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ARTICLES

CONVERGENCE OF ADAPTIVE LEARNING AND EXPECTATIONAL STABILITY: THE CASE OF MULTIPLE RATIONAL-EXPECTATIONS EQUILIBRIA

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This paper analyzes the relationship between the expectational stability of rational expectations solutions and the possible convergence of adaptive learning processes. Both concepts are used as selection criteria in the case of multiple rational expectations solutions. Results obtained using recursive least squares lead to the conjecture that there exists a general one-to-one correspondence between these two selection criteria. On the basis of a simple linear model and a stochastic gradient algorithm as an alternative learning procedure, it is demonstrated that such a conjecture would be incorrect: There are cases in which stochastic gradient learning converges to rational expectations solutions that are not expectationally stable.

Keywords: Multiple Equilibria, Learning, Expectational Stability

1. INTRODUCTION

Dynamic economic models, in which agents have to form expectations regarding the future, are usually closed by assuming that agents form rational expectations. Although this concept is formally elegant, it suffers from several problems. One of these problems is due to the fact that economic models may exhibit multiple rational expectations solutions. In such a case, it is impossible to select a specific solution without imposing additional restrictions. This is especially unpleasant if these solutions differ with respect to their comparative-static properties. Thus, the economic literature [e.g., McCallum (1983) and Evans and Honkapohja (1992)] discusses some selection criteria that should help to select a unique rational expectations solution in such cases.

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One of these selection criteria requires the respective solution to be a possible result of adaptive learning processes on the side of the agents in the model. Under the usual assumption of bounded rational learning, agents use parametrically specified—but possibly misspecified—perceived laws of motion for the relevant variables and form their expectations on the basis of these models. Learning in this context means that agents recursively estimate the parameters of their perceived law of motion. A central result of this approach is that the possible convergence of learning processes is connected with the expectational stability (E-stability) of the respective rational expectations solutions. The concept of E-stability is another criterion proposed to select among multiple rational expectations solutions. It traces back to Lucas (1978) and DeCanio (1979), while the extension of this concept in the context of real-time learning is attributed to Evans (1989). Assuming that agents learn using recursive least squares, Marcet and sargent (1988, 1989) and Evans and Honkapohja (1994, 1995) show that only E-stable rational expectations solutions are possible outcomes of such learning processes.

This result suggests that the convergence analysis of learning processes and the selection of rational expectations solutions can be achieved simultaneously simply by checking the E-stability of the respective solutions. Such a result would be very comfortable because usually it is easier to check the E-stability of rational expectations solutions rather than the possible convergence of learning processes. So, for instance, the survey on "Learning Dynamics" by Evans and Honkapohja (1998a) heavily relies upon such a probable connection between E-stability conditions and convergence conditions for economic learning processes. However, as the authors themself state, "although the bulk of work suggests the validity of the E-stability principle, there is no fully general result which underpins our assertion. . . . To date only a small set of estimators has been examined. We believe that obtaining precise general conditions under which the E-stability principle holds is a key subject for future research" [Evans and Honkapohja (1998a, p. 25)].

The present paper takes up this point and presents an example in which, contrary to the usual case, there is no one-to-one correspondence between the convergence conditions of a learning process and E-stability conditions. The analysis is based upon a stochastic gradient (SG) algorithm that is used as an alternative to the familiar recursive least-squares (LS) algorithm. On the basis of a simple linear model exhibiting multiple rational expectations solutions, it is shown on the one hand that there exist perceived laws of motion, for which the possible convergence of the SG algorithm to rational expectations solutions is indeed governed by E-stability conditions. On the other hand, there also exist perceived laws of motion for which the possible convergence is independent of the E-stability of these solutions.[1] In this respect the main result of the paper contrasts with a result obtained by Evans and Honkapohja (1998b): Those authors show, for the case of the familiar cobweb model, that the convergence conditions for learning are not altered if the LS algorithm is replaced by the SG algorithm, meaning that convergence is again governed by E-stability conditions. However, as will become clear below, this is no general result but is due to the special structure of the model they discuss.

The paper is organized as follows: The next section introduces a simple linear dynamic model that exhibits multiple rational expectations solutions. The conditions for E-stability of these solutions are reviewed with respect to two alternative perceived laws of motion, the first one being an AR(1) model and the second one being an ARMA(1,1) model. Sections 3 and 4 then proceed by assuming that agents learn by estimating parameters of their perceived laws of motion. Regarding the AR(1) perceived law of motion, Section 3 first reviews the well-known results that establish a one-to-one correspondence between E-stability conditions and the conditions for convergence of the LS algorithm. Furthermore, it is shown that with respect to this perceived law of motion the asymptotic properties of the LS algorithm and the SG algorithm are identical in the sense that both algorithms will converge only to E-stable rational expectations solutions. Afterwards, in Section 4, however, it is shown that this correspondence between the concept of E-stability and the possible convergence of the two learning algorithms does not hold for the ARMA(1,1) perceived law of motion. Using analytical results and numerical simulations, it is shown that the SG algorithm can in some cases converge to E-unstable rational expectations solutions and in other cases fail to exhibit local convergence to E-stable solutions. Proofs of the analytical results are relegated to the Appendix.

2. E-STABILITY OF RATIONAL EXPECTATIONS SOLUTIONS IN A LINEAR MODEL

In what follows, a model is considered that is equivalent to the one used by Evans and Honkapohja (1995) to describe the concept of E-stability and its connection to the possible convergence of adaptive learning procedures. This model is also identical to the leading example discussed in the survey by Evans and Honkapohja (1998a).[2] It is assumed that the value of the endogenous variable y in period t depends on its expected values for the periods t and $t+1$ as well as an exogenous, serially uncorrelated disturbance x_t:

$$y_t = \beta_0 + \beta_1 y_t^e + \beta_2 y_{t+1}^e + x_t. \tag{1}$$

With respect to the exogenous variable, it is assumed that $E[x_t] = 0$ and $E[x_t]^2 = \mathrm{Var}[x_t] = \sigma_x^2$ for all t. The expectations appearing in (1) have to be formed before y_t is known and the relevant information set is given as $\Omega_t = \{y_{t-1}, y_{t-2}, \ldots; x_{t-1}, x_{t-2}, \ldots\}$. This implies that the exogenous disturbances x_t are observable for all t—although they are observable only after the relevant expectations in t have been formed.

As is shown by Evans (1985), the model exhibits multiple rational expectations solutions. The two solution sets are given by[3]

$$y_t = \frac{\beta_0}{1 - \beta_1 - \beta_2} + x_t, \tag{2a}$$

$$y_t = -\frac{\beta_0}{\beta_2} + \frac{1-\beta_1}{\beta_2} y_{t-1} + \phi x_{t-1} + x_t, \qquad \phi \in \mathbf{R}. \tag{2b}$$

Note that the white-noise solution (2a) is unique, whereas the ARMA(1,1) solution (2b) represents a continuum of solutions for (1) because ϕ can take any value.

The starting point for the concept of E-stability is a perceived law of motion for the endogenous variable. In the following equations, two different perceived laws of motion are considered:

$$y_t = \alpha + \psi y_{t-1} + x_t, \tag{3a}$$

$$y_t = \alpha + \psi y_{t-1} + \phi x_{t-1} + x_t. \tag{3b}$$

The crucial difference between these models is that, according to (3a), the perceived law of motion for the endogenous variable is an AR(1) process, whereas according to (3b), the perceived law of motion is an ARMA(1,1) process. Note that whereas both models are able to represent the two rational expectations solutions (2a) and (2b) for the model (1), the perceived law of motion (3a) is the model of smallest degree regarding its AR and MA terms that encompasses at least partially both rational expectations solutions of the underlying model.

Since the conditions for E-stability of the rational expectations solutions (2a) and (2b) are equivalent with respect to the perceived laws of motion (3a) and (3b), the following formal exposition is based upon the perceived law of motion (3b).[4] On the basis of (3b), expectations regarding the future values of the endogenous variable are given by

$$\begin{aligned} y_t^e &= E[y_t \mid y_{t-1}, x_{t-1}] = \alpha + \psi y_{t-1} + \phi x_{t-1}, \\ y_{t+1}^e &= E[y_{t+1} \mid y_{t-1}, x_{t-1}] = \alpha + \psi E[y_t \mid y_{t-1}, x_{t-1}] \\ &= (1+\psi)\alpha + \psi^2 y_{t-1} + \psi\phi x_{t-1}. \end{aligned}$$

Substituting these equations into the model (1) yields the following actual process for the endogenous variable y_t:

$$\begin{aligned} y_t &= \beta_0 + \alpha[\beta_1 + \beta_2(1+\psi)] + \left(\beta_1\psi + \beta_2\psi^2\right) y_{t-1} + (\beta_1\phi + \beta_2\psi\phi) x_{t-1} + x_t \\ &= T_\alpha(\boldsymbol{\theta}) + T_\psi(\boldsymbol{\theta}) y_{t-1} + T_\phi(\boldsymbol{\theta}) x_{t-1} + x_t. \end{aligned}$$

Here, $\boldsymbol{\theta}' = (\alpha, \psi, \phi)$ denotes the vector of parameters of the perceived law of motion and the operator $T(\boldsymbol{\theta})' = [T_\alpha(\boldsymbol{\theta}), T_\psi(\boldsymbol{\theta}), T_\phi(\boldsymbol{\theta})]$ maps the parameter vector $\boldsymbol{\theta}$ of the perceived law of motion (3b) to the parameters of the resulting actual process for y_t. Hence, the operator $T(\boldsymbol{\theta})$ is given by

$$T(\boldsymbol{\theta}) = T\begin{pmatrix} \alpha \\ \psi \\ \phi \end{pmatrix} = \begin{pmatrix} T_\alpha(\boldsymbol{\theta}) \\ T_\psi(\boldsymbol{\theta}) \\ T_\phi(\boldsymbol{\theta}) \end{pmatrix} = \begin{pmatrix} \beta_0 + \alpha[\beta_1 + \beta_2(1+\psi)] \\ \beta_1\psi + \beta_2\psi^2 \\ \beta_1\phi + \beta_2\phi\psi \end{pmatrix}, \tag{4}$$

and from (4) we can compute the equilibrium points of $T(\boldsymbol{\theta})$:

$$\boldsymbol{\theta}_a'^{*} = \left[\beta_0(1-\beta_1-\beta_2)^{-1}, 0, 0\right], \tag{5a}$$

$$\Theta_b^{*} = \left\{\boldsymbol{\theta} \mid \boldsymbol{\theta}' = \left(-\beta_0\,\beta_2^{-1}, (1-\beta_1)\beta_2^{-1}, \phi\right), \qquad \phi \in \mathbf{R}\right\}. \tag{5b}$$

It is apparent that the equilibrium points (5a) and (5b) imply that the perceived law of motion (3b) coincides with the rational expectations solutions of (1).

