O_{p-\rho_2}^{p-\rho_1} \end{bmatrix} , M^{-1} = \begin{bmatrix} N_{\rho_1}^{\rho_2} & N_{\rho_1}^{p-\rho_2} \\ N_{p-\rho_1}^{\rho_2} & O_{p-\rho_1}^{p-\rho_2} \end{bmatrix} \tag{8.5.13}$$

Then , since M^{-1} is unimodular it is clear that $N_{p-\rho_1}^{\rho_2}$ is right unimodular . Let U_{ρ_2} be a unimodular matrix such that :

$$N_{p-\rho_1}^{\rho_2} \cdot U_{\rho_2}^{-1} = [\, I_{p-\rho_1} \vdots\, O^{\rho_1+\rho_2-p} \,] \tag{8.5.14}$$

Let $B_{\rho_2}^{\rho_1+\rho_2-p}$ be a base of $\mathcal{N}\{N_{p-\rho_1}^{\rho_2}\}$. A parametrization of the elements of $C_{(P_1,P_2)}$ in terms of (P_1 , P_2) is described by the following proposition .

Proposition (8.5.2) *: All the elements (F_1 , F_2) of $C_{(P_1,P_2)}$ are parametrized in terms of (P_1 , P_2) as :*

$$F_1 = P_1 \cdot W^{-1} , \; F_2 = P_2 \cdot Q^{-1} \tag{8.5.15}$$

where ,

$$W = \begin{bmatrix} W_{\rho_1} & W_{\rho_1}^{p-\rho_1} \\ O & W_{p-\rho_1} \end{bmatrix} = M^{-1} \cdot Q \cdot M \;, \; unimodular \tag{8.5.16}$$

$$Q = \begin{bmatrix} Q_{\rho_2} & Q_{\rho_2}^{p-\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix} , \; unimodular \tag{8.5.17}$$

with ,

$$Q_{\rho_2} = U_{\rho_2}^{-1} \cdot \begin{bmatrix} A_{p-\rho_1} & O \\ C_{\rho_1+\rho_2-p}^{p-\rho_1} & D_{\rho_1+\rho_2-p} \end{bmatrix} \cdot U_{\rho_2} \;, \; Q_{\rho_2}^{p-\rho_2} = B_{\rho_2}^{\rho_1+\rho_2-p} \cdot \Omega_{\rho_1+\rho_2-p}^{p-\rho_2} \tag{8.5.18}$$

and $Q_{p-\rho_2}$, $A_{p-\rho_1}$, $D_{\rho_1+\rho_2-p}$ *are arbitrary unimodular* , $C^{p-\rho_1}_{\rho_1+\rho_2-p}$, $\Omega^{p-\rho_2}_{\rho_1+\rho_2-p}$ *are arbitrary parametric* .

Proof

($\Rightarrow$) Let (F_1 , F_2) be an element of $C_{(P_1,P_2)}$. We shall show that unimodular matrices W , Q exist such that , (8.5.15) , (8.5.16) , (8.5.17) , (8.5.18) hold true . (F_1 , F_2) satisfies (8.5.5.*i*) and :

$$F_2^{-1}\cdot G\cdot F_1 = M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} \tag{8.5.19}$$

(P_1 , P_2) as a representative of $C_{(P_1,P_2)}$ satisfies (8.5.5.*i*) and

$$P_2^{-1}\cdot G\cdot P_1 = M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} \tag{8.5.20}$$

(8.5.19) , (8.5.20) combined result to :

$$(F_2^{-1}\cdot P_2)\cdot M\cdot (P_1^{-1}\cdot F_1) = M \tag{8.5.21}$$

Set

$$W^{-1} = P_1^{-1}\cdot F_1 \ , \ Q = F_2^{-1}\cdot P_2 \tag{8.5.22}$$

and (8.5.21) can be written as :

$$Q\cdot M\cdot W^{-1} = M \Leftrightarrow W = M^{-1}\cdot Q\cdot M \tag{8.5.23}$$

The unimodular matrices $W = M^{-1}\cdot Q\cdot M$, Q have the upper triangular structure of (8.5.16) , (8.5.17) , since :

$$W = F_1^{-1}\cdot P_1 = \begin{bmatrix} F^{-1}_{\rho_1} & -F^{-1}_{\rho_1}\cdot F^{p-\rho_1}_{\rho_1}\cdot F^{-1}_{p-\rho_1} \\ O & F^{-1}_{p-\rho_1} \end{bmatrix}\cdot\begin{bmatrix} P_{\rho_1} & P^{p-\rho_1}_{\rho_1} \\ O & P_{p-\rho_1} \end{bmatrix} \tag{8.5.24}$$

$$Q = F_2^{-1}\cdot P_2 = \begin{bmatrix} F^{-1}_{\rho_2} & -F^{-1}_{\rho_2}\cdot F^{p-\rho_2}_{\rho_2}\cdot F^{-1}_{p-\rho_2} \\ O & F^{-1}_{p-\rho_2} \end{bmatrix}\cdot\begin{bmatrix} P_{\rho_2} & P^{p-\rho_2}_{\rho_2} \\ O & P_{p-\rho_2} \end{bmatrix} \tag{8.5.25}$$

Using the partition of M , M^{-1} as in (8.5.13) ; (8.5.23) , (8.5.24) , (8.5.25) imply :

$$\begin{bmatrix} W_{\rho_1} & W_{\rho_1}^{p-\rho_1} \\ O & W_{p-\rho_1} \end{bmatrix} = \begin{bmatrix} N_{\rho_1}^{\rho_2} & N_{\rho_1}^{p-\rho_2} \\ N_{p-\rho_1}^{\rho_2} & O_{p-\rho_1}^{p-\rho_2} \end{bmatrix} \cdot \begin{bmatrix} Q_{\rho_2} & Q_{\rho_2}^{p-\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix} \cdot \begin{bmatrix} M_{\rho_2}^{\rho_1} & M_{\rho_2}^{p-\rho_1} \\ M_{p-\rho_2}^{\rho_1} & O_{p-\rho_2}^{p-\rho_1} \end{bmatrix} \tag{8.5.26}$$

Carrying out the operations in (8.5.26) with respect to the partitioning of the matrices it is implied that :

$$[\ O_{p-\rho_1}^{\rho_1} : W_{p-\rho_1}\ ] = [\ N_{p-\rho_1}^{\rho_2} : O_{p-\rho_1}^{p-\rho_2}\ ] \cdot \begin{bmatrix} Q_{\rho_2} & Q_{\rho_2}^{p-\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix} \cdot \begin{bmatrix} M_{\rho_2}^{\rho_1} & M_{\rho_2}^{p-\rho_1} \\ M_{p-\rho_2}^{\rho_1} & O_{p-\rho_2}^{p-\rho_1} \end{bmatrix} \tag{8.5.27}$$

If (8.5.27) is multiplied on the right by M^{-1} it follows that :

$$[\ O_{p-\rho_1}^{\rho_1} : W_{p-\rho_1}\ ] \cdot \begin{bmatrix} N_{\rho_1}^{\rho_2} & N_{\rho_1}^{p-\rho_2} \\ N_{p-\rho_1}^{\rho_2} & O_{p-\rho_1}^{p-\rho_2} \end{bmatrix} = [\ N_{p-\rho_1}^{\rho_2} : O_{p-\rho_1}^{p-\rho_2}\ ] \cdot \begin{bmatrix} Q_{\rho_2} & Q_{\rho_2}^{p-\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix} \tag{8.5.28}$$

$$[\ W_{p-\rho_1} \cdot N_{p-\rho_1}^{\rho_2} : O_{p-\rho_1}^{p-\rho_2}\ ] = [\ N_{p-\rho_1}^{\rho_2} \cdot Q_{\rho_2} : N_{p-\rho_1}^{\rho_2} \cdot Q_{\rho_2}^{p-\rho_2}\ ] \tag{8.5.29}$$

and (8.5.29) finally implies that :

$$\begin{cases} N_{p-\rho_1}^{\rho_2} \cdot Q_{\rho_2}^{p-\rho_2} = O_{p-\rho_1}^{p-\rho_2} & (8.5.30) \\ W_{p-\rho_1} \cdot N_{p-\rho_1}^{\rho_2} = N_{p-\rho_1}^{\rho_2} \cdot Q_{\rho_2} & (8.5.31) \end{cases}$$

If $B_{\rho_2}^{\rho_1+\rho_2-p}$ is a base of $\mathcal{N}\{N_{p-\rho_1}^{\rho_2}\}$, then by (8.5.30) it is clear that a parametric matrix $\Omega_{\rho_1+\rho_2-p}^{p-\rho_2}$ exists such that :

$$Q_{\rho_2}^{p-\rho_2} = B_{\rho_2}^{\rho_1+\rho_2-p} \cdot \Omega_{\rho_1+\rho_2-p}^{p-\rho_2} \tag{8.5.32}$$

If U_{ρ_2} is the matrix defined in (8.3.14) , then (8.5.31) can be viewed as :

$$[\ I_{p-\rho_1} : O^{\rho_1+\rho_2-p}\ ] \cdot \begin{bmatrix} W_{p-\rho_1} & O \\ O & I_{\rho_1+\rho_2-p} \end{bmatrix} \cdot U_{\rho_2} = [\ I_{p-\rho_1} : O^{\rho_1+\rho_2-p}\ ] \cdot U_{\rho_2} \cdot Q_{\rho_2} \tag{8.5.33}$$

or equivalently ,

$$[\ I_{p-\rho_1} : O^{\rho_1+\rho_2-p}\ ] \cdot \begin{bmatrix} W_{p-\rho_1} & O \\ O & I_{\rho_1+\rho_2-p} \end{bmatrix} \cdot U_{\rho_2} \cdot Q_{\rho_2}^{-1} \cdot U_{\rho_2}^{-1} = [\ I_{p-\rho_1} : O^{\rho_1+\rho_2-p}\ ] \tag{8.5.34}$$

(8.5.34) clearly implies that there exist matrices $H_{\rho_1+\rho_2-p}$ unimodular , $R^{p-\rho_1}_{\rho_1+\rho_2-p}$ parametric such that :

$$\begin{bmatrix} W_{p-\rho_1} & O \\ O & I_{\rho_1+\rho_2-p} \end{bmatrix} \cdot U_{\rho_2} \cdot Q^{-1}_{\rho_2} \cdot U^{-1}_{\rho_2} = \begin{bmatrix} I_{p-\rho_1} & O \\ R^{p-\rho_1}_{\rho_1+\rho_2-p} & H_{\rho_1+\rho_2-p} \end{bmatrix} \tag{8.5.35}$$

(8.3.35) finally implies that :

$$Q_{\rho_2} = U^{-1}_{\rho_2} \cdot \begin{bmatrix} W_{p-\rho_1} & O \\ -H^{-1}_{\rho_1+\rho_2-p} \cdot R^{p-\rho_1}_{\rho_1+\rho_2-p} & H^{-1}_{\rho_1+\rho_2-p} \end{bmatrix} \cdot U_{\rho_2} \tag{8.5.36}$$

Set $Q_{p-\rho_2} = F^{-1}_{p-\rho_2} \cdot P_{p-\rho_2}$, $A_{p-\rho_1} = W_{p-\rho_1}$, $D_{\rho_1+\rho_2-p} = H^{-1}_{\rho_1+\rho_2-p}$, $R^{p-\rho_1}_{\rho_1+\rho_2-p} = -H^{-1}_{\rho_1+\rho_2-p} \cdot R^{p-\rho_1}_{\rho_1+\rho_2-p}$. (8.5.22) , (8.5.23) , (8.5.24) , (8.5.25) , (8.5.32) , (8.5.36) imply that unimodular matrices W , Q exist such that (8.5.15) , (8.5.16) , (8.5.17) , (8.5.18) hold true .

($\Leftarrow$) Let unimodular matrices W , Q exist such that (8.5.15) , (8.5.16) , (8.5.17) , (8.5.18) hold true . We shall show that the pair of matrices (F_1 , F_2) defined in (8.5.15) belongs to $C_{(P_1,P_2)}$. In order to do so , we must prove that the pair (F_1 , F_2) satisfies (8.5.5.*i*) and (8.5.6) for L = M , (in other words $F_2^{-1} \cdot G \cdot F_1 = M$) . Since the pair ($P_1$, P_2) is a representative of $C_{(P_1,P_2)}$ it satisfies (8.5.5.*i*) . The latter and (8.5.15) , (8.5.16), (8.5.17) imply that :

$$F_1 = P_1 \cdot W^{-1} = \begin{bmatrix} P_{\rho_1} & P^{p-\rho_1}_{\rho_1} \\ O & P_{p-\rho_1} \end{bmatrix} \cdot \begin{bmatrix} W^{-1}_{\rho_1} & -W^{-1}_{\rho_1} \cdot W^{p-\rho_1}_{\rho_1} \cdot W^{-1}_{p-\rho_1} \\ O & W^{-1}_{p-\rho_1} \end{bmatrix} \tag{8.5.37}$$

$$F_2 = P_2 \cdot Q^{-1} = \begin{bmatrix} P_{\rho_2} & P^{p-\rho_2}_{\rho_2} \\ O & P_{p-\rho_2} \end{bmatrix} \cdot \begin{bmatrix} Q^{-1}_{\rho_2} & -Q^{-1}_{\rho_2} \cdot Q^{p-\rho_2}_{\rho_2} \cdot Q^{-1}_{p-\rho_2} \\ O & Q^{-1}_{p-\rho_2} \end{bmatrix} \tag{8.5.38}$$

(8.5.37) , (8.5.38) clearly imply that the pair (F_1 , F_2) satisfies (8.5.5.*i*) . Consider now the matrix :

$$F_2^{-1} \cdot G \cdot F_1 \tag{8.5.39}$$

By (8.5.15) , (8.5.39) may be expressed as :

$$F_2^{-1} \cdot G \cdot F_1 = Q \cdot P_2^{-1} \cdot G \cdot P_1 \cdot W^{-1} \tag{8.5.40}$$

Because the pair (P_1 , P_2) is a representative of $C_{(P_1,P_2)}$ it satisfies (8.5.6) for L = M , (in other words $P_2^{-1} \cdot G \cdot P_1 = M$) . Thus , (8.5.40) results to :

$$F_2^{-1} \cdot G \cdot F_1 = Q \cdot M \cdot W^{-1} \tag{8.5.41}$$

equivalently if (8.5.16) is applied , then :

$$F_2^{-1} \cdot G \cdot F_1 = Q \cdot M \cdot W^{-1} = Q \cdot M \cdot M^{-1} \cdot Q^{-1} \cdot M = M \tag{8.5.41}$$

which clearly implies (8.5.6) for L = M . Thus the pair (F_1 , F_2) defined in (8.5.15) belongs to $C_{(P_1,P_2)}$. □

Combining the results of propositions (8.5.1) , (8.5.2) we can fully generate the set of matrices K_i which satisfy (8.5.5.*i*) , (8.5.5.*iii*) . This result is summarized below :

***Proposition (8.5.3)** : If (X_1 , X_2) is a solution of the DSP then :*
***i)** A pair of matrices (R_1 , R_2) exists such that , (8.5.5.i) , (8.5.5.iii) , or (8.5.6) hold true .*
***ii)** An arbitrary pair of matrices (K_1 , K_2) which satisfies (8.5.5.i) , (8.5.5.iii) , or , (8.5.6) is generated in terms of (R_1 , R_2) by :*

$$K_1 = (R_1 \cdot B_2) \cdot W^{-1} \ , \ K_2 = (R_2 \cdot B_1^{-1}) \cdot Q^{-1} \tag{8.5.42}$$

where , (B_1 , B_2) , (W , Q) are defined in propositions (8.5.1) , (8.5.2) respectively .

Proof

Let a solution (X_1 , X_2) exists . Then (X_1 , X_2) can be found using one of the already known methods e.g. in [Gün. 1] .
i) Following the steps (8.4.6) – (8.4.22) in the proof of theorem (8.4.1) we can construct a pair of matrices (R_1 , R_2) which satisfies (8.5.5.*i*) , (8.5.5.*iii*) , or (8.5.6) in an algorithmic way :

Step 1 : Set U the unimodular matrix [$T_1 \cdot X_1 \vdots T_2 \cdot X_2$] and partition U^{-1} as :

$$U^{-1} = \begin{bmatrix} U_{p_1}^{-1} \\ U_{p_2}^{-1} \end{bmatrix} \tag{8.5.43}$$

Step 2 : Using the results of section 8.3 a particular pair of matrices (A_1 , A_2) can be constructed such that the pair of matrices :

$$(Y_1 , Y_2) = ([X_1 \vdots A_1] , [X_2 \vdots A_2]) \tag{8.5.44}$$

is unimodular .

Step 3 : Set

$$\begin{cases} \Omega_{p_2}^{m_1} = U_{p_2}^{-1} \cdot T_1 \cdot A_1 \\ \Omega_{p_1}^{m_2} = U_{p_1}^{-1} \cdot T_2 \cdot A_2 \end{cases} \tag{8.5.45}$$

Step 4 : Construct the matrices L_1 , L_2 , V_1 , V_2 for which :

$$\begin{cases} S_1' = L_1 \cdot \Omega_{p_2}^{m_1} \cdot V_1 \text{ , is the Smith form of } \Omega_{p_2}^{m_1} \\ S_2' = L_2 \cdot \Omega_{p_1}^{m_2} \cdot V_2 \text{ , is the Smith form of } \Omega_{p_1}^{m_2} \end{cases} \tag{8.5.46}$$

Step 5 : Construct the matrices U_l^1 , U_r^1 , U_l^2 , U_r^2 for which :

$$\begin{cases} S_1 = (U_l^1)^{-1} \cdot T_1 \cdot (U_r^1)^{-1} \text{ , is the Smith form of } T_1 \\ S_2 = (U_l^2)^{-1} \cdot T_1 \cdot (U_r^2)^{-1} \text{ , is the Smith form of } T_2 \end{cases} \tag{8.5.47}$$

Step 6 : The pair of matrices (R_1 , R_2) in question can now be constructed by setting :

$$R_1 = (U_l^1)^{-1} \cdot U \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1^{-1} \end{bmatrix} , R_2 = (U_l^2)^{-1} \cdot U \cdot \begin{bmatrix} O & L_2^{-1} \\ I_{p_2} & O \end{bmatrix} \tag{8.5.48}$$

ii) (⇒) Let (K_1 , K_2) be an arbitrary pair of matrices which satisfy (8.5.5.*i*) , (8.5.5.*iii*) , or (8.5.6) for an appropriate matrix M . Definition (8.5.1) implies that (K_1 , K_2) belongs to an equivalence class characterized by M , or that a unimodular matrix :

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} \tag{8.5.49}$$

exists such that :

$$K_2^{-1} \cdot G \cdot K_1 = M \tag{8.5.50}$$

Proposition (8.5.1) implies that appropriate matrices B_1 , B_2 defined by L_1 , L_2 , M_1 , M_2 exist such that the pair of matrices (F_1 , F_2) defined by :

$$F_1 = R_1 \cdot B_2 \ , \ F_2 = R_2 \cdot B_1^{-1} \qquad (8.5.51)$$

is a representative for the equivalence class of (K_1 , K_2) . Proposition (8.5.2) implies that appropriate matrices W , Q defined by M exist such that :

$$K_1 = F_1 \cdot W^{-1} \ , \ K_2 = F_2 \cdot Q^{-1} \qquad (8.5.52)$$

(8.5.51) , (8.5.52) imply (8.5.42) .

($\Leftarrow$) Let (K_1 , K_2) be a pair of matrices generated by (8.5.42) , namely :

$$K_1 = (R_1 \cdot B_2) \cdot W^{-1} \ , \ K_2 = (R_2 \cdot B_1^{-1}) \cdot Q^{-1} \qquad (8.5.53)$$

where , (B_1 , B_2) , (W , Q) are defined in propositions (8.5.1) , (8.5.2) respectively . The structure of (B_1 , B_2) , (W , Q) clearly imply that (K_1 , K_2) satisfies (8.5.5.*i*) . (8.5.53) and the definition of (B_1 , B_2) , (W , Q) imply that a unimodular matrix :

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} \qquad (8.5.54)$$

exists such that :

$$K_2^{-1} \cdot G \cdot K_1 = Q \cdot B_1 \cdot R_2^{-1} \cdot G \cdot R_1 \cdot B_2 \cdot W^{-1} =$$

$$= Q \cdot \begin{bmatrix} M_1^{-1} \cdot L_1 & O \\ O & I_{p_1} \end{bmatrix} \cdot \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} \cdot \begin{bmatrix} L_2^{-1} \cdot M_2 & O \\ O & I_{p_2} \end{bmatrix} \cdot W^{-1} = M \qquad (8.5.55)$$

Thus (K_1 , K_2) belongs to an equivalence class characterized by M . □

***Corollary (8.5.1)** : Remark (8.5.1) and proposition (8.5.3) imply that the parametrization of solutions of the DSP in theorem (8.5.1) is in closed form .* □

Summarizing the results of case 1 we can express the parametrization of solutions to the DSP in closed form as shown next . Let the DSP has a solution ; (R_1 , R_2) be the pair of matrices constructed by the algorithm in part *i*) of proposition (8.5.3) . Also let :

$$R_2^{-1} \cdot G \cdot R_1 = \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} = L$$

Set M the arbitrary unimodular matrix :

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix}$$

where , M_1 , M_2 have dimensions p_2xp_2 , p_1xp_1 respectively . Let (K_1 , K_2) be the pairs of matrices generated in part *ii*) of proposition (8.5.3) :

$$K_1 = (R_1 \cdot B_2) \cdot W^{-1} = \begin{bmatrix} K_{\rho_1} & K_{\rho_1}^{p-\rho_1} \\ O & K_{p-\rho_1} \end{bmatrix}, K_2 = (R_2 \cdot B_1^{-1}) \cdot Q^{-1} = \begin{bmatrix} K_{\rho_2} & K_{\rho_2}^{p-\rho_2} \\ O & K_{p-\rho_2} \end{bmatrix} \quad (8.5.56)$$

Theorem (8.5.2) *: All the solutions X_i of the set of equations (8.5.1) are parametrized in closed form as :*

$$X_i = (U_r^i)^{-1} \cdot Z_i^{-1} \cdot [\, I_{p_i} \vdots O^{m_i} \,]^T \quad (8.5.57)$$

where ,

$$Z_i^{-1} = \begin{bmatrix} Z_{\rho_i} & O \\ Z_{(p_i+m_i)-\rho_i}^{\rho_i} & Z_{(p_i+m_i)-\rho_i} \end{bmatrix} \in \mathbb{R}_{\mathcal{P}}^{(p_i+m_i)x(p_i+m_i)}(s) \quad (8.5.58)$$

are unimodular , such that , $Z_{\rho_i} = K_{\rho_i}$, $Z_{(p_i+m_i)-\rho_i}^{\rho_i}$ is an arbitrary parametric matrix and $Z_{(p_i+m_i)-\rho_i}$ is an arbitrary unimodular matrix . □

CASE 2 : In the following , we study the parametrization of solution of the DSP when one of the matrices T_i in (8.5.1) is square . We assume that T_1 is square , (similar arguments apply in the case of T_2 square) . As in case 1 , the non square matrix T_2 is assumed to have Smith form given by :

$$S_2 = \begin{bmatrix} I_{\rho_2} & O \\ O & O \end{bmatrix} \quad (8.5.59)$$

Clearly when $\rho_2 = p$ or p_2+m_2 we have the generic case for T_2 , (lemma (8.2.1)) . Lemma (8.2.1) implies that T_1 is generically equivalent to the diag{ I_{p-1} , $| T_1 |$ } .

i) If $| T_1 | = 0$ the closed form parametrization of solutions of the DSP is described by theorem (8.5.2) for $\rho_1 = p-1$.

ii) If $| T_1 | = 1$ the closed form parametrization of solutions of the DSP is described by theorem (8.5.2) for $\rho_1 = p$.

iii) If $| T_1 | = a \in \mathbb{R}_{\mathcal{P}}(s)$, then $\rho_1 = p = (p_1+m_1)$ and the Smith form of T_1 over $\mathcal{P}$ is given by :

$$S_1 = \begin{bmatrix} I_{p-1} & O \\ O & \alpha \end{bmatrix} \tag{8.5.60}$$

Theorem (8.4.1) , appropriately adjusted to meet the assumptions in *iii)* , provides a parametrization for the solutions of the DSP for this case .

Theorem (8.5.3) *: All the solutions X_i of the set of equations (8.5.1) are parametrized in closed form as:*

$$X_i = (U_r^i)^{-1} \cdot Z_i^{-1} \cdot [\, I_{p_i} \vdots O^{m_i} \,]^T \tag{8.5.61}$$

where ,

$$Z_1^{-1} \in \mathbb{R}_{\mathcal{P}}^{p x p}(s) \ , \ Z_2^{-1} = \begin{bmatrix} Z_{\rho_2} & O \\ Z^{\rho_2}_{(p_2+m_2)-\rho_2} & Z_{(p_2+m_2)-\rho_2} \end{bmatrix} \in \mathbb{R}_{\mathcal{P}}^{(p_2+m_2)x(p_2+m_2)}(s) \tag{8.5.62}$$

are unimodular and such that , unimodular matrices $K_i \in \mathbb{R}_{\mathcal{P}}^{pxp}(s)$, $L_i \in \mathbb{R}_{\mathcal{P}}^{(p-p_i)x(p-p_i)}(s)$ exist and the following conditions hold true :

$$i) \ K_2 = \begin{bmatrix} K_{\rho_2} & K_{\rho_2}^{p-\rho_2} \\ O & K_{p-\rho_2} \end{bmatrix} \tag{8.5.63.i}$$

$$ii) \ K_1 \cdot S_1' = S_1 \cdot Z_1^{-1} \overset{(8.5.60)}{\Leftrightarrow} \begin{cases} \kappa_{ij}^1 = z_{ij}^1 \ , \ i \ , \ j = 1 \ , \ldots \ , \ p-1 \ , \ i = j = p & (8.5.61.ii) \\ \kappa_{ip}^1 \cdot \alpha = z_{ip}^1 \ , \ i = 1 \ , \ldots \ , \ p-1 & (8.5.62.ii) \\ \kappa_{pj}^1 = \alpha \cdot z_{pj}^1 \ , \ j = 1 \ , \ldots \ , \ p-1 & (8.5.63.ii) \end{cases}$$

where $K_1 = [\kappa_{ij}^1]$, $Z_1^{-1} = [z_{ij}^1]$

$$K_2 \cdot S_2 = S_2 \cdot Z_2^{-1} \overset{(8.5.59)}{\Leftrightarrow} K_{\rho_2} \cdot I_{\rho_2} = I_{\rho_2} \cdot Z_{\rho_2} \Leftrightarrow K_{\rho_2} = Z_{\rho_2} \tag{8.5.64.ii}$$

$$iii) \ U_l^1 \cdot K_1 \cdot \begin{bmatrix} I_{p_1} & O \\ O & L_1 \end{bmatrix} = U_l^2 \cdot K_2 \cdot \begin{bmatrix} O & I_{p_2} \\ L_2 & O \end{bmatrix} \Leftrightarrow (K_2)^{-1} \cdot G \cdot K_1 = \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} = L \tag{8.5.65.iii}$$

□

Remark (8.5.2) *: The parametrization described in theorem (8.5.3) is in closed form if and only if the family of parameters which satisfy the parametrization conditions (8.5.61) – (8.5.64.iii) is fully generated . Inspection of the parametrization conditions implies that :*

i) The matrices Z_1^{-1} can be generated by the unimodular matrices K_1 which satisfy

(8.5.65.iii) and the first $p-1$ entries of their last row are multiples of α . If all such matrices are parametrized then we use (8.5.61.ii) – (8.5.63.ii) to construct the Z_1^{-1} .
ii) The matrices Z_2^{-1} can be generated by setting $Z_{\rho_2} = K_{\rho_2}$, $Z^{\rho_2}_{(p_2+m_2)-\rho_2}$, an arbitrary parametric matrix , $Z_{(p_2+m_2)-\rho_2}$ an arbitrary unimodular matrix , for all the unimodular matrices K_2 which satisfy (8.5.64.ii) , (8.5.65.iii) .
It is clear that if the matrices (K_1 , K_2) mentioned above are fully generated then the family of parameters in theorem (8.5.3) can be fully described . □

***Definition (8.5.2)** : Let $\mathcal{F}$ be the set of matrix pairs (K_1 , K_2) such that :*
i) K_1 , K_2 satisfy (8.5.65.iii) .
ii) α / κ^1_{ij} , $j = 1 , \dots , p-1$, (α does not divide κ_{pp} , since K_1 is unimodular and α is not a unit) .
iii) K_2 satisfies (8.5.63.i) .
Denote $\sim$ the relation between the elements of $\mathcal{F}$ defined by :

$$(K_1 , K_2) \sim (P_1 , P_2) \Leftrightarrow (K_1 , K_2) , (P_1 , P_2) \text{ satisfy (8.5.65.iii) for the same } L$$

□

Clearly this is an equivalence relation and partitions $\mathcal{F}$ into equivalence classes . Each equivalence class is characterized by the matrix L :

$$L = \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} \tag{8.5.66}$$

If L changes then a new equivalence class is determined . The task set in remark (8.5.2) is to generate the elements of $\mathcal{F}$ or equivalently of $\mathcal{F}/\sim$. As in case 1 this task involves two steps : If (K_1 , K_2) is an element of $\mathcal{F}$, the first step is to determine representatives in terms of (K_1 , K_2) , for all the equivalence classes in $\mathcal{F}/\sim$. The second step is to parametrize the elements of an arbitrary equivalence class in terms of its representative determined in step 1 . This process parametrizes all the elements of $\mathcal{F}/\sim$ and thus of $\mathcal{F}$ in closed form .

