CONSTRUCTION OF CONSTANT SCALAR CURVATURE KÄHLER CONE METRICS

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Abstract. Over a compact Kähler manifold, we provide a Fredholm alternative result for the Lichnerowicz operator associated to a Kähler metric with conic singularities along a divisor. We deduce several existence results of constant scalar curvature Kähler metrics with conic singularities: existence result under small deformations of Kähler classes, existence result over a Fano manifold, existence result over certain ruled manifolds. In this last case, we consider the projectivisation of a parabolic stable holomorphic bundle. This leads us to prove that the existing Hermitian-Einstein metric on this bundle enjoys a regularity property along the divisor on the base.

1. Introduction

In this paper we investigate the construction of constant scalar curvature Kähler metrics (cscK in short) with conical singularities over a smooth compact Kähler manifold and provide several existence results. Starting with a model metric ω_D with conical singularity along a smooth divisor D of the compact Kähler manifold X (see definition in Section 2.2), we are interested in cscK cone metrics ω, i.e metrics of the form $\omega = \omega_D + \sqrt{-1}\partial\bar{\partial}\varphi$ such that

- ω is a Kähler cone metric,
- ω has constant scalar curvature over the regular part $M := X \setminus D$.

In our study, we will consider the linearization of the constant scalar curvature equation. This leads to consider the Lichnerowicz operator over functions u defined by

$$\mathbb{L}\mathrm{ic}_\omega(u) = \triangle^2_\omega u + u^{i\bar{j}} R_{i\bar{j}}(\omega)$$

and the associated *Lichnerowicz equation*

$$\mathbb{L}\mathrm{ic}_\omega(u) = f \tag{1.1}$$

for $f \in C^{,\alpha,\beta}$ with $\int_M f\omega^n = 0$, n being the complex dimension of the manifold and ω defined as above. Note that our study will require to work with certain Hölder spaces adapted to the singularities, the spaces $C^{\cdot,\alpha,\beta}$, that are described in details in Section 2.2. In particular, we say a Kähler potential φ is $C^{2,\alpha,\beta}$ cscK cone potential (resp. $C^{4,\alpha,\beta}$ cscK cone potential) if $\omega = \omega_D + \sqrt{-1}\partial\bar{\partial}\varphi$ is a cscK cone metric and additionally $\varphi \in C^{2,\alpha,\beta}$ (resp. $C^{4,\alpha,\beta}$). In the sequel when we speak of cscK cone metric, its potential is at least $C^{2,\alpha,\beta}$ as in [40, Definition 2.9].

We will need a certain restriction on the cone angle $2\pi\beta$ and the Hölder exponent α, namely that

$$0 < \beta < \frac{1}{2}; \quad \alpha\beta < 1 - 2\beta. \tag{C}$$

This restriction appeared in previous works, e.g. in [7, 42] and is required to have regularity results.

Our first theorem is an analytic result Fredholm alternative type. It provides a solution to the Lichnerowicz equation (1.1) over the Hölder spaces $C^{4,\alpha,\beta}$ for $C^{,\alpha,\beta}$ data.

Theorem 1.1 ((Linear theory))**.** *Let X be a compact Kähler manifold, $D \subset X$ a smooth divisor, ω a cscK cone metric with $C^{2,\alpha,\beta}$ potential such that the cone angle $2\pi\beta$ and the Hölder exponent α satisfy Condition (C). Assume that $f \in C^{,\alpha,\beta}$ with normalisation condition $\int_M f\omega^n = 0$. Then one of the following holds:*

- *Either the Lichnerowicz equation $\mathbb{L}\mathrm{ic}_\omega(u) = f$ has a unique $C^{4,\alpha,\beta}(\omega)$ solution.*
- *Or the kernel of $\mathbb{L}\mathrm{ic}_\omega$ has positive dimension and corresponds to the space of holomorphic vector fields tangent to D.*

Note that the solution furnished by the theorem can be extended continuously to the whole manifold X.

As in the smooth situation (see for instance Lebrun-Simanca's results in [33, Corollary 2]), this result provides an existence theorem by small deformations. To derive it, we just use the implicit function theorem together with the one-one correspondence between the kernel of the Lichnerowicz operator and the holomorphic vector fields on the manifold tangential to the divisor, see [42, Section 4]. We introduce some notations. Define the complex Lie group of automorphisms $Aut(X, [\omega])$ as the group of holomorphic diffeomorphisms of X preserving the class $[\omega]$. Let denote $Aut^0(X, [\omega])$ the identity component of $Aut(X, [\omega])$. Now, we consider $Aut_D(X, [\omega]) \subset Aut^0(X, [\omega])$ the subgroup composed of automorphisms that fix the divisor D. Then, $Lie(Aut_D(X, [\omega]))$ consists in the Lie algebra of holomorphic vector fields tangential to D with holomorphy potential. Recall that a holomorphy potential is a function whose complex gradient, with respect to the metric ω is a holomorphic vector field.

Corollary 1.2 ((CscK cone metric by deformation))**.** *Consider (X, ω) compact Kähler manifold endowed with ω a cscK cone metric along $D \subset X$, smooth divisor, with angle $2\pi\beta$ satisfying Condition (C). Assume that the Lie algebra $Lie(Aut_D(X, [\omega_B]))$ is trivial. Then the set* *of all Kähler classes around $[\omega]$ containing a cscK metric with cone singularities is non-empty and open.*

A direct application of this last result is the existence of cscK cone metrics close to Kähler-Einstein cone metrics on Fano manifolds. Before stating the result, we refer to [4,11] for a definition the α-invariant for general polarization and its relation with log-canonical thresholds. The next corollary is obtained from the results of Berman [4] and Li-Sun [35, Corollaries 2.19 and 2.21] on existence of a Kähler-Einstein cone metric over a Fano manifold and the non existence of holomorphic vector field tangent to D (when the parameter λ below is greater or equal to 1). The regularity of the Kähler-Einstein cone metric is also sufficient to apply Theorem 1.1 (the regularity issue is discussed in [41], see also references therein).

Corollary 1.3 ((CscK cone metrics for Fano manifolds))**.** *Assume that $\Omega_0 = c_1(-K_X)$ and D is a smooth divisor which is $\mathbb{Q}$-linearly equivalent to $-\lambda K_X$, where $\lambda \in \mathbb{Q}_+^*$. Denote L_D the line bundle associated to D.*

(i) *If $\lambda \geq 1$, then there is a constant δ such that in the nearby class Ω satisfying $|\Omega - \Omega_0| < \delta$ there exists a constant scalar curvature Kähler cone metric in Ω with cone angle $2\pi\beta$ satisfying*

(1.2)
$$0 < \beta < \min\left(\frac{1}{2}, \left(1 - \frac{1}{\lambda}\right) + \frac{n+1}{n}\min\left(\frac{1}{\lambda}\alpha(-K_X), \alpha(L_{D|D})\right)\right).$$

(ii) *If $\frac{2n}{2n+1} < \lambda < 1$, $Lie(Aut_D(X, \Omega_0))$ is trivial, then the same conclusion as in (i) holds for angle $2\pi\beta$ satisfying (1.2) and the extra condition $\beta > n\left(\frac{1}{\lambda} - 1\right)$.*

Note that the upper bound in (1.2) may not be optimal but has the advantage of being effective and calculable, we also refer to [49] on that point. A partial algebraic version of this result in terms of log K-stability can be found in [14, Theorem 1.3].

Our next main result is a construction theorem of cscK metrics with cone singularities in Kähler classes (that may not be integral) over projective bundles, which generalizes the main result of [28]. It is also an application of Theorems 1.1 and 3.27 but requires much more work. The notion of parabolic stability is explained in Sections 4.1 and 4.3.

Theorem 1.4 ((CscK cone metric for projective bundles))**.** *Let B be a base compact Kähler manifold endowed with a cscK metric ω_B with cone singularities along $D \subset B$, smooth divisor with trivial Lie algebra $Lie(Aut_D(B, [\omega_B]))$. Assume the Hölder exponent α and the angle $2\pi\beta$ of ω_B satisfy Condition (C). Let E be a parabolic stable vector bundle over B with respect to ω_B.*
Then, for $k \in \mathbb{N}^*$ large enough, there exists a cscK metric with cone

singularities on $X := \mathbb{P}E^*$ *in the class*

$$\Omega = [k\pi^*\omega_B + \hat{\omega}_E]$$

where $\pi : X \to B$ *and* $\hat{\omega}_E$ *represents the first Chern class of* $\mathcal{O}_{\mathbb{P}E^*}(1)$. *This cscK metric has its cone singularities along* $\mathcal{D} := \pi^{-1}(D)$ *with* $C^{4,\alpha,\beta}$ *potential.*

Remark 1.5. *In Theorem 1.4 the assumption on* E *could be replaced by saying that it is an indecomposable holomorphic vector bundle equipped with a parabolic structure and a Hermitian-Einstein cone metric compatible with this structure, providing a purely differential geometric statement.*

Let's do now some brief comments. CscK cone metrics constitute a natural generalization of Kähler-Einstein metrics with conical singularities (see Section 2.5) . The importance of the notion of Kähler-Einstein cone metric is now well established from the recent breakthrough for the celebrated Yau-Tian-Donaldson conjecture, when one restricts attention to Fano manifolds and the anticanonical class, see for instance the pioneering paper [20] or the survey [21] and references therein. One may expect that the study of cscK cone metrics may lead to new progress on Yau-Tian-Donaldson conjecture for general polarizations or may have applications for construction of moduli spaces or Chern number inequalities. Furthermore, a logarithmic version of Yau-Tian-Donaldson conjecture is expected to be also true in the context of cscK cone metrics. Nevertheless, as far as we know, only very few examples of cscK cone metrics that are not Kähler-Einstein appeared in the literature. CscK cone metrics are far from being well understood and for instance uniqueness results have only appeared very recently, cf. [42]. From the point of view of existence, the case of curves has been studied by R.C. McOwen, M. Troyanov and F. Luo–G.Tian in the late eighties. In higher dimension, Y. Hashimoto [26] has recently obtained momentum-constructed cscK cone metrics on the projective completion of a pluricanonical line bundle over a product of Kähler-Einstein Fano manifolds. This enabled him to give first evidence of the log Yau-Tian-Donaldson conjecture. Note that his definition of Kähler cone metrics is more restricted than the general usual definition that we consider here. A more general setup has been studied in [30] where it is shown morally that the notion of cscK cone metric is the most natural notion of Kähler metrics with special curvature properties for projective bundles over a curve, when the holomorphic bundle is irreducible and not Mumford stable (otherwise, the "right" notion would be the classical notions of smooth extremal/cscK metric). A related work for extremal Kähler cone metric, still on the projective completion of a line bundle over admissible manifolds, can also be found in [36].

Our results provide an effective method to construct plenty of cscK cone metrics on various manifolds and partially generalize previous results op. cit. We also expect that Theorem 1.1 will have many applications in a long range, including for studying log-K-stability (see Section 7).

We shall now explain the structure of the paper. In Section 2, we introduce the notion of metrics with singularities, together with the adapted Hölder spaces and recall some results about regularity of cscK cone metrics. Among other things we prove the vanishing of the log-Futaki invariant of a Kähler class endowed with a cscK cone metric. In Section 3, we introduce weighted Sobolev spaces and obtain Schauder type estimates for Laplace equation associated to a Kähler cone metric under half angle condition (Proposition 3.22) or without half angle condition but with weaker regularity (Proposition 3.23). This allows us to see that a weak solution u to the bi-Laplacian equation $\Delta^2 u - K\Delta u = f$ is actually $C^{4,\alpha,\beta}$. Using this result and introducing a new continuity method, we are able to prove Theorem 1.1 by proving the key estimate (Theorem 3.25) showing closeness. In Section 4, we introduce the notion of *Hermitian-Einstein cone metrics*, that are hermitian metrics over a parabolic vector bundle that satisfy the Einstein equation (with respect to a Kähler cone metric) together with a certain regularity property. Theorem 4.7 shows that a parabolic stable vector bundle can be equipped with a Hermitian-Einstein cone metric, improving results of Simpson [47, 48] and Li [38]. Using this result, we adapt the work of Hong [28, 29] for smooth cscK metrics to the conical setting and construct inductively almost cscK cone metrics (Proposition 5.4). Using now Theorem 3.27 and taking the adiabatic limit, we can deduce Theorem 1.4 in Section 5.5. In Section 6, we explain that the existence of a Kähler-Einstein cone metric on a manifold provides a Hermitian-Einstein cone metric on its tangent bundle, generalizing a well-known result in the smooth case. This could be used to provide extra concrete examples of applications of our Theorem 1.4. Eventually, in Section 7, we discuss natural generalizations of our work and some possible applications to other geometric questions. In the particular case of the projectivisation of a parabolic vector bundle over a curve, we formulate a conjecture between existence of cscK cone metric, log K-stability and parabolic stability.

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Contents

2. CscK metrics with cone singularities

Let (X, ω_0) be a Kähler manifold. We denote $[\omega_0]$ the Kähler class containing the smooth Kähler metric ω_0. We let D be a smooth divisor in X with $0 < \beta < \frac{1}{2}$.

