THE UNIVERSITY OF HULL
CARTOGRAPHIC INFORMATION SYSTEMS
RESEARCH GROUP C.I.S.R.G.

DEPARTMENT OF COMPUTER SCIENCE · HULL HU6 7RX · UNITED KINGDOM
TELEPHONE 0482 465295/465951 · FACSIMILE 0482 466666 · TELEX 9312134728 HUG

DR M VISVALINGAM
HONORARY CO-ORDINATOR

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C.I.S.R.G. DISCUSSION PAPER 14

Approximation, Generalisation and Deconstruction of Planar Curves

by

M Visvalingam

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The credit for using fractals as test data for comparing algorithms goes to Michael Harasimiuk and Roger Wyatt, students on our taught MSc course on Computer Graphics and Virtual Environments in session 1995/96. They used elements of Figures 2 and 3 in this paper, within their coursework, to propose that Visvalingam's line generalisation algorithm could not generalise simulated curves. The idea of de-constructors for fractals arose during my evaluation of their analyses and conclusions. I thank them for their data and for providing the stimulus for this research. I am also indebted to my PhD student, Chris Brown, for drawing my attention to the Cesaro constructions in Mandelbrot's book, for his Postscript program for pasting my separate diagrams into a single figure and for his participation in on-going research focusing on the deconstruction of other fractals.

1. Introduction

Algorithms for approximation and compact representation of planar curves have been the subject of interest in Pattern Recognition (Pikaz and Dinstein, 1995) and in Digital Cartography (Thapa, 1987). Polygonal approximation, which seeks to minimise the number of points needed for representing a curve, is just one of many tasks in Digital Cartography (Robinson et al, 1995). This was largely achieved using Ramer's (1972) algorithm, independently originated by Douglas within Digital Cartography (see Douglas and Peucker, 1973).

Although there have been numerous competitors, Douglas' algorithm is the most widely used. It is used with varying tolerance values for achieving different levels of line generalisation. Visvalingam and Whyatt (1990 & 1991) pointed out that the algorithm has some limitations for line generalisation. The objectives of cartographic generalisation vary with the scale of display and it was generally believed that the art of caricatural generalisation was beyond automation. Visvalingam and Whyatt (1993) showed that the novel iterative point elimination algorithm, designed by Visvalingam, is capable of caricaturing lines and producing results which compare favourably with manual output. Whyatt (1991) found that Visvalingam's algorithm produced better results than some other notable line generalisation algorithms. This algorithm has opened up new directions for research within Digital Cartography since it can eliminate minor features in their entirety and preserve only the more important in-line features. More recently, Pikaz and Dinstein (1995) independently described Visvalingam's iterative point elimination algorithm for polygonal approximation of curves (documented earlier in Pikaz 1992) and made a number of propositions regarding this method.

The primary aim of this paper is to propose the novel concept of ***Deconstruction***, as opposed to the approximation and generalisation, of curves. The next section reviews the current state-of-the-art. It focuses on the process of line filtering and describes the current practice of interactive control of filtering algorithms within proprietary mapping systems. Published observations on the algorithms by Douglas (Ramer) and by Visvalingam are then summarised before addressing the second aim of this paper, which is to assess some of the propositions made by Pikaz and Dinstein. A simulated fractal curve, namely the triadic Koch curve, is used as a test line. This inaugral use of fractals, as test lines in the evaluation of line filtering algorithms, in turn prompted the notion of deconstruction as a distinct process. The paper concludes with an account of current research.

2. Previous Work

Cartographic generalisation is concerned with defining suitable graphic precis of reality for a variety of scales of representation. Jenks' (1978) classification of perceptual thresholds for generalisation, based on psychophysical testing, is widely accepted in cartography (see Visvalingam and Whyatt, 1990). Line generalisation is just one of the many processes involved in map generalisation (Robinson et al, 1995). Line filtering, in turn, is just one approach used in line generalisation. In cartography, lines are filtered for a variety of purposes, namely to :

a) remove redundant digitised data which fall within the limits of the graphic line
b) approximate the original line with a smaller set of points at the original scale
c) typify the original line for depiction at smaller scale by using even fewer points so that it is perceived as a much simplified representation but of the same line
d) produce gross caricatural representations which can be recognised as abstractions of the source line.