STEP 1 : Generation of representatives for the elements of $\mathcal{F}/\sim$

The following arguments are similar to those in step 1 of case 1 . Let (K_1 , K_2) be an element of $\mathcal{F}$. Then the equivalence class $C_{(K_1,K_2)}$ is defined and a unimodular matrix L exists such that :

$$(K_2)^{-1} \cdot G \cdot K_1 = \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} = L \tag{8.5.67}$$

Proposition (8.5.4) *: A representative of an arbitrary equivalence class in $\mathcal{F}/\sim$ is expressed in terms of (K_1, K_2) as :*

$$P_1 = K_1 \cdot B_2 \ , \ P_2 = K_2 \cdot B_1^{-1} \tag{8.5.68}$$

where ,

$$B_1 = \begin{bmatrix} M_1^{-1} \cdot L_1 & O \\ O & I_{p_1} \end{bmatrix} , \ B_2 = \begin{bmatrix} L_2^{-1} \cdot M_2 & O \\ O & I_{p_2} \end{bmatrix} \tag{8.5.69}$$

and M_i are arbitrary $(p-p_i)x(p-p_i)$ unimodular matrices , L_i are defined by L in (8.5.67) .

Proof

Let B_1 , B_2 be two unimodular matrices defined as in (8.5.69) . Set $P_1 = K_1 \cdot B_2$, $P_2 = K_2 \cdot B_1^{-1}$. Then :

$$P_1 = \begin{bmatrix} \kappa_{11}^1 & \cdots & \kappa_{1p-1}^1 & \kappa_{1p}^1 \\ \vdots & & \vdots & \vdots \\ \alpha \cdot \lambda_{p1}^1 & \cdots & \alpha \cdot \lambda_{pp-1}^1 & \kappa_{pp}^1 \end{bmatrix} \cdot \left[\begin{array}{c:c} L_2^{-1} \cdot M_2 & \\ \hdashline & \\ & I_{p_2} \end{array} \right] \tag{8.5.70}$$

$$P_2 = \begin{bmatrix} K_{\rho_2} & K_{\rho_2}^{p-\rho_2} \\ O & K_{p-\rho_2} \end{bmatrix} \cdot \left[\begin{array}{cc:c} L_1^{-1} \cdot M_1 & & \\ & I_{\rho_2 - p_2} & \\ \hdashline & & I_{p-\rho_2} \end{array} \right] = \begin{bmatrix} P_{\rho_2} & P_{\rho_2}^{p-\rho_2} \\ O & P_{p-\rho_2} \end{bmatrix} \tag{8.5.71}$$

(8.5.70) , (8.5.71) imply that $(P_1$, $P_2)$ satisfy parts *ii)* , *iii)* of definition (8.5.2) . Furthermore ,

$$(P_2)^{-1} \cdot G \cdot P_1 = B_1 \cdot (K_2)^{-1} \cdot G \cdot K_1 \cdot B_2 = B_1 \cdot \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} \cdot B_2 = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} = M \tag{8.5.72}$$

For $(P_1$, $P_2)$, part *i)* of definition (8.5.2) holds true for $L = M$. Thus $(P_1$, $P_2)$ can be viewed as a representative of an equivalence class $C_{(P_1,P_2)}$ with elements all the pairs $(F_1$, $F_2)$ for which definition (8.5.2) holds true . Since the matrices M_i are arbitrarily selected , the matrix M which characterizes $C_{(P_1,P_2)}$ is arbitrary and thus $C_{(P_1,P_2)}$ is arbitrary . □

STEP 2 : Parametrization of the elements of $C_{(P_1,P_2)}$ in terms of (P_1 , P_2)

Surprisingly the parametrization of the elements of $C_{(P_1,P_2)}$ in terms of (P_1 , P_2) , in case 2 , turns out to be more tedious than its counterpart in case 1 . This is due to the existence of a nonunit element , α , in the Smith form of one of the matrices T_i . Consider now the arbitrary equivalence class $C_{(P_1,P_2)}$ characterized by the unimodular matrix M :

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix} = P_2^{-1} \cdot G \cdot P_1 \qquad (8.5.73)$$

and M_1 , M_2 have dimensions $p_2 x p_2$, $p_1 x p_1$ respectively . Let p^1_{pp} denotes the (p , p) entry of P_1 . Since (P_1 , P_2) belongs to $C_{(P_1,P_2)}$, P_1 satisfies part *ii*) of definition (8.5.2) and thus $\alpha \not| \, p^1_{pp}$, ($\not|$ means "does not divide") . Factorize α such that :

$$\alpha = \alpha' \cdot \alpha_p \qquad (8.5.74)$$

where , $\alpha_p \mid p^1_{pp}$ and $\alpha' \not| \, p^1_{pp}$. For each selection of arbitrary $v'_i \in \mathbb{R}_{\mathcal{P}}(s)$, $i = 1 , \ldots , p-1$ set :

$$\begin{cases} \underline{v}^T = [\ (\underline{v}^T)^{p_1} \vdots (\underline{v}^T)^{p_2}\ ] = [\ v_1 \ldots v_{p_1} \vdots v_{p_1+1} \cdots v_p\ ] \\ \underline{y}^T = [\ (\underline{v}^T)^{p_1} \cdot (M_2^{-1})^{\rho_2 - p_2} \vdots (\underline{v}^T)^{p_2}\ ] = [\ y_1 \ldots y_{\rho_2 - p_2} \vdots v_{p_1+1} \cdots v_p\ ] \end{cases} \qquad (8.5.75)$$

with ,

$$\left.\begin{cases} v_i = v'_i \cdot \alpha' , \ i = 1 , \ldots , p-1 \\ v_p \neq v'_p \cdot \alpha' \end{cases}\right\} \text{and such that } \underline{v}^T , \underline{y}^T \text{ are coprime over } \mathcal{P} \qquad (8.5.76)$$

For all such $\underline{v}^T$ set :

$$\underline{d}^T = \underline{v}^T \cdot \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot \begin{bmatrix} I_{\rho_2 - p_2} & \vdots & O & \vdots & O \\ \cdots & & \cdots & & \cdots \\ O & \vdots & O & \vdots & I_{p-\rho_2} \\ \cdots & & \cdots & & \cdots \\ O & \vdots & I_{p_2} & \vdots & O \end{bmatrix} \qquad (8.5.77)$$

If $\underline{d}^T = [\ (\underline{d}^T)^{\rho_2} \vdots (\underline{d}^T)^{p-\rho_2}\ ] = [\ d_1 \ldots d_{\rho_2} \vdots d_{\rho_2+1} \cdots d_p\ ]$, then clearly $(\underline{d}^T)^{\rho_2}$ is a coprime vector . Using the results of section 8.2 the family $\mathcal{S}$ of right unimodular matrices $E^{\rho_2}_{\rho_2-1}$ can be constructed such that the matrix :

$$E_{\rho_2} = \begin{bmatrix} E^{\rho_2}_{\rho_2-1} \\ (\underline{d}^T)^{\rho_2} \end{bmatrix} \qquad (8.5.78)$$

is unimodular . For all such unimodular matrices E_{ρ_2} and $\Theta^{p-\rho_2}_{\rho_2-1}$ arbitrary matrices ,

$A_{p-\rho_2}$ arbitrary unimodular matrices the matrix :

$$D' = \begin{bmatrix} O & A_{p-\rho_2} \\ \cdots & \cdots \\ E^{\rho_2}_{\rho_2-1} & \Theta^{p-\rho_2}_{\rho_2-1} \\ \cdots & \cdots \\ (\underline{d}^T)^{\rho_2} & (\underline{d}^T)^{p-\rho_2} \end{bmatrix} \tag{8.5.79}$$

is unimodular . Carrying out the appropriate permutations on the rows of D' we create the unimodular matrix D :

$$D = \begin{bmatrix} E^{\rho_2}_{\rho_2-p_2} & \Theta^{p-\rho_2}_{\rho_2-p_2} \\ O & A_{p-\rho_2} \\ E^{\rho_2}_{p_2-1} & \Theta^{p-\rho_2}_{p_2-1} \\ \cdots & \cdots \\ \multicolumn{2}{c}{\underline{d}^T} \end{bmatrix} \tag{8.5.80}$$

Let $\mathfrak{D}$ denote the family of all matrices D created by the process of steps (8.5.77) – (8.5.80) . $\mathfrak{D}$ is fully generated since the parameters involved in the construction of the matrices D , ($\underline{d}^T$, $E_{\rho_2} \in \mathcal{E}$, $\Theta^{p-\rho_2}_{\rho_2-1}$, $A_{p-\rho_2}$) are fully described during the process of steps (8.5.75) – (8.5.80) .

***Proposition (8.5.5)** : All the elements (F_1 , F_2) of $C_{(P_1, P_2)}$ are parametrized in terms of (P_1 , P_2) by :*

$$F_1 = P_1 \cdot W \, , \; F_2 = P_2 \cdot Q \tag{8.5.81}$$

where , W , Q are unimodular matrices and further more :

$$W = \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot D \cdot \begin{bmatrix} I_{\rho_2-p_2} & \vdots & O & \vdots & O \\ \cdots & & \cdots & & \cdots \\ O & \vdots & O & \vdots & I_{p_2} \\ \cdots & & \cdots & & \cdots \\ O & \vdots & I_{p-\rho_2} & \vdots & O \end{bmatrix} \cdot \begin{bmatrix} M_2 & O \\ O & I_{p_2} \end{bmatrix} \tag{8.5.82}$$

with D an arbitrary element of $\mathfrak{D}$.

$$Q = M \cdot W \cdot M^{-1} = \begin{bmatrix} Q_{\rho_2} & Q^{p-\rho_2}_{\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix} \tag{8.5.83}$$

Proof

($\Rightarrow$) Let (F_1, F_2) be an arbitrary element of $C_{(P_1,P_2)}$. We shall prove that unimodular matrices W , Q exist such that (8.5.81) – (8.5.83) hold true . Part *i*) of definition (8.5.2) implies that :

$$F_2^{-1}\cdot G\cdot F_1 = M \tag{8.5.84}$$

$$P_2^{-1}\cdot G\cdot P_1 = M \tag{8.5.85}$$

(8.5.84) , (8.5.85) combined together provide :

$$F_2^{-1}\cdot P_2\cdot M\cdot P_1^{-1}\cdot F_1 = M \tag{8.5.86}$$

Set W , Q the matrices :

$$W = P_1^{-1}\cdot F_1\ ,\ Q^{-1} = F_2^{-1}\cdot P_2 \tag{8.5.87}$$

Clearly W , Q are unimodular as the product of unimodular matrices . Furthermore Q has the structure required by (8.5.83) since (8.5.86) , (8.5.87) and part *iii*) of definition (8.5.2) for P_2 imply that $Q = M\cdot W\cdot M^{-1}$ and :

$$Q^{-1} = \begin{bmatrix} F_{\rho_2}^{-1} & -F_{\rho_2}^{-1}\cdot F_{\rho_2}^{p-\rho_2}\cdot F_{p-\rho_2}^{-1} \\ O & F_{p-\rho_2}^{-1} \end{bmatrix}\cdot\begin{bmatrix} P_{\rho_2} & P_{\rho_2}^{p-\rho_2} \\ O & P_{p-\rho_2} \end{bmatrix} \tag{8.5.88}$$

The structure of Q can be exploited to investigate the properties of W and we do so in the following . Since we have assumed that the DSP has a solution corollary (8.2.1) implies that $\rho_i \geq p_i$, $(p-\rho_2 \leq p_1)$, and thus the matrices M , M^{-1} can be partitioned as:

$$M = \begin{bmatrix} O & M_1^{-1} \\ (M_2)_{\rho_2-p_2} & O \\ (M_2)_{p-\rho_2} & O \end{bmatrix},\ M^{-1} = \begin{bmatrix} O & (M_2^{-1})^{\rho_2-p_2} & (M_2^{-1})^{p-\rho_2} \\ M_1 & O & O \end{bmatrix}$$

Similarly partition W as :

$$W = \begin{bmatrix} W_{p_1} & W_{p_1}^{p_2} \\ W_{p_2}^{p_1} & W_{p_2} \end{bmatrix}$$

The latter results and the expression of $Q = M\cdot W\cdot M^{-1}$ in (8.5.88) imply that :

$$\begin{bmatrix} O & M_1^{-1} \\ (M_2)_{\rho_2-p_2} & O \\ (M_2)_{p-\rho_2} & O \end{bmatrix} \cdot \begin{bmatrix} W_{p_1} & W_{p_1}^{p_2} \\ W_{p_2}^{p_1} & W_{p_2} \end{bmatrix} \cdot \begin{bmatrix} O & (M_2^{-1})^{\rho_2-p_2} & (M_2^{-1})^{p-\rho_2} \\ M_1 & O & O \end{bmatrix} = \begin{bmatrix} Q_{\rho_2} & Q_{\rho_2}^{p-\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix}$$

or equivalently ,

$$[\ (M_2)_{p-\rho_2} : O^{p_2}\ ] \cdot \begin{bmatrix} W_{p_1} & W_{p_1}^{p_2} \\ W_{p_2}^{p_1} & W_{p_2} \end{bmatrix} \cdot \begin{bmatrix} O & (M_2^{-1})^{\rho_2-p_2} \\ M_1 & O \end{bmatrix} = O_{p-\rho_2}^{\rho_2} \tag{8.5.89}$$

Carrying out the operations in (8.5.89) the following relations hold true :

$$\begin{cases} (M_2)_{p-\rho_2} \cdot W_{p_1}^{p_2} = O_{p-\rho_2}^{p_2} & (8.5.90) \\ (M_2)_{p-\rho_2} \cdot W_{p_1} \cdot (M_2^{-1})^{\rho_2-p_2} = O_{p-\rho_2}^{\rho_2-p_2} & (8.5.91) \end{cases}$$

Since $(M_2^{-1})^{\rho_2-p_2}$ is a base for the $\mathcal{N}_r\{(M_2)_{p-\rho_2}\}$, (8.5.90) implies that a matrix $E_{\rho_2-p_2}^{p_2}$ exists such that

$$W_{p_1}^{p_2} = (M_2^{-1})^{\rho_2-p_2} \cdot E_{\rho_2-p_2}^{p_2} \tag{8.5.92}$$

On the other hand (8.5.91) implies that a matrix $E_{\rho_2-p_2}$ exists such that :

$$W_{p_1} \cdot (M_2^{-1})^{\rho_2-p_2} = (M_2^{-1})^{\rho_2-p_2} \cdot E_{\rho_2-p_2} \tag{8.5.93}$$

or ,

$$M_2 \cdot W_{p_1} \cdot M_2^{-1} \begin{bmatrix} I_{\rho_2-p_2} \\ O_{p-\rho_2} \end{bmatrix} = \begin{bmatrix} E_{\rho_2-p_2} \\ O_{p-\rho_2} \end{bmatrix} \tag{8.5.94}$$

which clearly implies that matrices $\Theta_{\rho_2-p_2}^{p-\rho_2}$, $A_{p-\rho_2}$, exist :

$$M_2 \cdot W_{p_1} \cdot M_2^{-1} = \begin{bmatrix} E_{\rho_2-p_2} & \Theta_{\rho_2-p_2}^{p-\rho_2} \\ O & A_{p-\rho_2} \end{bmatrix} \tag{8.5.95}$$

or ,

$$W_{p_1} = M_2^{-1} \cdot B \cdot M_2 = M_2^{-1} \cdot \begin{bmatrix} E_{\rho_2-p_2} & \Theta_{\rho_2-p_2}^{p-\rho_2} \\ O & A_{p-\rho_2} \end{bmatrix} \cdot M_2 \tag{8.5.96}$$

(8.5.92) , (8.5.96) can be substituted into W and leads to :

$$W = \begin{bmatrix} W_{p_1} & W_{p_1}^{p_2} \\ W_{p_2}^{p_1} & W_{p_2} \end{bmatrix} = \begin{bmatrix} M_2^{-1} \cdot B \cdot M_2 & (M_2^{-1})^{\rho_2 - p_2} \cdot E_{\rho_2 - p_2}^{p_2} \\ W_{p_2}^{p_1} & W_{p_2} \end{bmatrix}$$

or ,

$$W = \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot \begin{bmatrix} E_{\rho_2 - p_2} & \Theta_{\rho_2 - p_2}^{p - \rho_2} & E_{\rho_2 - p_2}^{p_2} \\ O & A_{p - \rho_2} & O \\ E_{p_2 - 1}^{\rho_2 - p_2} & \Theta_{p_2 - 1}^{p - \rho_2} & W_{p_2 - 1}^{p_2} \\ & \underline{r}^T & \end{bmatrix} \cdot \begin{bmatrix} M_2 & O \\ O & I_{p_2} \end{bmatrix} \tag{8.5.97}$$

where , $[\, E_{p_2-1}^{\rho_2 - p_2} : \Theta_{p_2-1}^{p-\rho_2} \,] = W_{p_2-1}^{p_1} \cdot M_2^{-1}$,

$$\underline{r}^T = \underline{w}^T \cdot \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} , \ \underline{w}^T \text{ is the last row of } W \tag{8.5.98}$$

If we carry out the appropriate permutations on the columns of (8.5.97) it implied that :

$$W = \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot \begin{bmatrix} E_{\rho_2 - p_2}^{\rho_2} & \Theta_{\rho_2 - p_2}^{p - \rho_2} \\ O & A_{p - \rho_2} \\ E_{p_2 - 1}^{\rho_2} & \Theta_{p_2 - 1}^{p - \rho_2} \\ \cdots & \cdots \\ & \underline{d}^T \end{bmatrix} \cdot \begin{bmatrix} I_{\rho_2 - p_2} & O & O \\ O & O & I_{p_2} \\ O & I_{p - \rho_2} & O \end{bmatrix} \cdot \begin{bmatrix} M_2 & O \\ O & I_{p_2} \end{bmatrix} \tag{8.5.99}$$

with , $E_{\rho_2 - p_2}^{\rho_2} = [\, E_{\rho_2 - p_2} : E_{\rho_2 - p_2}^{p_2} \,]$, $E_{p_2-1}^{\rho_2} = [\, E_{p_2-1}^{\rho_2 - p_2} : W_{p_2-1}^{p_2} \,]$,

$$\underline{d}^T = \underline{r}^T \cdot \begin{bmatrix} I_{\rho_2 - p_2} & O & O \\ O & O & I_{p - \rho_2} \\ O & I_{p_2} & O \end{bmatrix} \tag{8.5.100}$$

Finally W is written as :

$$W = \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot D \cdot \begin{bmatrix} I_{\rho_2 - p_2} & O & O \\ O & O & I_{p_2} \\ O & I_{p - \rho_2} & O \end{bmatrix} \cdot \begin{bmatrix} M_2 & O \\ O & I_{p_2} \end{bmatrix} \tag{8.5.101}$$

In order W to satisfy the structure required by (8.5.82) D must be an element of $\mathfrak{D}$. Since , W is unimodular it is implied that D is unimodular . Consider now the matrix D' :

$$D' = \begin{bmatrix} O & A_{p-\rho_2} \\ \cdots & \cdots \\ E^{\rho_2}_{\rho_2-1} & \Theta^{p-\rho_2}_{\rho_2-1} \\ \cdots & \cdots \\ (\underline{d}^T)^{\rho_2} & (\underline{d}^T)^{p-\rho_2} \end{bmatrix} \tag{8.5.102}$$

constructed by the matrix blocks of D as follows :

$$E^{\rho_2}_{\rho_2-1} = \begin{bmatrix} E^{\rho_2}_{\rho_2-p_2} \\ E^{\rho_2}_{p_2-1} \end{bmatrix}, \ \Theta^{p-\rho_2}_{\rho_2-1} = \begin{bmatrix} \Theta^{p-\rho_2}_{\rho_2-p_2} \\ \Theta^{p-\rho_2}_{p_2-1} \end{bmatrix}, \ \underline{d}^T = [\ (\underline{d}^T)^{\rho_2} \vdots (\underline{d}^T)^{p-\rho_2}\ ]$$

In other words D' is constructed by carrying out appropriate permutations on the rows of D and vice versa . Thus D' is unimodular and subsequently the matrices :

$$E_{\rho_2} = \begin{bmatrix} E^{\rho_2}_{\rho_2-1} \\ (\underline{d}^T)^{\rho_2} \end{bmatrix}, \ A_{p-\rho_2}$$

are unimodular . The latter implies the fact that $(\underline{d}^T)^{\rho_2}$ is a coprime vector and $E^{\rho_2}_{\rho_2-1}$ belongs to the family , $\mathcal{E}$, of right unimodular matrices which complete $(\underline{d}^T)^{\rho_2}$ to a unimodular one . So far we have proved that the matrix D can be constructed by the matrix D' of (8.5.102) in the way steps (8.5.78) – (8.5.80) suggest . For D to belong to $\mathfrak{D}$ it remains to prove that the vector $\underline{d}^T$, (the last row of D') , satisfies (8.5.77) ; in other words that a vector $\underline{v}^T$ exists such that (8.5.76) , (8.5.77) hold true and $\underline{v}^T = \underline{w}^T$. Let $F_1 = [f^1_{ij}]$, $P_1 = [p^1_{ij}]$, $W = [w_{ij}]$. Then :

$$f^1_{pj} = \sum_{\kappa=1}^{p-1} p^1_{p\kappa} \cdot w_{\kappa j} + p^1_{pp} \cdot w_{pj} \ , \forall\ j = 1 , \ldots , p \tag{8.5.103}$$

Since F_1 , P_1 satisfy part *ii)* of definition (8.5.2) and thus :

$$\begin{cases} \alpha | f^1_{pj} \ , \ \alpha | p^1_{pj} \ , \ \forall\ j = 1 , \ldots , p-1 \\ \alpha \not| f^1_{pp} \ , \ \alpha \not| p^1_{pp} \end{cases} \tag{8.5.104}$$

Then (8.5.103) , (8.5.104) imply that :

$$\begin{cases} \alpha | p^1_{pp} \cdot w_{pj} \ , \ \forall\ j = 1 \ , \dots , \ p-1 \\ \\ \alpha \nmid p^1_{pp} \cdot w_{pp} \end{cases} \tag{8.5.105}$$

If α is factorized as in (8.5.74) and $\underline{w}^T = [w_{pj}]$ then :

$$\begin{cases} \alpha' | w_{pj} \ , \ \forall\ j = 1 \ , \dots , \ p-1 \\ \\ \alpha' \nmid w_{pp} \end{cases} \tag{8.5.106}$$

(8.5.98) , (8.5.100) combined together imply for the vector $\underline{d}^T$ that :

$$\underline{d}^T = [\ (\underline{d}^T)^{\rho_2} : (\underline{d}^T)^{p-\rho_2}\ ] = \underline{w}^T \cdot \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot \begin{bmatrix} I_{\rho_2 - p_2} & O & O \\ \cdots & \cdots & \cdots \\ O & O & I_{p-\rho_2} \\ \cdots & \cdots & \cdots \\ O & I_{p_2} & O \end{bmatrix} \tag{8.5.107}$$

The coprimeness of $(\underline{d}^T)^{\rho_2}$ together with (8.5.106) and the fact that $\underline{w}^T$ is the last row of a unimodular matrix imply that $\underline{w}^T$ satisfies (8.5.75) , (8.5.76) . Thus $\underline{d}^T$ satisfies (8.5.77) for $\underline{v}^T = \underline{w}^T$and finally we have proved that D is an element of $\mathfrak{D}$. Summarizing (8.5.87) , (8.5.88) , (8.5.101) and the latter analysis imply that for an arbitrary element (F_1 , F_2) of the equivalence $C_{(P_1, P_2)}$ relations (8.5.81) – (8.5.83) hold true .

($\Leftarrow$) Let a pair of matrices (F_1 , F_2) exists such that (8.5.81) – (8.5.83) hold true for some $D \in \mathfrak{D}$. Then we shall prove that (F_1 , F_2) belongs to $C_{(P_1, P_2)}$. In order to do so we must show that (F_1 , F_2) satisfies definition (8.5.2) .

i)

$$F_2^{-1} \cdot G \cdot F_1 \overset{(8.5.81)}{=} Q^{-1} \cdot P_2^{-1} \cdot G \cdot P_1 \cdot W = Q^{-1} \cdot M \cdot W \overset{(8.5.83)}{=} M \tag{8.5.108}$$

which clearly implies that (F_1 , F_2) satisfies part *i)* of definition (8.5.2) .

ii) Let $F_1 = [f^1_{ij}]$, $P_1 = [p^1_{ij}]$, $W = [w_{ij}]$. Then :

$$f^1_{pj} = \sum_{\kappa=1}^{p-1} p^1_{p\kappa} \cdot w_{\kappa j} + p^1_{pp} \cdot w_{pj} \ \ , \forall\ j = 1 \ , \dots , \ p \tag{8.5.109}$$

If $\underline{w}^T = [w_{pj}]$ denotes the last row of W , (8.5.82) implies that :

$$\underline{w}^T = \underline{d}^T \cdot \begin{bmatrix} I_{\rho_2 - p_2} & O & O \\ \cdots & \cdots & \cdots \\ O & O & I_{p_2} \\ \cdots & \cdots & \cdots \\ O & I_{p-\rho_2} & O \end{bmatrix} \cdot \begin{bmatrix} M_2 & O \\ O & I_{p_2} \end{bmatrix} \tag{8.5.110}$$

where , $\underline{d}^T$ is the last row of D and satisfies (8.5.77) ; in other words a vector $\underline{v}^T$ that satisfies (8.5.75) , (8.5.76) exists such that :

$$\underline{d}^T = \underline{v}^T \cdot \begin{bmatrix} M_2^{-1} & O \\ O & I_{p_2} \end{bmatrix} \cdot \begin{bmatrix} I_{\rho_2 - p_2} & \vdots & O & \vdots & O \\ \cdots & & \cdots & & \cdots \\ O & \vdots & O & \vdots & I_{p-\rho_2} \\ \cdots & & \cdots & & \cdots \\ O & \vdots & I_{p_2} & \vdots & O \end{bmatrix} \tag{8.5.111}$$

(8.5.110) , (8.5.111) combined together imply that $\underline{w}^T = \underline{v}^T$ and thus $\underline{w}^T$ satisfies (8.5.75) , (8.5.76) . (8.5.109) , the fact that P_1 satisfies part *ii*) of definition (8.5.2) and the latter imply that :

$$\begin{cases} \alpha | f^1_{pj} \, , \; \forall \; j = 1 \, , \ldots , \, p-1 \\ \\ \alpha \not| f^1_{pp} \end{cases} \tag{8.5.112}$$

and F_1 satisfies part *ii*) of definition (8.5.2) as well .

***iii*)** (8.5.83) and the fact that P_2 satisfies part *iii*) of definition (8.5.2) imply that :

$$F_2 = P_2 \cdot Q = \begin{bmatrix} P_{\rho_2} & P^{p-\rho_2}_{\rho_2} \\ O & P_{p-\rho_2} \end{bmatrix} \cdot \begin{bmatrix} Q_{\rho_2} & Q^{p-\rho_2}_{\rho_2} \\ O & Q_{p-\rho_2} \end{bmatrix} \tag{8.5.113}$$

and clearly F_2 satisfies part *iii*) of definition (8.5.2) as well .

i) , *ii*) , *iii*) imply that $(F_1 \, , \, F_2)$ belongs to $C_{(P_1, P_2)}$. □

Combining the results of propositions (8.5.4) , (8.5.5) together we are able to fully generate the set of matrices K_i which satisfy definition (8.5.2) . This result is stated in the following proposition :

Proposition (8.5.6) *: If a solution $(X_1 \, , \, X_2)$ of the DSP exists then :*

i) *A pair of matrices $(R_1 \, , \, R_2)$ exists such that definition (8.5.2) holds true .*

ii) *An arbitrary pair of matrices $(K_1 \, , \, K_2)$ which satisfies definition (8.5.2) is generated in terms of $(R_1 \, , \, R_2)$ by :*

$$K_1 = (R_1 \cdot B_2) \cdot W \, , \; K_2 = (R_2 \cdot B_1^{-1}) \cdot Q \tag{8.5.114}$$

where , $(B_1 \, , \, B_2)$, $(W \, , \, Q)$ are defined in propositions (8.5.4) , (8.5.5) respectively .