Given a point p in D, let $\{z^1\}$ be the local defining functions of the hypersurface where p locates. The local chart $(U_p, z^1, ..., z^n)$ centered at p is called *cone chart* at p.

Definition 2.1. *A* Kähler cone metric ω *of cone angle* $2\pi\beta$ *along* D*, is a closed positive* $(1,1)$ *current, which is also a smooth Kähler metric on the regular part*

$$M := X \setminus D.$$

In a local cone chart U_p*, the Kähler form is quasi-isometric to the standard cone flat metric, which is*

$$\omega_{cone} := \frac{\sqrt{-1}}{2}\left(\beta^2|z^1|^{2(\beta-1)}dz^1 \wedge dz^{\bar{1}} + \sum_{2\leq j\leq n} dz^j \wedge dz^{\bar{j}}\right). \tag{2.1}$$

The standard cone metric has nice properties. The Christoffel symbols of the connection of ω_{cone} under the holomorphic coordinate $\{z^1, \dots z^n\}$ are for all $2 \leq i, j, k \leq n$,

$$\Gamma^1_{1k}(\omega_{cone}) = \Gamma^i_{11}(\omega_{cone}) = \Gamma^1_{jk}(\omega_{cone}) = \Gamma^i_{1k}(\omega_{cone}) = \Gamma^i_{jk}(\omega_{cone}) = 0,$$

except $\Gamma^1_{11}(\omega_{cone}) = -\frac{1-\beta}{z^1}$. Also, the Riemannian curvature of ω_{cone} is identical to zero.

2.1. Hölder spaces in cone charts.

In this section, we start by recalling the definition of Donaldson's Hölder spaces [20], see also [7]. A quasi-isometric mapping W is well defined in the cone chart U_p as follows,

$$W(z^1, \cdots, z^n) := (w^1 = |z^1|^{\beta-1}z^1, z^2, \cdots, z^n). \tag{2.2}$$

We let $v(w^1, \cdots, z^n) = u(z^1, \cdots, z^n)$. A function $u(z) : U_p \to \mathbb{R}$ is said to be $C^{,\alpha,\beta}$, if $v(w^1, \cdots, z^n)$ is a C^α Hölder function in the classical sense. The space $C^{,\alpha,\beta}_{\{0\}}$ contains all functions $f \in C^{,\alpha,\beta}$ such that

$$f(0, z^2, \cdots, z^n) = 0.$$

In the cone charts U_p, the Hölder semi-norm $[u]_{C^{,\alpha,\beta}(U_p)}$ is defined to be $[v]_{C^\alpha(W(U_p))}$ and then the Hölder norm $|u|_{C^{,\alpha,\beta}(U_p)}$ is $\sup_{U_p}|u| + [u]_{C^{,\alpha,\beta}(U_p)}$ in the usual sense. The global semi-norm or norm on the whole manifold X is defined by using a partition of unity of X, since in the charts away from D, everything is defined in the classical sense. Together with the Hölder norm, $C^{,\alpha,\beta}$ becomes a Banach space. A $(1,1)$-form σ is said to be $C^{,\alpha,\beta}$, if for any $2 \le i, j \le n$,

$$\begin{cases} \sigma(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^{\bar{j}}}) \in C^{,\alpha,\beta}, & |z^1|^{2-2\beta}\sigma(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^{\bar{1}}}) \in C^{,\alpha,\beta}, \\ |z^1|^{1-\beta}\sigma(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^{\bar{j}}}) \in C^{,\alpha,\beta}, & |z^1|^{1-\beta}\sigma(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^{\bar{1}}}) \in C^{,\alpha,\beta}. \end{cases} \tag{2.3}$$

Similarly, we could define $C^{,\alpha,\beta}$ of higher order tensors. The Hölder space $C^{2,\alpha,\beta}$ is defined by

$$C^{2,\alpha,\beta} = \{u \mid u, \partial u, \sqrt{-1}\partial\bar{\partial}u \in C^{,\alpha,\beta}\} .$$

Note that the spaces $C^{,\alpha,\beta}$ and $C^{2,\alpha,\beta}$ are independent of the choice of the background Kähler cone metrics when they are equivalent. But we can see that the higher order spaces are more complicated, since the geometry of the background metric is a priori involved. The Hölder space $C^{3,\alpha,\beta}$ and $C^{4,\alpha,\beta}$ are introduced in [10] and further detailed computations can be found in [42]. The idea is that we first define the local model Hölder spaces in the cone charts, and then extend it to the whole manifold by using a global background Kähler cone metric.

2.2. **Model cone metric.** Let s be a section of the line bundle associated to D equipped with a smooth hermitian metric h_D. It is explained by Donaldson in [20] that for sufficiently small $\delta > 0$,

$$\omega_D = \omega_0 + \delta\frac{\sqrt{-1}}{2}\partial\bar{\partial}|s|^{2\beta}_{h_D} \tag{2.4}$$

is a Kähler cone metric and independent of the choices of ω_0, h_D, δ up to quasi-isometry. We call such ω_D the **model metric** with conical singularity. The model metric ω_D has rich geometric information. The detailed computation could be found in [7, 10]. We then compare the general Kähler cone metrics with the growth of the model metric and use the following definitions (see also Definition 2.9 for Christoffel symbols and 2.13 for curvature tensors in [42]).

Definition 2.2. *Assume that α and β satisfy Condition* (C). *We say a Kähler metric*

$$\omega = \frac{\sqrt{-1}}{2\pi}\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\overline{z^j}$$

has the 3rd model growth if for any $2 \le i, j, k \le n$, all the following elements are $C^{,\alpha,\beta}$,

$$\begin{cases} \frac{\partial g_{k\bar{l}}}{\partial z^i}, \quad |z^1|^{1-\beta}\frac{\partial g_{k\bar{1}}}{\partial z^i}, \quad |z^1|^{1-\beta}\frac{\partial g_{1\bar{l}}}{\partial z^i}, \quad |z^1|^{1-\beta}\nabla_1^{cone} g_{k\bar{l}}, \\ |z^1|^{2-2\beta}\frac{\partial g_{1\bar{1}}}{\partial z^i}, \quad |z^1|^{2-2\beta}\nabla_1^{cone} g_{k\bar{1}}, \quad |z^1|^{2-2\beta}\nabla_1^{cone} g_{1\bar{l}}, \quad |z^1|^{3-3\beta}\nabla_1^{cone} g_{1\bar{1}}. \end{cases} \tag{2.5}$$

We say a Kähler metric ω has the 4th model growth if for any $2 \leq i, j, l \leq n$, the following second order covariant derivatives of the model metric ω_D are $C^{,\alpha,\beta}$,

$$\begin{cases} \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial z^{\bar{j}}}, \quad |z^1|^{1-\beta} \frac{\partial^2 g_{1\bar{l}}}{\partial z^i \partial z^{\bar{j}}}, \quad |z^1|^{2-2\beta} \frac{\partial^2 g_{1\bar{1}}}{\partial z^i \partial z^{\bar{j}}}, \quad |z^1|^{2-2\beta} \nabla^{cone}_{\bar{j}} \nabla^{cone}_{1} g_{1\bar{l}}, \\ |z^1|^{3-3\beta} \nabla^{cone}_{\bar{1}} \nabla^{cone}_{i} g_{1\bar{1}}, \quad |z^1|^{4-4\beta} \nabla^{cone}_{\bar{1}} \nabla^{cone}_{1} g_{1\bar{1}}. \end{cases} \tag{2.6}$$

2.3. **Global set-up.** We could define the $C^{3,\alpha,\beta}$ and $C^{4,\alpha,\beta}$ spaces with respect to ω_D on the whole manifold X via replacing the metric ω_{cone} with ω_D in both the local Hölder spaces $C^{3,\alpha,\beta}(\tilde{U}; \omega_{cone})$ in (8.1) and $C^{4,\alpha,\beta}(\tilde{U}; \omega_{cone})$ in (8.2). In fact, we have a more general property. Details can also be found in Section 5.3 in [42].

Proposition 2.3 (([42])). *Assume that ω is a Kähler cone metric and its connection satisfies (2.5). Then the local $C^{3,\alpha,\beta}(\tilde{U}; \omega_{cone})$ function u could be extended to be global. Precisely, its 3rd order covariant derivatives belong to $C^{,\alpha,\beta}(X)$, i.e. letting ∇ denote the covariant derivative regarding to the Kähler cone metric ω, for any $2 \leq i, j, k, l \leq n$,*

$$\begin{cases} \nabla_k \nabla_{\bar{j}} \nabla_i u, \quad |z^1|^{1-\beta} \nabla_k \nabla_{\bar{1}} \nabla_i u, \quad |z^1|^{1-\beta} \nabla_k \nabla_{\bar{j}} \nabla_1 u, \\ |z^1|^{2-2\beta} \nabla_k \nabla_{\bar{1}} \nabla_1 u, \quad |z^1|^{2-2\beta} \nabla_1 \nabla_{\bar{j}} \nabla_1 u, \quad |z^1|^{3-3\beta} \nabla_1 \nabla_{\bar{1}} \nabla_1 u, \end{cases} \tag{2.7}$$

belong to $C^{,\alpha,\beta}(X)$.

Thus we can define the $C^{3,\alpha,\beta}(X; \omega)$ norm of a function u on the whole manifold X as the $C^{2,\alpha,\beta}$ norm of u plus the $C^{,\alpha,\beta}$ norms of all the elements in (2.7).

Proposition 2.4 (([42])). *Assume that ω is a Kähler cone metric and satisfies (2.5) and (2.6). Then the local $C^{4,\alpha,\beta}(\tilde{U}; \omega_{cone})$ function u could be extended to be global, i.e. its 4th order covariant derivatives are all $C^{,\alpha,\beta}(X)$, i.e. letting ∇ denote the covariant derivative regarding to the Kähler cone metric ω, for any $2 \leq i, j, k, l \leq n$,*

$$\begin{cases} \nabla_{\bar{l}} \nabla_k \nabla_{\bar{j}} \nabla_i u, \quad |z^1|^{1-\beta} \nabla_{\bar{l}} \nabla_1 \nabla_{\bar{j}} \nabla_i u, \quad |z^1|^{2-2\beta} \nabla_{\bar{1}} \nabla_1 \nabla_{\bar{j}} \nabla_i u, \\ |z^1|^{2-2\beta} \nabla_{\bar{l}} \nabla_1 \nabla_{\bar{j}} \nabla_1 u, \quad |z^1|^{3-3\beta} \nabla_{\bar{1}} \nabla_1 \nabla_{\bar{1}} \nabla_i u, \quad |z^1|^{4-4\beta} \nabla_{\bar{1}} \nabla_1 \nabla_{\bar{1}} \nabla_1 u \end{cases} \tag{2.8}$$

belong to $C^{,\alpha,\beta}(X)$.

The $C^{4,\alpha,\beta}(X; \omega)$ norm of a function u is defined in the same way, that is the $C^{3,\alpha,\beta}(X; \omega)$ norm of u plus the $C^{,\alpha,\beta}$ norms of all the elements in (2.8).

The proofs of Propositions 2.3 and 2.4 are carried out in [42]. Moreover both spaces $C^{3,\alpha,\beta}(X; \omega)$ and $C^{4,\alpha,\beta}(X; \omega)$ are Banach spaces, as proved in [42, Section 5].

Remark 2.5. *It is natural to use the model metric ω_D in both Propositions 2.3 and 2.4, since it satisfies all required conditions. The same* *scheme of ideas allows to define without difficulty higher order function spaces $C^{k,\alpha,\beta}$ for any $k \geq 5$.*

Next we consider the general Kähler cone metric

$$\omega_\varphi = \omega_D + \sqrt{-1}\partial\bar\partial\varphi.$$

When we have a Kähler potential $\varphi \in C^{4,\alpha,\beta}(X;\omega_D)$, we get $\log \omega_\varphi^n \in C^{2,\alpha,\beta}(X)$, and the Ricci curvature $-\sqrt{-1}\partial\bar\partial \log \omega_\varphi^n \in C^{,\alpha,\beta}(X)$, according to Corollary 2.20 in [42], under the restriction (C). Furthermore, we have the following information for the connection and curvature of ω_φ, which will be needed in the later sections.

Proposition 2.6 ((Proposition 2.10 in [42])). *Assume that the potential function φ of a Kähler cone metric belongs to $C^{3,\alpha,\beta}(X;\omega_D)$. Then ω_φ has the 3rd model growth. Actually, denoting for simplicity $g = g_\varphi$ as the Riemannian metric associated to ω_φ, we have that all the Christoffel symbols of the connection of g_φ for any $2 \leq i,j,k \leq n$ are $C^{,\alpha,\beta}$,*

$$\begin{cases} \Gamma^i_{jk}, & |z^1|^{1-\beta}\Gamma^i_{j1}, \quad |z^1|^{\beta-1}\Gamma^1_{jk}, \\ \Gamma^1_{j1}, & |z^1|^{2-2\beta}\Gamma^i_{11}, \quad |z^1|^{1-\beta}(\Gamma^1_{11} + \dfrac{1-\beta}{z^1}). \end{cases} \tag{2.9}$$

In addition, if we strengthen the condition on the cone potential φ, we have the following bound.