Level a) is enforced as a part of the data cleaning process. Level b) is the level for simplifying the contours of the land surface, for example. In Level c), the output line deviates from the original line to a noticeable extent. Level d) representations tend to make gross departures from the original line in order to bring out the larger structures of the phenomena being modelled. For example, the detailed coastline of a country will be severely generalised for depiction on a world map.

The most widely used algorithm for weeding and cartographic simplification of lines is the Douglas' algorithm. It is widely known that this method is unsuitable for achieving Level d) caricatural generalisations but it was believed that the art of caricature was outside the realms of automation since it required holistic human perception and judgement.

2.1 Tagging vertices for runtime filtering

Map data is normally vector digitised at the largest scale of mapping. Traditionally, maps were manually reduced for publication at specified scales by cartographers; some subtasks involved in the derivation of such multiple representations have been automated. The long-term aim of Digital Cartography and GIS is scale-free mapping. The aim is to present suitable depictions of data for interactive zoom. Packages for map generalisation already offer a range of generalisation operators which are driven by user-specified parameters (Robinson et al, 1995). However, algorithms are needed to identify, segment and tag features on lines so that they may be easily filtered at run time. The practice in the meantime is to tag each point on a line with a value corresponding to its significance so that the line may be filtered with a tolerance specified at run time (Whyatt and Wade, 1988). The Douglas' algorithm is used to tag points with the offset values which led to their selection. When using Visvalingam's algorithm, the value of the metric (such as area or distance) which resulted in its elimination is recorded with the point (Visvalingam and Whyatt, 1993).

Users can interactively experiment with tolerance values via a GUI towards desired results. This is particularly important with level (d) generalisation since the selection or omission of a single point can make an immense difference as demonstrated by Visvalingam and Whyatt (1993). The speed of the algorithm and resource usage during the tagging phase is of less concern than the efficacy of the algorithm in performing visually appropriate generalisations.

2.2 Douglas' algorithm

Although Douglas' algorithm is well known, Visvalingam and Whyatt (1991) identified a number of special cases which affect its implementation as a computer program. Douglas' algorithm is the most widely used method in Digital Cartography and GIS. McMaster (1987), for instance, concluded that it was mathematically and perceptually superior to several others that he evaluated. His main criteria for evaluation was minimal areal and vector displacement of the approximation from the original line. Douglas (and Ramer), like Pikaz and Dinstein, were primarily concerned with line approximation. The limitations of this algorithm for achieving higher levels of generalisation was noted by Visvalingam and Whyatt (1990) who also pointed out that the measure of minimal displacement was only appropriate up to Level (b) since higher levels of line reduction deliberately seek to depart from the original line.

2.3 Visvalingam's algorithm

Visvalingam's algorithm was primarily designed to achieve the higher levels of generalisation which rely on the progressive elimination of entire features, such as rivers (Visvalingam and Whyatt, 1993) and a hierarchy of scale-related features on road networks (Visvalingam and Williamson, 1995). Visvalingam and Whyatt suggested that the algorithm may be driven by different size and shape metrics but illustrated the algorithm using the concept of the "effective area" of each point. This corresponds to the area by which a line becomes displaced if that point was deleted from the line. The elimination of triangular features was seen as an indirect means of detecting geographic features. Pikaz and Dinstein noted that it is possible for the area to decrease and suggested that their algorithm may produce suboptimal results. Visvalingam and Whyatt (1993) noted that sequences of values, which are smaller than the largest rejected value, indicate line configurations; these include elongated smaller scale features, such as estuaries, rivers, sand spits, branch roads and so on. Visvalingam's algorithm (Visvalingam and Whyatt, 1993) deals explicitly with this special case as indicated in the following pseudocode.

```
CLEAR the output list
LET μ be the measure of significance, eg the effective area
CALCULATE μ for all intermediate points
WHILE (there are internal points)
{ ELIMINATE the point with the smallest μ
  OUTPUT this point with its sequence number and μ
  RECOMPUTE μ for its two neighbours
  WHILE (the smaller of the recomputed μ < the value for the last eliminated point)
        { RESET its μ to that of the last eliminated point
          OUTPUT this point with its sequence number and μ
             and ELIMINATE it from the input list
          RECOMPUTE μ for its two neighbours
        }
}
```

Visvalingam and Whyatt (1993) suggested that the sign of μ (area or height values) should be retained to steer alternative generalisations, for example, for land- and sea-based applications.