Proof

The proof of part *i*) is identical to the one in proposition (8.5.3) . Arguments similar to the ones in proof of part *ii*) of proposition (8.5.3) if – instead of propositions (8.5.1) , (8.5.2) – propositions (8.5.4) , (8.5.5) are used , can provide the proof part *ii*) of proposition (8.5.6) . □

Summarizing the results of case 2 , part *iii)* , we can express the parametrization of solutions to the DSP in closed form as shown next . Let the DSP has a solution ; (R_1 , R_2) be the pair of matrices constructed by the algorithm in part *i)* of proposition (8.5.6) Also let :

$$R_2^{-1}\cdot G\cdot R_1 = \begin{bmatrix} O & L_1^{-1} \\ L_2 & O \end{bmatrix} = L$$

Set M the arbitrary unimodular matrix :

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix}$$

where , M_1 , M_2 have dimensions $p_2 x p_2$, $p_1 x p_1$ respectively . Let (K_1 , K_2) be the pairs of matrices generated in part *ii)* of proposition (8.5.6) :

$$K_1 = [\kappa_{ij}^1] = (R_1\cdot B_2)\cdot W\ ,\ K_2 = (R_2\cdot B_1^{-1})\cdot Q = \begin{bmatrix} K_{\rho_2} & K_{\rho_2}^{p-\rho_2} \\ O & K_{p-\rho_2} \end{bmatrix} \tag{8.5.115}$$

Theorem (8.5.4) *: All the solutions X_i of the set of equations (8.5.1) are parametrized in closed form as:*

$$X_i = (U_r^i)^{-1}\cdot Z_i^{-1}\cdot\begin{bmatrix} I_{p_i} \\ O \end{bmatrix} \tag{8.5.116}$$

where ,

$$Z_1^{-1} \in \mathbb{R}_{\mathcal{P}}^{pxp}(s)\ ,\ Z_2^{-1} = \begin{bmatrix} Z_{\rho_2} & O \\ Z_{(p_2+m_2)-\rho_2}^{\rho_2} & Z_{(p_2+m_2)-\rho_2} \end{bmatrix} \in \mathbb{R}_{\mathcal{P}}^{(p_2+m_2)x(p_2+m_2)}(s) \tag{8.5.117}$$

are unimodular matrices such that :

i) $Z_{\rho_2} = K_{\rho_2}$, $Z_{(p_2+m_2)-\rho_2}^{\rho_2}$, an arbitrary parametric matrix , $Z_{(p_2+m_2)-\rho_2}$ an arbitrary unimodular matrix .

ii) $Z_1^{-1} = [z_{ij}^1]$ and :

$$\begin{cases} z_{ij}^1 = \kappa_{ij}^1\ ,\ i\ ,\ j = 1\ ,\ \dots\ ,\ p-1\ ,\ i = j = p & (8.5.118.ii) \\ z_{ip}^1 = \kappa_{ip}^1\cdot\alpha\ ,\ i = 1\ ,\ \dots\ ,\ p-1 & (8.5.119.ii) \\ z_{pj}^1 = (\kappa_{pj}^1/\alpha)\ ,\ j = 1\ ,\ \dots\ ,\ p-1 & (8.5.120.ii) \end{cases}$$

□

We illustrate the parametrization methods studied so far by the following example :

Example (8.5.1) : Consider the system with transfer function of the plant given by :

$$P = \begin{bmatrix} \frac{(s+1)^2}{s^2-3s+2} & \frac{s^2+s}{s^2-3s+2} & \frac{(s+1)^2}{s^2-3s+2} \\ \frac{s^2+s}{s^2-3s+2} & \frac{2s^4+s^3-3s^2+4s}{s^4-s^3-3s^2+s+2} & \frac{2s^3-s^2+3s}{s^3-2s^2-s+2} \\ \frac{(s+1)^2}{s^2-3s+2} & \frac{3s^3-3s^2+2s+2}{s^3-2s^2-s+2} & \frac{3s^3-3s^2+7s+1}{s^3-2s^2-s+2} \end{bmatrix} \quad (8.5.121)$$

then 1 , 2 are poles of P and the system is not stable . In this example we illustrate the closed form parametrization of decentralized controllers $C = diag\{C_1 , C_2\}$, $C_1 \in \mathbb{R}_{pr}^{1x1}(s)$ $C_2 \in \mathbb{R}_{pr}^{2x2}(s)$, which stabilize the plant P via a precompensator and unity output feedback scheme . A coprime left MFD , (D , N) of the plant P , over $\mathbb{R}_{\mathcal{P}}(s)$, is found to be represented by :

$$D = \begin{bmatrix} \frac{-5s^2+s}{(s+1)^3} & 1 & 0 \\ \frac{-5s+1}{(s+1)^2} & 0 & 1 \\ \frac{s^2-3s+2}{(s+1)^2} & 0 & 0 \end{bmatrix} , N = \begin{bmatrix} \frac{s}{s+1} & \frac{2s}{s+1} & \frac{2s}{s+1} \\ 1 & \frac{3s+1}{s+1} & \frac{3s+1}{s+1} \\ 1 & \frac{s}{s+1} & 1 \end{bmatrix} \quad (8.5.122)$$

Because of the structure of the controllers the inputs , outputs , (p , m) , are partitioned to local inputs , outputs , (p_1 , p_2) , (m_1 , m_2) , with $(p_1 , m_1) = (1 , 2)$, $(p_2 , m_2) = (1 , 2)$ respectively . All stabilizing controllers should satisfy equation (8.2.1) , or equivalently if (N_i , D_i) , $i = 1 , 2$ represent coprime right MFD's of the blocks C_i of the controllers , equations (8.2.2) and (8.2.3) must hold true for $\kappa = 2$ and the matrices T_i given by :

$$T_1 = \begin{bmatrix} \frac{-5s^2+s}{(s+1)^3} & \frac{s}{s+1} \\ \frac{-5s+1}{(s+1)^2} & 1 \\ \frac{s^2-3s+2}{(s+1)^2} & 1 \end{bmatrix} , T_2 = \begin{bmatrix} 1 & 0 & \frac{2s}{s+1} & \frac{2s}{s+1} \\ 0 & 1 & \frac{3s+1}{s+1} & \frac{3s+1}{s+1} \\ 0 & 0 & \frac{s}{s+1} & 1 \end{bmatrix} \quad (8.5.123)$$

T_1 , T_2 can be expressed via their Smith forms over $\mathbb{R}_{\mathcal{P}}(s)$ as :

$$T_1 = U_l^1 \cdot S_1 \cdot U_r^1 = \begin{bmatrix} 0 & \frac{s}{s+1} & \frac{1}{s+1} \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{s^2-3s+2}{(s+1)^2} & 1 \\ \frac{-5s+1}{(s+1)^2} & 1 \end{bmatrix} \tag{8.5.124}$$

$$T_2 = \begin{bmatrix} \frac{s-1}{s+1} & 1 & 0 \\ \frac{2s}{s+1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 1 & 1 \\ \frac{2s}{s+1} & \frac{1-s}{s+1} & 1 & 1 \\ 0 & 0 & \frac{s}{s+1} & 1 \\ 0 & 0 & \frac{-s}{s+1} & 1 \end{bmatrix} = $$

$$= U_l^2 \cdot S_2 \cdot U_r^2 \tag{8.5.125}$$

If ρ_i denotes the rank of T_i, then $\rho_1 = 2$, $\rho_2 = 3$. Corollary (8.2.1) implies that decentralized stabilizing controllers of the type examined in this example exist. Such a controller is given by, [Gün. 1],

$$C = diag\{C_1, C_2\} = \begin{bmatrix} \frac{5s-1}{(s+1)^2} & 0 & 0 \\ 0 & \frac{1-s}{s+3} & \frac{-2}{s+3} \\ 0 & \frac{s^2-s}{s^2+4s+3} & \frac{2s}{s^2+4s+3} \end{bmatrix} \tag{8.5.126}$$

If $C_i = N_i \cdot D_i^{-1}$ then C corresponds to a pair of solutions (X_1, X_2) of (8.2.3) given by :

$$X_1 = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5s-1}{(s+1)^2} \end{bmatrix}, X_2 = \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{s^2+2s-1}{(s+1)^2} & \frac{s^2+4s+1}{(s+1)^2} \\ \frac{2s^2+5s+1}{(s+1)^2} & \frac{s^2+5s+2}{(s+1)^2} \\ -1 & -1 \\ \frac{s}{s+1} & \frac{s}{s+1} \end{bmatrix} \tag{8.5.127}$$

For this pair of solutions equation (8.2.3) implies that :

$$[\, T_1 \cdot X_1 \,,\, T_2 \cdot X_2 \,] = U = \begin{bmatrix} 0 & \frac{s-1}{s+1} & 1 \\ 0 & \frac{2s}{s+1} & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad (8.5.128)$$

with U an $\mathbb{R}_{\mathcal{P}}(s)$ unimodular matrix . In order to parametrize the family of all decentralized stabilizing controllers of our example we have to apply proposition (8.5.3) or , in other words to find unimodular matrices $R_i \in \mathbb{R}_{\mathcal{P}}^{3x3}(s)$, $L_i \in \mathbb{R}_{\mathcal{P}}^{(3-p_i)x(3-p_i)}(s)$ $i = 1 , 2$ such that conditions (8.5.5*i* , *iii*) of theorem (8.5.1) hold true . In order to do so we apply the algorithm introduced in *i*) of proposition (8.5.3) :

Step 1 : Set U the unimodular matrix of (8.5.128) and partition U^{-1} as :

$$U^{-1} = \begin{bmatrix} U_1^{-1} \\ U_2^{-1} \end{bmatrix}, \text{ with } U_1^{-1} = [\, 0 \; 0 \; 1 \,] \,,\, U_2^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ \frac{2s}{s+1} & \frac{1-s}{s+1} & 0 \end{bmatrix} \qquad (8.5.129)$$

Step 2 : Using the results of section 8.3 a particular pair of matrices :

$$A_1 = \begin{bmatrix} -1 \\ \frac{s^2-3s+2}{(s+1)^2} \end{bmatrix}, \; A_2 = \begin{bmatrix} \frac{-2s}{s+1} & \frac{2s}{(s+1)^2} \\ \frac{-(3s+1)}{s+1} & \frac{3s+1}{(s+1)^2} \\ 0 & -1 \\ 1 & \frac{s}{s+1} \end{bmatrix} \qquad (8.5.130)$$

exists such that the pair of matrices $(Y_1 , Y_2) = ([\, X_1 \vdots A_1 \,] , [\, X_2 \vdots A_2 \,])$ is unimodular .

Step 3 : Set $\Omega_2^1 = U_2^{-1} \cdot T_1 \cdot A_1 = [\, 1/(s+1) \,,\, (s^2+1)/(s+1)^2 \,]^T$, $\Omega_1^2 = U_1^{-1} \cdot T_2 \cdot A_2 = [\, 1 \; 0 \,]$

Step 4 : Set

$$L_1 = \begin{bmatrix} \frac{s^2+2s-1}{(s+1)^2} & \frac{s+2}{s+1} \\ \frac{-(s^2+1)}{(s+1)^2} & \frac{1}{s+1} \end{bmatrix}, \; L_2 = [\, 1 \,] \,,\, V_1 = [\, 1 \,] \,,\, V_2 = I_2$$

Clearly $S_1' = [\ 1\ 0\ ]^T = L_1 \cdot \Omega_2^1 \cdot V_1$, $S_2' = [\ 1\ 0\ ] = L_2 \cdot \Omega_1^2 \cdot V_2$, are the Smith forms of Ω_2^1 , Ω_1^2 respectively

Step 6 : The pair of matrices $(R_1\ ,\ R_2)$ in question can now be constructed by setting :

$$R_1 = (U_l^1)^{-1} \cdot U \cdot \begin{bmatrix} 1 & O \\ O & L_1^{-1} \end{bmatrix} = I_2\ ,\ R_2 = (U_l^2)^{-1} \cdot U \cdot \begin{bmatrix} O & L_2^{-1} \\ I_2 & O \end{bmatrix} = I_3$$

Applying *ii*) of proposition (8.5.3) we find that a closed form parametrization of the pairs $(K_1\ ,\ K_2)$ which satisfy (8.5.5.*i* , *iii*) is given by :

$$K_1 = B_2 \cdot W^{-1} = \begin{bmatrix} M_2 & O \\ O & I_2 \end{bmatrix} \cdot M^{-1} \cdot Q^{-1} \cdot M\ ,\ K_2 = B_1^{-1} \cdot Q^{-1} = \begin{bmatrix} L_1^{-1} \cdot M_1 & O \\ O & 1 \end{bmatrix} \cdot Q^{-1}$$

$$M = \begin{bmatrix} O & M_1^{-1} \\ M_2 & O \end{bmatrix},\ Q = V_3^{-1} \cdot \begin{bmatrix} A_1 & O \\ C_2^1 & D_2 \end{bmatrix} \cdot V_3$$

for all the arbitrary unimodular matrices M_1 , $D_2 \in \mathbb{R}_{\mathcal{P}}^{2x2}(s)$, M_2 , $A_1 \in \mathbb{R}_{\mathcal{P}}^{1x1}(s)$, all arbitrary parametric matrices $C_2^1 \in \mathbb{R}_{\mathcal{P}}^{2x1}(s)$; $V_3 \in \mathbb{R}_{\mathcal{P}}^{3x3}(s)$ is unimodular and such that the last row of M^{-1} multiplied on the right by V_3^{-1} gives [1 0 0] . Now we can proceed with the parametrization of all solutions to equation (8.2.3) . Theorem (8.5.2) implies that all X_i are given by :

$$X_1 = (U_r^1)^{-1} \cdot Z_1^{-1} \cdot [\ 1 : 0\ ]^T\ ,\ X_2 = (U_r^2)^{-1} \cdot Z_2^{-1} \cdot [\ I_2 : O_2\ ]^T$$

where , $Z_1^{-1} = K_{\rho_1}$, (the first 2 *x* 2 block of K_1) ,

$$Z_2^{-1} = \begin{bmatrix} Z_3 & O \\ Z_1^3 & Z_1 \end{bmatrix} \in \mathbb{R}_{\mathcal{P}}^{4x4}(s)$$

such that , $Z_3 = K_{\rho_2}$, (the first 3 *x* 3 block of K_2) , Z_1^3 is an arbitrary parametric matrix Z_1 is an arbitrary unimodular matrix . □

8.6. CONCLUSIONS

Parametrization issues of the general Decentralized Stabilization Problem (DSP) have been studied . The DSP has been approached in an algebraic manner via the set of equations $T_i \cdot X_i = U_i$, X_i , left unimodular , $[U_1 \cdots U_\kappa]$ unimodular , T_i matrices defined by appropriately partitioning an $\mathbb{R}_{\mathfrak{P}}(s)$ – left coprime MFD of the plant . A parametrization of the family of solutions , X_i , which corresponds to $[U_1 \cdots U_\kappa]$ unimodular has been given by theorem (8.4.1) . The above parametrization requires the existence of a constructive method that enables us to generate the family of all unimodular matrices of given dimension , as well as the families of left , (right) unimodular matrices which complete given left , (right) , unimodular matrices to square unimodular ones . Such methods has been examined in section 8.3 . The families of parameters involved need to satisfy certain parametrization constraints . These constraints constitute a necessary and sufficient criterion that enables us to identify the admissible parameters . Particular cases where closed form parametrization is possible have been studied in sections (8.4) , (8.5) . In the case of two blocks decentralized controllers a full description of the set of parameters has been given , especially when T_i are considered generically and are either not square or , one of T_1 or T_2 are square . The study of closed form parametrization when T_1 , T_2 are simultaneously square as well as the generalization in the case of κ blocks decentralized controllers are still under investigation .

CHAPTER 9

THE DIAGONAL DECENTRALIZED STABILIZATION PROBLEM AND RELATED ISSUES

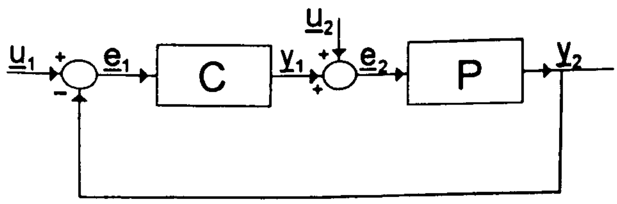
9.1. INTRODUCTION

A special case of decentralized stabilization of linear multivariable time invariant systems is the problem of diagonal stabilization , [Güc. 1] . [Kar. 2] . In this special case the problem is to determine a stabilizing compensator C = diag { c_i } , such that the plant P is internally stabilized by C . The internal stability requirement may be expressed in terms of transfer functions matrices , [Vid. 4] , and highlights the important role of fixed modes in decentralized stabilization . Various researchers have provided characterizations of "fixed modes" , [And. 1] , [Cor. 1] , [Wan. 1] , [And. 2] , [Gün. 1] , [Kar. 9] . It has been shown , [Wan. 1] , that the diagonal stabilization of P is possible if and only if it is free of unstable fixed modes . Recent algebraic synthesis methods for linear multivariable control problems have highlighted the importance of the set $\mathbb{R}_{\mathcal{P}}(s)$ of proper rational functions with no poles inside the region $\mathcal{P} = \Omega \cup \{\infty\}$, $(\Omega \subset \mathbb{C})$, [Des. 1] , [Sae. 1] , [Vid. 1] , [Fra. 1] , [Vid. 4] . These methods are based on what is termed the "fractional representation" approach to linear systems theory . The detailed structure of the set $\mathbb{R}_{\mathcal{P}}(s)$ has been studied in [Var. 3] , [Var. 5] , [Vid. 4] , [Mor. 1] .

Our aim in this chapter is to provide a closed form parametrization of solutions of the diagonal stabilization problem , by extending the results stated , for two inputs – outputs systems , in [Kar. 2] , to the general case . Our approach in doing so , differs from the study of the general decentralized stabilization problem in chapter 8 , in a way that makes the results established here easier to apply in the special case of diagonal stabilization . On the other hand the results of chapter 8 do not imply closed form parametrizations in the general case of diagonal stabilization yet , whereas those introduced here tackle the specific problem in a better fashion . In the following necessary and sufficient solvability conditions for the decentralized stabilization problem using diagonal controllers , factorized over $\mathbf{R}_{\mathcal{P}}(s)$, are given . The existence and characterization of solutions is intimately related to systems that exhibit the property of cyclicity , [Kar. 2] . The characterization is essential since it provides the means to define special type solutions such as proper , reliable , stable . A statement of the problem and its consequent formulation are introduced in section 9.2 . The notion of cyclicity is defined . Section 9.3 refers to an equivalent formulation of the problem which finally transforms it to the search of necessary and sufficient solvability conditions for a scalar Diophantine equation , over $\mathbf{R}_{\mathcal{P}}(s)$, the solutions of which must meet certain factorization constraints . The actual necessary and sufficient solvability conditions for the problem are introduced in section 9.4 . The connection between the cyclicity property of the plant and the existence of diagonal stabilizing controllers is established . The parametrization of all stabilizing controllers is studied in section 9.5 . It is reduced to determining what are termed mode T mutually stabilizing pairs and the existence of such pairs forms the basis of a complete parametrization . The rest of the chapter deals with the determination of proper , reliable , stable stabilizing diagonal controllers by making use of the parametrization introduced in section 9.5 .

9.2. THE DIAGONAL DECENTRALIZED STABILIZATION PROBLEM

Consider the standard feedback configuration associated with a lumped , linear , time invariant (continuous time) system :



where , $P \in \mathbb{R}_{pr}^{mxm}(s)$ is the plant transfer function and $C \in \mathbb{R}_{pr}^{mxm}(s)$ is the transfer function of the controller . It is assumed that both plant and controller are stabilizable and detectable .

Problem : Given a plant transfer function $P \in \mathbb{R}_{pr}^{mxm}(s)$ find a controller transfer function $C = diag\{c_1 , \dots , c_m\} \in \mathbb{R}_{pr}^{mxm}(s)$ such that the feedback system is internally stable . This is defined as the diagonal decentralized stabilization problem (DDSP) . □

If $\mathcal{P} = \mathbb{C}_+ \cup \{\infty\}$ and $\mathbb{R}_{\mathcal{P}}(s)$ denotes the ring of proper and $\mathcal{P}$ – stable functions ; consider an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of the plant $P = A_1^{-1} \cdot B_1$, where $A_1 \in \mathbb{R}_{\mathcal{P}}^{mxm}(s)$, $B_1 \in \mathbb{R}_{\mathcal{P}}^{mxm}(s)$ and (A_1 , B_1) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair ; and let $C = diag\{c_1 , \dots , c_m\} = N_2 \cdot D_2^{-1}$ be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of the diagonal controller , where , $c_i = n_i \, d_i^{-1}$, $i = 1 , 2 , \dots , m$, is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of c_i . Then $N_2 = diag\{n_1 , \dots , n_m\}$ and $D_2 = diag\{d_1 , \dots , d_m\}$. It is known that the controller internally stabilizes the feedback system , if and only if there exists some $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular matrix U such that :

$$A_1 D_2 + B_1 N_2 = U \tag{9.2.1}$$

By partitioning A_1 , B_1 in terms of columns , then (9.2.1) is expressed as :

$$[\, \underline{a}_1 , \underline{a}_2 , \dots , \underline{a}_m \,] \cdot \begin{bmatrix} d_1 & & O \\ & d_2 & \\ & & \ddots \\ O & & d_m \end{bmatrix} + [\, \underline{b}_1 , \underline{b}_2 , \dots , \underline{b}_m \,] \cdot \begin{bmatrix} n_1 & & O \\ & n_2 & \\ & & \ddots \\ O & & n_m \end{bmatrix} =$$

$$= [\ \underline{u}_1\ ,\ \underline{u}_2\ ,\ \dots\ ,\ \underline{u}_m\ ] \tag{9.2.2}$$

Or equivalently ,

$$[\ \underline{a}_i\ ,\ \underline{b}_i\ ]\cdot\begin{bmatrix} d_i \\ n_i \end{bmatrix} = \underline{u}_i\ ,\ i = 1\ ,\ 2\ ,\ \dots\ ,\ m \tag{9.2.3}$$

where , $P_i = [\ \underline{a}_i\ ,\ \underline{b}_i\ ] \in \mathbb{R}_{\mathcal{P}}^{mx2}(s)$ are matrices defined by the plant and the vectors $\underline{q}_i = [\ d_i\ ,\ n_i\ ]^T \in \mathbb{R}_{\mathcal{P}}^{2x1}(s)$ characterize the single input , single output (SISO) controllers . The vectors $\underline{u}_i$ are arbitrary vectors of $\mathbb{R}_{\mathcal{P}}^{mx1}(s)$, with the additional property that $U \triangleq [\ \underline{u}_1\ ,\ ,\ \underline{u}_2\ ,\ \dots\ ,\ \underline{u}_m\ ]$ is $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular . The latter condition implies that $\underline{u}_i$ are irreducible in $\mathcal{P}$ (have no zeros in $\mathcal{P}$) .

Remark (9.2.1) : *The solvability of (9.2.1) is independent of the particular $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD of the plant which is used . Indeed , if $(A_1\ ,\ B_1)$, $(A_2\ ,\ B_2)$ are two $\mathbb{R}_{\mathcal{P}}(s)$ – coprime MFD's of the plant then there exists $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular matrix U_l such that $(A_2\ ,\ B_2) = U_l\cdot(A_1\ ,\ B_1)$. From (9.2.1) we take :*

$$A_1\ D_2 + B_1\ N_2 = U_1 \Leftrightarrow U_l^{-1}A_2\ D_2 + U_l^{-1}B_2\ N_2 = U_1 \tag{9.2.4}$$

or ,

$$A_2\ D_2 + B_2\ N_2 = U_l\cdot U_1 = U_2 \tag{9.2.5}$$

where U_1 , U_2 are $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular matrices . The solvability of (9.2.5) implies the solvability of (9.2.4) and vice versa . □

The set $\{P_i\ ,\ i = 1\ ,\ \dots\ ,\ m\ \}$ is characteristic of the plant and for any other coprime MFD of the plant the corresponding set is $\{U_l\cdot P_i\ ,\ i = 1\ ,\ \dots\ ,\ m\ \}$, U_l is $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular .

Definition (9.2.1) [Kar. 2] : *A set $\mathcal{L} = \{P_i\ ,\ i = 1\ ,\ \dots\ ,\ m\ \}$ will be referred to as a representative decentralized matrix set (RDM) of the plant .* □

Definition (9.2.2) [Kar. 2] : *Let $T \in \mathbb{R}_{\mathcal{P}}^{mx\kappa}(s)$, $m \geq \kappa$, $rank_{\mathbb{R}(s)}\{T\} = \kappa$ and let $\mathcal{F}_T = \{f_i : f_i \in \mathbb{R}_{\mathcal{P}}(s)\ i = 1\ ,\ \dots\ ,\ m\ ,\ f_1/f_2/\ \dots\ /f_\kappa\}$ be the invariant functions of T over $\mathbb{R}_{\mathcal{P}}(s)$. T is cyclic if $f_1 = f_2 = \dots = f_{\kappa-1} = 1$; if more than one of the f_i is nontrivial , T will be called noncyclic. T will be called complete , if $f_i = 1$ for all $i = 1\ ,\ \dots\ ,\ m$.* □

Definition (9.2.3) [Kar. 2] : *An RDM set $\mathcal{L} = \{P_i\ ,\ i = 1\ ,\ \dots\ ,\ m\ \}$ of the plant P will be called cyclic if for all $i = 1\ ,\ \dots\ ,\ m$ the matrices P_i are cyclic ; if at least one P_i is noncyclic , then $\mathcal{L}$ will be called noncyclic . The set $\mathcal{L}$ will be called complete if for all i =*

= 1 , ... , m the matrices P_i are complete . □

Denote by $\mathcal{F}(P_i) = \{f_{1i}(s) , f_{2i}(s) : f_{1i}(s)/f_{2i}(s)\}$ the invariant functions of P_i and by $\mathcal{F}_{\mathcal{L}} = \{\mathcal{F}(P_1) , \mathcal{F}(P_2) , \dots , \mathcal{F}(P_m)\}$ the ordered set of invariant functions of $\mathcal{L}$. Further more let $Q = [P_1 , P_2 , \dots , P_m]$ and $\mathcal{R}_{\mathcal{L}} = [\mathbb{R}_{\mathcal{P}}(s)$ – row module of $\{Q\}]$. Then :

***Proposition** (9.2.1) : Let $\mathcal{L}$ and $\bar{\mathcal{L}}$ be any two RDM sets associated with the plant P . Then :*

$$\mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\bar{\mathcal{L}}} \text{ and } \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\bar{\mathcal{L}}} \tag{9.2.6}$$

□

The set $\mathcal{F}_{\mathcal{L}}$ and the module $\mathcal{R}_{\mathcal{L}}$ are thus invariants of the plant P and will be simply denoted by $\mathcal{F}_{\mathcal{L}}$, $\mathcal{R}_{\mathcal{L}}$. Clearly , the plant is cyclic if $f_{1i} = 1$ for all $i = 1 , \dots , m$ and complete if $f_{1i} = 1$, $f_{2i} = 1$ for all $i = 1 , \dots , m$.

***Proposition** (9.2.2) : If P is noncyclic , there exists no diagonal C that stabilizes the feedback system .*

Proof

Let $\mathcal{L}$ be an RDM set and assume P_j is noncyclic matrix . Also , assume that there exists a diagonal stabilizing controller . By (9.2.3) , $P_j \cdot \underline{q}_j = \underline{u}_j$, where $\underline{u}_j$ must be a coprime $\mathbb{R}_{\mathcal{P}}(s)$ vector (as a column of an $\mathbb{R}_{\mathcal{P}}(s)$ unimodular matrix) . Let U_l^{-1} , U_r^{-1} be a pair of $\mathbb{R}_{\mathcal{P}}(s)$ unimodular matrices that reduce P_j to its Smith form over $\mathbb{R}_{\mathcal{P}}(s)$. Then by partitioning U_l according to the partitioning of the Smith form we have :

$$U_l \cdot \begin{bmatrix} f_{1j} & 0 \\ 0 & f_{2j} \\ \dots & \dots \\ O & O \end{bmatrix} \cdot U_r \cdot \underline{q}_j = \underline{u}_j$$

or equivalently ,

$$[\, \underline{v}_1 \ \underline{v}_2 \ U_l' \,] \cdot \begin{bmatrix} f_{1j} & 0 \\ 0 & f_{2j} \\ \dots & \dots \\ O & O \end{bmatrix} \cdot \underline{q}_j' = \underline{u}_j \tag{9.2.7}$$

where , $\underline{q}_j' = U_r \cdot \underline{q}_j = [\, \hat{d}_j , \hat{n}_j \,]^T \in \mathbb{R}_{\mathcal{P}}^{2x1}(s)$. Thus ,

$$\underline{v}_1 \, f_{1j} \, \hat{d}_j + \underline{v}_2 \, f_{2j} \, \hat{n}_j = \underline{u}_j$$

Clearly , since $f_{1j}(s)/f_{2j}(s)$ for all the choices of $(\hat{n}_j , \hat{d}_j)$ and thus (n_j , d_j) , f_{1j} must divide $\underline{u}_j$ and $\underline{u}_j$ is not coprime $\mathbb{R}_{\mathcal{P}}(s)$ vector . □

***Corollary (9.2.1)** : A necessary condition for diagonal closed loop stabilization is that the plant P is cyclic .* □

Let $\mathcal{F}^1_{\mathcal{L}} = \{f_{11} , f_{12} , \dots , f_{1m}\}$ and $p(s) = \prod_{i=1}^{m} f_{1i}(s)$, p(s) will be called the first invariant function of P . The properties of p(s) are summarized below .