Proposition 2.7 ((Lemma 2.21 and 2.22 in [42])). *Assume that the potential function φ of a Kähler cone metric belongs to $C^{4,\alpha,\beta}(X;\omega_D)$. Then the properties (2.5) and (2.6) hold for g_φ. Actually, denoting for simplicity $g = g_\varphi$ as the hermitian metric associated to ω_φ, besides the conclusion of Corollary 2.6, we have that ω_φ has the 4th model growth and for any $2 \leq i,j,k,l \leq n$, the curvature tensor of g_φ is cone admissible i.e the terms*

$$\begin{cases} R_{i\bar j k\bar l}, & R_{1\bar j k\bar l}, \quad R_{i\bar 1 k\bar l}, \quad R_{1\bar 1 k\bar l}, \\ R_{i\bar j 1\bar 1}, & R_{1\bar j 1\bar 1}, \quad R_{i\bar 1 1\bar 1}, \quad R_{1\bar 1 1\bar 1}. \end{cases} \tag{2.10}$$

satisfy Definition 2.13 in [42].

Corollary 2.8. *Assume that the potential function φ of a Kähler cone metric belongs to $C^{4,\alpha,\beta}(X;\omega)$ and ω satisfies the properties (2.5) and (2.6). Then ω_φ satisfies (2.9) and (2.10).*

Proof. We apply Proposition 2.3 and Proposition 2.4 yo φ. Then the conclusion follows from direct computation and applying Proposition 2.6 and Proposition 2.7. □

2.4. Second order elliptic equations with conical singularities.

We first quote a proposition of the general linear elliptic equation which essentially uses Donaldson's estimates [20] (see also Brendle [7], Calamai-Zheng [10]). Consider the boundary value problem

$$\mathbb{L}u := g^{i\bar{j}}u_{i\bar{j}} + b^i u_i + cu = f + \partial_i h^i \text{ in } M = X \setminus D. \tag{2.11}$$

Here $g^{i\bar{j}}$ is the inverse matrix of a Kähler cone metric ω in $C^{,\alpha,\beta}$. We also denote the vector field $h^i\partial_i$ to be $\mathbf{h}$ and $b^i\partial_i$ to be $\mathbf{b}$. Moreover, we are given the following data.

$$\mathbf{h} \in C^{1,\alpha,\beta} \qquad \text{and} \qquad \mathbf{b}, c, f \in C^{,\alpha,\beta} . \tag{2.12}$$

Proposition 2.9 (([7, 10, 20]))**.** *Fix α with $0 < \alpha < \frac{1}{\beta} - 1$. Then there is a constant C depending on β, n, α, $|\mathbf{b}|_{C^{,\alpha,\beta}}$, $|c|_{C^{,\alpha,\beta}}$ such that for all the functions $f \in C^{,\alpha,\beta}$ and $\mathbf{h} \in C^{1,\alpha,\beta}$, we have the following Schauder estimate of the weak solution of equation (2.11),*

$$|u|_{C^{2,\alpha,\beta}} \leq C(\|u\|_{L^\infty} + |f|_{C^{,\alpha,\beta}} + |\mathbf{h}|_{C^{1,\alpha,\beta}}) .$$

2.5. **Some properties of cscK cone metrics.** In this section we review some recent progress on the theory of cscK cone metrics and show some extra properties. Recall the definition of the cscK cone metrics in [53].

Definition 2.10. *We say that ω_{cscK} is a cscK metric with conical singularities if*

- *ω_{cscK} is a cscK metric on the regular part M;*
- *ω_{cscK} is quasi isometric to the model metric ω_D;*
- *the potential of ω_{cscK} lies in $C^{2,\alpha,\beta}$.*

From the definition, the cscK cone metric satisfies the constant scalar curvature equation on the regular part M,

$$S(\omega_{cscK}) = \underline{S}_\beta. \tag{2.13}$$

Remark 2.11. *We only require the second order behavior of the cscK cone metric in this definition. There are different ways to define cscK metrics with conical singularities and different notions are compared in [40]. However, a crucial issue is the question of higher regularities of such metrics.*

We write the cscK cone metric ω_{cscK} using ω_D-potentials i.e

$$\omega_{cscK} := \omega_D + \sqrt{-1}\partial\bar{\partial}\varphi_{cscK}.$$

Because φ_{cscK} is $C^{2,\alpha,\beta}$, the 4th order equation (2.13) could be rewritten as the couple system of two second order elliptic equations

$$\begin{cases} \dfrac{\omega^n_{cscK}}{\omega^n_D} = e^P, \\ \triangle_{cscK} P = g^{i\bar{j}}_{cscK} R_{i\bar{j}}(\omega_D) - \underline{S}_\beta. \end{cases} \tag{2.14}$$

The following higher regularity theorem is proved in [42].

Theorem 2.5.1 (([42])). *Assume that φ_{cscK} is the potential of a cscK cone metric satisfying (2.14) with φ_{cscK} is $C^{2,\alpha,\beta}(X)$. Assume that the angle $2\pi\beta$ and the Hölder exponent α satisfy Condition (C). Then φ_{cscK} is actually in $C^{4,\alpha,\beta}(X;\omega_D)$ and the Ricci curvature of ω_{cscK} is $C^{,\alpha,\beta}(X)$.*

Next proposition appeared in [34] in the case of Kähler-Einstein cone metrics.

Proposition 2.12. *Suppose that $\omega_{cscK} \in [\omega]$ is a cscK cone metric with $C^{2,\alpha,\beta}$ potential under Condition (C). The average of the scalar average of the cscK cone metric is*

$$\underline{S}_\beta = n\frac{2\pi c_1(X)\cup[\omega]^{n-1}}{[\omega]^n} - n(1-\beta)\frac{2\pi c_1(D)\cup[\omega]^{n-1}}{[\omega]^n}. \tag{2.15}$$

Proof. Since ω_{cscK} has $C^{4,\alpha,\beta}$ potential thanks to Theorem 2.5.1, $\omega_{cscK}^n = |s|_h^{2\beta-2}\omega_0^n e^\psi$ for some ψ in $C^{2,\alpha,\beta}$ and h a smooth hermitian metric on $\mathcal{O}(D)$. Then we have, with $\Theta(h)$ curvature of the metric h,

$$Ric(\omega_{cscK}) = Ric(\omega_0) - (1-\beta)\Theta(h) + 2\pi(1-\beta)[D] - \sqrt{-1}\partial\bar\partial\psi, \tag{2.16}$$

as an equality of closed currents. By definition the scalar curvature is trace of Ricci curvature, and so

$$\begin{aligned}\int_M S(\omega_{cscK})\frac{\omega_{cscK}^n}{n!} &= \int_M Ric(\omega_{cscK})\wedge\frac{\omega_{cscK}^{n-1}}{(n-1)!},\\ &= \int_M Ric(\omega_0)\wedge\frac{\omega_{cscK}^{n-1}}{(n-1)!} - (1-\beta)\int_M \Theta(h)\wedge\frac{\omega_{cscK}^{n-1}}{(n-1)!}\\ &\quad - \int_M \sqrt{-1}\partial\bar\partial\psi\wedge\frac{\omega_{cscK}^{n-1}}{(n-1)!}.\end{aligned}$$

The first two terms are what we need, i.e equal to

$$\frac{2\pi c_1(X)\cup[\omega_{cscK}]^{n-1} - 2\pi(1-\beta)c_1(D)\cup[\omega_{cscK}]^{n-1}}{(n-1)!}.$$

The third term vanishes by integration by part. □

Remark 2.13. *Note if we had only a cscK cone metric ω with $C^{2,\alpha,\beta}$ potential and $Ric(\omega)\in C^{,\alpha,\beta}$, we could update our reasoning. Actually, we have $Ric(\omega_{cscK}) \le C_0\cdot\omega_{cscK}$ for some constant $C_0 > 1$. Moreover, there is a constant $C_1 > 1$ such that $Ric(\omega_0) - (1-\beta)\Theta(h) \ge -C_1\cdot\omega_{cscK}$ on X. We set $C_2 = C_0 + C_1$. Using now (2.16), we obtain*

$$C_2\cdot\omega_{cscK} + \sqrt{-1}\partial\bar\partial\psi \ge 0.$$

We notice that φ_{cscK} and $\varphi_{cscK} + \frac{1}{C_2}\psi$ are both ω_0-plurisubharmonic functions which are globally bounded on X. We only need to check the vanishing along the direction z^1. According to the integration by *part formula in* [9, Theorem 1.14], *we can do the integration by part as above.*

Proposition 2.14. *Consider X a smooth Fano manifold. Suppose that D is a smooth divisor which is $\mathbb{Q}$-linearly equivalent to $-\lambda K_X$, with $\lambda \in \mathbb{Q}_+^*$. Consider $\omega = \omega_{cscK}$ is a cscK cone metric in the class $2\pi c_1(X)$ along D with angle $2\pi\beta$ and Hölder exponent α. Assume that (α, β) satisfy Condition (C). Then ω is actually a Kähler-Einstein cone metric satisfying the equation*

$$Ric(\omega) = \nu\omega + 2\pi(1-\beta)[D],$$

with $\nu = 1-(1-\beta)\lambda$. Conversely, such a Kähler-Einstein cone metric is also a cscK cone metric.

Proof. Using same notations as above, the function z^1 is the local defining function of the divisor D. The Poincaré-Lelong equation tells us that

$$2\pi[D] = \sqrt{-1}\partial\bar{\partial}\log|z^1|^2,$$

so the trace reduces to $g^{1\bar{1}}\delta_{\{z^1=0\}}$. Note that ω is $C^{,\alpha,\beta}$ and quasi-isometric to the standard cone metric

$$\omega_0 = |z^1|^{2\beta-2}\sqrt{-1}dz^1 \wedge d\bar{z}^1 + \sqrt{-1}\sum_{i=2}^{n} dz^i \wedge d\bar{z}^i.$$

Since δ function is a generalized function of order 0 (i.e. its action can be continuously extended to C^0 functions), this implies $g^{1\bar{1}}\delta_{\{z^1=0\}} = 0$. Consequently, we have

$$tr_\omega[D] = 0. \tag{2.17}$$

From Equation (2.16), $Ric(\omega)$ is actually a representative of $2\pi c_1(X) - 2\pi(1-\beta)c_1(D) = \nu c_1(X)$. By considering cohomology classes, we can find a smooth real valued function f such that

$$Ric(\omega) - \nu\omega - 2\pi(1-\beta)[D] = \sqrt{-1}\partial\bar{\partial}f. \tag{2.18}$$

Taking trace with respect to ω and using (2.17), we have

$$\triangle_\omega f = S(\omega) - n\nu.$$

Now, using the cscK condition, we obtain $\triangle_\omega f = 0$. Thus f is constant and we can conclude using again (2.18). □

Consider V_f a holomorphic vector field on X with holomorphy potential $f \in C^\infty(X,\mathbb{C})$ i.e $\iota_{V_f^{1,0}}\omega = -\bar{\partial}f$. Given the Kähler class $[\omega]$ and the vector field V_f, one can define the Futaki invariant as

$$Fut_{[\omega]}(V_f) = \frac{1}{2\pi}\int_X f\,(S(\omega) - \underline{S})\,\frac{\omega^n}{n!}$$

where $\underline{S}$ is the average of the scalar curvature of any Kähler form in the class $[\omega]$ and the log-Futaki invariant for vector fields V_f that are furthermore tangent to D as

$$\begin{aligned}Fut_{D,\beta,[\omega]}(V_f) =&\frac{1}{2\pi}\int_X f\left(S(\omega)-n\frac{2\pi c_1(X)\cup[\omega]^{n-1}}{[\omega]^n}\right)\frac{\omega^n}{n!}\\ &-(1-\beta)\left(\int_D f\frac{\omega^{n-1}}{(n-1)!}-n\frac{c_1(D)\cup[\omega]^{n-1}}{[\omega]^n}\int_X f\frac{\omega^n}{n!}\right),\\ =&Fut_{[\omega]}(V_f)-(1-\beta)\left(\int_D f\frac{\omega^{n-1}}{(n-1)!}-\frac{\mathrm{Vol}_{[\omega]}(D)}{\mathrm{Vol}_{[\omega]}(X)}\int_X f\frac{\omega^n}{n!}\right).\end{aligned}$$

Both Futaki invariants depend only on the class $[\omega]$. As pointed out in [26], the log-Futaki invariant is the differential-geometric interpretation of the algebraic log Donaldson-Futaki invariant that can be defined using test configurations, see [20] and [44].

Next corollary is known for Kähler-Einstein cone metrics on Fano manifolds, see for instance [49], or in the smooth case for the classical Futaki invariant. Consider $\omega_{cscK}\in[\omega]$ a cscK cone metric. It satisfies globally in the sense of distributions

$$S(\omega_{cscK})=\underline{S}_\beta+2\pi(1-\beta)\mathrm{tr}_{\omega_{cscK}}[D].$$

Applying Proposition (2.12), we obtain the following result.