3. Comparison of the two algorithms

Visvalingam and Whyatt (1993) compared Douglas' with Visvalingam's algorithm using small scale data for coastlines digitised from 1:50 000 scale maps. They demonstrated that it was possible to filter satisfactory caricatures of the line for target scales of 1:10 million scale using just 4 of the original 1581 points in one case. The tolerance values had to be manually selected since Topfer's Law (Robinson et al, 1995) is no longer applicable at this level. The use of coastline data did not enable Whyatt (1991) and Visvalingam and Whyatt (1993) to investigate many properties of this algorithm. Visvalingam and Williamson's (1995) comparison of these two algorithms, using large-scale data for roads digitised form 1:1250 scale maps, provided further insights. This section summarises some of the main findings to date.

3.1 Minimal simplification

Visvalingam and Williamson noted that Douglas' algorithm appears to be the better of the two algorithms for polygonal approximation. There is clearly a need for further research since their conclusions may be data dependent. McMaster's (1987) areal and vector displacement metrics for measuring the goodness of fit of the approximation to the original curve may be used for comparing the performance of algorithms at this level.

3.2 The Concept of a Fixed Hierarchy of Points

Douglas' algorithm was used by Ballard (1981) to construct strip trees to reduce search time. Jones and Abraham (1987) used such ranking of points in their scale-free databases. Douglas' algorithm cannot achieve a balanced generalisation with a single tolerance value and Buttenfield's (1986) attempt, to segment lines so as to achieve aesthetically more balanced output at higher levels of generalisation by using multiple tolerances, was unsuccessful. Visvalingam's algorithm produces balanced generalisations.

3.3 Features versus Points

In Visvalingam's algorithm, point significance is assessed relative to the part-processed, rather than the original, line. This results in a truncation of curves (see Visvalingam and Williamson). The Figures in Pikaz and Dinstein also show this effect. Since smooth curves become detached from the line, they can be easily detected and represented by parametric curves (for example those used in the original design of the highway) so that a suitable number of points can be generated on the fly to suit the scale of display. Visvalingam & Whyatt (1990) had pointed out that the emphasis on points was misplaced and that the concept of a fixed hierarchy of points, in particular, was inappropriate. The long term aim of scale free mapping requires a data model based on the concept of a hierarchy of features. However, the computationally simpler concept of a hierarchy of points is still favoured.

3.4 Typification and Caricatural Generalisation of Entire Maps

Douglas' algorithm was designed to reduce the number of points needed to approximate stream digitised lines; the title of their paper has unfortunately encouraged others to use it for higher levels of generalisation. Visvalingam's algorithm can produce satisfactory typifications of features (Visvalingam and Whyatt, 1993). Typification occurs when a set of many detailed features, such as fjords, are replaced by a few exaggerated examples to indicate the character of that part of the (coast) line. The progressive elimination of scale-related features by Visvalingam's algorithm seems to provide cartographically acceptable caricatural generalisations of not only individual lines but also of a whole map consisting of a set of lines (Visvalingam and Williamson, 1995). The scope for caricatural generalisation facilitates pattern recognition since the algorithm can abstract meaningful structures within complex and even convoluted geographic data.

3.5 Area versus Height

Pikaz and Dinstein suggested that area is preferable if smoothing is the objective but that perpendicular distance is better for preserving shapes. This conclusion is valid at the level of curve approximation. Extensive experiments with a variety of 2D lines suggests that the area metric tends to truncate features (Figures 2 and 4 in Pikaz and Dinstein also show this tendency) while perpendicular height produces results which are more like those produced by Douglas' algorithm which, as noted earlier, provides a better approximation. However, their conclusion does not hold at higher levels of generalisation.