***Proposition (9.2.3)** : Let $P \in \mathbb{R}_{pr}^{m x m}(s)$ be the transfer function of a plant and p(s) be its first invariant function . Then :*
i) p(s) is an invariant of the plant .
ii) The zeros in $\mathcal{P}$ of p(s) are fixed closed loop poles of any closed loop system obtained by diagonal precompensation and unity feedback .

Proof
i) It follows from proposition (9.2.1) .
ii) From the proof of proposition (9.2.2) it is clear that for a solution to exist , $\underline{u}_j = f_{1j} \cdot \underline{u}'_j$ for all $i = 1 , \dots , m$. Then :

$$[\, \underline{u}_1 , \underline{u}_2 , \dots , \underline{u}_m \,] = \begin{bmatrix} f_{11} & & O \\ & f_{12} \ddots & \\ O & & f_{1m} \end{bmatrix} \cdot [\, \underline{u}'_1 , \underline{u}'_2 , \dots , \underline{u}'_m \,] \qquad (9.2.8)$$

and for all choices of $C = \text{diag}\{c_1 , \dots , c_m\}$ the $|\, \text{diag}\{f_{11} , \dots , f_{1m}\} \,|$ will be a factor of the determinant of the denominator of the closed loop system . Thus the zeros of p(s) define fixed unstable closed loop poles . □

***Remark (9.2.2)** : If $p_f(s)$ denotes the fixed pole function of the closed loop system obtained under any diagonal precompensation and unity output feedback , then $p(s)/p_f(s)$.* □

***Remark (9.2.3)** : The transfer function P is cyclic if and only if for every fixed i , i = 1, … , m the elements of P_i are $\mathbf{R}_{\mathcal{P}}(s)$ – coprime .* □

***Definition (9.2.4)** : A cyclic plant P will be called diagonally stabilizable (D stabilizable) if condition (9.2.1) holds true for some $\mathbf{R}_{\mathcal{P}}(s)$ – unimodular matrix U and if in :*

$$N_2 = diag\{n_1 , \dots , n_m\} \text{ and } D_2 = diag\{d_1 , \dots , d_m\}$$

the pairs (n_i , d_i) are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime . □

From equations (9.2.1) and (9.2.3) it is clear that the problem is reduced to the following one. Given a set of cyclic matrices $P_i \in \mathbb{R}_{\mathcal{P}}^{m\times 2}(s)$, $i = 1 , \ldots , m$, determine the solvability of the following over $\mathbb{R}_{\mathcal{P}}(s)$:

$$P_i \cdot \underline{q}_i = \underline{u}_i \ , \ \underline{q}_i = \begin{bmatrix} d_i \\ n_i \end{bmatrix} \in \mathbb{R}_{\mathcal{P}}^{2\times 1}(s) \ , d_i \neq 0 \ , \ i = 1 , 2 , \ldots , m \qquad (9.2.9)$$

where , (n_i , d_i) are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime , $\underline{u}_i$ are arbitrary vectors of $\mathbb{R}_{\mathcal{P}}^{m\times 1}(s)$, with the additional property that $U \triangleq [\underline{u}_1 , , \underline{u}_2 , \ldots , \underline{u}_m]$ is $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular .

***Definition (9.2.5)** : The problem defined by (9.2.9) will be referred to as the D – stabilization problem (DDSP) .* □

9.3. THE D – STABILIZATION PROBLEM

In the following we consider some alternative transformation for the general case of DDSP . Notice that (9.2.9) may be expressed as :

$$[P_1 , \ldots , P_m] \cdot X_m = U \ , \ \widetilde{P}_m = [P_1 , \ldots , P_m] \ , \ X_m = \text{diag}\{\underline{q}_i , i = 1 , 2 , \ldots , m\} \qquad (9.3.1)$$

where , $\underline{q}_i = [d_i , n_i]^T = [x_{i1} , x_{i2}]^T$ and U , $\mathbf{R}_{\mathcal{P}}(s)$ – unimodular . By the Binet – Cauchy theorem we have :

$$| U | = u = C_m(\widetilde{P}_m) \cdot C_m(X_m) \ , \ u \text{ is } \mathbf{R}_{\mathcal{P}}(s) \text{ unit} \qquad (9.3.2)$$

The above equation is multilinear in the parameters x_{ij} , $i = 1 , 2 , \ldots , m$, $j = 1 , 2$ in $C_m(X_m)$. The structure of X_m leads to a number of fixed zero entries in $C_m(X_m)$. To demonstrate the form the above equation takes , we consider first the simple case $m = 2$, [Kar. 2] . Then :

$$X_2 = \begin{bmatrix} x_{11} & \vdots & 0 \\ x_{12} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & x_{21} \\ 0 & \vdots & x_{22} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ \\ 3 \\ 4 \end{matrix} = [\underline{x}_1 , \underline{x}_2] \qquad (9.3.3)$$

and $\rho_1 = \{1 , 2\}$, $\rho_2 = \{3 , 4\}$.

$$C_2(X_2) = \underline{x}_1 \wedge \underline{x}_2 = \begin{bmatrix} 0 \\ x_{11}\ x_{21} \\ x_{11}\ x_{22} \\ x_{12}\ x_{21} \\ x_{12}\ x_{22} \\ 0 \end{bmatrix} = \begin{bmatrix} 0_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ 0_{34} \end{bmatrix}$$

If $\widetilde{P}_2 = [\ \underline{p}_{11}\ \underline{p}_{12} \vdots \underline{p}_{21}\ \underline{p}_{22}\ ]$, then $C_2(\widetilde{P}_2) = [\ \alpha_{12}\ ,\ \alpha_{13}\ ,\ \alpha_{14}\ ,\ \alpha_{23}\ ,\ \alpha_{24}\ ,\ \alpha_{34}\ ]$, where $\alpha_{12} = |\ \underline{p}_{11}\ \underline{p}_{12}\ |$, $\alpha_{13} = |\ \underline{p}_{11}\ \underline{p}_{21}\ |$, $\alpha_{14} = |\ \underline{p}_{11}\ \underline{p}_{22}\ |$, $\alpha_{23} = |\ \underline{p}_{12}\ \underline{p}_{21}\ |$, $\alpha_{24} = |\ \underline{p}_{12}\ \underline{p}_{22}\ |$, $\alpha_{34} = |\ \underline{p}_{21}\ \underline{p}_{22}\ |$. Equation (9.3.2) may thus be expressed as :

$$\alpha_{13}\,\lambda_{13} + \alpha_{14}\,\lambda_{14} + \alpha_{23}\,\lambda_{23} + \alpha_{24}\,\lambda_{24} = u \qquad (9.3.4)$$

The above equation is defined by the nonzero entries in $C_2(X_2)$. Note that the elements of $C_2(X_2)$ are indexed by the sequences $\omega \in Q_{2,4}$, where $Q_{\kappa,n}$ denotes the set of lexicographically ordered strictly increasing sequences $\omega = (i_1\ ,\ \dots\ ,\ i_\kappa)$ of κ integers from $1\ ,\ 2\ ,\dots,\ n$. If the integers 1 , 2 , 3 , 4 are grouped as $\{\rho_1 = (1\ ,\ 2)\ ,\ \rho_2 = (3\ ,\ 4)\}$ then an element λ_ω in $C_2(X_2)$, $\omega \in Q_{2,4}$, will be zero if and only if more than one indices in $\omega = (i_1\ ,\ i_2)$ are taken from the same ρ_i . The location of nonzero elements is defined by the sequences $\omega \in Q_{2,4}$ for which only one index is taken from ρ_1 , ρ_2 respectively . The set of indices that characterizes the nonzero elements in $C_2(X_2)$ is $\Gamma_{2,2} = \{(1\ ,\ 3)\ ,\ (1\ ,\ 4)\ ,\ (2\ ,\ 3)\ ,\ (2\ ,\ 4)\}$ and will be referred to as the essential subset of $Q_{2,4}$. To generate the above observations we introduce some useful notation .

Definition (9.3.1) *[Kar. 2] , [Kar. 9] : Let $Q_{m,2m}$ denote the set of strictly increasing and lexicographically ordered sequences of m integers taken from $\{1\ ,\ 2\ ,\ \dots\ ,\ 2\,m\}$. For the set of integers $\{1\ ,\ 2\ ,\ \dots\ ,\ 2\,m\}$ a pair partitioning is defined as the set of ordered pairs $\Phi = \{\rho_1 = (1,\ 2)\ ,\ \rho_2 = (3\ ,\ 4)\ ,\ \dots\ ,\ \rho_m = (2\,m-1\ ,\ 2\,m)\}$. A sequence $\omega = (i_1\ ,\ \dots\ ,\ i_m) \in Q_{m,2m}$ will be called Φ – prime if there is no pair of indices $(i_j\ ,\ i_\kappa) \in \omega$ which is taken from the same $\rho_a \in \Phi$. The set of all Φ – prime sequences of $Q_{m,2m}$ will be denoted $\Gamma_{m,2}$ and referred to as the $(m\ ,\ 2)$ – prime set of $Q_{m,2m}$.* □

Proposition (9.3.1) *: Let $X_m \in \mathbb{R}_{\mathfrak{sp}}^{2m\times m}(s)$, $C_m(X_m) = [\ \dots\ ,\ \lambda_\omega\ ,\ \dots\ ]^T$, $\omega \in Q_{m,2m}$, $\Gamma_{m,2}$ be the $(m\ ,\ 2)$ – prime set of $Q_{m,2m}$ and $\Gamma^c_{m,2}$ be the complement of $\Gamma_{m,2}$ in $Q_{m,2m}$. Then :*

i) A coordinate λ_ω is zero for generic values of the nonzero elements in X_m if and only if $\omega \in \Gamma^c_{m,2}$.

ii) The nonzero coordinates λ_ω that correspond to generic values of the elements in X_m

are those corresponding to $\Gamma_{m,2}$. □

The following algorithm can be used to compute the set $\Gamma_{m,2}$ for any $m \geq 2$.

COMPUTATION OF $\Gamma_{m,2}$

Step 1 : Set $m = 2$. Then the set $\Gamma_{2,2}$ is clearly :

$$\Gamma_{2,2} = \{(1 , 3) , (1 , 4) , (2 , 3) , (2 , 4)\}$$

Step 2 : For every sequence $\omega_2 = (i_1 , i_2) \in \Gamma_{2,2}$ generate the two sequences of $\Gamma_{3,2}$ as $\{(i_1 , i_2 , 5) , (i_1 , i_2 , 6)\}$. This process generates all sequences in $\Gamma_{3,2}$.

⋮

Step m : For every $\omega_{m-1} = (i_1 , \dots , i_{m-1}) \in \Gamma_{m-1,2}$, generate two sequences of $\Gamma_{m,2}$ as $\{(i_1 , \dots , i_{m-1} , 2m-1) , (i_1 , \dots , i_{m-1} , 2m)\}$. This process generates all sequences in $\Gamma_{m,2}$. □

Note that the cardinality of $\Gamma_{m,2}$ is 2^m . The form that equation (9.3.2) takes may be simplified by setting :

$$y_{2i} \triangleq x_{ij} \text{ , when } j = 2 \text{ and } y_{2i-1} \triangleq x_{ij} \text{ , when } j = 1 \qquad (9.3.5)$$

With this notation , for every $\sigma = (i_1 , \dots , i_m) \in \Gamma_{m,2}$ we take $\lambda_\sigma \triangleq y_{i_1} y_{i_2} \cdots y_{i_m}$ and the fixed zeros in $C_m(X_m)$ appear in the $\Gamma^c_{m,2}$ locations . Equation (9.3.2) may then be expressed as :

$$\sum_\sigma \alpha_\sigma \lambda_\sigma = u \text{ , u is } \mathbb{R}_{\mathfrak{P}}(s) \text{ unit , } \sigma \in \Gamma_{m,2} \qquad (9.3.6)$$

the above is a Diophantine equation over $\mathbb{R}_{\mathfrak{P}}(s)$ with parameters $\mathcal{A}_m = \{\alpha_\sigma \in \mathbb{R}_{\mathfrak{P}}(s) , \sigma \in \Gamma_{m,2}\}$ and unknowns $\mathfrak{G} = \{\lambda_\sigma \in \mathbb{R}_{\mathfrak{P}}(s) , \sigma \in \Gamma_{m,2}\}$. For the set $\mathcal{A}_m$ we have the following property .

Proposition (9.3.2) : *The parametric set* $\mathcal{A}_m$ ***is invariant of the plant*** *P* ***modulo*** $\mathbb{R}_{\mathfrak{P}}(s)$ *units.*

Proof

If (A_1 , B_1) , (A_1' , B_1') are two $\mathbb{R}_{\mathfrak{P}}(s)$ – left coprime MFD pairs of P , then there exists an $\mathbb{R}_{\mathfrak{P}}(s)$ – unimodular matrix U such that $[A_1' , B_1'] = U \cdot [A_1 , B_1]$ and thus :

$$\widetilde{P}_m' = [P_1' , \dots , P_m'] = U \cdot [P_1 , \dots , P_m] = U \cdot \widetilde{P}_m \qquad (9.3.7)$$

Clearly $C_m(\widetilde{P}'_m) = |U| \cdot C_m(\widetilde{P}_m) = u \cdot C_m(\widetilde{P}_m)$, where u is $\mathbb{R}_{\mathfrak{P}}(s)$ unit . □

The set $\mathcal{A}_m$ characterizes the plant [modulo $\mathbb{R}_{\mathfrak{P}}(s)$ units] and will be referred to as a generator set of DDSP . A greatest common divisor of $\mathcal{A}_m$ will be denoted by f_g and referred to as a prime invariant function of the plant P .

Proposition (9.3.3) *: Let P be a plant and p(s) , $f_g(s)$ be the first and prime invariant functions respectively . Then :*
i) p(s) divides $f_g(s)$.
ii) The zeros of $f_g(s)$ are fixed modes of any closed loop system under diagonal precompensation and unity output feedback .

Proof
i) It suffices to show that $p(s) = \prod_{j=1}^{m} f_{1j}(s)$ is a common divisor of all the elements of $\mathcal{A}_m$. The nonzero elements of $\mathcal{A}_m$ are those elements α_σ of $C_m(\widetilde{P}_m)$ which correspond to $\sigma = (i_1 , \dots , i_m) \in \Gamma_{m,2}$, or equivalently the nonzero α_σ are the $m \; x \; m$ minors $| \underline{p}_{i_1} \underline{p}_{i_2} \dots \underline{p}_{i_m} |$ of $\widetilde{P}_m = [P_1 , \dots , P_m]$, where each $\underline{p}_{i_j}$ is taken from the corresponding P_j , $j = 1 , \dots , m$. f_{1j} is the greatest common divisor of the elements of P_j and hence a common divisor of the elements of $\underline{p}_{i_j}$. A common divisor of $\alpha_\sigma = | \underline{p}_{i_1} \underline{p}_{i_2} \cdots \underline{p}_{i_m} |$, $\underline{p}_{i_j} \in P_j$, is $p(s) = \prod_{j=1}^{m} f_{1j}(s)$. Hence , p(s) divides $f_g(s)$.
ii) By inspection of equation (9.3.6) we conclude that for each selection of (n_i , d_i) (and thus $c_i = \underline{n}_i \cdot d_i^{-1}$) the greatest common divisor of the elements of $\mathcal{A}_m$ is a factor of the determinant of the denominator of the closed loop system under diagonal precompensation and unity output feedback . Thus , the zeros of f_g are fixed modes of any closed loop system obtained under diagonal precompensation and unity output feedback . □

Corollary (9.3.1) *: If P is noncyclic , then the set $\mathcal{A}_m$ is not $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime .* □

Definition (9.3.2) *: A system for which f_g in an $\mathbb{R}_{\mathfrak{P}}(s)$ unit will be called strongly cyclic* □

Remark (9.3.1) *: If f_g is not an $\mathbb{R}_{\mathfrak{P}}(s)$ unit i.e. $\delta_\infty(f_g) > 0$ (there exist zeros at infinity), then all closed loop systems obtained under diagonal precompensation and unity output feedback have fixed poles at infinity with the total number defined by $\delta_\infty(f_g)$. In this case the closed loop system is unstable and exhibits impulsive behavior for all compensator schemes of the above type .* □

9.4. NECESSARY AND SUFFICIENT SOLVABILITY CONDITIONS OF DDSP

We consider the general case of DDSP and examine necessary and sufficient solvability conditions .

Remark (9.4.1) : The necessary and sufficient solvability conditions for equation (9.3.6) (including the decomposition of λ_σ as in (9.3.5)) are necessary and sufficient solvability conditions for equation (9.3.1) and hence , for (9.2.9) (DDSP) . □

Remark (9.4.1) implies that it suffices to find necessary and sufficient solvability conditions for equation (9.3.6) (including the decomposition of λ_σ as in (9.3.5)) . First we state the following useful lemma :

Lemma (9.4.1) : Let $A \in \mathbb{R}_{\mathfrak{P}}^{t x 2}(s)$, $t \geq 2$ and the greatest common divisor of all the entries of A be an $\mathbb{R}_{\mathfrak{P}}(s)$ unit . Let H denote the row Hermite form of A , namely :

$$H = \begin{bmatrix} b & w \\ 0 & z \\ \hline & O & \end{bmatrix} \tag{9.4.1}$$

Factorize b , w as , $b = g \cdot b'$, $w = g \cdot w'$, with (b' , w') an $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime pair . Then the family of $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime pairs (n , d) such that the vector :

$$\underline{r} = [\, r_1 \, , \, \ldots \, , \, r_t \,]^T = A \cdot [\, d \;\; n \,]^T \tag{9.4.2}$$

is $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime is given by all pairs :
i) (n , d) $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime , such that (n , b) is $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime , $(n , d) \neq h \cdot (b' , -w')$ for all $h - \mathbb{R}_{\mathfrak{P}}(s)$ units , when A is nondegenerate noncomplete .
ii) (n , d) $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime , when A is nondegenerate complete .
iii) (n , d) $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime , solutions of the scalar Diophantine equation :

$$h = \underline{v}^T \cdot [\, d \, , \, n \,]^T \tag{9.4.3}$$

for all $h - \mathbb{R}_{\mathfrak{P}}(s)$ units , when A is degenerate and $\underline{v}$ is a minimal McMillan degree and $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime base for the row [$\mathbb{R}_{\mathfrak{P}}(s)$ – module of A] .

Proof
From the hypothesis is clear that A is a cyclic matrix . The cyclicity of A implies the cyclicity of H and thus b , w , z are $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime .

i) Let A be a nondegenerate noncomlete matrix .

($\Rightarrow$) Let (n , d) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair such that (9.4.2) holds truc . Then :

$$\underline{r} = A \cdot [\ d\ n\ ]^T = U_l \cdot H \cdot [\ d\ n\ ]^T \tag{9.4.4}$$

with $\underline{r}$ an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector and U_l an $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular matrix . (9.4.4) implies that :

$$\begin{bmatrix} b & w \\ 0 & z \\ \cdots & \cdots \\ & \mathbf{O} \end{bmatrix} \cdot \begin{bmatrix} d \\ n \end{bmatrix} = U_l^{-1} \cdot \underline{r} = \underline{v} \tag{9.4.5}$$

With , $\underline{v} = [\ v_1\ v_2\ 0 \ldots 0\ ]^T = U_l^{-1} \cdot \underline{r}$, an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector . The latter implies that (v_1 , v_2) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair . Equation (9.4.5) can be expressed as :

$$\begin{cases} b \cdot d + w \cdot n = v_1 \\ \\ z \cdot n = v_2 \end{cases} \tag{9.4.6}$$

Then (n , b) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair , else an $s_0 \in \mathcal{P}$ would exist such that $n(s_0) = b(s_0) = 0$. But then (9.4.6) would imply that $b(s_0) \cdot d(s_0) + w(s_0) \cdot n(s_0) = v_1(s_0) = 0$ and $z(s_0) \cdot n(s_0) = v_2(s_0) = 0$, which contradicts the fact that (v_1 , v_2) is a coprime pair. Additionally , $(n , d) \neq h \cdot (b' , -w')$ for all h – $\mathbb{R}_{\mathcal{P}}(s)$ units , else :

$$v_1 = g \cdot \{b' \cdot d + w' \cdot n\} = g \cdot h \cdot \{b' \cdot (-w') + w' \cdot b'\} = 0 \ , \forall\, s \in \mathbb{C}$$

In that case the pair $(v_1 , v_2) = (0 , v_2)$ would be coprime if and only if v_2 was an $\mathbb{R}_{\mathcal{P}}(s)$ unit or equivalently , (9.4.6) , z , n were $\mathbb{R}_{\mathcal{P}}(s)$ units simultaneously . But if z was an $\mathbb{R}_{\mathcal{P}}(s)$ unit then (9.4.1) and the cyclicity of A would imply that A was a complete matrix something that contradicts the truth . Thus the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs (n , d) such that (9.4.2) holds true must satisfy the constraints of *i*) .

($\Leftarrow$) Let (n , d) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime , such that (n , b) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair and $(n , d) \neq h \cdot (b' , -w')$ for all h – $\mathbb{R}_{\mathcal{P}}(s)$ units . Then we shall show that (n , d) satisfies (9.4.2) . Consider the vector $\underline{r} = A \cdot [\ d\ n\ ]^T$. Then an $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular matrix U_l exists such that :

$$\begin{bmatrix} b & w \\ 0 & z \\ \cdots & \cdots \\ & \mathbf{O} \end{bmatrix} \cdot \begin{bmatrix} d \\ n \end{bmatrix} = U_l^{-1} \cdot \underline{r} = \underline{v} \tag{9.4.7}$$

or equivalently ,

$$\begin{cases} b \cdot d + w \cdot n = v_1 \\ z \cdot n = v_2 \end{cases} \tag{9.4.6}$$

It suffices to show that (v_1 , v_2) is an $\mathbb{R}_{\mathcal{P}}(s)$ coprime pair and thus $U_l \cdot \underline{v} = \underline{r}$ is an $\mathbb{R}_{\mathcal{P}}(s)$ coprime vector . Let $s_0 \in \mathcal{P}$ be an arbitrary zero of v_2 , then the following three alternatives may happen :

$$z(s_0) = 0 \ , \ n(s_0) \neq 0 \tag{9.4.7}$$
$$z(s_0) = 0 \ , \ n(s_0) = 0 \tag{9.4.8}$$
$$z(s_0) \neq 0 \ , \ n(s_0) = 0 \tag{9.4.9}$$

If (9.4.7) holds true then (9.4.6) implies that :

$$v_1(s_0) = g(s_0) \cdot \{b'(s_0) \cdot d(s_0) + w'(s_0) \cdot n(s_0)\} \tag{9.4.10}$$

$g(s_0) \neq 0$, since b , w , z are $\mathbb{R}_{\mathcal{P}}(s)$ coprime . We distinguish the following three cases :
1) $b'(s_0) = 0$, $w'(s_0) \neq 0$. Then (9.4.10) gives $v_1(s_0) = g(s_0) \cdot w'(s_0) \cdot n(s_0) \neq 0$ and thus the pair (v_1 , v_2) is $\mathbb{R}_{\mathcal{P}}(s)$ coprime .
2) $b'(s_0) \neq 0$, $w'(s_0) = 0$. Since $d \neq -h \cdot w'$, (h an $\mathbf{R}_{\mathcal{P}}(s)$ unit) , is implied that $d(s_0) \neq -h(s_0) \cdot w'(s_0) = 0$. Then (9.4.10) gives $v_1(s_0) = g(s_0) \cdot b'(s_0) \cdot d(s_0) \neq 0$ and thus the pair (v_1 , v_2) is $\mathbb{R}_{\mathcal{P}}(s)$ coprime .
3) $b'(s_0) \neq 0$, $w'(s_0) \neq 0$. Since $(n , d) \neq h \cdot (b' , -w')$ for all $h - \mathbf{R}_{\mathcal{P}}(s)$ units is implied that $\{b'(s_0) \cdot d(s_0) + w'(s_0) \cdot n(s_0)\} \neq 0$. Then (9.4.10) gives $v_1(s_0) \neq 0$ and thus the pair (v_1 , v_2) is $\mathbb{R}_{\mathcal{P}}(s)$ coprime .

If (9.4.8) holds true then (9.4.6) implies that :

$$v_1(s_0) = b(s_0) \cdot d(s_0) \tag{9.4.11}$$

Since (n , d) , (n , b) are $\mathbb{R}_{\mathcal{P}}(s)$ coprime pairs is implied that $d(s_0) \neq 0$, $b(s_0) \neq 0$. Thus $v_1(s_0) \neq 0$ and the pair (v_1 , v_2) is $\mathbf{R}_{\mathcal{P}}(s)$ coprime .
If (9.4.9) holds true then (9.4.6) implies the same result as above . Thus we have proved that an $\mathbf{R}_{\mathcal{P}}(s)$ – coprime pair (n , d) that satisfies the constraints of *i*) satisfies (9.4.2) as well .
***ii*)** Let A be nondegenerate complete . Then A is an $\mathbf{R}_{\mathcal{P}}(s)$ left unimodular matrix and *ii*) follows immediately .
***iii*)** Let A be degenerate . Then it is well known that A can be written as :

$$A = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix} \cdot [\, v_1 \,,\, v_2 \,] = \underline{u} \cdot \underline{v}^T \tag{9.4.12}$$

where , $\underline{u}$, $\underline{v}$ are minimal Mc Millan degree bases for the column [$\mathbb{R}_{\mathcal{P}}(s)$ – module of A] row [$\mathbb{R}_{\mathcal{P}}(s)$ – module of A] , respectively . Hence , $\underline{u}$, $\underline{v}$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors unique [modulo $\mathbb{R}_{\mathcal{P}}(s)$ units] .

(⇒) Let (n , d) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair such that (9.4.2) holds true . Then $\underline{r}$ is an $\mathbb{R}_{\mathcal{P}}(s)$ coprime vector and :

$$A \cdot \begin{bmatrix} d \\ n \end{bmatrix} = \underline{u} \cdot \underline{v}^T \cdot \begin{bmatrix} d \\ n \end{bmatrix} = \underline{u} \cdot h = \underline{r} \tag{9.4.13}$$

where , $h = \underline{v}^T \cdot [\, d \,,\, n \,]^T$. Since $\underline{u}$, $\underline{r}$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors h must be an $\mathbb{R}_{\mathcal{P}}(s)$ unit . Thus (n , d) is a solution of the scalar Diophantine equation $h = \underline{v}^T \cdot [\, d \,,\, n \,]^T$ with h an $\mathbb{R}_{\mathcal{P}}(s)$ unit and the constrain of *iii)* is satisfied .

(⇐) Let (n , d) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime , solution of the scalar Diophantine equation :

$$h = \underline{v}^T \cdot [\, d \,,\, n \,]^T \tag{9.4.14}$$

with h an $\mathbb{R}_{\mathcal{P}}(s)$ unit and $\underline{v}$ a minimal McMillan degree and $\mathbb{R}_{\mathcal{P}}(s)$ – coprime base for the row [$\mathbb{R}_{\mathcal{P}}(s)$ – module of A] . Then a $\underline{u}$ minimal McMillan degree and $\mathbb{R}_{\mathcal{P}}(s)$ – coprime base for the column [$\mathbb{R}_{\mathcal{P}}(s)$ – module of A] exists such that $A = \underline{u} \cdot \underline{v}^T$ and thus :

$$A \cdot \begin{bmatrix} d \\ n \end{bmatrix} = \underline{u} \cdot \underline{v}^T \cdot \begin{bmatrix} d \\ n \end{bmatrix} = \underline{u} \cdot h = \underline{r}$$

and $\underline{r}$ is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector since $\underline{u}$ is and h an $\mathbb{R}_{\mathcal{P}}(s)$ unit . □

***Theorem (9.4.1)** : Let $\mathcal{A}_m = \{\alpha_\sigma \in \mathbb{R}_{\mathcal{P}}(s) \,,\, \sigma \in \Gamma_{m,2}\}$ be a generator set of DDSP defined on the plant P . A necessary and sufficient condition for solvability of equation (9.3.6) (including the decomposition of λ_σ as in (9.3.5)) and hence for solvability of DDSP is that the system is strongly cyclic .*

Proof

(⇒) Let a solution of DDSP exists . Then by (9.2.9) , (9.3.1) , (9.3.2) , (9.3.5) equation (9.3.6) has a solution and thus the greatest common divisor f_g of the generator set $\mathcal{A}_m$ must be an $\mathbb{R}_{\mathcal{P}}(s)$ unit . Definition(9.3.2) implies that the system is strongly cyclic .