Corollary 2.15. *Under assumption (C), the log-Futaki invariant $Fut_{D,\beta,[\omega]}$ vanishes on Kähler classes $[\omega]$ which contain cscK cone metric with cone singularities along D with cone angle $2\pi\beta$.*

Remark 2.16 ((Expansion close to the divisor))**.** *For any cone angle $0<\beta<1$, general expansion formulas for Kähler-Einstein cone metrics appear in [52]. They come from the study of a singular Monge-Ampère equation. Similar expansion formulas hold for the cscK cone metrics in [54].*

Remark 2.17 ((Uniqueness of cscK cone metrics))**.** *The study of uniqueness of cscK cone metrics has been initiated and proved in [41, 42, 53, 54].*

Remark 2.18. *As in Remark 2.13, we notice that under assumption of $C^{2,\alpha,\beta}$ potential and $Ric(\omega)\in C^{,\alpha,\beta}$, Proposition 2.14 and Corollary 2.15 still hold.*

3. Lichnerowicz equations with conical singularities

The Lichnerowicz operator at a cscK metric ω_{cscK} is defined on functions u,

$$\mathbb{L}\mathrm{ic}_{cscK}(u)=\triangle^2_{cscK}u+u^{i\bar{j}}R_{i\bar{j}}(\omega_{cscK}). \tag{3.1}$$

We remark that when a Kähler metric has constant scalar curvature, the first variation of the scalar curvature is given by the Lichnerowicz operator.

We say that a Kähler cone metric ω has bounded Christoffel symbols of the connection, if for any $2 \leq i, j, k \leq n$, the following items are bounded,

$$\begin{cases} \Gamma^i_{jk}, \quad |z^1|^{1-\beta}\Gamma^i_{j1}, \quad |z^1|^{\beta-1}\Gamma^1_{jk}, \\ \Gamma^1_{j1}, \quad |z^1|^{2-2\beta}\Gamma^i_{11}, \quad |z^1|^{1-\beta}(\Gamma^1_{11} + \dfrac{1-\beta}{z^1}); \end{cases}$$

We say that a Kähler potential φ is a $C^{4,\alpha,\beta}(\omega_D)$ *(or* $C^{3,\alpha,\beta}(\omega_D)$ *) cscK potential* if $\omega = \omega_D + \sqrt{-1}\partial\bar\partial\varphi$ is a cscK metric and also $\varphi \in C^{4,\alpha,\beta}(\omega_D)$ (or $C^{3,\alpha,\beta}(\omega_D)$ respectively).

Consider a metric $\omega = \omega_D + \sqrt{-1}\partial\bar\partial\varphi$ with $C^{4,\alpha,\beta}(\omega_D)$ cscK potential φ. In this section, we are going to solve the equation for $f \in C^{,\alpha,\beta}$

$$\mathbb{L}\mathrm{ic}_\omega(u) = f, \tag{3.2}$$

with solution $u \in C^{4,\alpha,\beta}(\omega)$ and deduce a Fredholm alternative of the Lichnerowicz operator Theorem 1.1.

3.1. **Sobolev spaces for cone metrics.** Since the volume element of the reference cone metric ω is an L^p function (for some $p \geq 1$) with respect to any global smooth Kähler form, it gives rise to a measure ω^n on M. Thus, we can introduce the following Sobolev spaces with respect to ω^n. We shall use the following Banach spaces on the whole manifold X.

Definition 3.1 ((Sobolev spaces $W^{1,p,\beta}(\omega)$))**.** *For a Kähler cone metric ω with cone angle $2\pi\beta$, the Sobolev spaces $W^{1,p,\beta}(\omega)$ for $p \geq 1$ are defined to be the completion of the space of smooth functions with finite $W^{1,p,\beta}(\omega)$ norm where this norm is defined by*

$$\|u\|_{W^{1,p,\beta}(\omega)} = \left(\int_M (|u|^p + |\nabla u|^p_\omega)\, \omega^n\right)^{1/p}.$$

Definition 3.2 ((Sobolev spaces $W^{2,p,\beta}(\omega)$))**.** *For a Kähler cone metric ω with bounded Christoffel symbols of the connection, the Sobolev spaces $W^{2,p,\beta}(\omega)$ for $p \geq 1$ are defined to be the completion of the space of smooth functions with finite $W^{2,p,\beta}(\omega)$ norm which is the combination of $W^{1,p,\beta}(\omega)$ norm and $W^{2,p,\beta}(\omega)$ semi-norm. The $W^{2,p,\beta}(\omega)$ semi-norm is defined with respect to the reference Kähler cone metric ω with cone angle $2\pi\beta$,*

$$\begin{aligned}[u]_{W^{2,p,\beta}(\omega)} = &\sum_{1\leq a,b\leq n} \|\partial_a\partial_{\bar b}u\|_{L^p(\omega)} + \sum_{2\leq j\leq n} \|\partial_1\partial_j u\|_{L^p(\omega)} \\ &+ \sum_{2\leq j,k\leq n} \|\partial_j\partial_k u\|_{L^p(\omega)}.\end{aligned} \tag{3.3}$$

Definition 3.3 ((Strong Sobolev spaces $W^{2,p,\beta}_{\mathbf{s}}(\omega)$)). *For a Kähler cone metric ω with bounded Christoffel symbols of the connection, the Sobolev spaces $W^{2,p,\beta}_{\mathbf{s}}(\omega)$ for $p \geq 1$ are defined to be the completion of the space of smooth functions with finite $W^{1,p,\beta}(\omega)$ norm and $W^{2,p,\beta}_{\mathbf{s}}(\omega)$ semi-norm. The $W^{2,p,\beta}_{\mathbf{s}}(\omega)$ semi-norm is defined with respect to the reference Kähler cone metric ω,*

$$[u]_{W^{2,p,\beta}_{\mathbf{s}}(\omega)} = \left(\int_M (|\partial\bar{\partial} u|^p_\omega + |\partial\partial u|^p_\omega)\, \omega^n \right)^{\frac{1}{p}}. \tag{3.4}$$

In both (3.3) and (3.4), the second order pure covariant derivatives mean, for any $1 \leq a, b \leq n$,

$$\partial_a \partial_b u := \nabla_a \nabla_b u = \frac{\partial^2 u}{\partial z^a \partial z^b} - \sum_{c=1}^n \Gamma^c_{ab}(\omega) \frac{\partial u}{\partial z^c}. \tag{3.5}$$

The Christoffel symbols $\Gamma^c_{ab}(\omega)$ of the connection satisfy the properties of Proposition 2.7.

Remark 3.4. *From (3.5), we could see clearly that why the bounded Christoffel symbols of the connection of the background metric ω are required in the global definitions of the higher order Sobolev spaces.*

Definition 3.5. *We define the Sobolev space $H^{2,\beta} := W^{2,2,\beta}(\omega)$ and $H^{2,\beta}_0 = \{u \in H^{2,\beta} | \int_M u\, \omega^n = 0\}$. The Sobolev norm remains the same. The strong spaces $H^{2,\beta}_{\mathbf{s}} := W^{2,2,\beta}_{\mathbf{s}}(\omega)$ and $H^{2,\beta}_{\mathbf{s},0}$ are defined in a similar way.*

Lemma 3.6. *Assume that $u \in W^{1,p,\beta}(\omega)$. If $p < 2n$, then there exists a constant C such that*

$$\|u\|_{L^q(\omega)} \leq C \|u\|_{W^{1,p,\beta}(\omega)},$$

for any $q \leq \frac{2np}{2n-p}$.

Proof. We consider the function which is supported in a cone chart, we use the map W defined in (2.2) and the Sobolev inequality in Euclidean space to obtain the desired inequality. The general case follows from a partition of unity. $\square$

Similarly, we have Morrey's inequality.

Lemma 3.7. *Assume that $u \in W^{1,p,\beta}(\omega)$. If $p > 2n$, then there exists a constant C and $\alpha = 1 - 2n/p$ such that*

$$\|u\|_{C^{,\alpha,\beta}(\omega)} \leq C \|u\|_{W^{1,p,\beta}(\omega)}.$$

Lemma 3.8 ((Sobolev embedding theorem)). *Assume that $u \in W^{2,p,\beta}_{\mathbf{s}}(\omega)$, $p < 2n$ and $q \leq \frac{2np}{2n-p}$. There exists a constant C independent of u such that*

$$\|u\|_{W^{1,q,\beta}(\omega)} \leq C \|u\|_{W^{2,p,\beta}_{\mathbf{s}}(\omega)}. \tag{3.6}$$

Proof. From the lemma above,

$$\|\nabla u\|_{L^q(\omega)} \leq C(\|\nabla|\nabla u|\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\omega)}).$$

So the conclusion follows from applying classical Kato inequality that provides the inequality $\|\nabla|\nabla u|\|_{L^p(\omega)} \leq \|\nabla\nabla u\|_{L^p(\omega)}$. The R.H.S is controlled by the term $\|u\|_{W^{2,p,\beta}_{\mathbf{s}}(\omega)}$. □

Lemma 3.9 ((Kondrakov compactness theorem)). *Assume $p < 2n$ and $q < \frac{2np}{2n-p}$.*

- *The Sobolev embedding $W^{1,p,\beta}(\omega) \subset L^q(\omega)$ is compact.*
- *The Sobolev embedding $W^{2,p,\beta}_{\mathbf{s}}(\omega) \subset W^{1,q,\beta}(\omega)$ is compact.*

Proof. We cover the manifold X by a finite number of coordinates charts $\{U_i, \psi; 1 \leq i \leq N\}$ and let ρ_i be the smooth partition of unity subordinate to $\{U_i\}$. Let f_m be a bounded sequence in $W^{2,p,\beta}_{\mathbf{s}}(\omega)$. In the charts which do not intersect with the divisor, we let $\tilde{f}_m = (\rho_i f_m) \circ \psi_i^{-1}$. While, in the cone chart U among $\{U_i\}$ we let $\tilde{f}_m = (\rho_i f_m) \circ \psi_i^{-1} \circ W^{-1}$. Then we are able to pick Cauchy subsequence of f_m in each charts for $i = 1, 2, ..., N$ because of precompactness of $\tilde{f}_m$ in each U_i. □

Proposition 3.10 ((Interpolation inequality)). *Suppose that $\epsilon > 0$ and $1 < p < \infty$. There exists a constant C such that for all $u \in W^{2,p,\beta}_{\mathbf{s}}$, we have*

$$\|u\|_{W^{1,p,\beta}(\omega)} \leq \epsilon\|u\|_{W^{2,p,\beta}_{\mathbf{s}}(\omega)} + C\|u\|_{L^p(\omega)}.$$

Proof. It follows from Lemma 3.9 by using standard contradiction argument. We assume that the conclusion fails and for each $C_i = i$, there exists a $u_i \in W^{2,p,\beta}_{\mathbf{s}}$ with $\|u_i\|_{W^{2,p,\beta}_{\mathbf{s}}(\omega)} = 1$ such that

$$\|u_i\|_{W^{1,p,\beta}(\omega)} > \epsilon + i \cdot \|u_i\|_{L^p(\omega)}.$$

Thus $\|u_i\|_{W^{1,p,\beta}(\omega)} \geq \epsilon > 0$ and $\|u_i\|_{L^p(\omega)} \to 0$, as $i \to +\infty$. On one hand, from Kondrakov compactness (Lemma 3.9), after taking a subsequence, u_i converges to u_∞ in $W^{1,p,\beta}(\omega)$ norm. Also $\|u_\infty\|_{W^{1,p,\beta}(\omega)} \geq \epsilon > 0$. On the other hand, from Sobolev embedding (Lemma 3.8), $\|u_i\|_{W^{1,p,\beta}(\omega)}$ is uniformly bounded, and then by Lemma 3.6, u_i converges to zero in $L^p(\omega)$ as $i \to \infty$. Thus $u_\infty = 0$, contradiction!

□

Following the same proof, we actually obtain the following interpolation inequality.

Corollary 3.11. *Suppose that $\epsilon > 0$ and $2 \leq p < \infty$. There exists a constant C such that for all $u \in W^{2,p,\beta}_{\mathbf{s}}$, we have*

$$\|u\|_{W^{1,p,\beta}(\omega)} \leq \epsilon\|u\|_{W^{2,p,\beta}_{\mathbf{s}}(\omega)} + C\|u\|_{L^2(\omega)}.$$

Remark 3.12. *One may wonder whether we could replace the* $W^{2,p,\beta}_{\mathbf{s}}$ *with the partial norm* $W^{2,p,\beta}$ *in all Lemma 3.8, Lemma 3.9 and Proposition 3.10. But, after examining the proof of Lemma 3.8, it is obvious that a different Kato inequality would be needed.*

We set $H^{1,\beta} = W^{1,2,\beta}(\omega)$. The following lemma will be very useful.