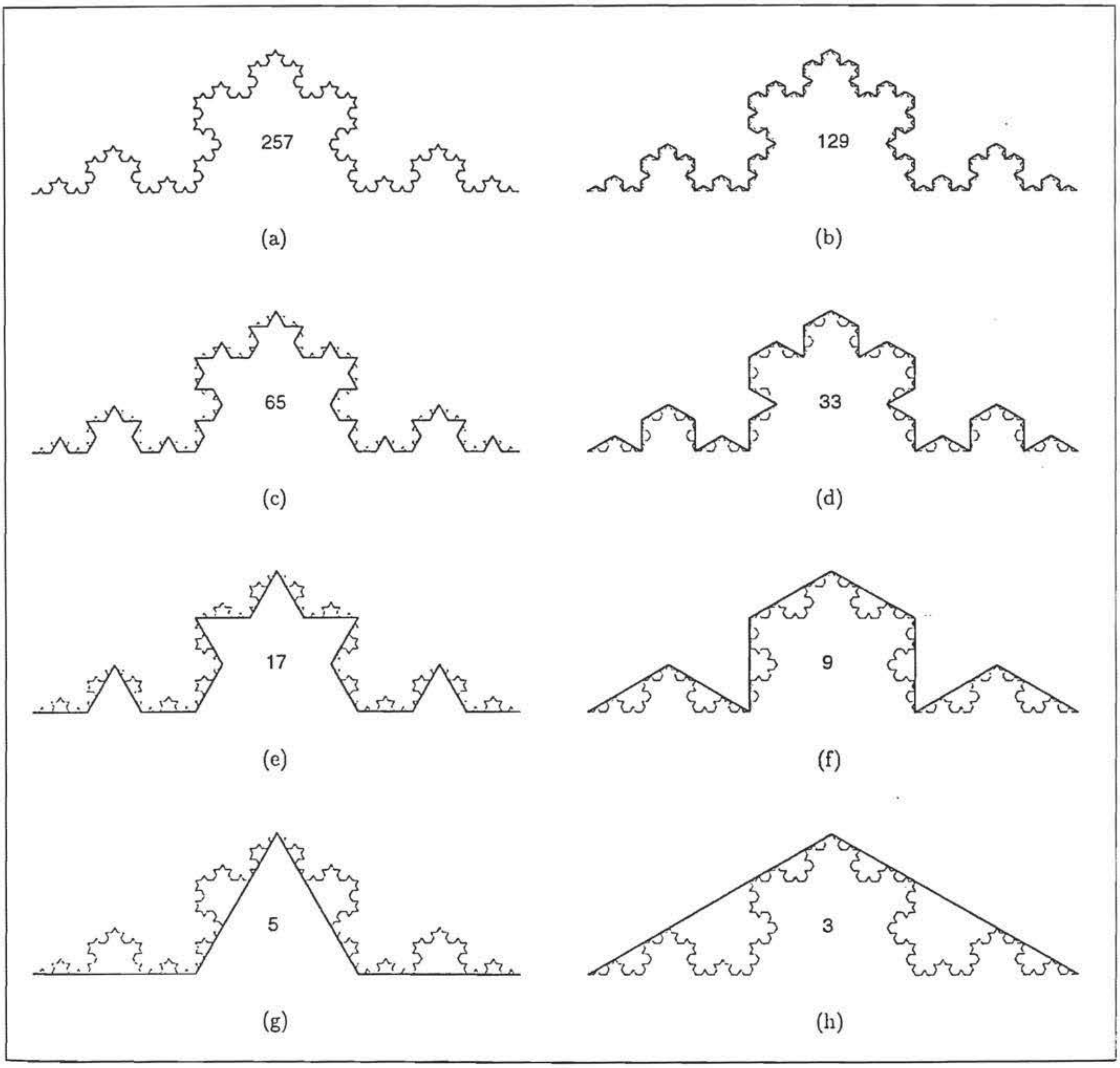
Visvalingam and Williamson pointed out that road data portray various types of detail; for example, random irregularities on the road which do not significantly alter the shape of lines at source scale; tight curves at filleted junctions and roundabouts; minor features such as lay-byes and entrances to drives; small branch roads; major branch roads; and large features such as car parks and large central reservations on roads. Whereas the use of the distance metric retains elements of smaller scale features at the expense of shape, the area metric tends to eliminate smaller scale features in their entirety and retains the shape of the larger features giving usable generalised information. This suggests that Jenks levels of generalisation differ in type and not just degree. The current view in cartography is that we need a variety of representational and processing models and metrics for scale-free zoom.