(⇐) Let the system be strongly cyclic . Then a greatest common divisor of the set $\mathcal{A}_m$ is an $\mathbb{R}_{\mathcal{P}}(s)$ unit and thus $\{\alpha_\sigma \in \mathbb{R}_{\mathcal{P}}(s) \,,\, \sigma \in \Gamma_{m,2}\}$ are coprime . Without loss of generality we can assume that u in equation (9.3.6) is 1 . Consider equation (9.3.6) :

$$\sum_{\sigma} \alpha_\sigma \lambda_\sigma = 1 \ , \ \sigma \in \Gamma_{m,2} \tag{9.4.15}$$

We shall prove that for all $l = 1, 2, \dots, m$, (n_l, d_l) $\mathbb{R}_{\mathcal{P}}(s)$ – coprime exist such that $\lambda_\sigma = y_{i_1} \cdot y_{i_2} \cdots y_{i_m}$:

$$y_{i_l} = \begin{cases} d_l \ , \text{ when } i_l = 1, 3, \dots, 2m-1 \ , (\text{or } 2l-1) \\ n_l \ , \text{ when } i_l = 2, 4, \dots, 2m \ , (\text{or } 2l) \end{cases} \tag{9.4.16}$$

If $\gamma_{m-\kappa}$ denotes the set $\gamma_{m-\kappa} = (i_1, \dots, i_{m-\kappa}) \in \Gamma_{m-\kappa,2}$, $\gamma_m = \sigma = (\gamma_{m-\kappa}, i_{m-\kappa+1}, \dots, i_m)$ $\kappa = 0, 1, \dots, m-1$, then :

Step 1 : Since the set $\mathcal{A}_m = \{\alpha_\sigma \in \mathbb{R}_{\mathcal{P}}(s) \ , \ \sigma \in \Gamma_{m,2}\}$ is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime is implied that the matrix $A_m = [\alpha_{i,j}] \in \mathbb{R}_{\mathcal{P}}^{t x 2}(s)$, $\forall \ i = \gamma_{m-1} \in \Gamma_{m-1,2}$, $j = 2m-1, 2m$, $t = 2^{m-1}$, is cyclic and , (lemma(9.4.1)) , thus $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\dots, \lambda_{\gamma_{m-1}}, \dots]$, $\underline{\alpha}_m = [\alpha_i]$, (y_{2m}, y_{2m-1}) , $\forall \ i = \gamma_{m-1} \in \Gamma_{m-1,2}$ exist such that :

$$A_m \cdot \begin{bmatrix} y_{2m-1} \\ y_{2m} \end{bmatrix} = \underline{\alpha}_m = [\alpha_{\gamma_{m-1}}] \tag{9.4.17}$$

and

$$\sum_{\gamma_{m-1}} \lambda_{\gamma_{m-1}} \alpha_{\gamma_{m-1}} = 1 \ , \ \gamma_{m-1} \in \Gamma_{m-1,2} \tag{9.4.18}$$

or,

$$\sum_{\gamma_{m-1}} \lambda_{\gamma_{m-1}} \{\alpha_{\{\gamma_{m-1}, 2m-1\}} \ y_{2m-1} + \alpha_{\{\gamma_{m-1}, 2m\}} \ y_{2m}\} = 1 \ , \ \gamma_{m-1} \in \Gamma_{m-1,2} \tag{9.4.19}$$

Clearly each solution of equation (9.4.19) :

$$\lambda_{\gamma_m} = \lambda_{\gamma_{m-1}} \cdot y_{i_m} \ , \ i_m \text{ in } \{2m-1, 2m\} \tag{9.4.20}$$

is a solution of equation (9.4.15) .

Step 2 : Since the set $\mathcal{A}_{m-1} = \{\alpha_{\gamma_{m-1}} \in \mathbb{R}_{\mathcal{P}}(s) \ , \ \gamma_{m-1} \in \Gamma_{m-1,2}\}$ is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime is implied that the matrix $A_{m-1} = [\alpha_{i,j}] \in \mathbb{R}_{\mathcal{P}}^{t x 2}(s)$, $\forall \ i = \gamma_{m-2} \in \Gamma_{m-2,2}$, $j = 2m-3$, $2m-2$, $t = 2^{m-2}$, is cyclic and , (by lemma(9.4.1)) , thus $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\dots, \lambda_{\gamma_{m-2}}, \dots]$, $\underline{\alpha}_{m-1} = [\alpha_i]$, (y_{2m-3}, y_{2m-2}) , $\forall \ i = \gamma_{m-2} \in \Gamma_{m-2,2}$, exist such that :

$$A_{m-1} \cdot \begin{bmatrix} y_{2m-3} \\ y_{2m-2} \end{bmatrix} = \underline{\alpha}_{m-1} = [\alpha_{\gamma_{m-2}}] \tag{9.4.21}$$

and

$$\sum_{\gamma_{m-2}} \lambda_{\gamma_{m-2}} \alpha_{\gamma_{m-2}} = 1 \ , \ \gamma_{m-2} \in \Gamma_{m-2,2} \tag{9.4.22}$$

or,

$$\sum_{\gamma_{m-2}} \lambda_{\gamma_{m-2}} \{\alpha_{\{\gamma_{m-2}, 2m-3\}} \, y_{2m-3} + \alpha_{\{\gamma_{m-2}, 2m-2\}} \, y_{2m-2}\} = 1 \ , \ \gamma_{m-2} \in \Gamma_{m-2,2} \tag{9.4.23}$$

Clearly each solution of equation (9.4.23) :

$$\lambda_{\gamma_{m-1}} = \lambda_{\gamma_{m-2}} \cdot y_{i_{m-1}} \ , \ i_{m-1} \text{ in } \{2\,m-3 \ , \ 2\,m-2\} \tag{9.4.24}$$

is a solution of (9.4.18) and thus :

$$\lambda_{\gamma_m} = \lambda_{\gamma_{m-2}} \cdot y_{i_{m-1}} \cdot y_{i_m} \ , \ (i_{m-1} \ , \ i_m) \text{ in } \{(2m-3 \ , \ 2m-2)x(2m-1 \ , \ 2m)\} \tag{9.4.25}$$

is a solution of (9.4.15) .

Now it is clear that if for $\kappa = 2 \ , \ldots , \ m-2$ we repeat the above process successively for the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime sets $\mathcal{A}_{m-\kappa} = \{\alpha_{\gamma_{m-\kappa}} \in \mathbb{R}_{\mathcal{P}}(s) \ , \ \gamma_{m-\kappa} \in \Gamma_{m-\kappa,2}\}$ we can construct $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\ \ldots \ , \ \lambda_{\gamma_{m-\kappa-1}} \ , \ \ldots\ ]$, $\underline{\alpha}_{m-\kappa} = [\alpha_i]$, $(y_{2m-2\kappa} \ , \ y_{2m-2\kappa-1})$, $\forall \ i = \gamma_{m-\kappa-1} \in \Gamma_{m-\kappa-1,2}$, such that :

$$A_{m-\kappa} \cdot \begin{bmatrix} y_{2m-2\kappa-1} \\ y_{2m-2\kappa} \end{bmatrix} = \underline{\alpha}_{m-\kappa} = [\alpha_{\gamma_{m-\kappa-1}}] \tag{9.4.26}$$

where , $A_{m-\kappa} = [\alpha_{i,j}\,] \in \mathbb{R}_{\mathcal{P}}^{tx2}(s)$, $\forall \ i = \gamma_{m-\kappa-1} \in \Gamma_{m-\kappa-1,2}$, $j = 2\,m-2\,\kappa-1 \ , \ 2\,m-2\,\kappa$, $t = 2^{m-\kappa-1}$, is a cyclic matrix . Furthermore :

$$\sum_{\gamma_{m-\kappa-1}} \alpha_{\gamma_{m-\kappa-1}} \lambda_{\gamma_{m-\kappa-1}} = 1 \ , \ \gamma_{m-\kappa-1} \in \Gamma_{m-\kappa-1,2} \tag{9.4.27}$$

or,

$$\sum_{\gamma_{m-\kappa-1}} \lambda_{\gamma_{m-\kappa-1}} \{\alpha_{\{\gamma_{m-\kappa-1}, 2m-2\kappa-1\}} \, y_{2m-2\kappa-1} + \alpha_{\{\gamma_{m-\kappa-1}, 2m-2\kappa\}} \, y_{2m-2\kappa}\} = 1 \ ,$$

$$\gamma_{m-\kappa-1} \in \Gamma_{m-\kappa-1,2} \tag{9.4.28}$$

Clearly each solution of equations (9.4.28) :

$$\lambda_{\gamma_{m-\kappa}} = \lambda_{\gamma_{m-\kappa-1}} \cdot y_{i_{m-\kappa}} \ , \ i_{m-\kappa} \text{ in } \{2\,m-2\,\kappa-1 \ , \ 2\,m-2\,\kappa\} \tag{9.4.29}$$

are solutions of equtions :

$$\sum_{\gamma_{m-\kappa}} \lambda_{\gamma_{m-\kappa}} \alpha_{\gamma_{m-\kappa}} = 1 \ , \ \gamma_{m-\kappa} \in \Gamma_{m-\kappa,2} \tag{9.4.30}$$

and thus it is implied that :

$$\lambda_\sigma = \lambda_{\gamma_m} = \lambda_{\gamma_1} \cdot y_{i_2} \cdots y_{i_m} = y_{i_1} \cdot y_{i_2} \cdots y_{i_m} \qquad (9.4.31)$$

is a solution of (9.4.15) . □

9.5. PARAMETRIZATION OF SOLUTIONS OF THE DDSP

In the following we introduce a parametrization of solutions to DDSP . First we state some useful preliminary results , [Kar. 2] .

Definition (9.5.1) *: Let $T \in \mathbb{R}_{\mathcal{P}}^{2x2}(s)$, cyclic . Then a pair of (n_1 , d_1) , (n_2 , d_2) , (coprime) n_i , $d_i \in \mathbb{R}_{\mathcal{P}}(s)$, that satisfy equation :*

$$[\ d_1\ n_1\ ] \cdot T \cdot \begin{bmatrix} d_2 \\ n_2 \end{bmatrix} = 1 \qquad (9.5.1)$$

is called a mode T mutually stabilizing pair . □

Assume that the Smith form of T over $\mathbb{R}_{\mathcal{P}}(s)$ is $S_T = \text{diag}\{1 , \phi(s)\}$ and let $\Lambda_T = \{\lambda_i \in \mathcal{P} : \phi(\lambda_i) = 0\}$ be the distinct values of the zeros of $\phi(s)$ in $\mathcal{P}$. Λ_T may be referred to as the root range of T over $\mathbb{R}_{\mathcal{P}}(s)$.

Definition (9.5.2) *: Let $T \in \mathbb{R}_{\mathcal{P}}^{2x2}(s)$ be a nondegenerate cyclic matrix and let (n , d) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair . Then (n , d) will be called mode T (mode T^T) $\mathbb{R}_{\mathcal{P}}(s)$ – coprime if the pair $(\tilde{n} , \tilde{d})$ $((\hat{n} , \hat{d}))$ is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime , where :*

$$[\tilde{d} , \tilde{n}] = [\ d , n\ ] \cdot T \ , \quad \begin{bmatrix} \hat{d} \\ \hat{n} \end{bmatrix} = T \cdot \begin{bmatrix} d \\ n \end{bmatrix}$$

□

The set of mode T (mode T^T) $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors is characterized by the following result :

Proposition (9.5.1) *: Let $T \in \mathbb{R}_{\mathcal{P}}^{2x2}(s)$ be a nondegenerate cyclic matrix and Λ_T its root range . An $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair (n , d) is :*

i) *mode T $\mathbb{R}_{\mathcal{P}}(s)$ – coprime if and only if $\forall\ \lambda_i \in \Lambda_T$:*

$$[\ d(\lambda_i) , n(\lambda_i)\ ] \cdot T(\lambda_i) \neq \underline{0}^T \qquad (9.5.2)$$

ii) *mode T^T $\mathbb{R}_{\mathcal{P}}(s)$ – coprime if and only if $\forall\ \lambda_i \in \Lambda_T$*

$$T(\lambda_i)\cdot\begin{bmatrix} d(\lambda_i) \\ n(\lambda_i) \end{bmatrix} \neq \underline{0} \qquad (9.5.3)$$

Proof

i) Since T is nondegenerate its Smith form is $S_T = \text{diag}\{1 \ , \ \phi(s)\}$. If $\phi(s)$ is a unit then $\forall$ (n , d) $\mathbb{R}_{\mathcal{P}}(s)$ – coprime $\Leftrightarrow$ [d , n]$\cdot$T is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and hence , (n , d) is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime . Let now $\phi(s)$ not be a unit . Then :

($\Rightarrow$) Let (n , d) be mode T $\mathbb{R}_{\mathcal{P}}(s)$ – coprime . Then $\forall\ \mu \in \mathcal{P} \Rightarrow [\ \tilde{d}(\mu)\ ,\ \tilde{n}(\mu)\ ] \neq \underline{0}^T \Leftrightarrow$
$\Leftrightarrow [\ d(\mu)\ ,\ n(\mu)\ ]\cdot T(\mu) \neq \underline{0}^T$. Hence , $\forall\ \lambda_i \in \Lambda_T \Rightarrow [\ d(\lambda_i)\ ,\ n(\lambda_i)\ ]\cdot T(\lambda_i)\ \neq \underline{0}^T$.

($\Leftarrow$) Let $\forall\ \lambda_i \in \Lambda_T$, $[\ d(\lambda_i)\ ,\ n(\lambda_i)\ ]\cdot T(\lambda_i)\ \neq \underline{0}^T$. Then $[\ \tilde{d}(\mu)\ ,\ \tilde{n}(\mu)\ ] \neq \underline{0}^T$, $\forall\ \mu \in \mathcal{P}$. (If $[\ \tilde{d}(\mu)\ ,\ \tilde{n}(\mu)\ ] = \underline{0}^T$, for some $\mu \in \mathcal{P} - \Lambda_T$, then $[\ d(\mu)\ ,\ n(\mu)\ ]\cdot T(\mu) = \underline{0}^T$ for some $\mu \in \mathcal{P} - \Lambda_T$. But since $|\ T(\mu)\ | \neq 0$, is implied that $[\ d(\mu)\ ,\ n(\mu)\ ] = 0$, for some $\mu \in \mathcal{P}$, which contradicts the fact that (n , d) is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime) .

ii) Can be proved in a similar way . □

Remark (9.5.1) *: By the proof of proposition(9.5.1) is concluded that when T is complete then all the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs (n , d) are mode T $\mathbb{R}_{\mathcal{P}}(s)$ – coprime .* □

Lemma (9.5.1) *: Let $A \in \mathbb{R}_{\mathcal{P}}^{tx4}(s)$, $t \geq 2$ and the greatest common divisor of the entries of A is an $\mathbb{R}_{\mathcal{P}}(s)$ unit . Then there always exist pairs $(\underline{b}^T\ ,\ \underline{\mu})$, $\underline{b}^T = [\ b_1\ ,\ \dots\ ,\ b_t\ ]$, $\underline{\mu} = [\mu_1\ ,\ \mu_2\ ,\ \mu_3\ ,\ \mu_4\ ]^T$, $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors respectively , such that :*

$$\underline{b}^T\cdot A\cdot\underline{\mu} = 1 \qquad (9.5.4)$$

Proof

By the hypothesis rank{A} can either be 1 , 2 , 3 , 4 . We prove the lemma for *rank*{A}=4 . Then the rest of the cases are direct results of it . The Smith form of A over $\mathbb{R}_{\mathcal{P}}(s)$ can be written as :

$$A = U_l\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & \phi_3 \\ \cdots & \cdots & \cdots & \cdots \\ O & O & O & O \end{bmatrix}\cdot U_r \ , \ U_l \ , \ U_l \text{ are } \mathbb{R}_{\mathcal{P}}(s)\text{ – unimodular}$$

Equation (9.5.4) can be written as : $\underline{b}^T\cdot U_l\cdot[\ \text{diag}\{1\ ,\ \phi_1\ ,\ \phi_2\ ,\ \phi_3\}\ \vdots\ O\ ]^T\cdot U_r\cdot\underline{\mu} = 1$, or ,

$$\underline{c}^T\cdot[\ \text{diag}\{1\ ,\ \phi_1\ ,\ \phi_2\ ,\ \phi_3\}\ \vdots\ O\ ]^T\cdot\underline{\nu} = 1 \qquad (9.5.5)$$

with , $\underline{c}^T = \underline{b}^T \cdot U_l$, $\underline{\nu} = U_r \cdot \underline{\mu}$, $\mathbb{R}_{\mathcal{P}}(s)$ coprime vectors whenever $\underline{b}^T$, $\underline{\mu}$ are $\mathbb{R}_{\mathcal{P}}(s)$ coprime vectors and vice versa . Equation (9.5.4) can now be written as :

$$c_1 \nu_1 + \phi_1 c_2 \nu_2 + \phi_2 c_3 \nu_3 + \phi_3 c_4 \nu_4 = 1 \tag{9.5.6}$$

For each selection of $\{\nu_i , i = 1 , 2 , 3 , 4\}$, $\mathbb{R}_{\mathcal{P}}(s)$ – coprime , such that the set $\{\nu_1 , \phi_1 \nu_2 , \phi_2 \nu_3 , \phi_3 \nu_4\}$ is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime , sets of $\{c_i\ i = 1 , 2 , 3 , 4\}$, $\mathbb{R}_{\mathcal{P}}(s)$ – coprime always exist such that (9.5.6) holds true . This implies that there always exist $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs $(\underline{b}^T , \underline{\mu}) = (\underline{c}^T \cdot U_l^{-1} , U_r^{-1} \cdot \underline{\nu})$, such that equation (9.5.4) holds true. The rest of the cases , namely , $rank\{A\} = 1 , 2 , 3$ can be derived by the previous analysis if we successively set $\phi_i = 0$ in (4.6) , for $i = 3 , \dots ,$ rank{A} . □

Now we can proceed with the parametrization of solutions to DDSP . For technical reasons we consider first the case $m = 2\rho$.

9.5.1. PARAMETRIZATION OF SOLUTIONS OF DDSP – CASE $m = 2\rho$.

As it is implied by the proof of theorem(9.4.1) the solutions of DDSP can be obtained by solving equation :

$$\sum_{\sigma} \alpha_\sigma \lambda_\sigma = 1 , \lambda_\sigma = y_{i_1} y_{i_2} \cdots y_{i_m} , \sigma \in \Gamma_{m,2} \tag{9.5.7}$$

where , $\mathcal{A}'_m = \{\alpha_\sigma \in \mathbb{R}_{\mathcal{P}}(s) , \sigma \in \Gamma_{m,2}\}$ is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime set (corresponds to a strongly cyclic system) . Let $\gamma_{m-\kappa} = (i_1 , \dots , i_{m-\kappa}) \in \Gamma_{m-\kappa,2}$, $\gamma_m = \sigma = (\gamma_{m-\kappa} , i_{m-\kappa+1} , \dots , i_m)$, and y_{i_l} as in (9.4.16) .

Algorithm for the Parametrization of solution of DDSP – Case $m = 2\rho$

Step 1 : Set $i_{m-1} = 2m-3 , 2m-2$, $i_m = 2m-1 , 2m$. Since $\mathcal{A}_m = \{\alpha_\sigma \in \mathbb{R}_{\mathcal{P}}(s) , \sigma \in \Gamma_{m,2}\}$ is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime set is implied that the matrix $A_{m-2} \in \mathbb{R}_{\mathcal{P}}^{tx4}(s)$, $t = 2^{m-2}$:

$$A_{m-2} = [\ \alpha_{\{\gamma_{m-2}, 2m-3, 2m-1\}} , \alpha_{\{\gamma_{m-2}, 2m-3, 2m\}} , \alpha_{\{\gamma_{m-2}, 2m-2, 2m-1\}} , \alpha_{\{\gamma_{m-2}, 2m-2, 2m\}}\ ]$$

has an $\mathbb{R}_{\mathcal{P}}(s)$ unit as a gcd of its entries and thus , by lemma(9.5.1) , we can find all $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\dots , \lambda_{\gamma_{m-2}} , \dots]$, $[\ \mu_1^\rho , \mu_2^\rho , \mu_3^\rho , \mu_4^\rho\ ]^T$ such that :

$$[\ \dots , \lambda_{\gamma_{m-2}} , \dots] \cdot A_{m-2} \cdot [\ \mu_1^\rho , \mu_2^\rho , \mu_3^\rho , \mu_4^\rho\ ]^T = 1 , \gamma_{m-2} \in \Gamma_{m-2,2} \tag{9.5.8}$$

or ,

$$\sum_{\gamma_{m-2}} \lambda_{\gamma_{m-2}} \alpha_{\gamma_{m-2}} = 1 \ , \ \gamma_{m-2} \in \Gamma_{m-2,2} \tag{9.5.9}$$

where ,

$$[\alpha_{\gamma_{m-2}}] = A_{m-2} \cdot [\ \mu_1^{\rho} \ , \ \mu_2^{\rho} \ , \ \mu_3^{\rho} \ , \ \mu_4^{\rho} \ ]^{T} \ , \ \gamma_{m-2} \in \Gamma_{m-2,2}$$

Now set $M_{\gamma_{m-2}}$ to be the matrices :

$$M_{\gamma_{m-2}} = \begin{bmatrix} \alpha_{\{\gamma_{m-2}, 2m-3, 2m-1\}} & \alpha_{\{\gamma_{m-2}, 2m-3, 2m\}} \\ \alpha_{\{\gamma_{m-2}, 2m-2, 2m-1\}} & \alpha_{\{\gamma_{m-2}, 2m-2, 2m\}} \end{bmatrix} , \ \gamma_{m-2} \in \Gamma_{m-2,2} \tag{9.5.10}$$

and T_ρ to be the cyclic matrices :

$$T_\rho = \sum_{\gamma_{m-2}} \lambda_{\gamma_{m-2}} \cdot M_{\gamma_{m-2}} \in \mathbb{R}_{\mathcal{P}}^{2x2}(s) \ , \ \gamma_{m-2} \in \Gamma_{m-2,2} \tag{9.5.11}$$

For each cyclic matrix T_ρ constructed by the above process the family of controllers that stabilize the $m-1$, m channels of the systems is given by the set of solutions of equation :

$$[\ d_{m-1} \ , \ n_{m-1} \ ] \cdot T_\rho \cdot \begin{bmatrix} d_m \\ n_m \end{bmatrix} = 1 \tag{9.5.12}$$

Step 2 : $i_{m-3} = 2m-7$, $2m-6$, $i_{m-2} = 2m-5$, $2m-4$. Since $\mathcal{A}_{m-2} = \{\alpha_{\gamma_{m-2}} \in \mathbb{R}_{\mathcal{P}}(s)$, $\gamma_{m-2} \in \Gamma_{m-2,2}\}$, $\mathbb{R}_{\mathcal{P}}(s)$ – coprime set is implied that the matrix $A_{m-4} \in \mathbb{R}_{\mathcal{P}}^{tx4}(s)$, $t = 2^{m-4}$:

$$A_{m-4} = [\ \alpha_{\{\gamma_{m-4}, 2m-7, 2m-5\}} \ , \ \alpha_{\{\gamma_{m-4}, 2m-7, 2m-4\}} \ , \ \alpha_{\{\gamma_{m-4}, 2m-6, 2m-5\}} \ \alpha_{\{\gamma_{m-4}, 2m-6, 2m-4\}} \ ]$$

has an $\mathbb{R}_{\mathcal{P}}(s)$ unit as a gcd of its entries and thus , by lemma(9.5.1) , we can find all $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\dots , \lambda_{\gamma_{m-4}} , \dots]$, $[\ \mu_1^{\rho-1} \ , \ \mu_2^{\rho-1} \ , \ \mu_3^{\rho-1} \ , \ \mu_4^{\rho-1} \ ]^T$ such that :

$$[\ \dots , \lambda_{\gamma_{m-4}} , \dots \ ] \cdot A_{m-4} \cdot [\ \mu_1^{\rho-1} \ , \ \mu_2^{\rho-1} \ , \ \mu_3^{\rho-1} \ , \ \mu_4^{\rho-1} \ ]^T = 1 \ , \ \gamma_{m-4} \in \Gamma_{m-4,2} \tag{9.5.13}$$

or ,

$$\sum_{\gamma_{m-4}} \lambda_{\gamma_{m-4}} \alpha_{\gamma_{m-4}} = 1 \ , \ \gamma_{m-4} \in \Gamma_{m-4,2} \tag{9.5.14}$$

where ,

$$[\alpha_{\gamma_{m-4}}] = A_{m-4} \cdot [\ \mu_1^{\rho-1} \ , \ \mu_2^{\rho-1} \ , \ \mu_3^{\rho-1} \ , \ \mu_4^{\rho-1} \ ]^T \ , \ \gamma_{m-4} \in \Gamma_{m-4,2}$$

Now set $M_{\gamma_{m-4}}$ to be :

$$M_{\gamma_{m-4}} = \begin{bmatrix} \alpha_{\{\gamma_{m-4}, 2m-7, 2m-5\}} & \alpha_{\{\gamma_{m-4}, 2m-7, 2m-4\}} \\ \alpha_{\{\gamma_{m-4}, 2m-6, 2m-5\}} & \alpha_{\{\gamma_{m-2}, 2m-6, 2m-4\}} \end{bmatrix} , \ \gamma_{m-4} \in \Gamma_{m-4,2} \tag{9.5.15}$$

and $T_{\rho-1}$ to be the cyclic matrices :

$$T_{\rho-1} = \sum_{\gamma_{m-4}} \lambda_{\gamma_{m-4}} \cdot M_{\gamma_{m-4}} \in \mathbb{R}_{\mathcal{P}}^{2x2}(s) \ , \ \gamma_{m-4} \in \Gamma_{m-4,2} \tag{9.5.16}$$

For each cyclic matrix T_ρ constructed by the above process the family of controllers that stabilize the $m-3$, $m-2$ channels of the systems is given by the set of solutions of equation :

$$[\ d_{m-3}\ ,\ n_{m-3}\ ] \cdot T_{\rho-1} \cdot \begin{bmatrix} d_{m-2} \\ n_{m-2} \end{bmatrix} = 1 \tag{9.5.1.17}$$

Repeat the above process successively for all $j = 2\ ,\ 3\ ,\ \dots\ ,\ \rho-1$, $(\kappa = 2\ j)$, $i_{m-\kappa-1} = 2m-2\kappa-3$, $2m-2\kappa-2$, $i_{m-\kappa} = 2m-2\kappa-1$, $2m-2\kappa$. Since , $\mathcal{A}_{m-\kappa} = \{\alpha_{\gamma_{m-\kappa}} \in \mathbb{R}_{\mathcal{P}}(s),$ $\gamma_{m-\kappa} \in \Gamma_{m-\kappa,2}\}$, $\mathbb{R}_{\mathcal{P}}(s)$ – coprime set is implied that the matrix $A_{m-\kappa-2} \in \mathbb{R}_{\mathcal{P}}^{tx4}(s)$, $t = 2^{m-\kappa-2}$:

$$A_{m-\kappa-2} = [\ \alpha_{\{\gamma_{m-\kappa-2},p,q\}}\ ,\ \alpha_{\{\gamma_{m-\kappa-2},p,q+1\}}\ ,\ \alpha_{\{\gamma_{m-\kappa-2},p+1,q\}}\ ,\ \alpha_{\{\gamma_{m-\kappa-2},p+1,q+1\}}\ ]$$

where , $\gamma_{m-\kappa-2} \in \Gamma_{m-\kappa-2,2}$, $p = 2m-2\kappa-3$, $q = 2m-2\kappa-1$, has an $\mathbb{R}_{\mathcal{P}}(s)$ unit as a gcd of its entries and by lemma (9.5.1) we can find all $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\ \dots\ ,\ \lambda_{\gamma_{m-\kappa-2}}\ ,\ \dots\ ]$, $[\ \mu_1^{\rho-j}\ ,\ \dots\ ,\ \mu_4^{\rho-j}\ ]^T$ such that :

$$[\ \dots\ ,\ \lambda_{\gamma_{m-\kappa-2}}\ ,\ \dots\ ] \cdot A_{m-\kappa-2} \cdot [\ \mu_1^{\rho-j}\ ,\ \dots\ ,\ \mu_4^{\rho-j}\ ]^T = 1\ ,\ \gamma_{m-\kappa-2} \in \Gamma_{m-\kappa-2,2} \tag{9.5.18}$$

or ,

$$\sum_{\gamma_{m-\kappa-2}} \lambda_{\gamma_{m-\kappa-2}}\ \alpha_{\gamma_{m-\kappa-2}} = 1\ ,\ \gamma_{m-\kappa-2} \in \Gamma_{m-\kappa-2,2} \tag{9.5.19}$$

where ,

$$[\alpha_{\gamma_{m-\kappa-2}}] = A_{m-\kappa-2} \cdot [\ \mu_1^{\rho-j}\ ,\ \dots\ ,\ \mu_4^{\rho-j}\ ]^T\ ,\ \gamma_{m-\kappa-2} \in \Gamma_{m-\kappa-2,2}$$

Now set $M_{\gamma_{m-\kappa-2}}$ to be :

$$M_{\gamma_{m-\kappa-2}} = \begin{bmatrix} \alpha_{\{\gamma_{m-\kappa-2},2m-2\kappa-3,2m-2\kappa-1\}} & \alpha_{\{\gamma_{m-\kappa-2},2m-2\kappa-3,2m-2\kappa\}} \\ \alpha_{\{\gamma_{m-\kappa-2},2m-2\kappa-2,2m-2\kappa-1\}} & \alpha_{\{\gamma_{m-\kappa-2},2m-2\kappa-2,2m-2\kappa\}} \end{bmatrix}$$

and $T_{\rho-j}$ to be the cyclic matrices :

$$T_{\rho-j} = \sum_{\gamma_{m-\kappa-2}} \lambda_{\gamma_{m-\kappa-2}} \cdot M_{\gamma_{m-\kappa-2}} \in \mathbb{R}_{\mathcal{P}}^{2x2}(s)\ ,\ \gamma_{m-\kappa-2} \in \Gamma_{m-\kappa-2,2} \tag{9.5.20}$$

For each cyclic matrix T_ρ constructed by the above process the family of controllers that stabilize the $m-\kappa-1$, $m-\kappa$ channels of the systems is given by the set of solutions of equation :

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9.5.21)$$

CASE 1 $T_{\rho-j}$ is degenerate : Then by *iii*) of lemma(9.4.1) , (t = 2) , (9.5.21) can be written as :

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \underline{u} \cdot \underline{v}^T \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9.5.22)$$

where , $\underline{u}$, $\underline{v}^T$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors uniquely defined modulo $\mathbb{R}_{\mathcal{P}}(s)$ units .