Lemma 3.13 ((Poincaré inequality))**.** *There is a constant* C_P *such that for any* $u \in H^{1,\beta}_0 = \{u \in H^{1,\beta} | \int_M u\,\omega^n = 0\}$*, one has*

$$\|u\|_{L^2(\omega)} \leq C_P \|\nabla u\|_{L^2(\omega)}. \tag{3.7}$$

Proof. The Poincaré inequality follows from the compactness theorem i.e. the inclusion $W^{1,p,\beta}(\omega) \subset L^q(\omega)$ with $q < \frac{2np}{2n-p}$ is compact. Actually, any minimizing sequence u_i of $\|\nabla u\|_{L^2(\omega)}$ over $H = \{u \in H^{1,\beta}_0 \text{ s.t } \|u\|_{L^2(\omega)} = 1\}$, converges strongly in L^2 and weakly in $H^{1,\beta}_0$, to a limit v. So $\inf_{u \in H} \|\nabla u\|_{L^2(\omega)}$ is realized by v and has to be positive. □

3.2. **A partial L^p estimate.** In [13] (Definition 2.1), it is defined a local Sobolev space $W^{2,p,\beta}_{loc}(\mathtt{B};\omega_{cone})$ over a ball $\mathtt{B}$ contained in a cone chart U. It contains the functions $u \in W^{1,p,\beta}(\mathtt{B};\omega_{cone})$ such that for all $2 \leq i,j \leq n$,

- $|z^1|^{2(1-\beta)} \frac{\partial^2 u}{\partial z^1 \partial z^{\bar 1}} \in L^p(\mathtt{B};\omega_{cone})$;
- $|z^1|^{1-\beta} \frac{\partial^2 u}{\partial z^1 \partial x^j} \in L^p(\mathtt{B};\omega_{cone})$, for all $2 \leq j \leq 2n$, with $z^i = x^i + \sqrt{-1}x^{n+i}$, for all $2 \leq i \leq n$;
- $\frac{\partial^2 u}{\partial x^j \partial x^k} \in L^p(\mathtt{B};\omega_{cone})$, for all $2 \leq j,k \leq 2n$;
- $u \in W^{2,2}(\mathtt{B} \setminus \mathtt{N}_\epsilon;\omega)$ for any $\mathtt{N}_\epsilon \subset \mathtt{B}$, ϵ-tubular neighbourhood of the divisor $D \cap \mathtt{B}$.

The semi-norm is defined to be

$$\begin{aligned}
[u]_{W^{2,p,\beta}_{loc}(\mathtt{B},\omega_{cone})} = & \left\| |z^1|^{2(1-\beta)} \frac{\partial^2 u}{\partial z^1 \partial z^{\bar 1}} \right\|_{L^p(\mathtt{B};\omega_{cone})} \\
& + \sum_{2 \leq j \leq 2n} \left\| |z^1|^{1-\beta} \frac{\partial^2 u}{\partial z^1 \partial x^j} \right\|_{L^p(\mathtt{B};\omega_{cone})} \\
& + \sum_{2 \leq j,k \leq 2n} \left\| \frac{\partial^2 u}{\partial x^j \partial x^k} \right\|_{L^p(\mathtt{B};\omega_{cone})}.
\end{aligned} \tag{3.8}$$

It is also proved in [13, Theorem 4.1] a L^p estimate over U with respect to the flat metric ω_{cone}. Note that we define $\mathtt{B}_r$ the balls of radius r times a small radii r_0 with respect to the cone metric ω_{cone}.

Lemma 3.14 (([13], Theorem 4.1))**.** *Assume that* $u \in W^{2,p,\beta}_{loc}(\mathtt{B}_1;\omega_{cone})$ *for* $2 \leq p < \infty$*, and* $\triangle_{\omega_{cone}} u \in L^p(\mathtt{B}_1;\omega_{cone})$*. Then there exists a constant* C *depending on* n, p, β *such that*

$$[u]_{W^{2,p,\beta}_{loc}(\mathtt{B}_1;\omega_{cone})} \leq C \cdot \|\triangle_{\omega_{cone}} u\|_{L^p(\mathtt{B}_1;\omega_{cone})}.$$

In order to extend the local definition to the global manifold, we need the following lemmas. We could see that if we restrict the semi-norm $W^{2,p,\beta}$ defined by (3.3) over U, it is controlled by patching up these local $W^{2,p,\beta}_{loc}$ semi-norms on coverings.

Lemma 3.15. *Let $\omega = \omega_D + \sqrt{-1}\partial\bar{\partial}\varphi$ be a Kähler cone metric with bounded Christoffel symbols of the associated connection. There exists a constant $C > 0$ such that for all function $u \in W^{2,p,\beta}_{loc}(\mathtt{B}_1; \omega_{cone})$, it holds*

$$[u]_{W^{2,p,\beta}(\mathtt{B}_1;\omega)} \leq C \cdot [u]_{W^{2,p,\beta}_{loc}(\mathtt{B}_1;\omega_{cone})}.$$

Proof. Recall the definition

$$[u]_{W^{2,p,\beta}(\mathtt{B}_1;\omega)} = \sum_{1\leq a,b\leq n} \|\partial_a\partial_{\bar{b}} u\|_{L^p(\mathtt{B}_1;\omega)} + \sum_{j=2}^{n} \|\partial_1\partial_j u\|_{L^p(\mathtt{B}_1;\omega)} + \sum_{2\leq j,k\leq n} \|\partial_j\partial_k u\|_{L^p(\mathtt{B}_1;\omega)}.$$

In the cone charts, ω is equivalent to ω_{cone}. We then examine term by term. $\partial_1\partial_1 u$ is already in (3.8). Then

$$\partial_1\partial_{\bar{j}} u = \frac{1}{2}\Big(\frac{\partial^2 u}{\partial z^1 \partial x^j} - i\frac{\partial^2 u}{\partial z^1 \partial x^{n+j}}\Big),$$

$\partial_k\partial_{\bar{1}}$ and $\partial_k\partial_{\bar{j}}$ are also $L^p(\omega_{cone})$. Meanwhile, the second term

$$\partial_1\partial_j u = \frac{1}{2}\Big(\frac{\partial^2 u}{\partial z^1 \partial x^j} - i\frac{\partial^2 u}{\partial z^1 \partial x^{n+j}}\Big) - \sum_{c=1}^{n}\Gamma^c_{1j}(\omega)\frac{\partial u}{\partial z^c}.$$

From assumption, the Christoffel symbols $\Gamma^1_{1j}(\omega)$ and $|z^1|^{1-\beta}\Gamma^c_{1j}(\omega)$ for $2 \leq c \leq n$ are all bounded. Therefore $\partial_1\partial_j u$ is $L^p(\omega_{cone})$ and so is the third term $\partial_j\partial_k u$. □

Then we consider the $W^{2,p,\beta}(\omega)$ solution of the linear equation on M,

$$\triangle_\omega u = f, \tag{3.9}$$

where $f \in L^p(\omega)$.

Proposition 3.16. *Let ω be a Kähler cone metric with bounded Christoffel symbols of the associated connection. Suppose that $u \in W^{2,p,\beta}(\omega)$ for $2 \leq p < \infty$ is a classical solution of Equation (3.9) for $f \in L^p(\omega)$. Then there exists a constant C depending on n, p, β, M, ω such that*

$$\|u\|_{W^{2,p,\beta}(\omega)} \leq C(\|f\|_{L^p(\omega)} + \|u\|_{W^{1,p,\beta}(\omega)}).$$

Proof. We choose a small ball $\mathtt{B}_R$ centered at a point in D (otherwise the argument is standard) and with small radius R such that over $\mathtt{B}_R$, $|\omega - \omega_{cone}|_{L^\infty(\omega_{cone})}$ is less than a small constant $\epsilon > 0$. Here we are using the fact that ω is assumed to be $C^{,\alpha,\beta}$. We also choose $0 < R_1 < R_2 < R$

and let the cutoff function ρ such that $\rho = 1$ in $\mathtt{B}_{R_1}$ and $\rho = 0$ outside $\mathtt{B}_{R_2}$.
We apply Lemma 3.14 over $\mathtt{B}_R$ to the equation of $v = \rho u$,

$$\triangle_{\omega_{cone}} v = (\triangle_{\omega_{cone}} - \triangle_\omega) v + \rho \triangle_\omega u + 2(\partial\rho, \partial u)_\omega + \triangle_\omega \rho u := \tilde{f}$$

to obtain,

$$[v]_{W^{2,p,\beta}_{loc}(\mathtt{B}_R;\omega_{cone})} \leq C \|\tilde{f}\|_{L^p(\mathtt{B}_R;\omega_{cone})}$$

Then the RHS of the previous equation is controlled by

$$C\left(\epsilon [v]_{W^{2,p,\beta}_{loc}(\mathtt{B}_R;\omega_{cone})} + \|\rho f + 2(\partial\rho, \partial u)_\omega + \triangle_\omega \rho u\|_{L^p(B_R;\omega_{cone})}\right).$$

Thus, provided we took $\epsilon < \frac{1}{2C}$, we have by $|\nabla\rho|_{\omega_{cone}} \leq \frac{1}{R_2 - R_1}$ and $|i\partial\bar{\partial}\rho|_{\omega_{cone}} \leq \frac{1}{(R_2-R_1)^2}$,

$$\frac{1}{2}[v]_{W^{2,p,\beta}_{loc}(\mathtt{B}_R;\omega_{cone})} \leq C\left(\|f\|_{L^p(B_R;\omega_{cone})} + \frac{\|\nabla u\|_{L^p(B_{R_2};\omega_{cone})}}{R_2 - R_1} + \frac{\|u\|_{L^p(B_{R_2};\omega_{cone})}}{(R_2 - R_1)^2}\right).$$

We further choose $R_1 = \frac{1}{2}R$ and $R_2 = \frac{3}{4}R$, apply Lemma 3.15 to the LHS over $\mathtt{B}_{\frac{R}{2}}$ and enlarge the terms on the RHS over the whole M, so u solution of (3.9) satisfies for a new uniform constant $C > 0$,

$$[u]_{W^{2,p,\beta}(\mathtt{B}_{\frac{R}{2}};\omega)} \leq C[u]_{W^{2,p,\beta}_{loc}(\mathtt{B}_{\frac{R}{2}};\omega_{cone})} \leq C(\|f\|_{L^p(\omega)} + \|u\|_{W^{1,p,\beta}(\omega)}).$$

Then the conclusion follows from covering the whole M by balls with radius $\frac{R}{2}$ and applying partition of unity: we choose a covering of the manifold M by a finite number of coordinates charts $\{U_i, \psi; 1 \leq i \leq N\}$ such that each $U_i \subset \mathtt{B}_{\frac{R}{2}}(p_i)$ for some p_i. Then we let ρ_i be the smooth partition of unity, subordinate to $\{U_i\}$ and supported on U_i for each i. We put together all estimates over each U_i, i.e

$$\begin{aligned}
\|u\|_{W^{2,p,\beta}(\omega)} &= \|u\|_{W^{1,p,\beta}(\omega)} + [\textstyle\sum_{i=1}^N \rho_i u]_{W^{2,p,\beta}(\omega)} \\
&\leq \|u\|_{W^{1,p,\beta}(\omega)} + \textstyle\sum_{i=1}^N [\rho_i u]_{W^{2,p,\beta}(\mathtt{B}_{\frac{R}{2}}(p_i);\omega)} \\
&\leq C(\|u\|_{W^{1,p,\beta}(\omega)} + \|f\|_{L^p(\omega)}),
\end{aligned}$$

for a new uniform constant $C > 0$.

□

3.3. **Weak solutions to bi-Laplacian equations: existence.** For a constant $K > 0$, we are looking for weak solutions to the following K-bi-Laplacian equation in $H^{2,\beta}_0(\omega)$,

$$\triangle^2_\omega u - K\triangle_\omega u = f. \tag{3.10}$$

The positive constant K will be determined later in this section. We recall the definition of the semi-norm

$$[u]_{H^{2,\beta}(\omega)} = \sum_{1\leq a,b\leq n} \|\partial_a \partial_{\bar{b}} u\|_{L^2(\omega)} + \sum_{2\leq j\leq n} \|\partial_1 \partial_j u\|_{L^2(\omega)} + \sum_{2\leq j,k\leq n} \|\partial_j \partial_k u\|_{L^2(\omega)},$$

which does not involve the term $\|\partial_1\partial_1 u\|_{L^2(\omega)}$. We introduce the following bilinear form.

Definition 3.17. *Define the bilinear form on $H_0^{2,\beta}(\omega)$ given by*

$$\mathcal{B}^K(u,\eta) := \int_M [\triangle_\omega u \triangle_\omega \eta + K g^{i\bar{j}} u_i \eta_{\bar{j}}]\omega^n$$

for all $u, \eta \in H_0^{2,\beta}(\omega)$.

Then we introduce the weak solution to (3.10) , whose leading coefficients are conical.