4. Filters as Deconstructors

The use of the triadic Koch curve as a test line in this paper serves two purposes. Firstly, it serves to evaluate propositions relating to Visvalingam's algorithm. More importantly, it encourages the definition of deconstruction as a distinctive cognitive process, having aims and objectives which are somewhat different from those of approximation and generalisation. Douglas' and Visvalingam's filters may well be members of a subclass of a larger **family of deconstructors** of fractal and natural curves.

Figure 1: Deconstruction of a Triadic Koch Curve with 257 points

(The numbers within subfigures refer to the number of points retained)



Whereas weeding and minimal simplification seek to represent the original pattern with a reduced set of points, deconstruction (like generalisation) is aimed at abstracting new patterns. The manual process of generalisation is knowledge-based and attaches semantic meanings to the in-line features being generalised. In contrast, deconstruction may be viewed as an entirely mechanical process, based only on (potentially different) geometric interpretations of input patterns, for abstracting new scale-related patterns. This additional concept provides opportunities for further research as noted in the conclusion. In this section, the Koch curve is used to assess some assumptions relating to Visvalingam's algorithm.

4.1 Effect of Start and End Points

It is well known that Douglas' algorithm is affected by the start and end positions. However, Visvalingam and Whyatt (1990) pointed out that these only affect the first few points to be selected and that the algorithm is relatively robust. Pikaz and Dinstein (p 561) proposed that the point elimination method is independent of these points. The fact that both algorithms serve as de-constructors of the triadic Koch curve (Figure 1) may be used to evaluate this proposition.

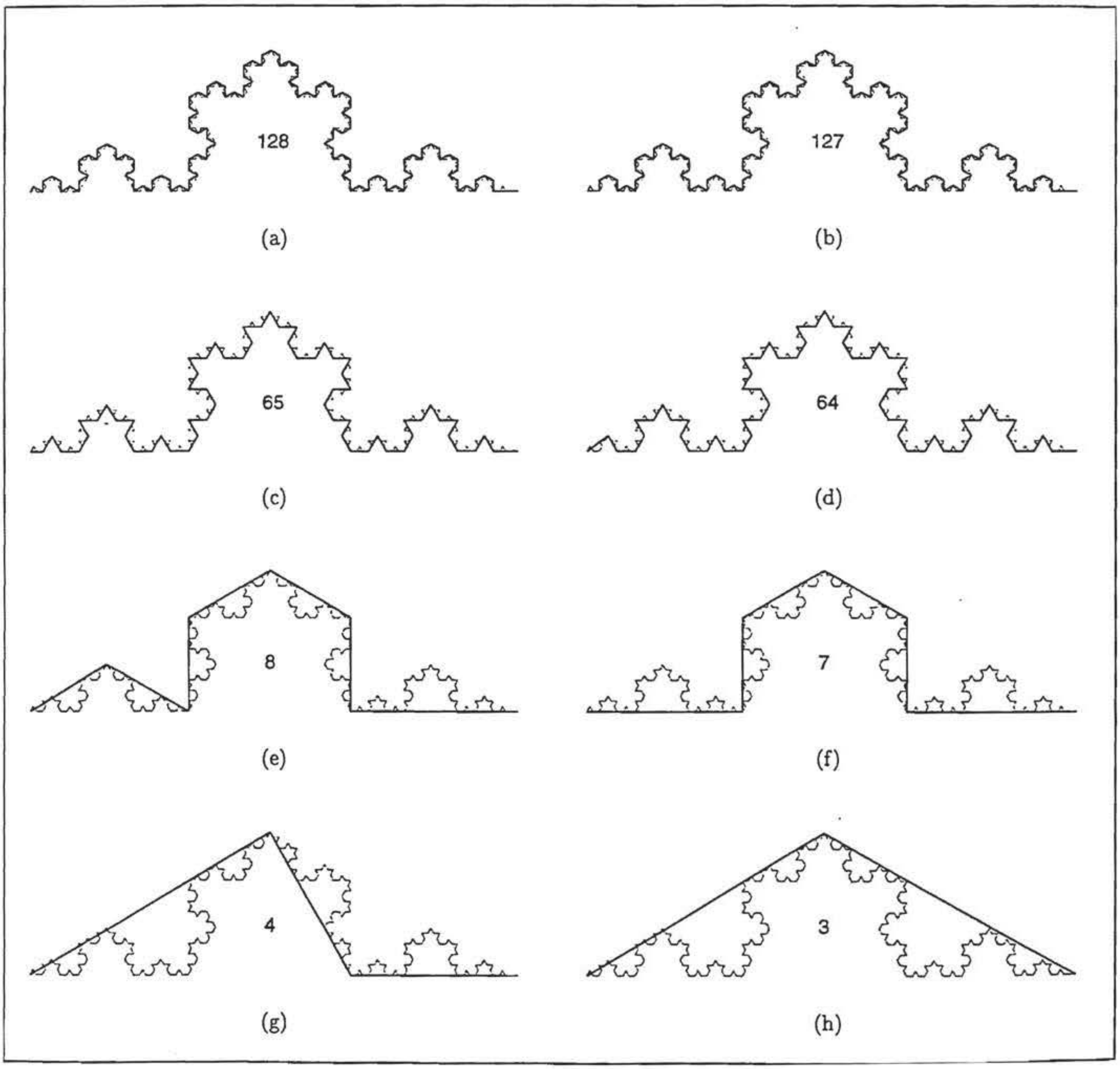
Given the extreme location of the generator's midpoint, such deconstruction is not difficult; not all fractals may be deconstructed using such simple filters (Visvalingam and Brown, forthcoming).
The profile of sorted μ for this curve is distinctly stepped and correlates exactly with the Koch construction. The odd (on the right) and even (on the left) steps of the deconstruction eliminate negative and positive μ values to give the Cesaro and Koch constructions alternatively. It is interesting to note that Mandelbrot's (1983, p 42 - 3) midpoint displacement interpretation effectively uses positive and negative displacements to alternatively generate the Cesaro and Koch teragons.