Following Evans (1989), a rational expectations solution is E-stable if this solution is asymptotically stable with respect to the differential equation

$$\frac{d\boldsymbol{\theta}}{d\tau} = \dot{\boldsymbol{\theta}} = T(\boldsymbol{\theta}) - \boldsymbol{\theta} \tag{6}$$

based on the operator $T(\boldsymbol{\theta})$. This in turn requires the associated equilibrium points (5a) and (5b) to be asymptotically stable equilibrium points of the differential equation (6). Taking into account that the Jacobian matrix of the differential equation (6) evaluated at these equilibrium points is given by $J(\boldsymbol{\theta}^*) = \nabla_\theta T(\boldsymbol{\theta}^*) - I_3$, it is possible to derive the following E-stability conditions [Evans and Honkapohja (1992)]:

(1) The rational expectations solution (2a) is E-stable with respect to the perceived laws of motion (3a) and (3b) if

$$\beta_1 < 1, \qquad \beta_1 + \beta_2 < 1.$$

(2) The rational expectations solutions (2b) are E-stable with respect to the perceived laws of motion (3a) and (3b) if[5]

$$\beta_1 > 1, \qquad \beta_2 < 0.$$

As Figure 1 makes clear, these stability conditions imply that the parameter space of the model is partitioned in a way that, given the perceived laws of motion (3a) and (3b), either one of the solutions (2a) and (2b) or none of these solutions is E-stable. Hence, depending on the values of the parameters β_1 and β_2, we can distinguish three regions in Figure 1. In region 1, the only E-stable solution is (2a), whereas in region 2 only the ARMA solutions (2b) are E-stable. Finally, in region 3 neither of the two solutions is E-stable.

3. LEARNING WITH AN AR(1) PERCEIVED LAW OF MOTION

In what follows, it is assumed that agents in the model use perceived law of motion (3a) for the endogenous variable y_t in order to form the relevant expectations. Furthermore, it is assumed that the parameters of this perceived law of motion are comprised in the vector $\boldsymbol{\theta}' = (\alpha, \psi)$. In every period t, agents learn by modifying the relevant vector $\boldsymbol{\theta}_t$ by means of new observations of the exogenous and endogenous variables that become available to them in this period.

With $\mathbf{z}_t' = (1, y_{t-1})$, expectations formed on the basis of the perceived law of motion (3a) imply that the actual value for the endogenous variable will be

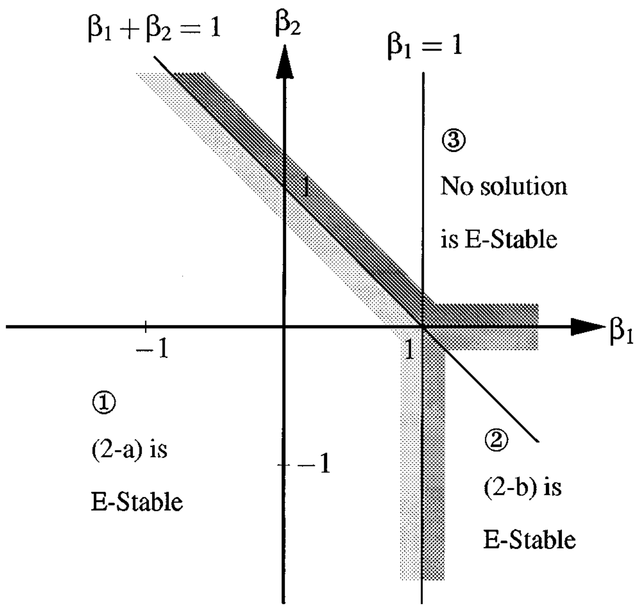

FIGURE 1. E-stability of (2a) and (2b).

$y_t = \mathbf{z}_t' T(\boldsymbol{\theta}_t) + x_t$, where $T(\boldsymbol{\theta})' = [T_\alpha(\boldsymbol{\theta}), T_\psi(\boldsymbol{\theta})]$ results from (4) by dropping the last row. This results in the following stochastic process on the vector $\mathbf{z}_t$:

$$\mathbf{z}_{t+1} = \begin{pmatrix} 1 \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T_\alpha(\boldsymbol{\theta}_t) & T_\psi(\boldsymbol{\theta}_t) \end{pmatrix} \begin{pmatrix} 1 \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_t. \tag{7}$$

Note that, given $\boldsymbol{\theta}_t$, the stochastic process (7) is stationary only if $|T_\psi(\boldsymbol{\theta}_t)| < 1$. Stationarity of (7) is an important assumption for the analysis of convergence of learning processes that are presented in the following.

Regarding the estimation procedures or learning algorithms that agents use to adapt their parameter vector $\boldsymbol{\theta}_t$ in the light of past experience, two alternative procedures are considered[6]:

(1) *SG procedure or algorithm:* The SG algorithm modifies the parameter vector on the basis of the gradient of the forecast $\hat{y}_t$ of the endogenous variable in period t (after x_t has been observed), $\hat{y}_t = \mathbf{z}_t' \boldsymbol{\theta}_t + x_t$, with respect to $\boldsymbol{\theta}_t$ and the forecast error $y_t - \hat{y}_t = \mathbf{z}_t'[T(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t]$. We get

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \gamma_{t+1}\mathbf{z}_t\{\mathbf{z}_t'[T(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t]\}, \tag{8}$$

where $\gamma_t = t^{-\kappa}$ with $0 < \kappa \leq 1$ as a time-dependent and declining learning rate. Such algorithms are quite common, especially for recursive estimation of nonlinear models such as neural networks [see Sargent (1993) and Kuan and White (1994)].

(2) *Recursive LS algorithm.* The main difference between the SG algorithm and recursive LS algorithm is that the latter uses the inverse of the estimated moment matrix of

the explanatory variables to determine the step size of the parameter modification. The LS algorithm thus is given as[7]

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \left(\frac{1}{t+1}\right) R_{t+1}^{-1} \mathbf{z}_t \{\mathbf{z}_t' [T(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t]\}, \tag{9a}$$

$$R_{t+1} = R_t + \left(\frac{1}{t+1}\right)[\mathbf{z}_t \mathbf{z}_t' - R_t]. \tag{9b}$$

Both algorithms are quite similar: Boundedly rational learning on the side of the agents in the model means that they take the stochastic process generating y_t (and thus $\mathbf{z}_t$) as given. This implies that the gradient of the squared prediction error $e_t^2 = (y_t - \hat{y}_t)^2$ with respect to $\boldsymbol{\theta}_t$ is given for them as $-2\mathbf{z}_t\mathbf{z}_t'[T(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t]$. In both algorithms, this term determines the direction of adjustments of the parameter vector $\boldsymbol{\theta}_t$ during the iterations. Indeed, with $\gamma_t = 1/t$, the only difference between these two algorithms is that the LS algorithm in equation (9a) modifies the step size of the parameter modification according to the inverse of the estimated Hessian of the squared prediction error $(\mathbf{z}_t\mathbf{z}_t')^{-1}$. Thus, the LS algorithm can be interpreted as a modification of the SG algorithm that results in a stochastic Newton algorithm [Ljung and Söderström (1983)].

Under certain assumptions, following Ljung (1977), the asymptotic properties of learning algorithms such as (8) and (9) can be described by means of an associated deterministic differential equation.[8] Evans and Honkapohja (1995) show that these assumptions are satisfied for the model considered here. Thus, we can proceed directly by analyzing the respective differential equations.

The differential equation associated with the SG algorithm (8) can be specified as

$$\frac{d\boldsymbol{\theta}}{d\tau} = \dot{\boldsymbol{\theta}} = E[\mathbf{z}_t\mathbf{z}_t'][T(\boldsymbol{\theta}) - \boldsymbol{\theta}] = M_z(\boldsymbol{\theta})[T(\boldsymbol{\theta}) - \boldsymbol{\theta}]. \tag{10}$$

Regarding the LS algorithm, the resulting associated differential equations are

$$\frac{d\boldsymbol{\theta}}{d\tau} = \dot{\boldsymbol{\theta}} = R^{-1} M_z(\boldsymbol{\theta})[T(\boldsymbol{\theta}) - \boldsymbol{\theta}], \tag{11a}$$

$$\frac{dR}{d\tau} = \dot{R} = M_z(\boldsymbol{\theta}) - R. \tag{11b}$$

In both differential equations, $M_z(\boldsymbol{\theta})$ refers to the (2×2) moment matrix $E[\mathbf{z}_t\mathbf{z}_t']$. From (7) and the above-made assumptions, it then follows that $M_z(\boldsymbol{\theta})$ is bounded and given by[9]

$$M_z(\boldsymbol{\theta}) = E[\mathbf{z}_t\mathbf{z}_t'] = \begin{bmatrix} 1 & \dfrac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})} \\ \dfrac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})} & \left[\dfrac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})}\right]^2 + \dfrac{1}{1 - T_\psi(\boldsymbol{\theta})^2}\sigma_x^2 \end{bmatrix}. \tag{12}$$

The essential result of Ljung (1977) states that the parameter vector $\boldsymbol{\theta}_t$ resulting from learning algorithms such as (8) and (9) will only converge—if at all—to equilibrium points of these associated differential equations. Moreover, the probability for such a convergence to occur is positive only if the respective equilibrium points are asymptotically stable. Convergence with probability 1 can be guaranteed only if additional restrictions are employed: $\boldsymbol{\theta}_t$ almost surely will converge to an equilibrium point of the associated differential equation if the learning algorithm is equipped with a so-called projection facility. This projection guarantees that the estimated parameter vector almost surely will stay infinitely often in the domain of attraction of that equilibrium point. Given a stable equilibrium point of the associated differential equation, it is always possible to find a nontrivial set (i.e., a set containing not only the equilibrium point itself) having this property, such that one can always find a nontrivial projection facility ensuring almost sure convergence [see Marcet and Sargent (1989)].