Definition 3.18 ((Weak solution))**.** *We say that u is the $H_0^{2,\beta}(\omega)$-weak solution of the K-bi-Laplacian equation* (3.10)*, if it satisfies the following identity for all $\eta \in H_0^{2,\beta}(\omega)$,*

$$\mathcal{B}^K(u,\eta) = \int_M f\eta\omega^n.$$

Lemma 3.19. *The bilinear form $\mathcal{B}^K$ is bounded and coercive for $K > C_P + 1$. Here C_P is the Poincaré constant of ω.*

Proof. For all $u, \eta \in H_0^{2,\beta}$, the boundedness follows from using Cauchy-Schwarz inequality,

$$\mathcal{B}^K(u,\eta) \leq \|u\|_{H^{2,\beta}(\omega)}\|\eta\|_{H^{2,\beta}(\omega)} + K\|u\|_{H^{1,\beta}(\omega)}\|\eta\|_{H^{1,\beta}(\omega)}.$$

Then we prove coercivity of the bilinear form. We first use the partial L^2 estimate of the cone metrics for the standard linear operator $\triangle_\omega$ (Proposition 3.16), i.e. that there exists a constant $C > 0$ such that,

$$\|u\|^2_{H^{2,\beta}(\omega)} \leq C\int_M (|\triangle_\omega u|^2 + |\nabla u|^2_\omega + |u|^2)\omega^n.$$

Then, using the definition of the bilinear form, the R.H.S above is

$$C\left(\mathcal{B}^K(u,u) + \int_M [(1-K)|\nabla u|^2_\omega + |u|^2]\omega^n\right).$$

We use Poincaré inequality (Lemma 3.13) and denoting the Poincaré constant by C_P,

$$\|u\|^2_{H^{2,\beta}(\omega)} \leq C \cdot (\mathcal{B}^K(u,u) + (1-K+C_P)\|\nabla u\|^2_{L^2(\omega)}).$$

Thus choosing $K > C_P + 1$, we have

$$\mathcal{B}^K(u,u) \geq \frac{1}{C}\|u\|^2_{H^{2,\beta}(\omega)}.$$

□

The next proposition proves the existence of weak solution.

Proposition 3.20. *Let ω be a Kähler cone metric with bounded Christoffel symbols of the connection. Suppose that $K > C_P + 1$ and f is in the dual space $(H_0^{2,\beta}(\omega))^*$. Then the K-bi-Laplacian equation* (3.10) *has a unique weak solution $u \in H_0^{2,\beta}(\omega)$.*

Proof. According to Lemma 3.19, the bilinear form $\mathcal{B}^K$ is bounded and coercive. Then the Lax-Milgram theorem tells us that there is a unique weak solution $v \in H_0^{2,\beta}$ to equation (3.10). □

We could define another 2nd Sobolev space $H^{2,\beta}_{\mathbf{w}}$ with semi-norm

$$[u]_{H^{2,\beta}_{\mathbf{w}}(\omega)} = \sum_{1\leq a,b\leq n} \|\partial_a \partial_{\bar{b}} u\|_{L^2(\omega)}$$

and norm

$$||u||_{H^{2,\beta}_{\mathbf{w}}(\omega)} = ||u||_{H^{1,\beta}(\omega)} + [u]_{H^{2,\beta}_{\mathbf{w}}(\omega)}.$$

Then following the same argument as above, we get a "very weak" solution, that lies in $H^{2,\beta}_{\mathbf{w},0}$, the space of functions of $H^{2,\beta}_{\mathbf{w}}$ with vanishing integral assuming that ω is merely a Kähler cone metric.

Proposition 3.21. *Assume that ω is a Kähler cone metric. Suppose that $K > C_P + 1$ and f is in the dual space $(H^{2,\beta}_{\mathbf{w},0}(\omega))^*$. Then the K-bi-Laplacian equation (3.10) has a unique weak solution $u \in H^{2,\beta}_{\mathbf{w},0}(\omega)$.*

3.4. **Weak solutions to bi-Laplacian equations: regularity.** According to the existence theorem (Proposition 3.20), we now already have a weak solution

$$u \in H_0^{2,\beta}(\omega).$$

We shall see we can obtain that the weak solution is actually $C^{4,\alpha,\beta}$ if additionally we impose more regularity on f.

Proposition 3.22 ((Schauder estimate))**.** *With previous notations with (α,β) satisfying Condition (C) and $f \in C^{,\alpha,\beta}$, the weak solution $u \in H_0^{2,\beta}$ to equation (3.10) is actually $C^{4,\alpha,\beta}(\omega)$. Moreover, there exists a constant C such that*

$$|\triangle_\omega u|_{C^{2,\alpha,\beta}} \leq C(\|u\|_{H_0^{2,\beta}(\omega)} + |f|_{C^{,\alpha,\beta}}).$$

Proof. We rewrite (3.10) as

$$(\triangle_\omega - K)\triangle_\omega u = f. \tag{3.11}$$

According to the Schauder estimate for second order equation (see Section 2.4), we have proved that $\triangle_\omega u$ is in $C^{2,\alpha,\beta}$ from (3.11). Then we could use the angle restriction to conclude that $u \in C^{4,\alpha,\beta}(\omega)$, according to Proposition 4.3 in [42]. □

Furthermore, we actually could weaken the condition on ω using Proposition 3.21, however we do not use it in this paper.

Proposition 3.23 ((Schauder estimate))**.** *Let $\omega = \omega_D + \sqrt{-1}\partial\bar\partial\varphi$ be a Kähler cone metric with $\varphi \in C^{2,\alpha,\beta}$, $f \in C^{,\alpha,\beta}$ and the Hölder exponent* *satisfies* $\alpha\beta < 1-\beta$. *Then the weak solution* $u \in H^{2,\beta}_{\mathbf{w},0}$ *to equation* (3.10) *is actually* $C^{2,\alpha,\beta}$. *Moreover, there exists a constant* C *such that*

$$|u|_{C^{2,\alpha,\beta}} \leq C(\|u\|_{H^{2,\beta}_{\mathbf{w},0}(\omega)} + |f|_{C^{,\alpha,\beta}}).$$

Proof. We use (3.11) again, we only need ω to be Kähler cone metric to conclude $\triangle_\omega u$ is in $C^{\alpha,\beta}$. Then, 2nd order linear elliptic theory [20] tells us that $u \in C^{2,\alpha,\beta}$. □

3.5. **Fredholm alternative for Lichnerowicz operator.** We now use continuity method in the Hölder spaces. We define the operator

$$\mathbb{L}\mathrm{ic}^K_\omega(u) = \mathbb{L}\mathrm{ic}_\omega(u) - K\triangle_\omega u$$

and then define the continuity path $L^K_t : C^{4,\alpha,\beta}(\omega) \to C^{,\alpha,\beta}$ with $0 \leq t \leq 1$,

$$\begin{aligned} L^K_t u &= t\mathbb{L}\mathrm{ic}^K_\omega(u) + (1-t)(\triangle^2_\omega u - K\triangle_\omega u), \\ &= \triangle^2_\omega u + tu^{i\bar{j}}R_{i\bar{j}}(\omega) - K\triangle_\omega u. \end{aligned} \tag{3.12}$$

Multiplying the equation with u and integrating over the manifold M, we obtain the bilinear form

$$\mathcal{B}^K_t(u,u) = \int_M uL^K_t u\,\omega^n = \int_M [|\triangle_\omega u|^2 + tu^{i\bar{j}}R_{i\bar{j}}(\omega)u + K|\nabla u|^2_\omega]\omega^n.$$

We will need the following lemma.

Lemma 3.24. *Assume that* $u \in C^{2,\alpha,\beta}$ *and* $Ric(\omega)$ *is bounded. Then it holds*

$$\int_M u^{i\bar{j}}R_{i\bar{j}}(\omega)u\,\omega^n = -\int_M u^i R_{i\bar{j}}(\omega)u^{\bar{j}}\omega^n.$$

Proof. We shall use the cutoff function χ_ϵ constructed in [41, Section 3.3, p.18]. Our argument follows essentially the lines of [42, Lemma 4.10]. Actually, by dominated convergence theorem, we have

$$\lim_{\epsilon\to 0}\int_M u^{i\bar{j}}R_{i\bar{j}}(\omega)u\chi_\epsilon\omega^n = \int_M u^{i\bar{j}}R_{i\bar{j}}(\omega)u\,\omega^n.$$

On the other hand, using $\nabla Ric = 0$ on M,

$$\begin{aligned} \int_M u^{i\bar{j}}R_{i\bar{j}}(\omega)u\chi_\epsilon\omega^n &= -\int_M u^i R_{i\bar{j}}(\omega)(u\chi_\epsilon)^{\bar{j}}\omega^n, \\ &= -\int_M u^i R_{i\bar{j}}(\omega)u^{\bar{j}}\chi_\epsilon\omega^n - \int_M u^i R_{i\bar{j}}(\omega)\chi^{\bar{j}}_\epsilon u\,\omega^n. \end{aligned}$$

The first term converges under the assumption on u and $Ric(\omega)$. The second term also converges, since for $2 \leq i,j \leq n$,

$$\begin{aligned} u^1 R_{1\bar{1}}(\omega)\chi^{\bar{1}}_\epsilon &= \epsilon.\mathrm{o}(\rho^{-\beta}), \quad u^1 R_{1\bar{j}}(\omega)\chi^{\bar{j}}_\epsilon = \epsilon.\mathrm{o}(1), \\ u^i R_{i\bar{1}}(\omega)\chi^{\bar{1}}_\epsilon &= \epsilon.\mathrm{o}(\rho^{-\beta}), \quad u^i R_{i\bar{j}}(\omega)\chi^{\bar{j}}_\epsilon = \epsilon.\mathrm{o}(1). \end{aligned}$$

This allows us to conclude the proof. □

When $t = 0$, $L_0 u = \triangle^2_\omega u - K\triangle_\omega u$. We could solve $L_0 u = f$ for any $f \in C^{,\alpha,\beta}$ and obtain an solution $u \in C^{4,\alpha,\beta}(\omega)$ thanks to Propositions 3.20 and 3.22.

In order to apply the continuity method to our linear PDE (see e.g. Theorem 5.2 in [23] for Banach space setting), we need to prove the following key estimate.

Theorem 3.25. *Assume ω is a cscK cone metric with $C^{4,\alpha,\beta}(\omega_D)$ potential, (α, β) satisfy the condition (C). Assume that $K > 0$ is large enough, i.e. $K > 1 + 2\|Ric(\omega)\|_{L^\infty} + 3C_P$. There is constant C_1 such that for any $u \in C^{4,\alpha,\beta}(\omega)$ along the continuity path (3.12) with $0 \leq t \leq 1$, we have*

$$|u|_{C^{4,\alpha,\beta}(\omega)} \leq C_1 |L^K_t u|_{C^{,\alpha,\beta}}.$$

Proof. Applying Schauder estimate for 2nd PDE to $\triangle_\omega u$ of the equation

$$\triangle^2_\omega u - K\triangle_\omega u = L^K_t u - tu^{i\bar{j}} R_{i\bar{j}}(\omega),$$

we have for some constant A depending on ω such that

$$|\triangle_\omega u|_{C^{2,\alpha,\beta}} \leq A \left(|L^K_t u - tu^{i\bar{j}} R_{i\bar{j}}(\omega)|_{C^{,\alpha,\beta}} + |\triangle_\omega u|_{C^{,\alpha,\beta}} \right).$$

Since ω is a cscK cone metric with $C^{4,\alpha,\beta}(\omega_D)$ potential, it has $C^{,\alpha,\beta}$ Christoffel symbols and Riemannian curvature, thanks to Proposition 2.7.

In order to obtain the $C^{4,\alpha,\beta}(\omega)$ estimate, we need to control all derivatives up to 4th order regarding to metric ω, according to Proposition 2.7. The idea is to obtain the following inequality, by direct computation (see Lemma 4.5, Lemma 4.7, Lemma 2.16, Lemma 2.17 in [42])

$$|u|_{C^{4,\alpha,\beta}(\omega)} \leq C_1 \left(|\triangle_\omega u|_{C^{2,\alpha,\beta}} + |u|_{C^{2,\alpha,\beta}} \right).$$

Note that both ω and $Ric(\omega)$ are $C^{,\alpha,\beta}$, combining the inequalities above, we have

$$|u|_{C^{4,\alpha,\beta}(\omega)} \leq C_2 \left(|L^K_t u|_{C^{,\alpha,\beta}} + |u|_{C^{2,\alpha,\beta}} \right).$$

By using Proposition 2.7 and the proof as Lemma 6.32 in [23], we have the ϵ-interpolation inequality of the Hölder spaces,

$$|u|_{C^{2,\alpha,\beta}} \leq \epsilon |u|_{C^{4,\alpha,\beta}(\omega)} + C(\epsilon) |u|_{C^{,\alpha,\beta}}.$$

So

$$|u|_{C^{4,\alpha,\beta}(\omega)} \leq C_3 \left(|L^K_t u|_{C^{,\alpha,\beta}} + |u|_{C^{,\alpha,\beta}} \right). \tag{3.13}$$

In order to replace $|u|_{C^{,\alpha,\beta}}$ by $||u||_{L^2(\omega)}$, we claim the following.