Elements of Figure 2 were used by students (see acknowledgements) to propose that Visvalingam's algorithm was incapable of providing appropriate generalisations of the Koch curve while Douglas' algorithm was able to do so. It was not immediately apparent that their computer program for interpreting L-grammars was incrementing the loop counter before rather than after output, causing the start and end points to be shifted by just one step.

The distribution of areal μ for Figure 2 is also stepped; but the steps have rounded corners. The filter values therefore are no longer obvious and their choice has to be guided by knowledge of the Koch curve. The odd, slightly smaller, μ value at each step has to be omitted for Cesaro (see Figures 2a & b) and retained for Koch constructions (eg Figures 2c & d). Even with this minor displacement of the end points, the Cesaro curve can no longer be recovered correctly. This effect, which is negligible at the level of minimal simplification, could adversely affect caricatural generalisation. The height metric produced similar suboptimal, but not identical, results.

However, the start and end points only pose a problem for open lines.; strictly speaking, there are no end points as such in closed curves. Ramer (1972) indicated how the problem of the starting point could be addressed when his algorithm is applied to closed loops. Similarly, one run through with

Figure 2: The impact of start and end points on the deconstruction of the Koch curve



Visvalingam's algorithm would reveal a more suitable starting point for closed curves as demonstrated by Visvalingam and Brown (forthcoming). Note also that the triangular Koch curve is symmetrical. Visvalingam and Whyatt (1991) pointed out that the direction in which a line is parsed can also affect the output of Douglas' algorithm; this is also true of Visvalingam's algorithm as demonstrated by Visvalingam and Brown.

4.2 Impact of Rounding and Digitising Errors

Visvalingam and Whyatt (1991) pointed out that the implementation of Douglas' algorithm must take account of special geometric cases and errors. Variations in machine representation of floating point numbers has produced different results on different computers. Given that digitising errors are considerably larger, it is possible to have several 'equally' extreme points from the base line; there appears to be no unique or universally applicable solution.

Pikaz and Dinstein considered the corresponding problem arising in Visvalingam's algorithm of a number of points in a neighbourhood having the same minimal value. They considered and then rejected a modification of the algorithm, which on each pass eliminates all points with the minimal value. Analysis of the Koch curve illustrates that such a modification will reduce the utility of Visvalingam's algorithm since this curve will be reduced to a straight line in just one iteration and it would be impossible to deconstruct the various generations of the Koch curve.