Now let Θ^R be the set of all $\boldsymbol{\theta} \in \mathbf{R}^2$, such that $T(\boldsymbol{\theta}) = \boldsymbol{\theta}$. Obviously, $\Theta^R = \{\boldsymbol{\theta}_a^*, \boldsymbol{\theta}_b^*\}$, where $\boldsymbol{\theta}_b^*$ is the one element from Θ_b^*, for that $\phi = 0$. Thus, all $\boldsymbol{\theta}^* \in \Theta^R$ are equilibrium points of the operator $T(\boldsymbol{\theta})$. Recall that this means that the perceived law of motion (3a) coincides with one of the rational expectations solutions (2a) and (2b) for all $\boldsymbol{\theta}^* \in \Theta^R$.

Looking at (10) and (11), we see that we can state the following proposition without any formal proof[10]:

PROPOSITION 1. *Any $\boldsymbol{\theta}^* \in \Theta^R$ is an equilibrium point of the differential equations* (10) *and* (11).

As already noted, rational expectations solutions are defined to be equilibrium points of the operator $T(\boldsymbol{\theta})$. Thus, Proposition 1 states that any rational expectations solution might be a possible outcome of the learning procedures considered here.

The following proposition now establishes the well-known correspondence between the E-stability of rational expectations solutions and the possible convergence of the LS algorithm.

PROPOSITION 2 [Marcet and Sargent (1989)]. *A necessary and sufficient condition for $\boldsymbol{\theta}^* \in \Theta^R$ and an associated $R^* = M_z(\boldsymbol{\theta}^*)$ to be a stable equilibrium point of the differential equation* (11) *is that $\boldsymbol{\theta}^*$ be a stable equilibrium point of the differential equation $\dot{\boldsymbol{\theta}} = T(\boldsymbol{\theta}) - \boldsymbol{\theta}$.*

The proof is based upon showing that, for $\boldsymbol{\theta}^* \in \Theta^R$, all eigenvalues of the Jacobian matrix of (11) that do not equal -1 coincide with the eigenvalues of the Jacobian $J(\boldsymbol{\theta})$ of $T(\boldsymbol{\theta}) - \boldsymbol{\theta}$ evaluated at $\boldsymbol{\theta}^*$. To check local stability, it is therefore sufficient to look at the subsystem (11a). Since $T(\boldsymbol{\theta}^*) = \boldsymbol{\theta}^*$, for all $\boldsymbol{\theta}^* \in \Theta^R$ the Jacobian matrix $J_{KQ}(\boldsymbol{\theta})$ of (11) evaluated at $\boldsymbol{\theta}^*$ is given by $J_{KQ}(\boldsymbol{\theta}^*) = (R^*)^{-1} M_z(\boldsymbol{\theta}^*) J(\boldsymbol{\theta}^*)$. Because an equilibrium point of (11) implies that $R^* = M_z(\boldsymbol{\theta}^*)$, we get $J_{KQ}(\boldsymbol{\theta}^*) = J(\boldsymbol{\theta}^*)$ such that the eigenvalues of $J_{KQ}(\boldsymbol{\theta}^*)$ coincide with that of $J(\boldsymbol{\theta}^*)$.

Note that the validity of this proposition is quite general and not restricted to the AR(1) perceived law of motion used here: Because any equilibrium point of the differential equations (11) must be a fixed point of the operator $T(\boldsymbol{\theta})$, such an equilibrium point will be stable only if the corresponding rational expectations solution is E-stable. An immediate consequence is that the LS algorithm will never converge to rational expectations solutions that are not E-stable with respect to the perceived law of motion under consideration. Thus, to make predictions regarding the possible convergence of this learning procedure and in order to select among multiple rational expectations solutions, it is sufficient to check whether the respective solutions are E-stable or not. This is a quite comfortable result because the formal proof of E-stability is easier to check than the possible convergence of the LS algorithm.

Let us now turn to the SG algorithm. In this case the stability analysis of the equilibrium points Θ^R is a more complicated matter: Note that the correspondence between the concept of E-stability and the possible convergence of the LS algorithm is due to the fact that any equilibrium point of (11) implies that $R = M_z(\Theta)$, such that (11a) reduces to $\dot{\boldsymbol{\theta}} = T(\boldsymbol{\theta}) - \boldsymbol{\theta}$. However, because the step size of the SG algorithm is independent of R, the argument underlying Proposition 2 does not carry over to the SG algorithm.

So, the local asymptotic stability of the equilibrium points of (10) has to be checked explicitly. It is quite easy to show that the equilibrium points of (10) coincide with the equilibrium points of $T(\boldsymbol{\theta}) - \boldsymbol{\theta}$.[11] Thus, the Jacobian matrix of (10) evaluated at an equilibrium point $\boldsymbol{\theta} \in \Theta^R$ is given as

$$J_{SG}(\boldsymbol{\theta}) = M_z(\boldsymbol{\theta})[\nabla_\theta T(\boldsymbol{\theta}) - I_2]. \tag{13}$$

This equation already reveals that if there is any connection between E-stability conditions and the convergence conditions of the SG algorithm, this connection will not be an obvious one. Of course, E-stability guarantees that the eigenvalues of $\nabla_\theta T(\boldsymbol{\theta}) - I_2$ are negative. However, no general conclusions regarding the eigenvalues of J_{SG} can be drawn from this. Nevertheless, at least for the AR(1) perceived law of motion, it is now possible to state the following proposition:

PROPOSITION 3.

(i) *A necessary and sufficient condition for $\boldsymbol{\theta}_a^*$ to be an asymptotically stable equilibrium point of the differential equation* (10) *is that the corresponding white-noise solution be E-stable.*
(ii) *Given stationarity of the ARMA*(1,1) *solutions* (2*b*), *a necessary and sufficient condition for $\boldsymbol{\theta}_b^*$ to be an asymptotically stable equilibrium point of the differential equation* (10) *is that the corresponding ARMA*(1,1) *solution be E-stable.*

Throughout this paper, attention is restricted to rational expectations solutions that are stationary, and so, Proposition 3 establishes a one-to-one correspondence between the SG algorithm and E-stability conditions. Thus, if the agent's perceived law of motion is given by the AR(1) model (3a), we get the following results: First, the asymptotic properties of the SG algorithm and the LS algorithm are identical in the sense that both algorithms will converge to identical parameter vectors and thus to identical rational expectations solutions. Second, the necessary conditions for the convergence of both algorithms toward one of the two solutions coincide and are identical to the conditions for E-stability of these solutions.

4. LEARNING WITH AN ARMA(1,1) PERCEIVED LAW OF MOTION

It is now assumed that the perceived law of motion used by the agents is given by the ARMA(1,1) model (3b). Given such a perceived law of motion, we have $\mathbf{z}_t' = (1, y_{t-1}, x_{t-1})$ and the stochastic process generating $\mathbf{z}_t$ is given as

$$\mathbf{z}_{t+1} = \begin{pmatrix} 1 \\ y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ T_\alpha(\boldsymbol{\theta}_t) & T_\psi(\boldsymbol{\theta}_t) & T_\phi(\boldsymbol{\theta}_t) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x_t, \tag{14}$$

where $T(\boldsymbol{\theta})$ now coincides with (4). As before, stationarity requires $|T_\psi(\boldsymbol{\theta})| < 1$ and, from (14), the (3×3) moment matrix $E[\mathbf{z}_t\mathbf{z}_t'] = M_z(\boldsymbol{\theta})$ for the vector $\mathbf{z}_t$ follows as[12]

$$M_z(\boldsymbol{\theta}) = E[\mathbf{z}_t\mathbf{z}_t']$$
$$= \begin{pmatrix} 1 & \dfrac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})} & 0 \\ \dfrac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})} & \left[\dfrac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})}\right]^2 + \dfrac{1 + T_\phi(\boldsymbol{\theta})^2 + 2T_\psi(\boldsymbol{\theta})T_\phi(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})^2}\sigma_x^2 & \sigma_x^2 \\ 0 & \sigma_x^2 & \sigma_x^2 \end{pmatrix}. \tag{15}$$

As before, the aim is to analyze the possible convergence of the two alternative learning procedures described in the preceding section. However, in presence of an ARMA(1,1) perceived law of motion, this analysis becomes more complicated.

4.1. Learning Via the SG Algorithm—a Formal Analysis

Consider first the SG algorithm and its associated differential equation (10). Equilibrium points of this differential equation satisfy the equation $M_z(\boldsymbol{\theta})[T(\boldsymbol{\theta}) - \boldsymbol{\theta}] = 0$. Obviously, all $\boldsymbol{\theta}^* \in \Theta_b^*$ as well as $\boldsymbol{\theta}_a^*$ are equilibrium points of this differential equation because all of these $\boldsymbol{\theta}^*$ imply that $T(\boldsymbol{\theta}^*) = \boldsymbol{\theta}^*$. However, as stated in the following proposition, there are other equilibrium points as well: If we define the set $\Theta^S = \{\boldsymbol{\theta}_a^*, \Theta_b^*\}$ as the set of all equilibrium points of $T(\boldsymbol{\theta}) - \boldsymbol{\theta}$ and the set Θ^F as the set containing all equilibrium points of (10), we get Proposition 4.

PROPOSITION 4. *Define* $\Theta^N = \Theta^F \backslash \Theta^S$. *Then* Θ^N *is non-empty and given by*

$$\Theta^N = \left\{\boldsymbol{\theta} \mid \boldsymbol{\theta}' = \left[(1-\psi)\beta_0(1-\beta_1-\beta_2)^{-1}, \psi, -\psi\right], \qquad \psi \in \mathbf{R}, \psi \neq 0\right\}.$$

Proposition 4 says that the differential equation (10) possesses equilibrium points that are not simultaneously equilibrium points of the operator $T(\boldsymbol{\theta})$ and therefore do not represent rational expectations solutions in the usual sense.