***Theorem (9.5.1)** : For strongly cyclic systems with $T_{\rho-j}$ degenerate the family of solutions to (9.5.22) is given by the family of solutions to the following scalar Diophantine equations :*

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \underline{u} = 1 \quad , \quad \underline{v}^T \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9.5.23)$$

Proof

Let $(n_{m-2j-1} \,,\, d_{m-2j-1})$, $(n_{m-2j} \,,\, d_{m-2j})$ be a solution of (9.5.22) . Then :

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \underline{u} = p \quad , \quad \underline{v}^T \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = q \qquad (9.5.24)$$

By (9.5.22) we have that $[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \underline{u} \cdot q = 1$ and thus q must be a divisor of 1 or equivalently q is an $\mathbb{R}_{\mathcal{P}}(s)$ unit . On the same token p is an $\mathbb{R}_{\mathcal{P}}(s)$ unit . This proves the necessity . The proof of sufficiency is obvious ; (the solutions of (9.5.23) are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and satisfy (9.5.22)) . □

CASE 2 : $T_{\rho-j}$ is nondegenerate : By making use of definitions (9.5.1) , (9.5.2) , proposition(9.5.1) and remark(9.5.1) we state the following theorem :

***Theorem (9.5.2)** : Let $T_{\rho-j} \in \mathbb{R}_{\mathcal{P}}^{2x2}(s)$ be a cyclic nondegenerate matrix .*

***a)** The following statements are equivalent :*

***i)** An $\mathbb{R}_{\mathcal{P}}(s)$ coprime pair $(n_{m-2j-1} \,,\, d_{m-2j-1})$, $(n_{m-2j} \,,\, d_{m-2j})$ is a solution of (9.5.21)* *ii)* (n_{m-2j-1}, d_{m-2j-1}), (n_{m-2j}, d_{m-2j}) *are mode* $T_{\rho-j}$ *mutually stabilizing pair*

iii) (n_{m-2j-1}, d_{m-2j-1}) *is mode* $T_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$*– coprime and* (n_{m-2j}, d_{m-2j}) *stabilizes* $(\widetilde{n}_{m-2j-1}, \widetilde{d}_{m-2j-1})$. *Equivalently*, (n_{m-2j}, d_{m-2j}) *is mode* $T^{T}_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$*– coprime and* (n_{m-2j-1}, d_{m-2j-1}) *stabilizes* $(\widehat{n}_{m-2j}, \widehat{d}_{m-2j})$.

b) The family of $\mathbb{R}_{\mathcal{P}}(s)$*– coprime pairs* (n_{m-2j-1}, d_{m-2j-1}), (n_{m-2j}, d_{m-2j}) *is defined as follows :*

i) For any (n_{m-2j-1}, d_{m-2j-1}) *mode* $T_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$*– coprime pair a subfamily of* (n_{m-2j}, d_{m-2j}) *that together with* (n_{m-2j-1}, d_{m-2j-1}) *fixed are solutions of (9.5.21) is given by the solutions of :*

$$\widetilde{d}_{m-2j-1}\, d_{m-2j} + \widetilde{n}_{m-2j-1}\, n_{m-2j} = 1\ ,\ [\widetilde{d}_{m-2j-1}\ ,\ \widetilde{n}_{m-2j-1}] = [d_{m-2j-1}\ ,\ n_{m-2j-1}]\cdot T_{\rho-j} \qquad (5.1.25)$$

ii) For any (n_{m-2j}, d_{m-2j}) *mode* $T^{T}_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$*– coprime pair a subfamily of* (n_{m-2j-1}, d_{m-2j-1}) *that together with* (n_{m-2j}, d_{m-2j}) *fixed are solutions of (9.5.21) is given by the solutions of :*

$$\widehat{d}_{m-2j}\, d_{m-2j-1} + \widehat{n}_{m-2j}\, n_{m-2j-1} = 1\ ,\ [\widehat{d}_{m-2j}\ ,\ \widehat{n}_{m-2j}] = [d_{m-2j}\ ,\ n_{m-2j}]\cdot T^{T}_{\rho-j} \qquad (9.5.26)$$

Proof

a) *i)* ⇒ *ii)* By definition(9.5.1) and (9.5.21) (n_{m-2j-1}, d_{m-2j-1}), (n_{m-2j}, d_{m-2j}) are a mode $T_{\rho-j}$ mutual stabilizing pair .

ii)* ⇒ *iii) Consider $(\widetilde{n}_{m-2j-1}, \widetilde{d}_{m-2j-1})$; in order to be $\mathbb{R}_{\mathcal{P}}(s)$– coprime , by proposition(9.5.1) it suffices to show that $[\ \widetilde{d}_{m-2j-1}(s)\ ,\ \widetilde{n}_{m-2j-1}(s)\ ] \neq \underline{0}^{T}$, for all s in the root range of $T_{\rho-j}$. Let an s in the root range of $T_{\rho-j}$ exists such that $[\ \widetilde{d}_{m-2j-1}(s)\ ,\ \widetilde{n}_{m-2j-1}(s)\ ] = \underline{0}^{T}$. Then :

$$[\ d_{m-2j-1}(s)\ ,\ n_{m-2j-1}(s)\ ]\cdot T_{\rho-j}(s)\cdot \begin{bmatrix} d_{m-2j}(s) \\ n_{m-2j}(s) \end{bmatrix} = 0 \neq 1 \qquad (9.5.27)$$

which contradicts the fact that (n_{m-2j-1}, d_{m-2j-1}) , (n_{m-2j}, d_{m-2j}) are a mode $T_{\rho-j}$ mutual stabilizing pair . Hence , $(\widetilde{n}_{m-2j-1}, \widetilde{d}_{m-2j-1})$ is $\mathbb{R}_{\mathcal{P}}(s)$– coprime pair and (n_{m-2j-1}, d_{m-2j-1}) is mode $T_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$– coprime . By (9.5.21) (n_{m-2j}, d_{m-2j}) stabilizes $(\widetilde{n}_{m-2j-1}, \widetilde{d}_{m-2j-1})$.

iii)* ⇒ *i) Consider $\mathbb{R}_{\mathcal{P}}(s)$– coprime pairs (n_{m-2j-1}, d_{m-2j-1}) , (n_{m-2j}, d_{m-2j}) such that (n_{m-2j-1}, d_{m-2j-1}) is mode $T_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$– coprime and (n_{m-2j}, d_{m-2j}) stabilizes $(\widetilde{n}_{m-2j-1}, \widetilde{d}_{m-2j-1})$. Then :

$$\widetilde{d}_{m-2j-1}\, d_{m-2j} + \widetilde{n}_{m-2j-1}\, n_{m-2j} = 1\ ,\ [\widetilde{d}_{m-2j-1}\ ,\ \widetilde{n}_{m-2j-1}] = [d_{m-2j-1}\ ,\ n_{m-2j-1}]\cdot T_{\rho-j}$$

and is obvious that (9.5.21) holds true .

b) ***i)*** By using lemma(9.4.1) , (t = 2) , $A = T^{T}_{\rho-j}$, $[\ d\ ,\ n\ ]^{T} = [\ d_{m-2j-1}\ ,\ n_{m-2j-1}\ ]^{T}$ we can find $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime such that :

$$[\ \tilde{d}_{m-2j-1}\ ,\ \tilde{n}_{m-2j-1}\ ] = [\ d_{m-2j-1}\ ,\ n_{m-2j-1}\ ] \cdot T_{\rho-j}$$

and $(\tilde{n}_{m-2j-1}\ ,\ \tilde{d}_{m-2j-1})$ is $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime . Thus , for each $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ fixed the family of solutions of $\tilde{d}_{m-2j-1}\, d_{m-2j} + \tilde{n}_{m-2j-1}\, n_{m-2j} = 1$ satisfy (9.5.21) . Now we must show that all solutions of (9.5.21) are generated by this process . Let (n_{m-2j-1}, d_{m-2j-1}), $(n_{m-2j}\ ,\ d_{m-2j})$ be a pair satisfying (9.5.21) . Then by a) *iii)* $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ is mode $T_{\rho-j}$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime and $(n_{m-2j}\ ,\ d_{m-2j})$ stabilizes $(\tilde{n}_{m-2j-1}\ ,\ \tilde{d}_{m-2j-1})$. Hence , (9.5.25) holds true .

ii) It can be proved in a similar fashion to *i)* . □

Corollary (9.5.1) : *Consider equation (9.5.21) with $T_{\rho-j}$ cyclic , nondegenerate and $\Lambda_{T_{\rho-j}}$ be the root range of $T_{\rho-j}$.*

a) *If $T_{\rho-j}$ is complete i.e. $\Lambda_{T_{\rho-j}} = \emptyset$, then :*

i) *For any $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime pair $(n_{m-2j-1}\ ,\ d_{m-2j-1})$, (mode $T_{\rho-j}$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime , by remark(9.5.1)) , the family of $(n_{m-2j}\ ,\ d_{m-2j})$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime which together with $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ are solutions of (9.5.21) are given by :*

$$\begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = T^{-1}_{\rho-j} \cdot \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + t \cdot T^{-1}_{\rho-j} \cdot \begin{bmatrix} -n_{m-2j-1} \\ d_{m-2j-1} \end{bmatrix} ,\ t \in \mathbb{R}_{\mathfrak{sp}}(s) \text{ arbitrary} \qquad (9.5.28)$$

where , $(b_1\ ,\ a_1)$ is a SISO plant that stabilizes $(n_{m-2j-1}\ ,\ d_{m-2j-1})$.

ii) *For any $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime pair $(n_{m-2j}\ ,\ d_{m-2j})$, (mode $T^{T}_{\rho-j}$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime , by remark(9.5.1)) , the family of $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime which together with $(n_{m-2j}\ ,\ d_{m-2j})$ are solutions of (9.5.1.24) are given by :*

$$[\ d_{m-2j-1}\ ,\ n_{m-2j-1}\ ] = T^{-1}_{\rho-j} \cdot [\ a_2\ ,\ b_2\ ] + t \cdot T^{-1}_{\rho-j} \cdot [-n_{m-2j}\ ,\ d_{m-2j}\ ] \qquad (9.5.29)$$

where , $t \in \mathbb{R}_{\mathfrak{sp}}(s)$ arbitrary , $(b_2\ ,\ a_2)$ is a SISO plant that stabilizes $(n_{m-2j}\ ,\ d_{m-2j})$.

b) *If $T_{\rho-j}$ is noncomplete i.e. $\Lambda_{T_{\rho-j}} \neq \emptyset$, then :*

i) *For any $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime pair $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ such that :*

$$[\ d_{m-2j-1}(s)\ ,\ n_{m-2j-1}(s)\ ] \cdot T_{\rho-j}(s) \neq \underline{0}^{T}\ ,\ \forall\ s \in \Lambda_{T_{\rho-j}} \qquad (9.5.30)$$

there exists a family of $(n_{m-2j}\ ,\ d_{m-2j})$ $\mathbb{R}_{\mathfrak{sp}}(s)$ – coprime defined by (9.5.25) , which

together with $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ *are solutions of (9.5.21) .*
ii) For any $\mathbb{R}_{\mathcal{P}}(s)$ *– coprime pair* $(n_{m-2j}\ ,\ d_{m-2j})$ *such that :*

$$[\ d_{m-2j}(s)\ ,\ n_{m-2j}(s)\ ]\cdot T^{\mathrm{T}}_{\rho-j}(s) \neq \underline{0}^{\mathrm{T}}\ ,\ \forall\ s \in \Lambda_{T_{\rho-j}} \tag{9.5.31}$$

there exists a family of $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ $\mathbb{R}_{\mathcal{P}}(s)$ *– coprime defined by (9.5.26) , which together with* $(n_{m-2j}\ ,\ d_{m-2j})$ *are solutions of (9.5.21) .* □

It is clear that using the parametrization of solutions of (9.5.21) for all $j = 0\ ,\ 1\ ,\ \dots\ ,\ \rho-1\ ,\ \mathrm{T}_{\rho-j}$, we achieve a parametrization of the solutions to DDSP when $m = 2\ \rho$.

9.5.2. PARAMETRIZATION OF SOLUTIONS OF DDSP – CASE $m = 2\ \rho + 1$

As it is implied by the proof of theorem(9.4.1) the solutions of DDSP can be obtained by solving the equation :

$$\sum_{\sigma} \alpha_\sigma\ \lambda_\sigma = 1\ ,\ \lambda_\sigma = \mathrm{y}_{i_1}\ \mathrm{y}_{i_2} \cdots\ \mathrm{y}_{i_m}\ ,\ \sigma \in \Gamma_{m,2} \tag{9.5.32}$$

where , $\mathcal{A}_m = \{\alpha_\sigma \in \mathbb{R}_{\mathcal{P}}(\mathrm{s})\ ,\ \sigma \in \Gamma_{m,2}\}$ is an $\mathbb{R}_{\mathcal{P}}(\mathrm{s})$ – coprime set (corresponds to a strongly cyclic system) . Following the same steps as in section 9.5.1 , for $j = 0\ ,\ 1\ ,\ \dots\ ,\ \rho-1$ the following set of equations is generated :

$$[\ \mathrm{d}_{m-2j-1}\ ,\ \mathrm{n}_{m-2j-1}\ ]\cdot \mathrm{T}_{\rho-j}\cdot \begin{bmatrix} \mathrm{d}_{m-2j} \\ \mathrm{n}_{m-2j} \end{bmatrix} = 1 \tag{9.5.33}$$

where , $\mathrm{T}_{\rho-j}$ are identical to the ones in (9.5.21) for $j = 0\ ,\ 1\ ,\ \dots\ ,\ \rho-2$, whereas for $j= \rho-1\ ,\ \mathrm{T}_1$ is :

$$\mathrm{T}_1 = \mathrm{d}_1\cdot \begin{bmatrix} \alpha_{135} & \alpha_{136} \\ \alpha_{145} & \alpha_{146} \end{bmatrix} + \mathrm{n}_1\cdot \begin{bmatrix} \alpha_{235} & \alpha_{236} \\ \alpha_{245} & \alpha_{246} \end{bmatrix},\ \text{cyclic} \tag{9.5.34}$$

and $\{\ \alpha_{135}\ ,\ \alpha_{136}\ ,\ \alpha_{145}\ ,\ \alpha_{146}\ ,\ \alpha_{235}\ ,\ \alpha_{236}\ ,\ \alpha_{245}\ ,\ \alpha_{246}\ \}$ is an $\mathbb{R}_{\mathcal{P}}(\mathrm{s})$ – coprime set . There exists $\mathbb{R}_{\mathcal{P}}(\mathrm{s})$ – coprime set $\{\ \mu^1_i\ ,\ i = 1\ ,\ 2\ ,\ 3\ ,\ 4\ \}$ such that :

$$\mathrm{d}_1\{\mu^1_1\ \alpha_{135}+\mu^1_2\ \alpha_{136}+\mu^1_3\ \alpha_{145}+\mu^1_4\ \alpha_{146}\} + \mathrm{n}_1\{\mu^1_1\ \alpha_{235}+\mu^1_2\ \alpha_{236}+\mu^1_3\ \alpha_{245}+\mu^1_4\ \alpha_{246}\} = 1$$

or equivalently :

$$[\ \mathrm{d}_1\ ,\ \mathrm{n}_1\ ]\cdot \mathrm{A}_1\cdot[\ \mu^1_1\ ,\ \mu^1_2\ ,\ \mu^1_3\ ,\ \mu^1_4\ ]^{\mathrm{T}} = 1 \tag{9.5.35}$$

Using lemma(9.5.1) ($A = A_1$, $t = 2$, $\underline{b}^T = [\ d_1\ ,\ n_1\ ]$, $\underline{\mu} = [\ \mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1\ ]^T$) , we can parametrize all the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs (n_1 , d_1) such that equation (9.5.35) holds true . Hence , the parametrization of solutions to DDSP in this case is given by the parametrization of solutions of the following set of equations :

$$[\ d_{m-2j-1}\ ,\ n_{m-2j-1}\ ]\cdot T_{\rho-j}\cdot\begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1\ ,\ j = 0\ ,\ 1\ ,\ \ldots\ ,\ \rho-1$$

$$[\ d_1\ ,\ n_1\ ]\cdot A_1\cdot[\ \mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1\ ]^T = 1 \tag{9.5.36}$$

The parametrization of solutions to DDSP allows the searching for proper , strictly proper, biproper , reliable solutions as well as stable diagonal decentralized controllers . First we study the case of proper solutions to DDSP .

Example (9.5.1) : In the following example we illustrate the parametrization method described above for an unstable strongly cyclic plant $P \in \mathbb{R}_{pr}^{3x3}(s)$. In this case a generator set of DDSP is given by :

$$\mathcal{A} = \{\alpha_{135}\ ,\ \alpha_{136}\ ,\ \alpha_{145}\ ,\ \alpha_{146}\ ,\ \alpha_{235}\ ,\ \alpha_{236}\ ,\ \alpha_{245}\ ,\ \alpha_{246}\}$$

and is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime set . Following the parametrization process introduced in the case $m = 2\,\rho + 1 = 3$, $\rho = 1$, $j = 0$, set M_1 , M_2 to be the matrices :

$$M_1 = \begin{bmatrix} \alpha_{135}\ \alpha_{136} \\ \alpha_{145}\ \alpha_{146} \end{bmatrix},\ M_2 = \begin{bmatrix} \alpha_{235}\ \alpha_{236} \\ \alpha_{245}\ \alpha_{246} \end{bmatrix},\ A_1 = \begin{bmatrix} \alpha_{135}\ \alpha_{136}\ \alpha_{235}\ \alpha_{236} \\ \alpha_{145}\ \alpha_{146}\ \alpha_{245}\ \alpha_{246} \end{bmatrix}$$

Applying lemma(9.5.1) for $t = 2$, $A = A_1$, we construct $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors $[\lambda_1$, $\lambda_2]$, $[\mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1]^T$ and the family of cyclic matrices :

$$T_1 = \sum_{\gamma_1} \lambda_{\gamma_1}\cdot M_{\gamma_1}\ ,\ \gamma_1 \in (1\ ,\ 2)$$

The family of diagonal decentralized stabilizing controllers (n_1 , d_1) , (n_2 , d_2) , (n_3 , d_3) for the channels 1 , 2 , 3 respectively are given by the families of coprime solutions of the set of equations :

$$[\ d_2\ ,\ n_2\ ]\cdot T_1\cdot\begin{bmatrix} d_3 \\ n_3 \end{bmatrix} = 1$$

$$[\ d_1\ ,\ n_1\ ]\cdot A_1\cdot[\mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1]^T = 1$$

□

9.6. PROPER SOLUTIONS OF THE DDSP

The searching for proper solutions to DDSP can be restricted to the searching of $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs (n_i , d_i) , $i = 1 , \dots , m$ such that the corresponding stabilizing controller for the channel i is $c_i = n_i \cdot d_i^{-1}$, proper . In other words :

a) When $m = 2\rho$, we can search for $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs $(n_{m-2j-1} , d_{m-2j-1})$, (n_{m-2j} , d_{m-2j}) , $j = 0 , 1 , \dots , \rho - 1$, which are solutions of (9.5.21) and $c_{m-2j-1} = n_{m-2j-1} \cdot d^{-1}_{m-2j-1}$, $c_{m-2j} = n_{m-2j} \cdot d^{-1}_{m-2j}$ the stabilizing controllers for channels $m - 2j - 1$, $m - 2j$ are proper .

b) When $m = 2\rho + 1$, we can search for $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs $(n_{m-2j-1} , d_{m-2j-1})$, (n_{m-2j} , d_{m-2j}) , $j = 0 , 1 , \dots , \rho - 1$, which are solutions of (9.5.32) and $c_{m-2j-1} = n_{m-2j-1} \cdot d^{-1}_{m-2j-1}$, $c_{m-2j} = n_{m-2j} \cdot d^{-1}_{m-2j}$ the stabilizing controllers for channels $m - 2j - 1$, $m - 2j$ are proper.

Cases a) , b) reveal an intimate relation to results concerning the properness of solutions to scalar Diophantine equations over $\mathbb{R}_{\mathcal{P}}(s)$.

9.6.1. PROPERNESS OF SOLUTIONS OF SCALAR DIOPHANTINE EQUATIONS

Let (b , a) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair . The pair (b , a) will be called proper nonproper , strictly proper , if the transfer function $p = b \cdot a^{-1}$ is respectively proper , nonproper , strictly proper . For the general given pair (coprime) we define the scalar Diophantine equation :

$$b\,n + a\,d = 1 \tag{9.6.1}$$

where , the solution (n , d) over $\mathbb{R}_{\mathcal{P}}(s)$ always exists because of the $\mathbb{R}_{\mathcal{P}}(s)$ – coprimeness of (b , a) . The solution pairs (n , d) are always $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and if (n_0 , d_0) is a particular solution then the general solution is expressed by :

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ -b \end{bmatrix} , \; t \in \mathbb{R}_{\mathcal{P}}(s) \text{ , arbitrary} \tag{9.6.2}$$

In the study of DDSP , Diophantine equations of the type (9.6.1) always emerge , where (b , a) is not necessarily proper ; however , since (n , d) represents controllers the question of properness is always an important aspect to be examined .

***Definition (9.6.1)** : A pair {(b , a) , (n , d)} that satisfies (9.6.1) will be referred to as mutually stabilizing pair ; in particular (n , d) , (or (b , a)) will be called dual of (b , a), (or (n , d)) .* □

The existence of proper dual pairs for a given (b , a) is examined next . The following result establishes a useful general property of mutually stabilizing pairs .

Lemma (9.6.1) *: Let (b , a) , (n , d) be $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and mutually stabilizing pairs . Then :*

$$\min \{ \delta_\infty(b) + \delta_\infty(n) , \delta_\infty(a) + \delta_\infty(d) \} = 0 \qquad (9.6.3)$$

Proof

Since b n + a d = 1 by taking valuations we have : $\delta_\infty(\mathrm{b\,n + a\,d}) = 0$. By the properties of $\delta_\infty(\cdot)$ valuation it follows that :

$$0 = \delta_\infty(\mathrm{b\,n + a\,d}) \geq \min \{ \delta_\infty(\mathrm{b\,n}) , \delta_\infty(\mathrm{a\,d}) \} =$$

$$= \min \{\delta_\infty(\mathrm{b}) + \delta_\infty(\mathrm{n}) , \delta_\infty(\mathrm{a}) + \delta_\infty(\mathrm{d})\} \geq 0$$

Since (b , a) , (n , d) are from $\mathbb{R}_{\mathcal{P}}(s)$ and thus have nonnegative valuation . The last condition clearly implies (9.6.3) . □

Remark (9.6.1) *: Let (b , a) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair . Then the following three cases concerning (b , a) properness are the only possible :*

i) *(b , a) is nonproper .*

ii) *(b , a) is strictly proper .*

iii) *(b , a) is biproper .* □

Using lemma (9.6.1) for the case of nonproper pairs (b , a) we have .

Proposition (9.6.1) *: Let (b , a) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime nonproper pair . Then :*

i) *For all (n , d) dual pairs , $\delta_\infty(n) = 0$.*

ii) *If a proper dual exists , it has to be biproper .*

iii) *There always exists a family of biproper duals (n , d) .*

Proof

i) Since (b , a) is coprime and nonproper , it follows that $\delta_\infty(\mathrm{b}) = 0$, $\delta_\infty(\mathrm{a}) = \epsilon > 0$. Thus by condition (9.6.3) we have $\min \{ \delta_\infty(\mathrm{n}) , \epsilon + \delta_\infty(\mathrm{d}) \} = 0$. Clearly , since $\epsilon > 0$ $\Rightarrow \delta_\infty(\mathrm{d}) \geq 0$ follows that $\delta_\infty(\mathrm{n}) = 0$.

ii) Since , for all duals (n , d) , $\delta_\infty(\mathrm{n}) = 0$, if a proper dual exists , we must have :

$$0 \leq \delta_\infty(\mathrm{c}) = \delta_\infty(\mathrm{n \cdot d^{-1}}) = \delta_\infty(\mathrm{n}) - \delta_\infty(\mathrm{d}) = 0 - \delta_\infty(\mathrm{d}) = -\delta_\infty(\mathrm{d})$$

and thus $\delta_\infty(d) = 0$. Thus if a proper dual exists it must be biproper .
iii) Consider the family of duals as defined by (9.6.2) . At $s = \infty$ we have :

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty \\ d_0^\infty \end{bmatrix} + t_\infty \cdot \begin{bmatrix} a_\infty \\ -b_\infty \end{bmatrix}, \; t_\infty = t(\infty) \;,\; t \in \mathbb{R}_{\mathfrak{P}}(s) \;,\; \text{arbitrary}$$

where , $\delta_\infty(b) = 0$, $\delta_\infty(a) > 0$. Then it follows that $b_\infty = \beta \neq 0$, and $a_\infty = 0$. Furthermore by part *ii)* $n_0^\infty = \alpha \neq 0$ and thus the above may be written as

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} \alpha \\ d_0^\infty - t_\infty \beta \end{bmatrix}, \; t_\infty = t(\infty) \;,\; t \in \mathbb{R}_{\mathfrak{P}}(s) \;,\; \text{arbitrary} \qquad (9.6.4)$$

We may distinguish the following cases :
a) Particular solution (n_0 , d_0) is nonproper .
b) Particular solution (n_0 , d_0) is biproper .

a) If particular solution is nonproper , then $\delta_\infty(d_0) > 0$ and thus $d_\infty^0 = 0$. By (9.6.4) we have :

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} \alpha \\ -t_\infty \beta \end{bmatrix}, \; t_\infty = t(\infty) \;,\; t \in \mathbb{R}_{\mathfrak{P}}(s) \;,\; \text{arbitrary} \qquad (9.6.5)$$

and thus for any biproper $t \in \mathbb{R}_{\mathfrak{P}}(s)$, i.e. $\delta_\infty(t) = 0$, $d_\infty \neq 0$ and the corresponding d has $\delta_\infty(d) = 0$, i.e. there exist biproper duals for all biproper parameters $t \in \mathbb{R}_{\mathfrak{P}}(s)$.
b) If particular solution is biproper , then $\delta_\infty(d_0) = 0$, and $d_0^\infty = \gamma \neq 0$. By (9.6.1.4) we have :

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma - t_\infty \beta \end{bmatrix}, \; t_\infty = t(\infty) \;,\; t \in \mathbb{R}_{\mathfrak{P}}(s) \;,\; \text{arbitrary} \qquad (9.6.6)$$

Clearly for all $t \in \mathbb{R}_{\mathfrak{P}}(s)$ parameters such that :

$$\gamma - t_\infty \beta \neq 0 \qquad (9.6.7)$$

$d(\infty) \neq 0$ and $\delta_\infty(d) = 0$, i.e. solution (n , d) is biproper . □

An important remark that follows immediately from the above proof is stated next .

Remark (9.6.2) *: If (b , a) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime nonproper pair , then there exist non strictly proper duals .* □

Corollary (9.6.1) *: Let (b , a) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime nonproper pair .*

a) *There always exists a biproper dual (n_0 , d_0) .*

b) *Let $b_\infty = \beta \neq 0$, $n_0^\infty = \alpha \neq 0$, $d_0^\infty = \gamma \neq 0$. The family of biproper duals is defined by :*

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ -b \end{bmatrix}, \; t \in \mathbb{R}_{\mathcal{P}}(s) \text{ , arbitrary} \tag{9.6.8}$$

where , t is constrained by the condition :

$$\gamma - t_\infty \beta \neq 0 \text{ , } t_\infty = t(\infty) \text{ , } t \in \mathbb{R}_{\mathcal{P}}(s) \text{ , arbitrary} \tag{9.6.9}$$

□

Remark (9.6.3) *: The duals of $\mathbb{R}_{\mathcal{P}}(s)$ – coprime nonproper pairs (b , a) are generically biproper . Indeed , $t(\infty) = t_\infty \in \mathbb{R}$. Those , $t_\infty = (\gamma/\beta)$ form a hyperplane (the set $\{\gamma/\beta\}$) of the line $\mathbb{R}$. Thus the set $\mathcal{T} = \{ t_\infty \in \mathbb{R} : t_\infty = (\gamma/\beta) \}$ is of measure zero , which implies that generically each $t \in \mathbb{R}_{\mathcal{P}}(s)$ has $t_\infty \neq (\gamma/\beta)$.* □

The case of strictly proper pairs (b , a) is considered next .