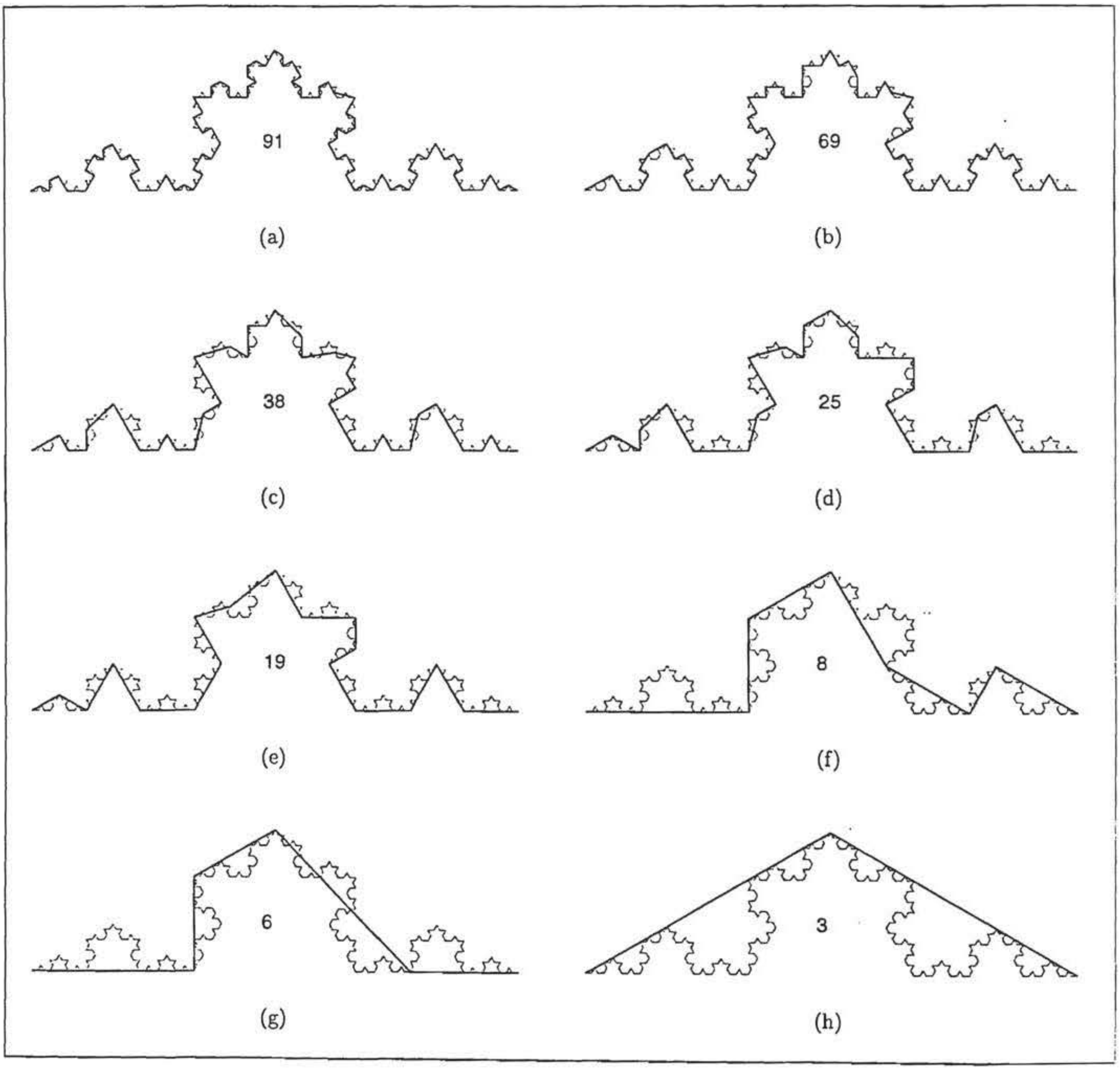
Given the inexact nature of number representation on computers, equal minima present an implementation problem. The presence of random errors does not affect the output of the Douglas' algorithm in this case since there is only one extreme point on each iteration of the Koch curve. In computing Figures 1 and 2, allowance was made for rounding errors in the input data and the inexact representation of floating point calculation of areas. Figure 3, based on Visvalingam's algorithm using areas, shows the effect of ignoring such errors. The distribution of sorted μ values for Figure 3 is much more confusing and bears only a vague relationship to the structure of the Koch curve. It was impossible to filter the same cascading number of points as in Figure 1 and optimal cut-offs were chosen by inspection. The implementation of the pseudocode statement:, **ELIMINATE the point with the smallest μ**, therefore should interpret the phrase, 'smallest μ ', in fuzzy terms. The height metric produced similar suboptimal, but not identical, results.