Given a $\boldsymbol{\theta} \in \Theta^N$, the perceived law of motion results in

$$y_t = (1-\psi)\beta_0(1-\beta_1-\beta_2)^{-1} + \psi y_{t-1} - \psi x_{t-1} + x_t. \tag{16}$$

As can be seen, this process is generated from the white-noise solution (2a) by multiplication with the lag polynomial $(1-\psi L)$. Note that the overparameterized white-noise process (16) and the solution (2a) are equivalent for all t only if $y_0 = x_0 + [\beta_0/(1-\beta_1-\beta_2)]$ [Evans and Honkapohja (1986)].[13] If this initial condition is not satisfied, both processes are at least asymptotically equivalent if $|\psi| < 1$, that is, if (16) is a stationary process. Regarding the expectations based upon such a perceived law of motion, it must be noted, however, that these expectations imply an actual process for the endogenous variable y_t that is not given by (16). Substitution of $\boldsymbol{\theta} \in \Theta^N$ into (14) reveals that the actual process for y_t is given by

$$\begin{aligned} y_t = {} & \frac{\beta_0}{1-\beta_1-\beta_2}\left(1-\beta_1\psi-\beta_2\psi^2\right) + \left[\beta_1\psi+\beta_2\psi^2\right]y_{t-1} \\ & - \left[\beta_1\psi+\beta_2\psi^2\right]x_{t-1} + x_t. \end{aligned} \tag{17}$$

For any initial values, y_0 and x_0, the process (17) is asymptotically equivalent to (2a) only if $|\beta_1\psi + \beta_2\psi^2| < 1$.[14] This condition is equivalent to $|T_\psi(\boldsymbol{\theta})| < 1$; that is, the stochastic process (14) is stationary for all $\boldsymbol{\theta} \in \Theta^N$ that in addition satisfy $|\beta_1\psi + \beta_2\psi^2| < 1$. Note that, for this condition to hold, it is neither necessary nor sufficient that $|\psi| < 1$.

The main difference between perceived laws of motion that are parameterized by $\boldsymbol{\theta} \in \Theta^N$ and those parameterized by $\boldsymbol{\theta} \in \Theta^S$ is that the latter coincide with rational expectations solutions. Contrary to this, perceived laws of motion parameterized by $\boldsymbol{\theta} \in \Theta^N$ will coincide with the rational expectations solution (2a) only asymptotically and even this occurs only if $|T_\psi(\boldsymbol{\theta})| < 1$. In what follows, I refer to perceived laws of motion that are parameterized by $\boldsymbol{\theta} \in \Theta^N$ and satisfy the condition $|T_\psi(\boldsymbol{\theta})| < 1$ as overparameterized white-noise solutions and subsume these solutions under the rational expectations solutions of the underlying model (1).

Thus, given the perceived law of motion (3b) and the learning procedure (8), the model (1) exhibits two distinct sets of rational expectations solutions, each corresponding to a continuum of parameter vectors $\boldsymbol{\theta}$: The set Θ_b^* from (5b) generates ARMA solutions of the form (2b), and the set

$$\Theta_a^* = \left\{\boldsymbol{\theta} \mid \boldsymbol{\theta}' = \left[(1-\psi)\beta_0(1-\beta_1-\beta_2)^{-1}, \psi, -\psi\right], \qquad |T_\psi(\boldsymbol{\theta})| < 1\right\}$$

generates, at least asymptotically, the white-noise solution (2a). In what follows, the set $\Theta^F = \{\Theta_a^*, \Theta_b^*\}$ refers to these vectors.

The stationarity condition $|T_\psi(\boldsymbol{\theta})| < 1$ is important in the following respects: First, stationarity of the process for $\mathbf{z}_t$ resulting from (14) is necessary to perform the convergence analysis of the learning process. Second, this condition ensures that the perceived law of motion (3b) parameterized by $\boldsymbol{\theta} \in \Theta^N$ coincides with the rational expectations solution (2a) at least asymptotically. Furthermore, a closer look at the moment matrix $M_z(\boldsymbol{\theta})$ in equation (15) reveals that the differential equation $\dot{\boldsymbol{\theta}} = M_z(\boldsymbol{\theta})[T(\boldsymbol{\theta}) - \boldsymbol{\theta}]$ associated with the SG algorithm will exhibit singularities for all $\psi \in \mathbf{R}$ that imply $|T_\psi(\boldsymbol{\theta})| = 1$. This means that this differential equation may exhibit quite complicated dynamics that cannot be described solely by formal analysis.

An additional difficulty arises because convergence results for stochastic approximation algorithms using the approach of Ljung (1977) are not valid if the respective equilibrium points represent continua as is the case for Θ_a^* and Θ_b^* [Evans and Honkapohja (1994)]. Since the motivation for these convergence results is to show that algorithms like (8) converge to specific points in the parameter space, the respective theorems apply to isolated equilibrium points only. However, there is one result that can be used, even if we are confronted with a continuum of equilibrium points: Theorem 3 of Ljung (1977) states that the learning rate of the algorithm can be chosen such that for $t \to \infty$ the time path of $\boldsymbol{\theta}_t$ of the algorithm stays arbitrary close to an exponentially stable solution of the associated differential equation with an arbitrarily high probability. Hence, if we have trajectories of the associated differential equation that converge to points that belong to a continuum of equilibrium points, the probability that the learning algorithm converges to points that belong to this continuum cannot be zero. Moreover, irrespective of this, at least the following statements about the SG algorithm and its associated differential equation remain valid: The SG algorithm considered here will not converge to parameter vectors that are not equilibrium points of the associated differential equation and it will never converge to equilibrium points that are unstable.

Thus, to derive the conditions for possible convergence of the SG algorithm toward one of the rational expectations solutions, the stability of the equilibrium points $\boldsymbol{\theta}^* \in \Theta^F = \{\Theta_a^*, \Theta_b^*\}$ has to be checked. Unfortunately, a formal proof of stability is no easy task because the Jacobian matrix of (10) will possess at least one eigenvalue that equals zero for any $\boldsymbol{\theta}^* \in \Theta^F$. However, although we cannot use standard techniques for stability analysis in this case, it is possible to obtain at the least the following necessary condition for convergence[15]:

PROPOSITION 5. *One eigenvalue of the Jacobian matrix of* (10) *evaluated at* $\boldsymbol{\theta}^* \in \Theta^F$ *equals zero. In case of of stationarity of* $\mathbf{z}_t$, *that is, if* $|T_\psi(\boldsymbol{\theta}^*)| < 1$, *the remaining two eigenvalues are negative if and only if*

(i) *for the white-noise solutions* $\boldsymbol{\theta}^* \in \Theta_a^*$, *we have* $\beta_1 + \beta_2 < 1$ *as well as* $\psi < (1-\beta_1)/\beta_2$ *if* $\beta_2 > 0$, *or* $\psi > (1-\beta_1)/\beta_2$ *if* $\beta_2 < 0$;

(ii) *for the ARMA*(1,1) *solutions* $\boldsymbol{\theta}^* \in \Theta_b^*$, *we have* $\beta_2 < 0$ *as well as* $\phi \in (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ *if* $1-\beta_1 > 0$, *or* $\phi \notin (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ *if* $1-\beta_1 < 0$.

In Proposition 5, attention is restricted to stationary rational expectations solutions of the model (1). This is reasonable because we are particularly interested in convergence conditions for the SG algorithm itself rather than in the stability conditions for equilibrium points of the differential equation (10).

Note that Proposition 5 does not give a sufficient condition for the stability of equilibrium points belonging to Θ^F. Nevertheless, there is at least one result that can be derived immediately: There exist rational expectations solutions that may be E-stable with respect to the perceived law of motion under consideration, but that cannot be learned with the help of the SG algorithm.[16] If, for instance, the underlying model is such that $\beta_2 < 0$ and $1 - \beta_1 < 0$, any $\boldsymbol{\theta} \in \Theta_b^*$ is E-stable with respect to a perceived law of motion (3b) (cf. Figure 1), but for $\phi \in (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ the respective equilibrium point of the differential equation is unstable, such that these solutions cannot be reached via SG learning.

Although Proposition 5 allows us to identify parts of the continua of equilibrium points in Θ^F that are definitively unstable, there always remain parts where stability is not a priori excluded. Moreover, as is verified later by numerical simulations of the associated differential equation, the conditions stated in Proposition 5 also seem to be sufficient for stability of these equilibrium points. Taking into account that Proposition 5 does not require the respective rational expectations solution under consideration to be E-stable in order for the respective equilibrium point to be stable or at least not a priori unstable, we get the result that the differential equation associated with the SG algorithm may have equilibrium points that are not E-stable but that are nevertheless stable. Given such a result, it is therefore quite reasonable that the learning algorithm may converge to rational expectations solutions that are not E-stable. In Section 4.2, results from numerical simulations of the SG algorithm are presented to verify that the properties of the associated differential equation do indeed carry over to the learning algorithm. As a result, it is shown that learning via the SG algorithm may converge toward rational expectations solutions that are not E-stable with respect to the perceived law of motion (3b).

4.2. Simulation Results for the SG Algorithm and Its Associated Differential Equation

In this subsection, results from numerical simulations of the differential equation associated with the SG algorithm as well as the SG algorithm itself are presented to verify the earlier statements regarding the convergence properties of the SG algorithm based upon the perceived law of motion (3b). Such simulation results cannot be seen as formal proofs, but they give at least strong evidence in favor of the statements made earlier.

The simulations are based upon two specifications of the model, with alternative values for the parameters β_1 and β_2 and fixed values $\beta_0 = 1$ and $\sigma_x^2 = 1$. The first configuration specifies $\beta_1 = -0.25$ and $\beta_2 = -1.5$, such that only the white-noise solution (2a) is E-stable, whereas the second configuration specifies $\beta_1 = 1.25$ and

$\beta_2 = -1$, such that only the ARMA(1,1) solutions (2b) are E-stable (cf. Figure 1). Note that these values for β_1 and β_2 imply that, for each configuration, we have $|(1-\beta_1)/\beta_2| < 1$. Thus, the stochastic process (14) for $\boldsymbol{z}_t$ will be stationary for all $\boldsymbol{\theta} \in \Theta_b^*$. Stationarity of (14) for $\boldsymbol{\theta} \in \Theta_a^*$ requires no further restrictions because, for all β_1, β_2, there exists a non-empty interval for ψ, implying $|T_\psi(\boldsymbol{\theta})| < 1$.[17]

Given these two specifications of the model, the respective differential equation was solved numerically starting with initial vectors $\boldsymbol{\theta}(0)$ that are given by four predetermined values for the parameter ψ and a grid of points in the α, ϕ-plane for each value of ψ.[18] To avoid problems arising from the existence of singularities, ψ was restricted such that the stationarity condition $|T_\psi(\boldsymbol{\theta})| < 1$ is satisfied. For the first configuration $\beta_1 = -0.25$ and $\beta_2 = -1.5$, the predetermined values for ψ are given by $\psi = -0.9$, $\psi = (1-\beta_1)/\beta_2 = -0.833$, $\psi = -0.75$, and $\psi = -0.65$. The values for the parameters α and ϕ form a 50×50 grid of equidistant points with $-1.25 \leq \alpha \leq 2.75$ as well as $-1.25 \leq \phi \leq 2.75$. For the other configuration of the model, the predetermined values for ψ are given by $\psi = 0.1$, $\psi = (1-\beta_1)/\beta_2 = 0.25$, $\psi = 0.7$, and $\psi = 0.9$ and the values for the parameters α and ϕ form a 50×50 grid of equidistant points with $-0.5 \leq \alpha \leq 3.0$ as well as $-7 \leq \phi \leq 1$. So in all, for each configuration of the model, the differential equation has been solved for 10,000 different initial vectors $\boldsymbol{\theta}(0)$.