Proposition (9.6.2) *: Let (b , a) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime strictly proper pair . Then all duals (n , d) are proper .*

Proof

Since (b , a) is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime strictly proper pair it follows that $\delta_\infty(b) = \epsilon > 0$, $\delta_\infty(a) = 0$. By condition (9.6.3) (necessary condition which all duals must satisfy) we have $min\{ \epsilon + \delta_\infty(n) , \delta_\infty(d) \} = 0$. Clearly , since $\epsilon > 0$, $\delta_\infty(n) \geq 0$ follows that $\delta_\infty(d) = 0$, i.e. all duals have d biproper and thus they are proper . □

The case of (b , a) biproper pairs is considered next .

Proposition (9.6.3) *: Let (b , a) be an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime biproper pair ($b_\infty \neq 0$, $a_\infty \neq 0$)*

a) ***There always exists a family of biproper duals and a family of strictly proper duals .***

b) ***Let (n_0 , d_0) be a biproper dual :***

i) ***The family of biproper duals is defined by :***

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ -b \end{bmatrix}, \ t \in \mathbb{R}_{\mathfrak{P}}(s) \ , \ arbitrary \qquad (9.6.10)$$

where , t is constrained by the condition :

$$d_0^\infty - t_\infty \ b_\infty \neq 0 \ , \ n_0^\infty + t_\infty \ a_\infty \neq 0 \ , \ t_\infty = t(\infty) \ , \ t \in \mathbb{R}_{\mathfrak{P}}(s) \ , \ arbitrary \qquad (9.6.11)$$

ii) The family of strictly proper duals is defined by :

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ -b \end{bmatrix}, \ t \in \mathbb{R}_{\mathfrak{P}}(s) \ , \ arbitrary \qquad (9.6.12)$$

where , t is constrained by the condition :

$$d_0^\infty - t_\infty \ b_\infty \neq 0 \ , \ n_0^\infty + t_\infty \ a_\infty = 0 \ , \ t_\infty = t(\infty) \ , \ t \in \mathbb{R}_{\mathfrak{P}}(s) \ , \ arbitrary \qquad (9.6.13)$$

Proof

a) The general family of duals is given by :

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ -b \end{bmatrix}, \ t \in \mathbb{R}_{\mathfrak{P}}(s) \ , \ \text{arbitrary}$$

where from the coprimeness of every dual it follows that $(n_0 \ , \ d_0)$ may have one of the following properties :

1) $\delta_\infty(n_0) = 0 \ , \ \delta_\infty(d_0) > 0$: nonproper dual .
2) $\delta_\infty(n_0) > 0 \ , \ \delta_\infty(d_0) = 0$: strictly proper dual .
3) $\delta_\infty(n_0) = 0 \ , \ \delta_\infty(d_0) = 0$: biproper dual .

1) If $(n_0 \ , \ d_0)$ is nonproper dual then $n_0^\infty \neq 0$ and $d_0^\infty = 0$ and thus :

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty \\ d_0^\infty \end{bmatrix} + t_\infty \cdot \begin{bmatrix} a_\infty \\ -b_\infty \end{bmatrix}, \ t_\infty = t(\infty) \ , \ t \in \mathbf{R}_{\mathfrak{P}}(s) \ , \ \text{arbitrary}$$

or ,

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty + t_\infty \ a_\infty \\ d_0^\infty - t_\infty \ b_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty + t_\infty \ a_\infty \\ -t_\infty \ b_\infty \end{bmatrix}, \ t_\infty = t(\infty) \ , \ t \in \mathbf{R}_{\mathfrak{P}}(s) \ , \ \text{arbitrary} \qquad (9.6.14)$$

by selecting $t \in \mathbb{R}_{\mathcal{P}}(s)$ such that $n_0^\infty + t_\infty a_\infty \neq 0$, $t_\infty \neq 0$ then a biproper solution is defined . If $t \in \mathbb{R}_{\mathcal{P}}(s)$ is constrained by the condition $n_0^\infty + t_\infty a_\infty = 0$, then $n_\infty = 0$, $d_\infty \neq 0$ and a strictly proper solution exists .

2) If (n_0 , d_0) is strictly proper dual then at least a strictly proper solution exists . Further more $n_0^\infty = 0$ and $d_0^\infty \neq 0$ and thus :

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty + t_\infty a_\infty \\ d_0^\infty - t_\infty b_\infty \end{bmatrix} = \begin{bmatrix} t_\infty a_\infty \\ d_0^\infty - t_\infty b_\infty \end{bmatrix}, \; t_\infty = t(\infty) \; , \; t \in \mathbb{R}_{\mathcal{P}}(s) \; , \; \text{arbitrary} \qquad (9.6.15)$$

by selecting $t \in \mathbb{R}_{\mathcal{P}}(s)$ such that $d_0^\infty - t_\infty b_\infty \neq 0$, $t_\infty \neq 0$ then a biproper solution is defined .

3) If (n_0 , d_0) is biproper dual then at least a biproper solution exists . Further more $n_0^\infty \neq 0$ and $d_0^\infty \neq 0$ and thus :

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty + t_\infty a_\infty \\ d_0^\infty - t_\infty b_\infty \end{bmatrix}, \; t_\infty = t(\infty) \; , \; t \in \mathbb{R}_{\mathcal{P}}(s) \; , \; \text{arbitrary} \qquad (9.6.16)$$

by selecting $t \in \mathbb{R}_{\mathcal{P}}(s)$ such that $n_0^\infty + t_\infty a_\infty = 0$, then a strictly proper solution is defined (since , $n_\infty = 0$ and $d_\infty = ((d_0^\infty a_\infty + n_0^\infty b_\infty)/ a_\infty) = (1/ a_\infty) \neq 0)$.

b) The analysis of the above cases demonstrates that there always exists a biproper dual say (n_0 , d_0) . Using this , the whole family of duals is given by :

$$\begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} n_0 \\ d_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ -b \end{bmatrix}, \; t \in \mathbb{R}_{\mathcal{P}}(s) \; , \; \text{arbitrary}$$

At $s = \infty$, the above yields

$$\begin{bmatrix} n_\infty \\ d_\infty \end{bmatrix} = \begin{bmatrix} n_0^\infty \\ d_0^\infty \end{bmatrix} + t_\infty \cdot \begin{bmatrix} a_\infty \\ -b_\infty \end{bmatrix}, \; t_\infty = t(\infty) \; , \; t \in \mathbb{R}_{\mathcal{P}}(s) \; , \; \text{arbitrary}$$

where , n_0^∞ , d_0^∞ , a_∞ , b_∞ are nonzero . By restricting the parameters $t \in \mathbb{R}_{\mathcal{P}}(s)$, such that $n_0^\infty + t_\infty a_\infty \neq 0$, $d_0^\infty - t_\infty b_\infty \neq 0$, n_∞ , d_∞ becomes nonzero and (n , d) is biproper. This proves part *i*) . Part *ii*) follows along similar lines . □

Corollary (9.6.2) : Let $(b_\infty \neq 0 , a_\infty \neq 0)$. Then starting from a biproper dual (n_0 , d_0):

a) The condition for existence of nonproper duals is :

$$d_0^{\infty} - t_{\infty}\, b_{\infty} = 0\ ,\ t_{\infty} = t(\infty)\ ,\ t \in \mathbb{R}_{\mathcal{P}}(s)\ ,\ arbitrary \qquad (9.6.17)$$

b) The condition for existence of strictly proper duals is :

$$n_0^{\infty} + t_{\infty}\, a_{\infty} = 0\ ,\ t_{\infty} = t(\infty)\ ,\ t \in \mathbb{R}_{\mathcal{P}}(s)\ ,\ arbitrary \qquad (9.6.18)$$

Proof

a) By proposition(9.6.3) part b) , *i)* , *ii)* the conditions for existence of nonproper duals is :

$$d_0^{\infty} - t_{\infty}\, b_{\infty} = 0\ ,\ n_0^{\infty} + t_{\infty}\, a_{\infty} \neq 0\ ,\ t_{\infty} = t(\infty)\ ,\ t \in \mathbb{R}_{\mathcal{P}}(s)\ ,\ \text{arbitrary} \qquad (9.6.19)$$

But whenever $d_0^{\infty} - t_{\infty}\, b_{\infty} = 0$ is implied that $((n_0^{\infty}\, b_{\infty} + d_0^{\infty}\, a_{\infty})/b_{\infty}) = (1/b_{\infty}) \neq 0$ or equivalently $n_0^{\infty} + t_{\infty}\, a_{\infty} \neq 0$ and thus we may omit the second equation of (9.6.19) .

b) By proposition(9.6.3) part b) , *ii)* the conditions for existence of strictly proper duals is :

$$d_0^{\infty} - t_{\infty}\, b_{\infty} \neq 0\ ,\ n_0^{\infty} + t_{\infty}\, a_{\infty} = 0\ ,\ t_{\infty} = t(\infty)\ ,\ t \in \mathbb{R}_{\mathcal{P}}(s)\ ,\ \text{arbitrary} \qquad (9.6.20)$$

But whenever $n_0^{\infty} + t_{\infty}\, a_{\infty} = 0$ is implied that $((d_0^{\infty}\, a_{\infty} + n_0^{\infty}\, b_{\infty})/a_{\infty}) = (1/a_{\infty}) \neq 0$ or equivalently $d_0^{\infty} - t_{\infty}\, b_{\infty} \neq 0$ and thus we may omit the first equation of (9.6.20) . □

***Remark (9.6.4)** : The duals of an $(b\ ,\ a)$ $\mathbb{R}_{\mathcal{P}}(s)$ – coprime biproper pair are generically biproper . The existence of nonproper and strictly proper duals is nongeneric . The proof or this result follows along similar lines to the proof of remark(9.6.3) .* □

The above results are used next for the study of proper diagonal decentralized stabilizing controllers .

9.6.2. PARAMETRIZATION OF PROPER SOLUTIONS OF DDSP

The study of proper diagonal decentralized stabilizing controllers is equivalent to the study of proper $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs $(n_{m-2j-1}\ ,\ d_{m-2j-1})$, $(n_{m-2j}\ ,\ d_{m-2j})$ such that when $m = 2\,\rho$, the set of equations (9.5.21) holds true , $j = 0\ ,1\ ,\dots,\ \rho - 1$, whereas when $m = 2\,\rho + 1$ the set of equations (9.5.32) holds true , $j = 0\ ,1\ ,\dots,\ \rho$.

Parametrization of Proper Solutions of DDSP – Case $m = 2\rho$

Fix a j and a $T_{\rho-j}$.

i) If $T_{\rho-j}$ is degenerate then by theorem(9.5.1) the family of $\mathbb{R}_{\mathcal{P}}(s)$ – coprime solutions of:

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9.6.21)$$

is given by the family of solutions to the scalar Diophantine equations (9.5.23) :

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \underline{u} = 1 \quad , \quad [\, d_{m-2j} \,,\, n_{m-2j} \,] \cdot \underline{v} = 1 \qquad (9.6.22)$$

where , $\underline{u} = [\, u_{11} \,,\, u_{21} \,]^T$, $\underline{v}^T = [\, v_{11} \,,\, v_{12} \,]$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors uniquely defined modulo $\mathbb{R}_{\mathcal{P}}(s)$ units . By making use of the results of section 9.6.1 we can distinguish the following cases :

1) $(u_{21} \,,\, u_{11})$, $(v_{12} \,,\, v_{11})$ are nonproper . Then the duals $(n_{m-2j-1} \,,\, d_{m-2j-1})$, $(n_{m-2j} \,,\, d_{m-2j})$ satisfying (9.6.22) are generically biproper . The family of biproper duals of (9.6.22) is given by (9.6.8) .

2) $(u_{21} \,,\, u_{11})$ is nonproper , $(v_{12} \,,\, v_{11})$ is strictly proper . Then the duals $(n_{m-2j-1} \,,\, d_{m-2j-1})$ satisfying (9.6.22) are generically biproper . Their family is given by (9.6.8) . The duals $(n_{m-2j} \,,\, d_{m-2j})$ satisfying (9.6.22) are always proper . Their family is given by (9.6.2) .

3) $(u_{21} \,,\, u_{11})$ is strictly proper , $(v_{12} \,,\, v_{11})$ is nonproper proper . This is dual to case 2) .

4) $(u_{21} \,,\, u_{11})$ is nonproper , $(v_{12} \,,\, v_{11})$ is biproper . Then the duals $(n_{m-2j-1} \,,\, d_{m-2j-1})$ satisfying (9.6.22) are generically biproper . Their family is given by (9.6.8) . Whereas biproper (generically) and strictly proper (nongenerically) families of duals $(n_{m-2j} \,,\, d_{m-2j})$ satisfying (9.6.22) exist ; given by (9.6.10) and (9.6.12) respectively .

5) $(u_{21} \,,\, u_{11})$ is biproper , $(v_{12} \,,\, v_{11})$ is nonproper . This is dual to case 4) .

6) $(u_{21} \,,\, u_{11})$ is biproper , $(v_{12} \,,\, v_{11})$ is strictly proper . Then biproper (generically) and strictly proper (nongenerically) families of duals $(n_{m-2j-1} \,,\, d_{m-2j-1})$ satisfying (9.6.22) exist ; given by (9.6.10) and (9.6.12) respectively . Whereas the duals $(n_{m-2j} \,,\, d_{m-2j})$ satisfying (9.6.22) are always proper . Their family is given by (9.6.2) .

7) (u_{21}, u_{11}) is biproper , (v_{12}, v_{11}) is strictly proper . This is dual to case 6) .

8) (u_{21}, u_{11}) , (v_{12}, v_{11}) are strictly proper . Then the duals (n_{m-2j-1}, d_{m-2j-1}) , (n_{m-2j}, d_{m-2j}) satisfying (9.6.22) are always proper . Their family is given by (9.6.2) .

9) (u_{21}, u_{11}) , (v_{12}, v_{11}) are biproper . Then biproper (generically) and strictly proper (nongenerically) families of duals (n_{m-2j-1}, d_{m-2j-1}) , (n_{m-2j}, d_{m-2j}) satisfying (9.6.22) exist ; given by (9.6.10) and (9.6.12) respectively .

ii) If $T_{\rho-j}$ is nondegenerate then by theorem(9.5.2) the solutions of (9.6.21) are mode $T_{\rho-j}$ mutually stabilizing pairs . In other words for each stabilizing controller for the fixed channel $m-2j-1$ defined by (n_{m-2j-1}, d_{m-2j-1}) there exists a subfamily of stabilizing controllers for the channel $m-2j$ defined by (n_{m-2j}, d_{m-2j}) such that (9.6.21) holds true . A realizable controller for the fixed channel $m-2j-1$ is ensured if $c_{m-2j-1} = n_{m-2j-1} \cdot d^{-1}_{m-2j-1}$, $\delta_\infty(d_{m-2j-1}) \leq \delta_\infty(n_{m-2j-1})$ or , if $\delta_\infty(d_{m-2j-1}) = 0$ and either $\delta_\infty(n_{m-2j-1}) = 0$ or > 0 .

Consider channel $m-2j-1$ fixed ; select a realizable controller c_{m-2j-1} defined by (n_{m-2j-1}, d_{m-2j-1}) . This can be achieved as follows . By theorem(9.5.2) the stabilizing controllers for the channel $m-2j-1$ are mode $T_{\rho-j}$ $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs and can be found by solving equation :

$$[\, d_{m-2j-1} , n_{m-2j-1} \,] \cdot T_{\rho-j} = [\, u_1 , u_2 \,] \tag{9.6.23}$$

where , $\underline{u}^T := [\, u_1 , u_2 \,]$ is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector . Equation (9.6.23) can be viewed as similar to the one of lemma(9.4.1) , where $t = 2$, $A = T^T_{\rho-j}$, $\underline{r} = [\, u_1 , u_2 \,]^T$. Using the results of *i*) , *ii*) of lemma(9.4.1) we take that in order $\delta_\infty(d_{m-2j-1}) = 0$ we must add to the parametrization constraints concerning the selection of (n_{m-2j-1}, d_{m-2j-1}) that d_{m-2j-1} is arbitrarily selected to have $\delta_\infty(d_{m-2j-1}) = 0$.

The pair $(\tilde{n}_{m-2j-1}, \tilde{d}_{m-2j-1})$ defined by $[\, \tilde{d}_{m-2j-1} , \tilde{n}_{m-2j-1}] = [\, d_{m-2j-1} , n_{m-2j-1} \,] \cdot T_{\rho-j}$ will be called nonproper , proper , or strictly proper if its respective transfer function $\tilde{c}_{m-2j-1} = \tilde{n}_{m-2j-1} \cdot \tilde{d}^{-1}_{m-2j-1}$ is so defined . There are three cases which may be distinguished :

1) $(\tilde{n}_{m-2j-1}, \tilde{d}_{m-2j-1})$ is nonproper . If the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime plant $(\tilde{n}_{m-2j-1}, \tilde{d}_{m-2j-1})$ is nonproper , i.e. (n_{m-2j-1}, d_{m-2j-1}) selected to be realizable generates $(\tilde{n}_{m-2j-1}, \tilde{d}_{m-2j-1})$ nonproper then by proposition(9.6.1) there exists no strictly solution to :

$$[\, d_{m-2j-1} , n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = \tilde{d}_{m-2j-1}\, d_{m-2j} + \tilde{n}_{m-2j-1}\, n_{m-2j} = 1$$

If a solution exists then generically it will be biproper . The family of biproper solutions is defined by :

$$\begin{bmatrix} n_{m-2j} \\ d_{m-2j} \end{bmatrix} = \begin{bmatrix} n^0_{m-2j} \\ d^0_{m-2j} \end{bmatrix} + t \cdot \begin{bmatrix} \widetilde{d}_{m-2j-1} \\ -\widetilde{n}_{m-2j-1} \end{bmatrix}, \; t \in \mathbb{R}_{\mathcal{P}}(s) \text{ , arbitrary}$$

and $t \in \mathbb{R}_{\mathcal{P}}(s)$ is constrained such that $d^0_{m-2j}(\infty) - t(\infty)\, \widetilde{n}_{m-2j-1}(\infty) \neq 0$. Hence :
$(n_{m-2j-1} , d_{m-2j-1})$ realizable $\Rightarrow$ $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ nonproper $\Rightarrow$ (n_{m-2j} , d_{m-2j}) biproper (generically) and realizable .

2) $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ is strictly proper . If the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime plant $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ is strictly proper , i.e. $(n_{m-2j-1} , d_{m-2j-1})$ selected to be realizable generates $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ strictly proper then by proposition(9.6.2) all the solutions of :

$$[\, d_{m-2j-1} , n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = \widetilde{d}_{m-2j-1}\, d_{m-2j} + \widetilde{n}_{m-2j-1}\, n_{m-2j} = 1$$

are proper . Hence :
$(n_{m-2j-1} , d_{m-2j-1})$ realizable $\Rightarrow$ $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ strictly proper $\Rightarrow$ (n_{m-2j} , d_{m-2j}) proper and realizable .

3) $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ is biproper . If the $\mathbb{R}_{\mathcal{P}}(s)$ – coprime plant $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ is biproper , i.e. $(n_{m-2j-1} , d_{m-2j-1})$ selected to be realizable generates $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ biproper then by proposition(9.6.3) the solutions of :

$$[\, d_{m-2j-1} , n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = \widetilde{d}_{m-2j-1}\, d_{m-2j} + \widetilde{n}_{m-2j-1}\, n_{m-2j} = 1$$

are generically biproper and nongenerically strictly proper . The family of biproper solutions is defined by :

$$\begin{bmatrix} n_{m-2j} \\ d_{m-2j} \end{bmatrix} = \begin{bmatrix} n^0_{m-2j} \\ d^0_{m-2j} \end{bmatrix} + t \cdot \begin{bmatrix} \widetilde{d}_{m-2j-1} \\ -\widetilde{n}_{m-2j-1} \end{bmatrix}, \; t \in \mathbb{R}_{\mathcal{P}}(s) \text{ , arbitrary}$$

and $t \in \mathbb{R}_{\mathcal{P}}(s)$ is constrained such that $d^0_{m-2j}(\infty) - t(\infty)\, \widetilde{n}_{m-2j-1}(\infty) \neq 0$, $n^0_{m-2j}(\infty) + t(\infty)\, \widetilde{d}_{m-2j-1}(\infty) \neq 0$. Hence :
$(n_{m-2j-1} , d_{m-2j-1})$ realizable $\Rightarrow$ $(\widetilde{n}_{m-2j-1} , \widetilde{d}_{m-2j-1})$ biproper $\Rightarrow$ (n_{m-2j} , d_{m-2j}) biproper

(generically) strictly proper (nongenerically) and realizable .

Parametrization of Proper Solutions of DDSP – Case $m = 2\rho + 1$

Fix a j and a $T_{\rho-j}$. The searching for proper solutions to DDSP (n_{m-2j-1} , d_{m-2j-1}) , (n_{m-2j} , d_{m-2j}) when $m = 2\rho + 1$ is identical to the previous case $m = 2\rho$, for $j = 0$, ... , $\rho - 1$, apart from the fact that now we have to investigate equation (9.5.35) :

$$[\, d_1 , n_1 \,] \cdot A_1 \cdot [\, \mu_1^1 , \mu_2^1 , \mu_3^1 , \mu_4^1 \,]^T = 1 \qquad (9.6.24)$$

for proper stabilizing controllers defined by (n_1 , d_1) for the channel 1 . By the process of creating equation (9.5.35) (and thus (9.6.24)) A_1^T is cyclic , (n_1 , d_1) , $\underline{\mu} = [\, \mu_1^1 , \mu_2^1 , \mu_3^1 , \mu_4^1 \,]^T$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors . We can distinguish two cases :

1) A_1 is degenerate . Then $A_1 = \underline{u} \cdot \underline{v}^T$, where , $\underline{u}$, $\underline{v}$ are minimal Mc Millan degree bases for the column [$\mathbb{R}_{\mathcal{P}}(s)$ – module of A_1] , row [$\mathbb{R}_{\mathcal{P}}(s)$ – module of A_1] , respectively. Hence , $\underline{u}$, $\underline{v}$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors unique [modulo $\mathbb{R}_{\mathcal{P}}(s)$ units] . Then equation (9.6.24) becomes :

$$[\, d_1 , n_1 \,] \cdot \underline{u} \cdot \underline{v}^T \cdot [\, \mu_1^1 , \mu_2^1 , \mu_3^1 , \mu_4^1 \,]^T = 1 \qquad (9.6.25)$$

which clearly implies that stabilizing controllers defined by (n_1 , d_1) for channel 1 can be found by solving equation :

$$[\, d_1 , n_1 \,] \cdot \underline{u} = \lambda \,,\ \lambda \in \mathbb{R}_{\mathcal{P}}(s) \text{ , is an arbitrary unit} \qquad (9.6.26)$$

Applying the results introduced in section 9.6.1 for the scalar Diophantine equation (9.6.24) the searching for proper duals (n_1 , d_1) and hence , for proper stabilizing controllers defined by (n_1 , d_1) for channel 1 , is now straightforward .

2) A_1 is nondegenerate but complete . Then for all the selections of $\underline{\mu} = U\, [\underline{q}^T, \underline{w}^T]^T$, $\underline{q}^T$ $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector , with $A_1\, U = [I_2\ O]$, $A_1 \cdot \underline{\mu} = \underline{q}$ is always an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector . Hence , equation (9.6.24) can be written as :

$$[\, d_1 , n_1 \,] \cdot \underline{q} = 1 \qquad (9.6.27)$$

Applying the results introduced in section 9.6.1 for the scalar Diophantine equation (9.6.27) the searching for proper duals (n_1 , d_1) and hence , for proper stabilizing controllers defined by (n_1 , d_1) for channel 1 , is now straightforward .

3) A_1 is nondegenerate noncomplete . Then the column Hermite form of A_1 is :

$$A_1 \cdot U_r = \begin{bmatrix} k & 0 & 0 & 0 \\ w & z & 0 & 0 \end{bmatrix} \tag{9.6.28}$$

where , U_r is $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular . Equation (9.6.24) can now be written as :

$$[\, d_1 \,,\, n_1 \,] \cdot A_1 \cdot U_r \cdot U_r^{-1} \cdot [\, \mu_1^1 \,,\, \mu_2^1 \,,\, \mu_3^1 \,,\, \mu_4^1 \,]^T = 1$$

or ,

$$[\, d_1 \,,\, n_1 \,] \cdot \begin{bmatrix} k & 0 & 0 & 0 \\ w & z & 0 & 0 \end{bmatrix} \cdot \underline{r} = 1 \tag{9.6.29}$$

where , $\underline{r} = U_r^{-1} \cdot [\, \mu_1^1 \,,\, \mu_2^1 \,,\, \mu_3^1 \,,\, \mu_4^1 \,]^T$ is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime . (9.6.29) can finally be written as:

$$[\, d_1 \,,\, n_1 \,] \cdot \begin{bmatrix} k & 0 \\ w & z \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 1 \tag{9.6.30}$$

or ,

$$[\, d_1 \,,\, n_1 \,] \cdot H \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 1 \tag{9.6.2.31}$$

The searching for proper stabilizing controllers defined by $(n_1 \,,\, d_1)$ for channel 1 , has now been transferred to the searching for proper duals of the scalar Diophantine equation:

$$d_1 \,\hat{r}_1 + n_1 \,\hat{r}_2 = 1 \tag{9.6.32}$$

for each selection of $(r_2 \,,\, r_1)$ mode H^T $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and $[\, \hat{r}_1 \,,\, \hat{r}_2 \,] = [\, r_1 \,,\, r_2 \,] \cdot H^T$. Applying the results introduced in section 9.6.1 for the scalar Diophantine equation (9.6.32) the searching for proper duals $(n_1 \,,\, d_1)$ is now straightforward .

9.7. RELIABLE SOLUTIONS OF DDSP

Reliable stabilization is the ability of the system to maintain closed loop stability with the loss of one or more of its channels . Failure of channel i , $i = 1 , 2 , \dots , m$ is equivalent to the loss of a SISO controller $c_i = n_i \cdot d_i^{-1} \Rightarrow n_i = 0 \,,\, d_i \neq 0$.

Definition (9.7.1) : A strongly cyclic system is said to be reliable stabilized if :

a) The system is closed loop stable with a set of controllers defined by $(n_i\ ,\ d_i)$ *,* $i = 1\ ,\ 2\ ,\ \dots\ ,\ m$ *.*

b) The system remains stable with failures in channels $1\ ,\ 2\ ,\ \dots\ ,\ \kappa\ ,\ \kappa = 1\ ,\ \dots\ ,\ m$ *.* □

We have seen from the parametrization of the family of solutions to DDSP that condition a) is satisfied by selecting controllers to be mode $T_{\rho-j}$ $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime – mutually stabilizing , when $m = 2\rho$ and additionally the controllers defined by $(n_1\ ,\ d_1)$ when $m{=}2\rho + 1$ are solutions of (9.5.35) . The question that remains to be answered is , under what constraints such selected controllers satisfy condition b) of Definition(9.7.1) .

i) **CASE** $m = 2\rho$: Fix a j and a $T_{\rho-j}$, then the parametrization of stabilizing controllers for the channels $m-2j-1$, $m-2j$ is given by :

$$[\, d_{m-2j-1}\ ,\ n_{m-2j-1}\,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9.7.1)$$

Then we distinguish the cases :

1) Failure for channel $m-2j-1 \Rightarrow n_{m-2j-1} = 0$. Then equation (9.7.1) implies :

$$[\, d_{m-2j-1}\ ,\ 0\,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1$$

or ,

$$[\, d_{m-2j-1}\ ,\ 0\,] \cdot \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1$$

or ,

$$[\, d_{m-2j-1}\, t_{11}\ ,\ d_{m-2j-1}\, t_{12}\,] \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1$$

or ,

$$d_{m-2j-1} \cdot [\, t_{11}\ ,\ t_{12}\,] \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9.7.2)$$

(9.7.2) clearly implies that d_{m-2j-1} must be an $\mathbb{R}_{\mathfrak{P}}(s)$ unit . Thus the system remains closed loop stable with loss of channel $m-2j-1$ if d_{m-2j-1} is $\mathbb{R}_{\mathfrak{P}}(s)$ unit ($j = 0\ ,\ 1\ ,\ \dots$,

$\rho-1)$.

2) Failure for channel $m-2j \Rightarrow n_{m-2j}=0$. Then equation (9.7.1) implies :

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ 0 \end{bmatrix} = 1$$

or ,

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} d_{m-2j} \\ 0 \end{bmatrix} = 1$$

or ,

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \begin{bmatrix} t_{11}\, d_{m-2j} \\ t_{21}\, d_{m-2j} \end{bmatrix} = 1$$

or ,

$$[\, d_{m-2j-1} \,,\, n_{m-2j-1} \,] \cdot \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} \cdot d_{m-2j} = 1 \tag{9.7.3}$$

(9.7.3) clearly implies that d_{m-2j} must be an $\mathbb{R}_{\mathfrak{p}}(s)$ unit . Thus the system remains closed loop stable with loss of channel $m-2j$ if d_{m-2j} is $\mathbb{R}_{\mathfrak{p}}(s)$ unit $(j=0\,,1\,,\ldots,\rho-1)$.