If rounding errors are taken into account, the algorithm will produce results which are invariant under co-ordinate transformations so long as they do not alter the aspect ratio.

4.3 Effect of Pre-processing the Data

The fact that Douglas' algorithm appears to be better for approximation does not mean that lines should be pre-processed with this filter for storage purposes prior to use of Visvalingam's algorithm. Logic and empirical evidence suggest that this can affect the latter's performance.

Figure 3: The impact of rounding errors on the deconstruction of the Koch curve



5. Conclusion

Different algorithms are needed to achieve different levels of generalisation. Douglas' and Visvalingam' s algorithms represent two different approaches for modelling lines, based on the hierarchy of points and features respectively. Visvalingam's algorithm is useful for achieving previously impossible levels of caricatural generalisation of lines in cartography. However, empirical results in cartography still favour Douglas' algorithm for approximating the original line by a reduced set of points. Some of the conclusions to be drawn from experiments to date are as follows:

- The area metric is better at preserving the more significant shapes; it eliminates smaller features but tends to cut curves; the height metric can distort shapes because it retains extreme points.
- The end points of a line have a local and thus negligible effect on minimal simplification but they can pose problems at higher levels of generalisation making it more difficult to abstract the salient structure of the line. However, this is only a problem with open lines.
- Unless the implementation includes a data dependent error tolerance, rounding errors can distort the output of the point elimination method.

Digital Cartography has been mainly focused on approximation (data reduction) and generalisation (knowledge-based abstraction of typifying patterns). This paper put forward an original suggestion that deconstruction represents a distinctly different cognitive activity having aims and objectives which are somewhat different from those of approximation and generalisation. It is an entirely mechanical process, based only on (potentially different) geometric interpretations of input line configurations, for abstracting new scale-related patterns. By cutting the link between the form of the line and its assumed meanings, it has opened up new directions for research in both computer graphics and cartography.

Starting with a simple initiator a series of complex fractal curves, named teragons by Mandlebrot (1983), are generated by using recursive functions to geometrically transform and apply a generator pattern. In contrast, deconstruction seeks to highlight different geometric interpretations of complex lines by their systematic reduction into a series of simpler forms. The term, ***decogon***, is coined here to denote a decorative, rather than meaningful, pattern obtained through deconstruction. In this paper we showed how deconstruction revealed the Koch and Cesaro constructions of the triadic Koch curve but on-going research with some other complex fractals shows that it may only be possible to derive a very wide range of completely unexpected, and sometimes beautiful, series of decorative patterns. The series of patterns are providing new insights into the geometric properties of deconstructor functions and are assisting in the process of expert knowledge elicitation.

In Digital cartography, as in other knowledge-based expert systems, there is the problem of knowledge elicitation. One means of eliciting relevant knowledge and procedures is by asking cartographers to generalise complex fractals with justification. Although Mandlebrot (1983, p36) claimed that 'the Koch curve provides a rough but rigorous model of coastlines', cartographers see them either as meaningless or as ambivalent. Both their queries, when asked to generalise even relatively simple fractals, as well as their varied and subjective output point to the types of cognitive processes involved in manual generalisation. Similarly, their comments on decogons, produced by a range of deconstructors, identify desirable and undesirable outcomes. The results themselves are outside the scope of this paper and will be reported in due course, particularly since they generate testable hypotheses.

It is hoped that this first reported deconstruction of a fractal curve will start the hunt for a family of simple and elegant deconstructors. Such research may point to still unknown factors contributing to subjectivity in cartography and also guide the design of purpose oriented generalisations.

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