Figures 2 and 3 present some details of the simulation results to illustrate the quite complex dynamics of the differential equation and especially the sensitivity to initial conditions. For each of two configurations of the model, the figures show the respective grid of starting values in the α, ϕ-plane for one of the above-described predetermined values of ψ. A bold point indicates that the trajectory starting in this point converges to a $\boldsymbol{\theta}^* \in \Theta_b^*$; a thin point indicates convergence toward Θ_a^*. The

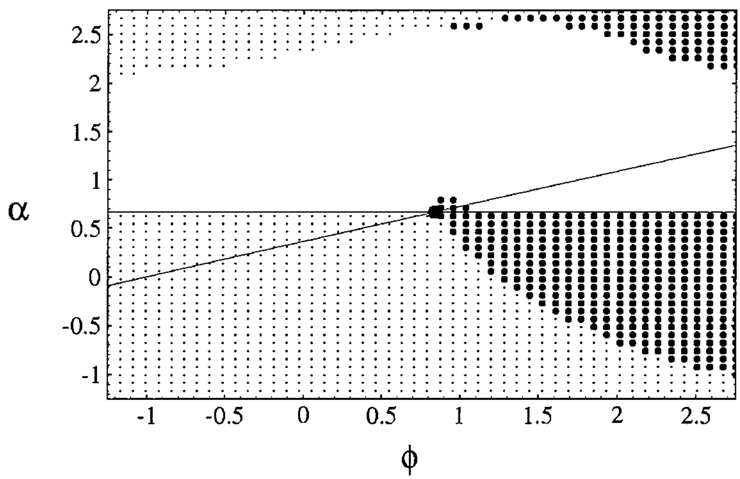

FIGURE 2. Convergence of the differential equation (10) with $\beta_0 = 1$, $\beta_1 = -0.25$, $\beta_2 = -1.5$, and $\sigma_x^2 = 1$ for $\psi = (1-\beta_1)/\beta_2 = -0.833$ and different initial values for α and ϕ.

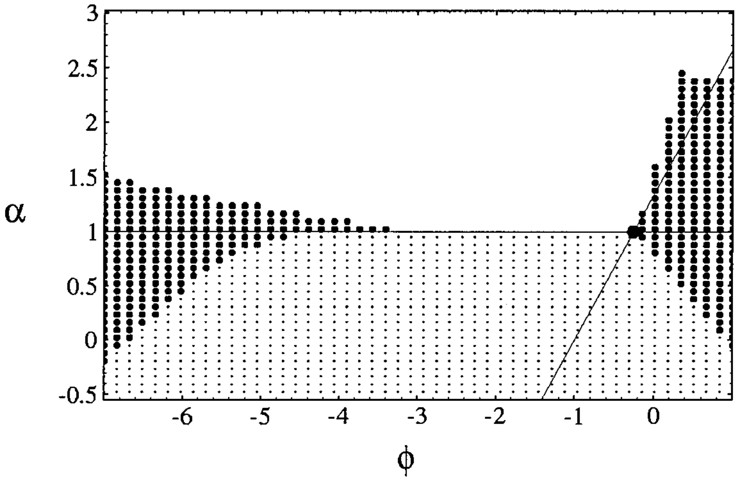

FIGURE 3. Convergence of the differential equation (10) with $\beta_0 = 1$, $\beta_1 = 1.25$, $\beta_2 = -1$, and $\sigma_x^2 = 1$ for $\psi = (1 - \beta_1)/\beta_2 = 0.25$ and different initial values for α and ϕ.

white regions without any points indicate starting values of diverging trajectories. In each figure, the solid lines represent projections of all stationary points onto the respective α, ϕ-plane, and the single bold point displays the stationary point in this plane.

The main result presented in Figure 2 is that there exist initial vectors $\boldsymbol{\theta}(0)$ that give rise to converging trajectories and that, depending on the initial values, convergence toward $\boldsymbol{\theta} \in \Theta_a^*$ as well as toward $\boldsymbol{\theta} \in \Theta_b^*$ might occur, although the configuration of the model implies that the ARMA(1,1) solutions (2b) are not E-stable. Thus, we have the result that the differential equation (10) possesses stable equilibrium points that do not correspond to E-stable rational expectations solutions. Figure 3 shows a similar result for the other configuration of the model. With the parameters of the model given by $\beta_1 = 1.25$ and $\beta_2 = -1$, the white-noise solution (2a) is not E-stable, but as the figure reveals, there nevertheless exist initial values that give rise to trajectories converging toward $\boldsymbol{\theta} \in \Theta_a^*$. As for the other configuration, it depends on the specific initial values whether diverging trajectories or convergence toward $\boldsymbol{\theta} \in \Theta_a^*$ as well as toward $\boldsymbol{\theta} \in \Theta_b^*$ occurs. Note that because of $1 - \beta_1 < 0$, all ARMA(1,1) solutions with $\phi \in (-(1 - \beta_1)/\beta_2, -\beta_2/(1 - \beta_1))$ are unstable. Thus, there are two separate areas with initial values converging toward the two distinct stable regions of Θ_b^*. Summarizing, we again get the result that the differential equation (10) possesses stable equilibrium points that do not correspond to E-stable rational expectations solutions.

Figure 4 shows the equilibrium points that resulted from the numerical simulations of the differential equation. For all converging trajectories depicted in Figures 2 and 3, the resulting parameters ψ and ϕ are shown.[19] The solid lines in the both figures correspond to the continua of equilibrium points Θ_a^* and Θ_b^*, the dashed lines indicate the interval for the parameter ϕ specified in Proposition 5. In

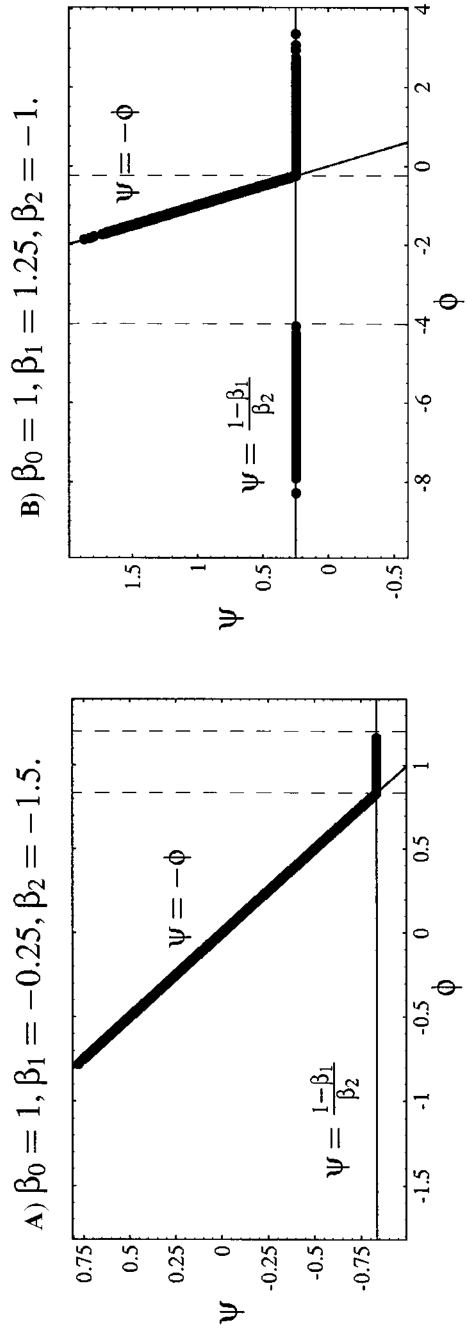

FIGURE 4. Equilibrium points of the differential equation (10) for the simulations presented in Figures 2 and 3.

Figure 4A, equilibrium points that resulted for the configuration $\beta_1 = -0.25$ and $\beta_2 = -1.5$ are depicted. As can bee seen, there are two branches of stable equilibrium points. Moreover, the figure makes evident that the necessary conditions formulated in Proposition 5 indeed seem to be sufficient for stability: Stable equilibrium points are located everywhere on the solid line corresponding to $\psi = -\phi$, where the condition $\psi > (1-\beta_1)/\beta_2$ is satisfied and, in addition, they are located everywhere on the solid line corresponding to $\psi = (1-\beta_1)/\beta_2$, where the condition $\phi \in (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ is satisfied. Figure 4B, corresponding to the configuration $\beta_1 = 1.25$ and $\beta_2 = -1$, gives the same findings. The only difference is that, because $\beta_1 > 1$, stable equilibrium points are located everywhere on the solid line corresponding to $\psi = (1-\beta_1)/\beta_2$, where the condition $\phi \notin (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ is satisfied.

From the figures presented so far, it cannot be inferred whether the equilibrium points depicted in Figures 4A and 4B are locally stable or merely semistable. However, additional simulations—with initial vectors $\boldsymbol{\theta}(0)$ as specified earlier but with smaller variations in ψ—indicate that initial vectors in a small neighborhood of the respective sets of equilibrium points result in convergence to these sets. This means that the respective equilibrium points are locally stable.

Let us now turn to the SG algorithm itself. If the stability of equilibrium points in the differential equation (10) does not necessarily correspond to the E-stability of the respective rational expectations solution, it is reasonable that the SG algorithm may converge to solutions that are not E-stable with respect to the perceived law of motion (3b). The simulation results for the SG algorithm presented in Figure 5 show that this is indeed the case and thus verify this presumption.