3) Failure for channels $m-2j-1$, $m-2j \Rightarrow n_{m-2j-1}=0$, $n_{m-2j}=0$. Then equation (9.7.1) implies :

$$[\, d_{m-2j-1} \,,\, 0 \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ 0 \end{bmatrix} = 1$$

or ,

$$[\, d_{m-2j-1} \,,\, 0 \,] \cdot \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} d_{m-2j} \\ 0 \end{bmatrix} = 1$$

or ,

$$d_{m-2j-1} \cdot t_{11} \cdot d_{m-2j} = 1 \tag{9.7.4}$$

(9.7.4) clearly implies that d_{m-2j-1} , d_{m-2j} must be $\mathbb{R}_{\mathfrak{p}}(s)$ units . Thus the system remains closed loop stable with loss of channels $m-2j-1$, $m-2j$ if d_{m-2j-1} , d_{m-2j} are $\mathbb{R}_{\mathfrak{p}}(s)$ units $(j=0\,,1\,,\ldots,\rho-1)$.

Any other combination of failing channels $1\,,\ldots,\kappa$, $\kappa=1\,,\ldots,m$ can be considered as a combination of the above three cases .

ii) CASE $m = 2\rho + 1$: The study of the constraints the stabilizing controllers must meet in order the system to be reliable stabilized is identical to the one when $m = 2\rho$, for channels $i = 2, \dots, m$, whereas for failure of channel 1 we proceed as follows :

The parametrization of stabilizing controllers for channel 1 is given by equation (9.5.36):

$$[\, d_1 ,\, n_1 \,] \cdot A_1 \cdot [\, \mu_1^1 ,\, \mu_2^1 ,\, \mu_3^1 ,\, \mu_4^1 \,]^T = 1 \qquad (9.7.5)$$

Failure of channel 1 $\Rightarrow n_1 = 0$. Equation (9.7.5) can be written as :

$$[\, d_1 ,\, 0 \,] \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \cdot [\, \mu_1^1 ,\, \mu_2^1 ,\, \mu_3^1 ,\, \mu_4^1 \,]^T = 1$$

or ,

$$d_1 \cdot \underline{a}^T \cdot \underline{\mu} = 1 \qquad (9.7.6)$$

where , $\underline{a}^T = [\, a_{11} ,\, a_{12} ,\, a_{13} ,\, a_{14} \,]$, $\underline{\mu} = [\, \mu_1^1 ,\, \mu_2^1 ,\, \mu_3^1 ,\, \mu_4^1 \,]^T$. (9.7.6) clearly implies that d_1 must be an $\mathbb{R}_{\mathcal{P}}(s)$ unit . Thus the system remains closed loop stable with loss of channel 1 if d_1 is $\mathbb{R}_{\mathcal{P}}(s)$ unit .

9.8. THE FAMILY OF STABLE DIAGONAL DECENTRALIZED STABILIZING CONTROLLERS OF A STRONGLY CYCLIC SYSTEM

Consider a strongly cyclic system . Then by theorem(9.4.1) there always exists a family of diagonal decentralized stabilizing controllers $\mathcal{C} = \{c_i = n_i \cdot d_i^{-1} ,\, i = 1, 2, \dots, m\}$. The controllers c_i are stable if and only if d_i is an $\mathbb{R}_{\mathcal{P}}(s)$ unit . Our task is to characterize the family of stable diagonal decentralized stabilizing controllers .

Definition (9.8.1) *: An $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pair (n, d) will be called stable if and only if d is an $\mathbb{R}_{\mathcal{P}}(s)$ unit .* □

a) Case $m = 2\rho$: When $m = 2\rho$ we recall from section 9.5.1 that the parametrization of diagonal decentralized stabilizing controllers is given by the parametrization of solutions to the set of equations :

$$[\, d_{m-2j-1} ,\, n_{m-2j-1} \,] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1 \qquad (9:8.1)$$

For a fixed j and $T_{\rho-j}$ we shall search for stabilizing controllers $c_{m-2j-1} = n_{m-2j-1} \cdot d_{m-2j-1}^{-1}$

$c_{m\text{-}2j} = n_{m\text{-}2j} \cdot d^{-1}_{m\text{-}2j}$, with $d_{m\text{-}2j\text{-}1}$, $d_{m\text{-}2j}$ $\mathbb{R}_{\mathcal{P}}(s)$ units respectively ($j = 0 , \ldots \rho - 1$) , or equivalently , we shall search for stable pairs $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$, $(n_{m\text{-}2j} , d_{m\text{-}2j})$ satisfying (9.8.1) . We can distinguish tow cases :

1) $T_{\rho\text{-}j}$ is degenerate . Then by theorem(9.5.1) the family of $\mathbb{R}_{\mathcal{P}}(s)$ – coprime solutions of (9.8.1) is given by the family of solutions to the scalar Diophantine equations (9.5.23):

$$[\, d_{m\text{-}2j\text{-}1} , n_{m\text{-}2j\text{-}1} \,] \cdot \underline{u} = 1 \;\; , \;\; [\, d_{m\text{-}2j} , n_{m\text{-}2j} \,] \cdot \underline{v} = 1 \tag{9.8.2}$$

where , $\underline{u} = [\, u_{11} , u_{21} \,]^T$, $\underline{v}^T = [\, v_{11} , v_{12} \,]$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors uniquely defined modulo $\mathbb{R}_{\mathcal{P}}(s)$ units . Thus , the parametrization of stable pairs $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$, $(n_{m\text{-}2j} , d_{m\text{-}2j})$ satisfying (9.8.1) is equivalent to the parametrization of stable pairs $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$, $(n_{m\text{-}2j} , d_{m\text{-}2j})$ satisfying (9.8.2) . The family of stable pairs $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$, $(n_{m\text{-}2j} , d_{m\text{-}2j})$ satisfying (9.8.2) define stable stabilizing SISO controllers for the SISO plants $p_{m\text{-}2j\text{-}1} = u_{11}^{-1} \cdot u_{21}$, $p_{m\text{-}2j} = v_{11}^{-1} \cdot v_{12}$ respectively . The parametrization of stable stabilizing SISO controllers is well known and can be found in [Vid. 4] . Hence, the family of stable stabilizing SISO controllers for the SISO plants $p_{m\text{-}2j\text{-}1}$, $p_{m\text{-}2j}$ defines the family of stable stabilizing controllers for the channels $m - 2j - 1$, $m - 2j$.

2) $T_{\rho\text{-}j}$ is nondegenerate . Then by theorem(9.5.2) the solutions of (9.8.1) are mode $T_{\rho\text{-}j}$ mutually stabilizing pairs . In other words for each stabilizing controller for the fixed channel $m - 2j - 1$ defined by $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ there exists a subfamily of stabilizing controllers for the channel $m - 2j$ defined by $(n_{m\text{-}2j} , d_{m\text{-}2j})$ such that (9.8.1) holds true. Thus , our first aim is to parametrize all the mode $T_{\rho\text{-}j}$ $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and stable pairs $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ and then for each $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ fixed to parametrize the subfamily of $(n_{m\text{-}2j} , d_{m\text{-}2j})$ stable pairs such that (9.8.1) holds true .

The family of mode $T_{\rho\text{-}j}$ $\mathbb{R}_{\mathcal{P}}(s)$ – coprime pairs $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ can be found by solving equation :

$$[\, d_{m\text{-}2j\text{-}1} , n_{m\text{-}2j\text{-}1} \,] \cdot T_{\rho\text{-}j} = [\, u_1 , u_2 \,] \tag{9.8.3}$$

where , $\underline{u}^T = [\, u_1 , u_2 \,]$ is an $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector . Equation (9.8.3) can be viewed as similar to the one of lemma(9.4.1) , where $t = 2$, $A = T^T_{\rho\text{-}j}$, $\underline{r} = [\, u_1 , u_2 \,]^T$. In order $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ to be stable pairs $d_{m\text{-}2j\text{-}1}$ must be selected to be an arbitrary unit which satisfies *i*) , *ii*) of lemma(9.4.1) .

Proposition (9.8.1) *: A mode $T_{\rho\text{-}j}$ $\mathbb{R}_{\mathcal{P}}(s)$ – **coprime** pair $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ **is stable** if and only if $(n_{m\text{-}2j\text{-}1} , d_{m\text{-}2j\text{-}1})$ is selected to satisfy i) , ii) of lemma(9.4.1) and $d_{m\text{-}2j\text{-}1}$ must be selected to be an $\mathbb{R}_{\mathcal{P}}(s)$ unit .* □

Theorem (9.8.1) *: Let $T_{\rho-j} \in \mathbb{R}_{\mathfrak{P}}^{2x2}(s)$ be a cyclic , nondegenerate matrix . Then for each selection of $(n_{m-2j-1}\ ,\ d_{m-2j-1})$ mode $T_{\rho-j}$ $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime and stable pair , defining a stabilizing controller $c_{m-2j-1} = n_{m-2j-1} \cdot d^{-1}_{m-2j-1}$ for the $m-2j-1$ channel , a subfamily of $(n_{m-2j}\ ,\ d_{m-2j})$ $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime pairs defining a stable stabilizing controller $c_{m-2j} = n_{m-2j} \cdot d^{-1}_{m-2j}$ for the $m-2\ j$, is given by the family of stable stabilizing controllers for the plant $p = \tilde{d}^{-1}_{m-2j-1} \cdot \tilde{n}_{m-2j-1}$, where :*

$$[\ \tilde{d}_{m-2j-1}\ ,\ \tilde{n}_{m-2j-1}\ ] = [\ d_{m-2j-1}\ ,\ n_{m-2j-1}\ ] \cdot T_{\rho-j} \qquad (9.8.4)$$

□

b) CASE $m = 2\ \rho + 1$: When $m = 2\ \rho + 1$ we recall from section 9.5.2 that the parametrization of diagonal decentralized stabilizing controllers is given by the parametrization of solutions to the set of equations :

$$\begin{gathered}[\ d_{m-2j-1}\ ,\ n_{m-2j-1}\ ] \cdot T_{\rho-j} \cdot \begin{bmatrix} d_{m-2j} \\ n_{m-2j} \end{bmatrix} = 1\ ,\ j = 0\ ,\ 1\ ,\ \dots\ ,\ \rho - 1 \\ [\ d_1\ ,\ n_1\ ] \cdot A_1 \cdot [\ \mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1\ ]^T = 1 \end{gathered} \qquad (9.8.6)$$

By (9.8.6) is clear that the parametrization of stable stabilizing controllers for the channels $i = 2\ ,\ 3\ ,\ \dots\ ,\ m$ is identical to the one described in case a) . It remains to study the parametrization of stable pairs $(n_1\ ,\ d_1)$, which define stabilizing controllers for channel 1 and thus satisfy (9.8.6) . Consider equation :

$$[\ d_1\ ,\ n_1\ ] \cdot A_1 \cdot [\ \mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1\ ]^T = 1 \qquad (9.8.7)$$

By its derivation equation (9.8.7) has A_1^T as an $\mathbb{R}_{\mathfrak{P}}^{4x2}(s)$ cyclic matrix . We can distinguish three cases :

1) A_1 is degenerate . Then $A_1 = \underline{u} \cdot \underline{v}^T$, where , $\underline{u}$, $\underline{v}$ are minimal Mc Millan degree bases for the column [$\mathbb{R}_{\mathfrak{P}}(s)$ – module of A_1] , row [$\mathbb{R}_{\mathfrak{P}}(s)$ – module of A_1] , respectively. Hence , $\underline{u}$, $\underline{v}$ are $\mathbb{R}_{\mathfrak{P}}(s)$ – coprime vectors unique [modulo $\mathbb{R}_{\mathfrak{P}}(s)$ units] . Equation (9.8.7) becomes :

$$[\ d_1\ ,\ n_1\ ] \cdot \underline{u} \cdot \underline{v}^T \cdot [\ \mu_1^1\ ,\ \mu_2^1\ ,\ \mu_3^1\ ,\ \mu_4^1\ ]^T = 1 \qquad (9.8.8)$$

All the stable pairs $(n_1\ ,\ d_1)$, defining stable stabilizing controllers for channel 1 , can be found as stable solutions to equation :

$$[\, d_1 \,,\, n_1 \,] \cdot \underline{u} = \lambda \,,\ \lambda \in \mathbb{R}_{\mathcal{P}}(s) \,,\ \text{is an arbitrary unit} \tag{9.8.9}$$

If $\underline{u} = [\, u_{11} \,,\, u_{21} \,]^T$, then the parametrization of stable solutions to equation (9.8.9) is equivalent to the parametrization of SISO stable stabilizing controllers for the plant p = $= u_{11}^{-1} \cdot u_{21}$. The latter parametrization is well known and can be found in [Vid. 4] .

2) A_1 is nondegenerate and complete . Then for all selections $\underline{\mu} = U\, [\underline{q}^T,\, \underline{w}^T]^T$, $\underline{q}^T$ $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vector , with $A_1\, U = [I_2\ O]$, $\underline{r} = A_1 \cdot \underline{\mu}$ are $\mathbb{R}_{\mathcal{P}}(s)$ – coprime vectors . Hence , equation (9.8.7) becomes :

$$[\, d_1 \,,\, n_1 \,] \cdot \underline{r} = 1 \tag{9.8.10}$$

All the stable pairs $(n_1 \,,\, d_1)$, defining stable stabilizing controllers for channel 1 , can be found as stable solutions to equation (9.8.10) . If $\underline{r} = [\, r_{11} \,,\, r_{21} \,]^T$, then the parametrization of stable solutions to equation (9.8.10) is equivalent to the parametrization of SISO stable stabilizing controllers for the plant $p = r_{11}^{-1} \cdot r_{21}$. The latter parametrization is well known and can be found in [Vid. 4] .

3) A_1 is nondegenerate and noncomplete . Then the column Hermite form of A_1 is :

$$A_1 \cdot U_r = \begin{bmatrix} k & 0 & 0 & 0 \\ w & z & 0 & 0 \end{bmatrix} \tag{9.8.11}$$

where , U_r is $\mathbb{R}_{\mathcal{P}}(s)$ – unimodular . Equation (9.8.7) can now be written as :

$$[\, d_1 \,,\, n_1 \,] \cdot A_1 \cdot U_r \cdot U_r^{-1} \cdot [\, \mu_1^1 \,,\, \mu_2^1 \,,\, \mu_3^1 \,,\, \mu_4^1 \,]^T = 1$$

or ,

$$[\, d_1 \,,\, n_1 \,] \cdot \begin{bmatrix} k & 0 & 0 & 0 \\ w & z & 0 & 0 \end{bmatrix} \cdot \underline{r} = 1 \tag{9.8.12}$$

where , $\underline{r} = U_r^{-1} \cdot [\, \mu_1^1 \,,\, \mu_2^1 \,,\, \mu_3^1 \,,\, \mu_4^1 \,]^T$ is $\mathbb{R}_{\mathcal{P}}(s)$ – coprime . (9.8.12) can be written as:

$$[\, d_1 \,,\, n_1 \,] \cdot \begin{bmatrix} k & 0 \\ w & z \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 1 \tag{9.8.13}$$

or ,

$$[\, d_1 \,,\, n_1 \,] \cdot H \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 1 \tag{9.8.14}$$

The searching for stable stabilizing controllers defined by (n_1 , d_1) for channel 1 , has now been transferred to the searching for stable solutions to the scalar Diophantine equation :

$$d_1 \hat{r}_1 + n_1 \hat{r}_2 = 1 \tag{9.8.15}$$

for each selection of (r_2 , r_1) mode H^T $\mathbb{R}_{\mathcal{P}}(s)$ – coprime and $[\hat{r}_1 , \hat{r}_2] = [r_1 , r_2] \cdot H^T$. The parametrization of stable solutions to equation (9.8.10) is equivalent to the parametrization of SISO stable stabilizing controllers for the plant $p = \hat{r}_1^{-1} \cdot \hat{r}_2$. The latter parametrization is well known and can be found in [Vid. 4] .

9.9. CONCLUSIONS

The diagonal stabilization problem (DDSP) has been defined over the ring $\mathbb{R}_{\mathcal{P}}(s)$ and necessary and sufficient conditions for its solvability have been described . The important relation between the cyclicity property that the plant may exhibits and the existence of stabilizing controllers has been established . The necessary and sufficient conditions for solvability of DDSP have been derived by the necessary and sufficient solvability conditions for a scalar Diophantine equation over $\mathbb{R}_{\mathcal{P}}(s)$ under certain factorization constrain of its solutions . A complete parametrization of the diagonal decentralized stabilizing controllers has been studied and its relation to what are termed T mutually stabilizing pairs , introduced . A parametrization of solutions to a scalar Diophantine equation over $\mathbb{R}_{\mathcal{P}}(s)$ which are defined as proper pairs , as it has been described in section 9.6.1 , in combination with the parametrization of diagonal stabilizing controllers has led us to a parametrization of proper diagonal stabilizing controllers . In section 9.7 reliable solutions to the DDSP have been studied . The use of the parametrization introduced in section 9.5 remains the basis from which these and the results of next section 9.8 has evolved . An interesting question that remains under consideration is the parametrization of minimal McMillan degree diagonal stabilizing controllers .

CHAPTER 10

CONCLUSIONS

Algebraic methods for solvability and characterization of solutions , (or special types of them) , of certain matrix equations over the ring of interest have been developed in this thesis . These equations are central to the formulation of various control synthesis problems concerning the stability and performance of linear , multivariable , time invariant systems , such as , the total finite settling time stabilization , (for discrete time systems) , the decentralized and diagonal stabilization , the disturbance decoupling noninteracting control and regulator problems with or without the internal stability requirement , (for continuous time systems) . More precisely , the matrix equations that have been studied are :

$$A \cdot X + B \cdot Y = C \, , \, (X \cdot A + Y \cdot B = C) \tag{10.1}$$

$$A \cdot X = B \, , \, (Y \cdot A = B) \tag{10.2}$$

$$A \cdot X \cdot B = C \tag{10.3}$$

$$\sum_{i=1}^{n} A_i \cdot X_i \cdot B_i = C \tag{10.4}$$

where , A , B , A_i , B_i , C , X , Y , X_i , are matrices over the ring of interest , i.e. a given Euclidean domain , (ED) , or principal ideal domain , (PID) . The procedure of reducing the solvability of the control synthesis problems under consideration to the solvability and characterization of solutions of the matrix equations (10.1) – (10.4) has been reviewed in Chapter 2 . There , after a brief survey of the concept of stability and especially the relation between internal and external stability of linear systems , each of the control synthesis problems in question has been presented and solvability conditions via the associated matrix equations have been established . The algebraic method of approaching such problems has been based on what is termed as matrix fractional representation over the ring of interest . From a control theory viewpoint the rings of importance are , $\mathbb{R}[s]$ – polynomials , $\mathbb{R}_{pr}(s)$ – proper rational functions , $\mathbb{R}_{\mathcal{P}}(s)$ – proper rational functions with no poles inside a prescribed region $\mathcal{P}$ of the complex plain .

The requirement of internal stability is central to all these control synthesis problems something that has motivated researchers to study thoroughly the properties of $\mathbb{R}_{\mathcal{P}}(s)$. In Chapter 3 we have concentrated on the study of the most important property of $\mathbb{R}_{\mathcal{P}}(s)$, i.e. the existence of a "Euclidean division" . A detailed analysis of a method for introducing unique – modulo $\alpha \in \mathbb{R}^-$ – factorization and hence a definition for exact division between two elements of $\mathbb{R}_{\mathcal{P}}(s)$ has been described . The important property of non uniqueness of Euclidean remainder in the Euclidean division in $\mathbb{R}_{\mathcal{P}}(s)$ leads to the need of characterization of the various families of remainders according to invariant characteristics as for example is the number of zeros in $\mathcal{P}$. The need for constructing the family of least "Euclidean degree" remainders of the "Euclidean division" in $\mathbb{R}_{\mathcal{P}}(s)$, has implied the transformation of this problem to the construction of a rational unit over the disc algebra of symmetric analytic functions which map the disc $\overline{((0 \, , \, \frac{1}{2}) \, , \, \frac{1}{2})}$ into the complex numbers , under certain interpolation constrains . A description of this disc algebra has been made and the interconnection between its units and the units of $\mathbb{R}_{\mathcal{P}}(s)$ has been given . An algorithmic construction of the required unit has been introduced and that has led to two algorithms for the construction of the family of least possible "Euclidean degree" remainders . These algorithms complete the results presented in [Vid. 4] where the existence of a least "Euclidean degree" remainder is established but not fully constructed . The knowledge of the least degree family of remainders in $\mathbb{R}_{\mathcal{P}}(s)$ has been used in the last section of Chapter 3 for the estimation of least unstable zeros stabilizing controllers . An extension of the Euclidean division in matrices over $\mathbb{R}_{\mathcal{P}}(s)$ has been mentioned .

An alternative characterization for the greatest common divisor (GCD) , f(s) , of a set of m polynomials , $\underline{p}(s)$, of maximal degree δ has been introduced in Chapter 4 by making use of the equivalent expression of relationship $\underline{p}(s) = \underline{q}(s) \cdot f(s)$ in terms of real matrices , (basis matrices (b.m.) P , Q of $\underline{p}(s)$, $\underline{q}(s)$ respectively) , and the Toeplitz representation of f(s) . The relation between the GCD and scalar Toeplitz bases , $\mathcal{W}$, of a subspace $\mathcal{V}$ of $\mathcal{N}_r\{P\}$ has been established . The additional property , that the nonzero entries of $\mathcal{W}$ should have a certain expression involving the coefficients of the gcd f(s) and $\mathcal{V}$ has the greatest possible dimension that the latter happens has appeared in section 4.3 . This has led to an algorithm for the construction of the coefficients of f(s) as a tuple taken from a certain affine variety . It has been shown that Groebner bases play an essential role in characterizing the GCD in terms of its Toeplitz representation . The present approach uses the notion of Groebner bases in an explicit manner . Although simpler methods for the computation of the GCD have already been given in the literature , (see [Mit. 2] and the closed form solution given in [Kar. 3]) , the present method has the advantage that may be extended to matrix divisors , whereas the others have considerable difficulties . Such an extension is under investigation .

In Chapter 5 we have investigated structural properties of matrices over a PID , $\mathcal{R}$. The matrices have been assumed to have entries over the field of fractions , $\mathcal{F}$, of $\mathcal{R}$. These properties have been used to generate algebraic tools that have enabled us to formulate a unifying framework to deal with solvability of matrix equations over $\mathcal{R}$. The existence and characterization of families of greatest left – right divisors , greatest extended left – right divisors , projectors , annihilators , left – right inverses , multiples and least multiples of the rows columns of matrices over $\mathcal{R}$ has been introduced . The relation between these algebraic tools and the column , row $\mathcal{R}$ – modules , maximum $\mathcal{R}$ – modules of the matrices under investigation has been established .

In Chapter 6 we have tackled the very important issue of formulating a unifying approach for solving the matrix equations (10.1) – (10.4) over the PID of interest , $\mathcal{R}$. In our attempt to do so we use the results have been derived in Chapter 5 . The given matrices A , B , A_i B_i , C , in (10.1) – (10.2) have been considered over the field of fractions , $\mathcal{F}$, of $\mathcal{R}$, whereas the unknown matrices X , Y , X_i are required to be over

$\mathcal{R}$. Conditions for the existence as well as parametrization of solutions of the equations in question have been provided in terms of greatest left – right divisors of the given matrices as well as parametric matrices over $\mathcal{R}$.

In Chapter 7 the standard polynomial matrix Diophantine equation , (PMDE) , (10.1) , (with (A , B) , (X , Y) coprime polynomial MFDs , C a unimodular matrix) , arising from many stabilization problems , like the total finite settling time stabilization , (TFSTS) , [Kar. 1] , [Mil. 1] of discrete – time linear systems , has been considered . Solutions of (10.1) , satisfying various constrains like minimal controllability index , least complexity , fixed complexity – PI controllers , minimal extended McMillan degree (EMD) , have been studied . The expression of [A , B] , $[X^T, Y^T]^T$ by composite matrices has led to the transformation of the PMDE to an equivalent one employing Toeplitz matrix representation of the product $[\, A , B \,]\cdot[\, X^T , Y^T \,]^T = C$. It has been showed in section 7.3 that certain solutions , (column reduced solutions) , of (10.1) have topological properties , (forms a nonempty dense but neither open nor closed set) , that allow the EMD of the controllers they define to serve as a reliable upper bound for the minimum one .

A characterization of the least column degrees solutions of (10.1) , as well as , equation $C_m\big([\, A , B \,]\big)\cdot C_m\big([\, X^T , Y^T \,]^T\big) = \textit{constant}$ has been examined in light of their Toeplitz matrix representations . This approach has led to a very simple algorithm involving only the computation of right , (left) , null spaces of real matrices . Thus upper and lower bounds for the minimum EMD of the stabilizing controllers have been introduced . It remains under investigation the construction of the set of least column degrees that occur among the family of sets of least column degrees of solutions of (10.1) for all $\mathbb{R}[s]$ – unimodular matrices C . Finally in section 7.5 the investigation of fixed complexity solutions of (10.1) , has provided necessary and sufficient conditions for the existence of a PI stabilizing controller for a discrete – time linear system .

In Chapter 8 parametrization issues of the general decentralized stabilization problem , (DSP) , have been studied . The problem of a closed form parametrization of the solutions of DSP studied previously in [Gün. 1] , [Özg. 1] still remains an open issue. We have approached the DSP in an algebraic manner via the set of equations $T_i\cdot X_i =$ $= U_i$, X_i , left unimodular , $[U_1 , \cdots , U_\kappa]$ unimodular , T_i matrices defined by appropriately partitioning an $\mathbb{R}_{\mathcal{P}}(s)$ – left coprime MFD of the plant . A parametrization of the family of solutions , X_i , which corresponds to $[U_1 , \cdots , U_\kappa]$ unimodular has been given by theorem(8.4.1) . The above parametrization requires the existence of a constructive method that enables us to generate the family of all unimodular matrices of given dimension , as well as the families of left , (right) unimodular matrices which complete given left , (right) , unimodular matrices to square unimodular ones . Such methods has been examined in section 8.3 . The families of parameters involved need to satisfy certain parametrization constrains . These constrains constitute a necessary and sufficient criterion that enables us to identify the admissible parameters . Particular cases where closed form parametrization is possible have been studied in sections (8.4) , (8.5) . In the case of two blocks decentralized controllers a full description of the set of parameters has been given , especially when T_i are considered generically and are either not square or , one of T_1 or T_2 are square . The study of closed form parametrization when T_1 , T_2 are simultaneously square as well as the generalization in the case of κ blocks decentralized controllers are still under investigation .

A special case of decentralized stabilization , the diagonal stabilization problem , (DDSP) , has been defined over the ring $\mathbb{R}_{\mathfrak{P}}(s)$ and necessary and sufficient conditions for its solvability have been described as an extension of the results in [Kar. 2] . The important relation between the cyclicity property that the plant may exhibits and the existence of stabilizing controllers has been established . The necessary and sufficient conditions for solvability of DDSP have been derived by the necessary and sufficient solvability conditions for a scalar Diophantine equation over $\mathbb{R}_{\mathfrak{P}}(s)$ under certain factorization constrain of its solutions .

A complete parametrization of the diagonal decentralized stabilizing controllers has been studied and its relation to what are termed T mutually stabilizing pairs , has been established . A parametrization of solutions to a scalar Diophantine equation over $\mathbb{R}_{\mathfrak{P}}(s)$ which are defined as proper pairs , as it has been described in section 9.6.1 , in combination with the parametrization of diagonal stabilizing controllers has led us to a parametrization of proper diagonal stabilizing controllers . In section 9.7 reliable solutions to the DSP have been studied . The use of the parametrization introduced in section 9.5 remains the basis from which these and the results of next section 9.8 has evolved . An interesting question that remains under consideration is the parametrization of minimal McMillan degree diagonal stabilizing controllers .

Many of the problems addressed in this thesis have not been solved completely . Open issues that still require further investigation have risen in chapters 6 , 7 , 8 and 9 . More precisely , in chapter 6 further investigation of necessary conditions for solvability over a PID of the matrix equation (6.1.4) is needed . This is equivalent to the study of special type of solutions over a PID , (block diagonal) , of the matrix equation (6.1.3) . In chapter 7 , issues like the parametrization of minimum EMD controllers , for discrete time linear systems , defined by solutions of the matrix Diophantine equation (7.2.3) as well as parametrization of solutions of (7.2.3) according to a fixed McMillan degree still remain open . Further study of topological properties of the family of solutions of (7.3.2) and especially of the non column reduced ones is needed . In chapter 8 we need to elaborate on the complete description of the family of parameters that satisfy the DSP parametrization constraints of theorem(8.4.1) . This will result to a closed form description of the family of solutions of DSP in the most general case . The reverse problem of selecting a decentralized scheme , when an unstable plant is given , such that decentralized stabilization is possible is worth studing . Finally in chapter 9 , a characterization of diagonal stabilizing controllers according to McMillan degree is a topic that needs to be addressed .

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