Claim 3.26. *There is a constant C_4 such that*

$$|u|_{C^{4,\alpha,\beta}(\omega)} \leq C_4 \left(|L^K_t u|_{C^{,\alpha,\beta}} + ||u||_{L^2(\omega)} \right). \tag{3.14}$$

of the claim. By Morrey inequality (Lemma 3.7), $C^{,\alpha,\beta}$ norm is bounded by $W^{1,p,\beta}$ norm for sufficient large $p > 2n$. We use Corollary 3.11 with sufficient small ϵ, then the $W^{1,p,\beta}$ norm is bounded by $\epsilon\|u\|_{W^{2,p,\beta}_{\mathbf{s}}(\omega)} + C(\epsilon)\|u\|_{L^2(\omega)}$. □

Since $u \in C^{4,\alpha,\beta}(\omega)$, we are able to apply the integration by parts to obtain a Gärding inequality as following. We first use the L^2 estimate of the cone metrics to the standard linear operator $\triangle_\omega u$ (Proposition 3.16), i.e. there exists a constant $C_5 > 0$ such that,

$$\|u\|^2_{H^{2,\beta}(\omega)} \leq C_5 \int_M (|\triangle_\omega u|^2 + |\nabla u|^2_\omega + |u|^2)\omega^n.$$

Here the integration by parts works since $u \in C^{4,\alpha,\beta}(\omega)$. Using the bilinear form $\mathcal{B}^K_t(u,u)$, the R.H.S of previous inequality is

$$C_5 \left(\mathcal{B}^K_t(u,u) + \int_M [-tu^{i\bar{j}} R_{i\bar{j}}(\omega)u + (1-K)|\nabla u|^2_\omega + |u|^2]\omega^n \right).$$

Then we use the simple inequality $|\mathcal{B}^K_t(u,u)| = |\int_M uL^K_t u\,\omega^n| \leq \frac{1}{2}\int_M |u|^2 + |L^K_t u|^2\,\omega^n$,

$$\|u\|^2_{H^{2,\beta}(\omega)} \leq C_5 \Big(\|L^K_t u\|^2_{L^2(\omega)} + \int_M [-tu^{i\bar{j}} R_{i\bar{j}}(\omega)u + (1-K)|\nabla u|^2_\omega + 2|u|^2]\omega^n \} \Big).$$

Integrating by parts the term containing $Ric(\omega)$ using Lemma 3.24, the R.H.S of last inequality becomes

$$C_5 \left(\|L^K_t u\|^2_{L^2(\omega)} + \int_M [tu^i R_{i\bar{j}}(\omega)u^{\bar{j}} + (1-K)|\nabla u|^2_\omega + 2|u|^2]\omega^n \right).$$

Thus,

$$\|u\|^2_{H^{2,\beta}(\omega)} \leq C_5 \Big(\|L^K_t u\|^2_{L^2(\omega)} + \int_M [(1-K+\|Ric(\omega)\|_{L^\infty})|\nabla u|^2_\omega + 2|u|^2]\omega^n \Big).$$

Then we apply the Poincaré inequality (Lemma 3.13) to the 3rd term again and set $K_0 = 1 - K + \|Ric(\omega)\|_{L^\infty} + 2C_P$. We obtain

$$\|u\|^2_{H^{2,\beta}(\omega)} \leq C_5 \left(\|L^K_t u\|^2_{L^2(\omega)} + K_0\|\nabla u\|^2_{L^2(\omega)} \right). \tag{3.15}$$

Now we use the special form of the Lichnerowicz operator to estimate the term $\partial\partial u$ (since $u \in C^{4,\alpha,\beta}(\omega)$), see [42, Lemma 4.10], i.e.

$$\int_M u\mathbb{L}\mathrm{ic}_\omega(u)\omega^n = \int_M |\partial\partial u|^2_\omega \omega^n.$$

Thus we use (3.12), integration by parts (Lemma 3.24) and Cauchy-Schwarz inequality as before,

$$\begin{aligned}\int_M |\partial\partial u|^2_\omega \omega^n &= \int_M u[L_t^K u + (1-t)u^{i\bar{j}} R_{i\bar{j}}(\omega) + K\triangle_\omega u]\omega^n, \\ &\leq \int_M (|L_t^K u|^2 + |u|^2 + \|Ric(\omega)\|_{L^\infty} |\nabla u|^2_\omega - K|\nabla u|^2_\omega)\omega^n.\end{aligned}$$

We apply the Poincaré inequality (Lemma 3.13) to the 2nd term again and set $K_1 = C_P + \|Ric(\omega)\|_{L^\infty} - K$,

$$\int_M |\partial\partial u|^2_\omega \leq \int_M (|L_t^K u|^2 + K_1 |\nabla u|^2_\omega)\omega^n. \tag{3.16}$$

Thus we add (3.15) and C_5 times (3.16) together and have that

$$\|u\|^2_{H^{2,\beta}_{\mathbf{s}}(\omega)} \leq C_5 \left(2\|L_t^K u\|^2_{L^2(\omega)} + (K_0 + K_1)\|\nabla u\|^2_{L^2(\omega)} \right).$$

We further choose $K_0 + K_1 < 0$ i.e. $2K > 1 + 2\|Ric(\omega)\|_{L^\infty} + 3C_P$, then we have

$$\|u\|^2_{H^{2,\beta}_{\mathbf{s}}(\omega)} \leq 2C_5 \|L_t^K u\|^2_{L^2(\omega)}.$$

Together with (3.14), this allows us to conclude that

$$|u|_{C^{4,\alpha,\beta}(\omega)} \leq C_1 |L_t^K u|_{C^{,\alpha,\beta}}. \tag{3.17}$$

□

of Theorem 1.1. We have just solved

$$L_1^K u = \mathbb{L}\mathrm{ic}^K_\omega(u) = \mathbb{L}\mathrm{ic}_\omega(u) - K\triangle_\omega u = f$$

and seen that the inverse map $(\mathbb{L}\mathrm{ic}^K_\omega)^{-1} : C^{2,\alpha,\beta} \to C^{4,\alpha,\beta}(\omega)$ is compact. Now, we can solve (1.1), i.e

$$\mathbb{L}\mathrm{ic}_\omega(u) = \mathbb{L}\mathrm{ic}^K_\omega(u) + K\triangle_\omega u = f. \tag{3.18}$$

Actually, this is equivalent, after taking $(\mathbb{L}\mathrm{ic}^K_\omega)^{-1}$, to

$$u + K(\mathbb{L}\mathrm{ic}^K_\omega)^{-1}\triangle_\omega u = (\mathbb{L}\mathrm{ic}^K_\omega)^{-1} f. \tag{3.19}$$

Since $\mathfrak{T} := -K(\mathbb{L}\mathrm{ic}^K_\omega)^{-1}\triangle_\omega : C^{4,\alpha,\beta}(\omega) \to C^{4,\alpha,\beta}(\omega)$ is compact, we can apply classical results of functional analysis and Riesz-Schauder theory (see [23, Theorem 5.3]) to the Lichnerowicz operator which is self-adjoint. Furthermore the reductivity of the automorphisms group of a Kähler manifold admitting cscK cone metric with $C^{4,\alpha,\beta}(\omega_D)$ potential is proved by [42, Corollary 1.4], showing the one-one correspondence between the kernel of the Lichnerowicz operator and the holomorphic vector fields tangential to the divisor. □

Theorem 1.1 admits a companion result for general Kähler cone metric. Note that contrarily to the previous case, the linearisation of the constant scalar curvature equation at a general Kähler cone metric involves a different operator than the $\mathbb{L}$ic operator.

Theorem 3.27. *Let X be a compact Kähler manifold, $D \subset X$ a smooth divisor, ω a Kähler cone metric with $C^{4,\alpha,\beta}(\omega_D)$ potential such that the cone angle $2\pi\beta$ and the Hölder exponent α satisfy Condition (C). Assume that $f \in C^{,\alpha,\beta}$ with normalisation condition $\int_M f\omega^n = 0$. Then one of the following holds:*

- *Either the Lichnerowicz equation $\mathbb{L}\mathrm{ic}_\omega(u) = f$ has a unique $C^{4,\alpha,\beta}(\omega)$ solution.*
- *Or the kernel of $\mathbb{L}\mathrm{ic}_\omega$ has positive dimension and corresponds to the space of holomorphic vector fields tangent to D.*

The proof of Theorem 3.27 is using exactly the same arguments as for the proof of Theorem 1.1, the crucial point being that the Kähler cone metric has $C^{4,\alpha,\beta}$ potential.

4. Hermitian-Einstein metrics with conical singularities

4.1. **Stable parabolic structures.** From now, we consider $E \to B$ a holomorphic vector bundle over a base B, compact Kähler manifold endowed with a smooth Kähler metric ω_0. Let $D = \sum_{i=1}^m D_i$ be a simple normal crossings divisor of B. We prefer to introduce notions associated with simple normal crossings divisors (rather than smooth divisors) in this subsection as we expect that the results we present in the next sections will hold under this general setting, cf. Section 7.1.

Definition 4.1. *A parabolic structure on E with respect to D consists of:*

- *a filtration of $E_{|D_i}$ for $1 \leq i \leq m$ such that*
$$E_{|D_i} = \mathcal{F}_i^1 \supsetneq .. \supsetneq \mathcal{F}_i^{l_i} \supsetneq \{0\}$$
with $\mathcal{F}_i^{p+1}$ proper subbundle of $\mathcal{F}_i^p$ and the flags satisfy a natural compatibility condition: for every $I = (i_1, ..., i_q)$, the restrictions $\{\mathcal{F}^p_{i_l|D_{i_1}...D_{i_q}}, 1 \leq l \leq q, 1 \leq p \leq l_{i_l}\}$ to $D_{i_1}...D_{i_q}$ yield to a flag of $E_{|D_{i_1}...D_{i_q}}$ which is a refined flag of $\{\mathcal{F}^p_{i_l|D_{i_1}...D_{i_q}}, 1 \leq p \leq l_{i_l}\}$ for every $1 \leq l \leq q$.
- *some real weights $\alpha_i^1, ..., \alpha_i^{l_i}$ attached to $\mathcal{F}_i^p$, $1 \leq p \leq l_i$ satisfying the inequalities $0 \leq \alpha_i^1 < ... < \alpha_i^{l_i} < 1$.*

We recall a classical definition.

Definition 4.2. *Given E a parabolic structure, one can define its parabolic degree with respect to ω_0 as*
$$\mathrm{par}\deg(E) = \deg(E) + \sum_{i=1}^m \sum_{p=1}^{l_i} \mathrm{rk}(\mathcal{F}_i^p/\mathcal{F}_i^{p+1})\alpha_i^p \deg(D_i)$$
and its parabolic slope as $\mathrm{par}\mu(E) = \mathrm{par}\deg(E)/\mathrm{rk}(E)$. Here the degree $\deg(E)$ is computed in the usual sense using the Kähler cone metric

ω_0 and depends only on the Kähler class. This definition extends to coherent subsheaves endowed with parabolic structures.

There exists a notion of stability for parabolic structures modeled on the notion of Mumford-Takemoto stability.

Definition 4.3. *Given a proper coherent subsheaf F of a parabolic vector bundle E along D, one can consider the induced parabolic structure for F. The only difficulty is to choose correctly the weights $\alpha_i^p(F)$. This is done by taking the maximum weights among the $\alpha_i^{p'}(E)$ that respect the flag structure, i.e $\mathcal{F}_i^p(F) \subset \mathcal{F}_i^{p'}(E)$. We say that E is **parabolic stable** if for all proper coherent subsheaf F of E, we have*

$$\mathrm{par}\mu(F) < \mathrm{par}\mu(E).$$

4.2. **Hölder spaces for bundle endomorphisms.** Let V be a holomorphic vector bundle over the base manifold B. Let us fix a holomorphic frame $F_V = \{e_\varsigma; 1 \leq \varsigma \leq \mathrm{rk}(V)\}$ and a finite covering $(U_i)_{i=0,..,N}$ of B composed of local cone charts around D. Consider a partition of unity $\{\psi_i\}_{i=0,..,N}$ where $\psi_i \in C^\infty(U_i)$ have compact support, associated to the finite covering $(U_i)_{i=0,..,N}$. Using the notations of subsection 2.1, we define the space $C^{,\alpha,\beta}(V)$ to be the space of sections s of V such that in F_V the decomposition of s is given by $\mathrm{rk}(V)$ functions $s_1, .., s_{\mathrm{rk}(V)}$ that lie in the space $C^{,\alpha,\beta}$. More precisely, s is a section in the Hölder space $C^{,\alpha}(V)$ such that if the local frame is defined close to the divisor D, over a cone chart U_i, each $\mathrm{rk}(V)$ complex valued functions defining s lie in the space $C^{,\alpha,\beta}(U_i)$. The advantage of fixing a frame and a partition of unity is that we can define now a norm $\|.\|_{C^{,\alpha,\beta}}$ by

$$\|s\|_{C^{,\alpha,\beta}} = \sum_{i=0}^{N} \sum_{j=1}^{\mathrm{rk}(V)} \|\psi_i s_j\|_{C^{,\alpha,\beta}(U_i)}.$$

Note that the space of sections with bounded $\|.\|_{C^{,\alpha,\beta}}$ norm is independent of the covering, the partition of unity and the holomorphic frame.

Similarly to subsection 2.1, we can define the vector spaces $C^{2,\alpha,\beta}(V)$, $C^{3,\alpha,\beta}(V)$, $C^{4,\alpha,\beta}(V)$ with respect to a Kähler cone metric, and also the associated norms by considering the analogue conditions on the decomposition of s. Eventually, all the spaces equipped with their natural norms are Banach spaces. With the previous reasoning, we can define this way the Hölder spaces of $End(E)$ that are denoted $C^{k,\alpha,\beta}(End(E))$.

The definition also applies to the space $\mathcal{H}^+(E)$ of hermitian metrics on the bundle E seen as sections of the frame bundle and for $Herm(E, h)$ the space of hermitian endomorphisms of E (over $M \setminus D$) with respect to the hermitian metric h.