These simulations are based upon the same configurations of the parameters β_1 and β_2 as well as the same initial vectors $\boldsymbol{\theta}_0$ that have been used for the numerical solution of the associated differential equation. In all of the simulation runs, the exogenous variable x_t was assumed to be the realization of a normally distributed random variable with $E[x_t] = 0$ and variance $\sigma_x^2 = 1.0$ and, regarding the learning rate γ_t, it was assumed that $\gamma_t = t^{-0.5}$, that is, $\kappa = 0.5$. This value for κ results in tolerably fast convergence, such that 15,000 iterations of the algorithm were sufficient to assess its convergence.[20] Note that, even if the initial vector $\boldsymbol{\theta}_0$ is given, the behavior of the SG algorithm depends on the initial values specified for the vector $\mathbf{z}_0$ as well as the realizations of the random variable x_t for $t = 0, 1, \ldots$. Because of this, for each of the 10,000 initial vectors, five simulation runs of the SG algorithm have been performed. Every single simulation run started with a randomly chosen vector $\mathbf{z}_0$ and different realizations for the exogenous variables x_t. Thus, all in all, the results for each configuration are based upon 50,000 different simulation runs of the SG algorithm.

As above, the bold points in Figure 5 show the resulting parameter values for ψ and ϕ for the two configurations of the model, for those simulation runs where convergence occurred.[21] Regarding the configuration $\beta_1 = -0.25$ and $\beta_2 = -1.5$, 2,713 of 50,000 simulations resulted in convergence [2,311 white noise/402 ARMA(1,1)]. As can be seen from Figure 5A, the SG algorithm converged

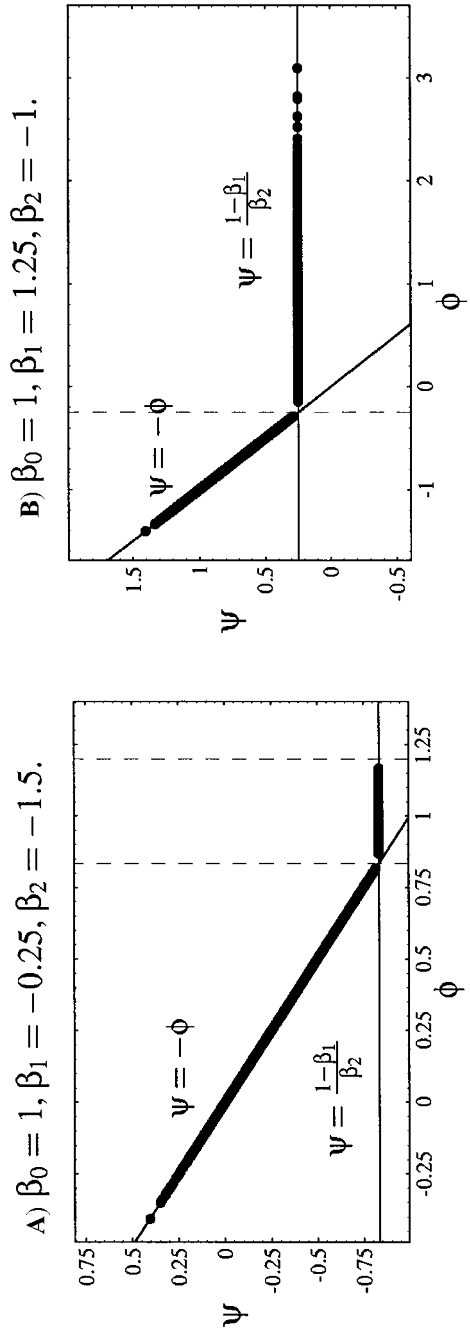

Figure 5. Convergence of the SG algorithm for two different configurations of the model [$x_t \sim N(0, 1.0)$, $\kappa = 0.5$, $t = 1, \ldots, 15{,}000$].

toward overparameterized white-noise solutions as well as toward ARMA(1,1) solutions (2b) that satisfy the stability conditions stated in Proposition 5. Because these ARMA(1,1) solutions are not E-stable, given the underlying configuration of the model, Figure 5A establishes the result that the SG algorithm indeed can converge toward rational expectations solutions that are not E-stable.

A similar result can be obtained with the other parameter configuration $\beta_1 = 1.25$ and $\beta_2 = -1$. Here, 4,335 of 50,000 simulations resulted in convergence [2,896 white-noise/1,439 ARMA(1,1)]. Although in this case the white-noise solution (2a) is not E-stable, Figure 5B reveals that the SG algorithm nevertheless converged toward overparameterized white-noise solutions.[22] As before, we get the result that the convergence of the SG algorithm toward rational expectations solutions is not one-to-one with E-stability conditions. Moreover, both figures show that, in the case of the perceived law of motion (3b), learning via the SG algorithm will not select a unique rational expectations solution.

Altogether, the simulation results reveal that the SG algorithm may indeed converge to parameter vectors that are not simultaneously equilibrium points of $T(\boldsymbol{\theta})$. Thus, it is by no means necessary that learning with an overparameterized model will result in values for the additional parameters that converge to zero, as would be the case for the differential equation (6) based on the operator $T(\boldsymbol{\theta})$. Although this result might not be very surprising, it nevertheless is worth noting because, as shown in the figures, a simulated learning algorithm will quite often converge to such overparameterized solutions.

4.3. A Remark on Learning via the LS Algorithm in Case of an ARMA(1,1) Perceived Law of Motion

Actually, one should not expect something new concerning LS learning with an ARMA(1,1) perceived law of motion because the results reviewed in Section 3 imply that the LS algorithm will converge only to equilibrium points that correspond to E-stable rational expectations solutions. However, given an ARMA(1,1) perceived law of motion, things become more complicated regarding the white-noise solution (2a): A look at the moment matrix (15) reveals that $M_z(\boldsymbol{\theta})$ becomes singular for any $\boldsymbol{\theta}^* \in \Theta_a^*$. The reason is that in the case of the white-noise solution (2a), the endogenous variable y_t contained in the perceived law of motion follows the process $y_t = \beta_0(1-\beta_1-\beta_2)^{-1} + x_t$. Thus, y_t is nothing more than a linear combination of the two exogenous variables that are already contained in the perceived law of motion and, consequently, $M_z(\boldsymbol{\theta})$ becomes singular for any $\boldsymbol{\theta}^* \in \Theta_a^*$. This implies that the differential equations (11) exhibit a singularity for all $\boldsymbol{\theta}^* \in \Theta_a^*$ and $R = M_z(\boldsymbol{\theta}^*)$ and, contrary to the SG algorithm, the set Θ_a^* does not represent equilibrium points of the differential equations (11).[23] Moreover, the LS algorithm (9) itself becomes undefined for all $\boldsymbol{\theta}_t^* \in \Theta_a^*$ and all associated $R_t = M_z(\boldsymbol{\theta}_t^*)$ because $M_z(\boldsymbol{\theta}_t^*)^{-1}$ will not exist.[24]

Even if formal analysis of the LS algorithm is not possible in the presence of such singularities, at least some evidence regarding the characteristics of the learning process can be given here. Numerical simulations of the differential equations (11) associated with the LS algorithm suggest that the E-stability conditions affect the properties of the singularity at Θ_a^*. In particular, there is evidence that the singularity attracts nearby trajectories if the white-noise solution is E-stable, and that it repels nearby trajectories if the white-noise solutions is not E-stable. Additional numerical simulations of the LS algorithm itself (not presented here) suggest that the connection between the learning algorithm and this differential equation still remains valid, meaning that the learning process steers toward $\boldsymbol{\theta}^* \in \Theta_a^*$ in the case of a white-noise solution that is E-stable. Clearly, in such a case, the moment matrix $M_z(\boldsymbol{\theta})$ approaches a singular matrix, such that—sooner or later—the learning algorithm will break down. Indeed, in the numerical simulations, this happened in finite time for all learning processes that were attracted to Θ_a^*.[25]

Given these findings and taking for granted that their interpretation is correct, one may state with due care that the convergence properties of the LS algorithm toward the rational expectations solutions of the model are again governed by E-stability conditions: We have the definite result that rational expectations solutions of the ARMA(1,1) type (2b) can be learned only if these solutions are E-stable with respect to the ARMA(1,1) perceived law of motion. Moreover, the simulation results suggest that the learning process will be attracted to overparameterized white-noise solutions only if the white-noise solution (2a) is E-stable.

5. SUMMARY OF THE RESULTS

The formal analysis as well as the numerical simulations carried out in the preceding sections have shown that whether there is a correspondence between the convergence of the SG algorithm and E-stability conditions depends on the perceived law of motion.

An AR(1) model is the model of smallest degree regarding its AR and MA terms that encompasses at least partially both rational expectations solutions of the linear model considered in this paper. Proposition 3 states that if the perceived law of motion is given by such an AR(1) model, SG learning is indeed governed by E-stability conditions. Thus, the asymptotic properties of the SG algorithm and the LS algorithm are in this respect identical.

If, however, the perceived law of motion is given by an ARMA(1,1) model, there is no such one-to-one correspondence between the convergence of the SG algorithm and E-stability conditions. Proposition 5 as well as the simulation results allow for the following conclusions: Dependent on the specification of the model, the E-stable rational expectations solutions may not be learnable using the SG algorithm. In addition, the SG algorithm can converge to rational expectations solutions that are E-unstable.

6. CONCLUDING REMARKS

The aim of this paper was to show that there may not be a full correspondence between the concept of E-stability and the possible convergence of adaptive learning procedures in linear economic models. Such a correspondence indeed exists if the recursive LS algorithm is considered. However, this algorithm represents merely one of many plausible adaptive learning procedures. Taking a SG algorithm as an alternative to the LS algorithm, there exist cases in which the correspondence is not one-to-one.

Assuming a simple linear rational expectations model, it has been shown that SG learning may converge to rational expectations solutions that are not E-stable with respect to the underlying perceived law of motion. Furthermore, even E-stable rational expectations solutions may not be stable under SG learning.

Thus, there is no fully general E-stability principle that can be used to determine the "learnability" of rational expectations solutions. However, to be fair, it must be emphasized that a general assertion that E-stability and the possible convergence of adaptive learning procedures correspond to each other is not made anywhere in the literature. So, the message of this paper is merely that one should not hope for such a general assertion to be valid. The specific conditions that have to be fulfilled for such a correspondence to be valid at least within limits still have to be determined.

The analysis of two distinct perceived laws motion also revealed that there exist perceived laws of motion where the correspondence between E-stability conditions and the possible convergence of the SG algorithm is indeed one-to-one. Regarding the SG algorithm and its possible convergence points, the main difference between the two perceived laws of motion considered in the paper is that the AR(1) model gives rise to locally unique rational expectations solutions, whereas the ARMA(1,1) model gives rise to continua of such solutions. So far, it is not clear whether the local uniqueness of rational expectations solutions in the case of the AR(1) perceived law of motion is the reason for the correspondence between E-stability and convergence conditions. This is a possible starting point for future research.

NOTES

1. Barucci and Landi (1997) make a similar point. They use the term "least mean squares algorithm" for the SG algorithm.

2. Evans and Honkapohja (1995) give a comprehensive analysis of this model with respect to E-stability and check the possible convergence of recursive least-squares learning for perceived laws of motion that will not be considered here.

3. Other solutions, which can be obtained by taking sunspot variables into account, are ignored. See Evans and Honkapohja (1992, 1995).

4. Conditions for E-stability depend on whether the underlying perceived law of motion is correctly parameterized or overparameterized with respect to the rational expectations solution at hand. Given the perceived laws of motion (3a) and (3b), it is obvious that both models are overparameterized with respect to the white-noise solution (2a) but correctly parameterized with respect to the ARMA(1,1) solutions (2b). However, regarding the perceived law of motion (3a), this last statement is only true if $\phi = 0$.

5. Because one eigenvalue of $J(\boldsymbol{\theta}^*)$ equals zero, the stability analysis is more difficult in this case. However, as shown by Evans and Honkapohja (1992), the conditions stated above ensure weak E-stability for all $\boldsymbol{\theta}^* \in \Theta_b^*$.

6. For a discussion of these and other algorithms, see Ljung and Söderström (1983).

7. In accordance with the SG algorithm, the LS algorithm can be modified by using the more general learning rate $\gamma_{t+1} = (t+1)^{-\kappa}$ with $0 < \kappa \le 1$ instead of $[1/(t+1)]$.

8. Comprehensive descriptions of this approach and the required assumptions are already available in economic literature [Marcet and Sargent (1989), Woodford (1990)].

9. Because $E[x_t] = 0$, we get $E[y_t x_t] = \sigma_x^2$ and $E[y_t] = T_\alpha/(1 - T_\psi)$. From the two equations

$$E[y_t y_{t-1}] = T_\alpha E[y_t] + T_\psi E\left[y_{t-1}^2\right], \qquad E\left[y_t^2\right] = T_\alpha E[y_t] + T_\psi E[y_t y_{t-1}] + \sigma_x^2,$$

it then follows that $(1 - T_\psi^2) E[y_t^2] = T_\alpha(1 + T_\psi) E[y_t] + \sigma_x^2$.

10. For all these $\boldsymbol{\theta}^*$, an equilibrium point of (11b) is given by $R^* = M_z(\boldsymbol{\theta}^*)$ such that this equation need not to be considered.

11. If we have α and ψ such that the first equation of the system $M_z(\boldsymbol{\theta})[T(\boldsymbol{\theta}) - \boldsymbol{\theta}]$ equals zero, then the second equation equals zero if $\sigma_x^2[T_\psi(\boldsymbol{\theta}) - \psi]/[1 - T_\psi(\boldsymbol{\theta})^2] = 0$. This requires that $\psi = 0$ or $\psi = (1 - \beta_1)/\beta_2$ and, from this, it follows that $\alpha = \beta_0/(1 - \beta_1 - \beta_2)$ or $\alpha = -\beta_0/\beta_2$.

12. Because $E[x_t] = 0$, we get $E[y_t x_t] = \sigma_x^2$, $E[y_t x_{t-1}] = (T_\psi + T_\phi)\sigma_x^2$ and $E[y_t] = T_\alpha/(1 - T_\psi)$. From the two equations

$$E[y_t y_{t-1}] = T_\alpha E[y_t] + T_\psi E\left[y_{t-1}^2\right] + T_\phi \sigma_x^2,$$

$$E\left[y_t^2\right] = T_\alpha E[y_t] + T_\psi E[y_t y_{t-1}] + T_\phi E[y_t x_{t-1}] + \sigma_x^2,$$

it then follows that $(1 - T_\psi^2) E[y_t^2] = T_\alpha(1 + T_\psi) E[y_t] + [1 + T_\phi^2 + 2T_\phi T_\psi]\sigma_x^2$.

13. Equation (16) implies that $y_t = \beta_0/(1 - \beta_1 - \beta_2) + x_t + \psi^t[y_0 - \beta_0/(1 - \beta_1 - \beta_2) - x_0]$.

14. From (17), we get

$$y_t = \frac{\beta_0}{1 - \beta_1 - \beta_2} + \left(\beta_1\psi + \beta_2\psi^2\right)^t \left[y_0 - \frac{\beta_0}{1 - \beta_1 - \beta_2} - x_0\right] + x_t.$$

If $y_0 \neq \beta_0/(1 - \beta_1 - \beta_2) + x_0$, y_t will converge to solution (2a) for $t \to \infty$ only if $|\beta_1\psi + \beta_2\psi^2| < 1$.

15. In case of zero eigenvalues, the stability of equilibrium points may be determined using the center manifold technique [cf. Guckenheimer and Holmes (1990)]. For the nonlinear system of differential equations given here, this turned out to be an extremely difficult task and so this approach was not pursued here.

16. I am indebted to an anonymous referee for drawing my attention to this point.

17. If $\psi = 0$, we get $T_\psi(\boldsymbol{\theta}) = 0$. Since $T_\psi(\boldsymbol{\theta})$ is quadratic in ψ and takes an extreme value at $\psi = 0$, there exists an open interval around $\psi = 0$, where we have $|T_\psi(\boldsymbol{\theta})| < 1$. Regarding the specification $\beta_1 = -0.25$ and $\beta_2 = -1.5$, stationarity of $\mathbf{z}_t$, that is, $|T_\psi(\boldsymbol{\theta})| < 1$, requires $\psi \in (-0.904, 0.737)$; with $\beta_1 = 1.25$ and $\beta_2 = -1$, all $\psi \in (-0.554, 1.804)$ implies $|T_\psi(\boldsymbol{\theta})| < 1$. If $\beta_2 > \frac{1}{4}\beta_1^2$ or $\beta_2 < -\frac{1}{4}\beta_1^2$, there exists another open interval, where the condition $|T_\psi(\boldsymbol{\theta})| < 1$ is also satisfied.

18. The numerical solutions were obtained with Mathematica 3.0 using the function `NDSolve[ ]`. The terminal time for the numerical solutions was set to $T = 100$.

19. The terminal time $T = 100$ appeared to be large enough in order to interpret these points as stable equilibrium points.

20. Although a smaller value of κ generally results in faster convergence, it also reduces the chance for convergence because the exogenous shocks will have an effect on the algorithm for a longer period of time.

21. Parameter values for ψ and ϕ are depicted only for those cases in which the SG algorithm converged. For those simulation runs in which the elements of the parameter vector became greater than 10^{16} in absolute value, the respective results were simply ignored. Because no projection facility was used in the simulations, the algorithm diverged quite often.

22. Interestingly, it was never the case that the algorithm converged toward ARMA(1,1) solutions with $\phi < -\beta_2/(1 - \beta_1)$, although as Figure 4B reveals, these solutions represent stable equilibrium points of the differential equation (10).

23. Nevertheless, any $\boldsymbol{\theta}^* \in \Theta_a^*$ is—as long as R is invertible—an equilibrium point of (11a).

24. Interestingly, it is just this case of a perceived law of motion that is overparameterized with respect to its AR *as well as* its MA terms that Evans and Honkapohja (1994) exclude explicitly from their analysis of the LS algorithm. There are good reasons for doing so, because a formal analysis of the LS algorithm becomes impossible in case of such a singularity.

25. There is no assumption in the model that allows prediction of agents' behavior in such a case, but it seems plausible to assume (ad hoc) that agents will treat this singularity as an econometrician would treat it. This means that agents will infer that the variable y_{t-1} in their perceived law of motion (3b) is a linear combination of the exogenous variables. Thus they will drop the variable y_{t-1} from the auxiliary model such that it coincides with the AR(1) model (3a). However, in such a case, agents would learn $\boldsymbol{\theta}_a^*$ and arrive at the "standard" white-noise solution (2a).

REFERENCES

Barucci, E. & L. Landi (1997) Least mean squares learning in self-referential linear stochastic models. *Economics Letters* 57, 313–317.

DeCanio, S. (1979) Rational expectations and learning from experience. *Quarterly Journal of Economics* 93, 47–57.

Evans, G. (1985) Expectational stability and the multiple equilibria problem in linear rational expectations models. *Quarterly Journal of Economics* 100, 1217–1233.

Evans, G. (1989) The fragility of sunspots and cycles. *Journal of Monetary Economics* 23, 147–157.

Evans, G. & S. Honkapohja (1986) A complete characterization of ARMA solutions to linear rational expectations models. *Review of Economic Studies* 53, 227–239.

Evans, G. & S. Honkapohja (1992) On the robustness of bubbles in linear RE models. *International Economic Review* 33, 1–14.

Evans, G. & S. Honkapohja (1994) Learning, convergence, and stability with multiple rational expectations equilibria. *European Economic Review* 38, 1071–1098.

Evans, G. & S. Honkapohja (1995) Adaptive learning and expectational stability: An introduction. In A. Kirman & M. Salmon (eds.), *Learning and Rationality in Economics*, pp. 102–126. Oxford: Basil Blackwell.

Evans, G. & S. Honkapohja (1998a) Learning dynamics. In J.B. Taylor & M. Woodford (eds.), *Handbook of Macroeconomics*. Amsterdam: North-Holland.

Evans, G. & S. Honkapohja (1998b) Stochastic gradient learning in the cobweb model. *Economics Letters* 61, 333–337.

Guckenheimer, J. & P. Holmes (1990) *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, 3rd Ed. Berlin: Springer-Verlag.

Kuan, C.-M. & H. White (1994) Artificial neural networks: An econometric perspective. *Econometric Reviews* 13, 1–93.

Ljung, L. (1977) Analysis of recursive stochastic algorithms. *IIIE Transactions on Automatic Control* AC-22, 551–575.

Ljung, L. & T. Söderström (1983) *Theory and Practice of Recursive Identification*. Cambridge, MA: MIT Press.

Lucas, R. (1978) Asset prices in an exchange economy. *Econometrica* 46, 1429–1445.

Marcet, A. & T. Sargent (1988) The fate of systems with "adaptive" expectations. *American Economic Association—Papers and Proceedings* 78, 168–172.

Marcet, A. & T. Sargent (1989) Convergence of least squares learning mechanisms in self referential linear stochastic models. *Journal of Economic Theory* 48, 337–368.

McCallum, B. (1983) On non-uniqueness in linear rational expectations models: An attempt at perspective. *Journal of Monetary Economics* 11, 139–168.

Sargent, T. (1993) *Bounded Rationality in Macroeconomics*. Oxford: Oxford University Press.

Woodford, M. (1990) Learning to believe in sunspots. *Econometrica* 58, 277–307.

APPENDIX

Proof of Proposition 3. (i) Given the vector $\boldsymbol{\theta}_a^*$ associated with the white-noise solution, the Jacobian matrix becomes

$$J_{SG} = \begin{pmatrix} \beta_1 + \beta_2 - 1 & -\beta_0 \\ -\beta_0 & \dfrac{\beta_0^2 - \sigma_x^2 (1-\beta_1)(\beta_1+\beta_2-1)}{\beta_1+\beta_2-1} \end{pmatrix}.$$

A necessary and sufficient condition for asymptotic stability of $\boldsymbol{\theta}_a^*$ is that all eigenvalues of J_{SG} are negative so that J_{SG} is negative definite. This is the case if and only if the principal minors of J_{SG} satisfy the following conditions:

$$\left|J_{SG_1}\right| = \beta_1 + \beta_2 - 1 < 0, \qquad \left|J_{SG_2}\right| = \det J_{SG} = (\beta_1 - 1)(\beta_1+\beta_2-1)\,\sigma_x^2 > 0.$$

From these two conditions, it can be seen easily that the eigenvalues of J_{SG} are negative only if $\beta_1+\beta_2-1 < 0$ and thus $\beta_1 - 1 < 0$. These are exactly the conditions for E-stability of the white-noise solution.

(ii) Given $\boldsymbol{\theta}_b^*$, the Jacobian matrix is given as

$$J_{SG} = \begin{pmatrix} \beta_2 & \dfrac{\beta_0\beta_2}{1-\beta_1-\beta_2} \\ \dfrac{\beta_0\beta_2}{1-\beta_1-\beta_2} & \dfrac{\beta_0^2}{\beta_1+\beta_2-1} + \dfrac{(1-\beta_1)\beta_2^2\sigma_x^2}{(\beta_2-\beta_1+1)(\beta_1+\beta_2-1)} + \dfrac{(1-\beta_1)\beta_0^2}{(\beta_1+\beta_2-1)^2} \end{pmatrix},$$

and the conditions regarding the principal minors of $J_{SG}(\boldsymbol{\theta}_b)$ are

$$\left|J_{SG_1}\right| = \beta_2 < 0, \qquad \left|J_{SG_2}\right| = \det J_{SG} = -\frac{(\beta_1-1)\beta_2^3\sigma_x^2}{(\beta_1+\beta_2-1)(\beta_2-\beta_1+1)} > 0.$$

Thus, the eigenvalues of J_{SG} are negative only if $\beta_2 < 0$. Restricting attention to stationary solutions, we must have $|(1-\beta_1)/\beta_2| < 1$. However, if $\beta_2 < 0$, this implies that the denominator of det J_{SG} is positive. So, stability requires $\beta_1 - 1 > 0$ and again the stability conditions coincide with the conditions for E-stability of the respective rational expectations solution. ■

Proof of Proposition 4. Given $M_z(\boldsymbol{\theta})$ according to (15), the last equation in the system of equations $M_z(\boldsymbol{\theta})[T(\boldsymbol{\theta}) - \boldsymbol{\theta}] = 0$ reads as $(\psi + \phi)[\beta_1 + \beta_2\psi - 1] = 0$. Therefore, the respective solutions are given by $\psi = -\phi$ and $\psi = (1-\beta_1)/\beta_2$, $\phi \in \mathbf{R}$. With $\phi = -\psi$, the second equation in this system of equations becomes

$$\frac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})}[T_\alpha(\boldsymbol{\theta}) - \alpha] + \left\{\left[\frac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})}\right]^2 + \sigma_x^2\right\}[T_\psi(\boldsymbol{\theta}) - \psi] + \sigma_x^2\,[T_\phi(\boldsymbol{\theta}) - \phi] = 0.$$

Since $\phi = -\psi$, we have $T_\phi(\boldsymbol{\theta}) = -T_\psi(\boldsymbol{\theta})$ and this equation is fulfilled if α and ψ satisfy the first equation

$$[T_\alpha(\boldsymbol{\theta}) - \alpha] + \frac{T_\alpha(\boldsymbol{\theta})}{1 - T_\psi(\boldsymbol{\theta})}[T_\psi(\boldsymbol{\theta}) - \psi] = \frac{T_\alpha\,(1-\psi)}{1 - T_\psi} - \alpha = 0.$$

Substituting for T_α and T_ψ from (4) then reveals that this requires $\alpha = (1-\psi)[\beta_0/(1 - \beta_1 - \beta_2)]$. ■

Proof of Proposition 5. (i) Consider first the case of equilibrium points $\boldsymbol{\theta}^* \in \Theta_a^*$. Specifying $c(\psi) = \beta_1\psi + \beta_2\psi^2 - 1$, $a(\psi) = \sigma_x^2(\beta_1 + \beta_2\psi - 1)$, and $b = 1 - \beta_1 - \beta_2$, the Jacobian matrix J_{SG} is given by

$$J_{SG} = \begin{pmatrix} b/c(\psi) & \beta_0/c(\psi) & 0 \\ \beta_0/c(\psi) & a(\psi) + \beta_0^2\big/[bc(\psi)] & a(\psi) \\ 0 & a(\psi) & a(\psi) \end{pmatrix}.$$

The characteristic equation $f(\lambda)$ of J_{SG} thus is given by

$$f(\lambda) = -\lambda^3 + \lambda^2\left\{2a(\psi) + b/c(\psi) + \beta_0^2\big/[bc(\psi)]\right\}$$
$$+ \lambda\left\{-2ba(\psi)/c(\psi) - \beta_0^2 a(\psi)\big/[bc(\psi)]\right\}.$$

Thus, one characteristic root equals zero. According to Descartes' rule, the remaining two roots are negative if and only if the terms in curly braces are negative. This gives rise to the following conditions:

$$b\left\{2\,a(\psi)/b + 1/c(\psi) + \beta_0^2\big/[b^2 c(\psi)]\right\} < 0, \qquad [a(\psi)/bc(\psi)]\left[-2b^2 - \beta_0^2\right] < 0.$$

The last condition is satisfied only if $a(\psi)/[bc(\psi)] > 0$. Stationarity implies that $c(\psi) < 0$, and this requires that $a(\psi)/b < 0$. Then, the first condition will be satisfied only if $b > 0$, which in turn requires that $a(\psi) < 0$. If $\beta_2 > 0$, $a(\psi) < 0$ is equivalent to $\psi < (1-\beta_1)/\beta_2$ if $\beta_2 < 0$, $\psi > (1-\beta_1)/\beta_2$ implies that $a(\psi) < 0$. Note that $b > 0$ is equivalent to the condition for (weak) E-stability of the white-noise solution.

(ii) For equilibrium points $\boldsymbol{\theta}^* \in \Theta_b^*$, define $d = (-\beta_0\beta_2)/(\beta_1 + \beta_2 - 1)$ and

$$h(\phi) = (1-\beta_1)\beta_2\left[1 + \phi^2 + \phi\left(\frac{\beta_1 - 1}{\beta_2} + \frac{\beta_2}{\beta_1 - 1}\right)\right].$$

With $H(\phi) = -\sigma_x^2\beta_2^2[h(\phi)/(\beta_1 - \beta_2 - 1)(\beta_1 + \beta_2 - 1)]$, the Jacobian matrix J_{SG} results in

$$J_{SG} = \begin{pmatrix} \beta_2 & d & 0 \\ d & d^2\big/\beta_2 + H(\phi) & 0 \\ 0 & 1 - \beta_1 + \beta_2\phi & 0 \end{pmatrix}.$$

In that case, the characteristic equation $f(\lambda)$ for J_{SG} is given by

$$f(\lambda) = -\lambda^3 + \lambda^2\left[d^2\big/\beta_2 + \beta_2 + H(\phi)\right] - \lambda\beta_2 H(\phi).$$

As before, one root of $f(\lambda)$ equals zero. The remaining two roots are negative if and only if

$$H(\phi)\left\{d^2\big/[\beta_2 H(\phi)] + \beta_2/H(\phi) + 1\right\} < 0, \qquad -\beta_2 H(\phi) < 0.$$

The last condition is satisfied only if $\beta_2 H(\phi) > 0$. This implies that the first condition will be satisfied only if $H(\phi) < 0$ and this in turn requires that $\beta_2 < 0$. Now, if $\beta_2 < 0$, stationarity of the ARMA(1,1) solutions requires that $\beta_1 + \beta_2 - 1 < 0$ and $\beta_1 - \beta_2 - 1 > 0$. So, $H(\phi) < 0$ requires $h(\phi) < 0$. The zeros of the function $h(\phi)$ are given by $-\beta_2/(1-\beta_1)$ and $-(1-\beta_1)/\beta_2$ and if $1-\beta_1 > 0$, all $\phi \in (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ imply that $h(\phi) < 0$. If $1-\beta_1 < 0$, all $\phi \notin (-(1-\beta_1)/\beta_2, -\beta_2/(1-\beta_1))$ imply that $h(\phi) < 0$.

Note that these results do not mean that stationarity is a necessary condition for stability. So, for instance, in case (i), stationarity implies $c(\psi) < 0$, but this condition also may be satisfied if $\beta_1\psi + \beta_2\psi^2 < -1$, that is, if the resulting process is nonstationary. Thus, nonstationarity of the resulting process for y_t does not necessarily imply instability of the respective equilibrium point. ■