4.3. **Existence of Hermitian-Einstein cone metrics.** Given ω_0 and the divisor D, we can consider a model metric ω_D as in Section 2.2. Moreover, we will denote ω_B a Kähler cone metric in the same class and $C^{2,\alpha,\beta}$ potential, which is quasi-isometric to ω_D.

We introduce now the Hermitian-Einstein equation and express it in coordinates. Let $H = \{H_{\varsigma\bar{\tau}}\}$ be the hermitian matrix induced by h in a local holomorphic frame $\{e_\varsigma; 1 \leq \varsigma \leq r\}$ for E of rank r.

Definition 4.4. *We say a hermitian metric h is $C^{k,\alpha,\beta}$ for $k \in \mathbb{Z}^+ = \{0, 1, 2, 3, \cdots\}$, if the associated hermitian matrix $H = \{H_{\varsigma\bar{\tau}}\}$ is $C^{k,\alpha,\beta}$, i.e. all its components $H_{\varsigma\bar{\tau}}$, $1 \leq \varsigma, \tau \leq r$ are all $C^{k,\alpha,\beta}$.*

The curvature is given as a 2 form

$$\frac{\sqrt{-1}}{2\pi} F^{\tau}_{\varsigma k\bar{j}} dz^k \wedge d\bar{z}^j = \frac{\sqrt{-1}}{2\pi} \sum_{j,k} F_{k,\bar{j}} dz^k \wedge d\bar{z}^j$$

with explicitly

$$F^{\tau}_{\varsigma k\bar{j}} = -\partial_{\bar{j}}(H^{\tau\bar{\gamma}} \partial_k H_{\gamma\bar{\varsigma}}).$$

Lowing down the index, we have

$$\begin{aligned} F_{\varsigma\bar{\tau}k\bar{j}} &= \sum H_{\gamma\tau} F^{\gamma}_{\varsigma k\bar{j}} \\ &= -\partial_{\bar{j}}\partial_k H_{\varsigma\bar{\tau}} + \sum_{\gamma,\nu=1}^{r} H^{\gamma\nu} \partial_k H_{\varsigma\bar{\nu}} \partial_{\bar{j}} H_{\gamma\bar{\tau}}. \end{aligned}$$

Fix $\omega_B = \frac{\sqrt{-1}}{2}(g_B)_{k\bar{j}} dz^k \wedge d\bar{z}^j$ a Kähler metric on the base B. The Hermitian-Einstein equation

$$\frac{\sqrt{-1}}{2\pi} \Lambda_{\omega_B} F_h = Cst \times Id_E$$

reads in coordinates, using the Kronecker symbol δ,

$$\begin{aligned} Cst \times \delta_{\varsigma\bar{\tau}} &= g_B^{k\bar{j}} F_{\varsigma\bar{\tau}k\bar{j}} \\ &= -\triangle_{\omega_B} H_{\varsigma\bar{\tau}} + g_B^{k\bar{j}} \sum_{\gamma,\nu=1}^{r} H^{\gamma\bar{\nu}} \partial_k H_{\varsigma\bar{\nu}} \partial_{\bar{j}} H_{\gamma\bar{\tau}}. \end{aligned}$$

Definition 4.5 ((Compatible metric with respect to parabolic structure))**.** *Let h be a metric on the parabolic bundle E. We say that h is compatible with the parabolic structure if the following holds. Given the parabolic structure, it is constructed by Li in [38] a model metric h_0 on E over $B \setminus D$, such that*

$$|\Lambda_{\omega_D} F_{h_0}|_{h_0} \in L^\infty(B \setminus D), \qquad |F_{h_0}|_{h_0} \in L^p(B \setminus D), p > 1. \tag{4.1}$$

This model metric on the bundle is natural. In a nutshell, the norm of a local section of $\mathcal{F}_i$ with respect to h_0 restricted to D has growth controlled by the weights of the filtration. To be compatible for h metric on E means that the 2 following conditions hold:

- *h, h_0 are mutually bounded;*
- *$|\bar{\partial}(h_0^{-1}h)|_{h_0} \in L^2(B, \omega_D)$.*

Given ω_D the model metric and h a metric on E, compatible with respect to the parabolic structure, it is possible to compute the analytic degree of E. It is given by the differential geometry as

$$\widetilde{\deg}(E) = \int_{B\setminus D} \operatorname{tr}\left(\frac{\sqrt{-1}}{2\pi}\Lambda_{\omega_D} F_h\right)\frac{\omega_D^n}{n!},$$

and a similar formula applies for the proper coherent subsheaves of E. In [38] it is checked that $\widetilde{\deg}(E)$ is actually proportional to the parabolic degree of $\operatorname{par}\deg(E)$ and is an invariant of the space of hermitian metrics compatible with the parabolic structure. In other words, the bundle E is parabolic stable if and only if it is stable with respect to the notion of slope induced by the analytic degree.
The same property holds if we replace ω_D by ω_B as we assumed it has $C^{2,\alpha,\beta}$ potential, and we have $\tilde{\mu}(E) = c \times \operatorname{par}\deg(E)$ for a certain constant $c > 0$. The property of compatibility can also be defined using the Kähler cone metric ω_B. In conclusion, we can speak of *parabolic stability of the parabolic bundle E with respect to ω_B* by using the analytic degree.

We are ready to present a theorem of C. Simpson improved by J. Li.

Theorem 4.3.1 (([47], [38, Theorem 6.3])). *Let B be a base compact Kähler manifold endowed with a Kähler metric ω_B with conical singularities along $D \subset B$, smooth divisor. Let E a parabolic stable vector bundle over B with respect to ω_B. There exists $\delta_0 > 0$ such that if the angle $2\pi\beta$ of ω_B satisfies $0 < \beta_i \leq \delta_0$, then there exists a Hermitian-Einstein metric h_E on E compatible with the parabolic structure over D. It satisfies outside D the Hermitian-Einstein equation,*

$$\frac{\sqrt{-1}}{2\pi}\Lambda_{\omega_B} F_{h_E} = \frac{\tilde{\mu}(E)}{\mathrm{Vol}} Id_E. \tag{4.2}$$

Here Id_E is the identity endomorphism of $E_{|B\setminus D}$ and Vol *the total volume of B with respect to ω_B.*

We introduce the following definition of Hermitian-Einstein cone metric.

Definition 4.6. *As above, let B be a base compact Kähler manifold endowed with a Kähler metric ω_B with conical singularities along D, smooth divisor. Let E a parabolic vector bundle with respect to D and h_E a hermitian metric on $E_{|B\setminus D}$. We say that h_E is a Hermitian-Einstein cone metric, if h_E satisfies the Hermitian-Einstein equation* (4.2) *pointwisely over $B \setminus D$, h_E is compatible with the parabolic structure and h_E lies in $C^{2,\alpha,\beta}(\mathcal{H}^+(E))$.*

Theorem 4.7. *Under same assumptions as in Theorem 4.3.1, the Hermitian-Einstein metric h_E is actually a Hermitian-Einstein cone metric in the sense of Definition 4.6.*
Moreover, if α and β satisfy the Condition (C), then $h_E \in C^{4,\alpha,\beta}(\mathcal{H}^+(E);\omega_B)$.

Remark 4.8. *Note that the converse is true and constitutes the easy sense of the correspondence: an indecomposable parabolic vector bundle equipped with a Hermitian-Einstein cone metric compatible with its parabolic structure is actually parabolic stable. We refer* [38, Theorem 6.3]*,* [47, Proposition 3.3].

Proof. We start the proof by noticing that we could take a partition of unity $\{\rho_p\}$ of the base manifold B subordinate to an open cover $\{U_p\}$ and construct the Hermitian-Einstein metric on each trivialization of the holomorphic vector bundle E over each cover. It suffices to consider the cone chart U, which intersects with the divisor D, since far from the divisor all the arguments are the same to [3, 16–18, 51].

The proof is divided in several steps. We first use Dirichlet problem for Donaldson's flow to produce the weak solution H to the Hermitian-Einstein equation away from the divisor and then prove the regularity of the Hermitian-Einstein limit metric. Of course, the new improvement with our theorem is the regularity of the weak Hermitian-Einstein metric. Note that we do not try to improve the regularity of the flow itself, which is a parabolic system and the Schauder estimate is not yet known in this case. But instead, we use the limit equation and improve the regularity by observing that the nonlinear term itself is Hölder. As a result, each equation in the system is independent, and we are able to apply the elliptic regularity theorem for second order equations with conical singularities to each single equation of the system.

Using h_0 the model metric fixed by Li and which satisfies (4.1), we write the endomorphism

$$H = h{h_0}^{-1}.$$

We denote by D_δ a δ tubular neighbourhood of the divisor D for small $0 < \delta \leq 1$. We also use

$$\lambda = \frac{\tilde{\mu}(E)}{\mathrm{Vol}}.$$

Eventually, we omit the factor $\frac{\sqrt{-1}}{2\pi}$ in from of the contraction operator Λ_ω to ease notations.

According to Donaldson [18], the Dirichlet problem for the following flow of hermitian metrics $h_\delta(t) = h_\delta(.,t)$ on $U_\delta = U \setminus D_\delta$,

$$\begin{cases} \dot{h}_\delta h_\delta^{-1} = -(\Lambda_{\omega_B} F_{h_\delta} - \lambda Id_E) \text{ over } U_\delta, \\ h_\delta(x,0) = h_0, \quad x \in U_\delta \\ h_\delta(x,t) = h_0, \quad x \in \partial U_\delta, t \geq 0 \end{cases} \tag{4.3}$$

has a unique global solution for time $0 \leq t < +\infty$. As further shown in [18], the convergence of (4.3) is irrelevant of the delicate conditions of stability. We will need that the approximation flow h_δ converges to a limit flow as $\delta \to 0$, while the limit flow converges to a Hermitian-Einstein metric h_∞ with conical singularities as $t \to +\infty$ and this limit metric h_∞ has higher order regularity across the divisor D. In order to achieves these goals, we need the following a priori estimates.

Step: Uniform bound of $A(x,t) := |\Lambda_{\omega_B} F_{h_\delta}|_{h_\delta}$. We have along the flow (4.3),

$$\begin{cases} (\partial_t - \triangle_{\omega_B})A(x,t) \leq 0 \text{ in } U_\delta, \\ A(x,0) = |\Lambda_{\omega_B} F_{h_0}|^2_{h_0}, \quad x \in U_\delta \\ A(x,t) = |\Lambda_{\omega_B} F_{h_0}|^2_{h_0}, \quad x \in \partial U_\delta, t \geq 0. \end{cases} \tag{4.4}$$

Let $A(t) = \sup_{U_\delta} A(x,t)$, then we apply the maximum principle,

$$\partial_t A(t) \leq 0. \tag{4.5}$$

So we prove that $\sup_{U_\delta} |\Lambda_{\omega_B} F_{h_\delta}|_{h_\delta}$ is non-increasing along the flow, and also it is uniformly bounded by the initial given data $|\Lambda_{\omega_B} F_{h_0}|_{h_0}$ and independent of δ and t.

Step: Zero order estimate. We are now aiming to prove that $h_\delta(t)$ converges to $h_\delta(T)$ in C^0 norm when $t \to T$, for any finite time $T < \infty$. We shall use Donaldson's distance between two Hermitian metrics, namely

$$\sigma(h,k) = \mathrm{tr} h^{-1}k + \mathrm{tr} k^{-1}h - 2\mathrm{rk}(E). \tag{4.6}$$

It is known that for any two flows of hermitian metrics $h(t)$ and $k(t)$,

$$\left(\frac{\partial}{\partial t} - \triangle_{\omega_B}\right) \sigma(h(t), k(t)) \leq 0. \tag{4.7}$$

For any $\epsilon > 0$, we choose $\kappa > 0$ such that in the κ-neighbourhood of $t = 0$, i.e. for all $0 \leq s, \tau < \kappa$,

$$\sup_{U_\delta} \sigma(h_\delta(s), h_\delta(\tau)) < \epsilon.$$

We now let $k_\delta(t) = h_\delta(t+s)$ in the inequality above, we see that $\sigma(h_\delta(t), k_\delta(t))$ is always zero on the boundary of the domain U_δ along the flow. From maximum principle, we see that in the κ-neighbourhood of T, i.e. when $T - \kappa < s', \tau' < T$,

$$\sup_{U_\delta} \sigma(h_\delta(s'), h_\delta(\tau')) \leq \epsilon.$$

Thus $h_\delta(t)$ is a uniform Cauchy sequence and converges in C^0 norm to $h_\delta(T)$.

Step: Gradient estimate. It follows from the contradiction method, see [48, Lemma 6.4], that $h_\delta(t)$ are bounded in C^1.

Step: $W^{2,p}(\omega_B)$ estimate. From the gradient estimates above, $|H_{\varsigma\bar{\tau}}|_{C^1}$ is bounded and also is $\Lambda_{\omega_B} F_{h_\delta}$. So $-\triangle_{\omega_B} H_{\varsigma\bar{\tau}}$ is bounded by using the equation

$$(\Lambda_{\omega_B} F_{h_\delta})_{\varsigma\bar{\tau}} = -\triangle_{\omega_B} H_{\varsigma\bar{\tau}} + g_B^{k\bar{j}} \sum_{\gamma,\nu=1}^{r} H^{\gamma\bar{\nu}} \partial_k H_{\varsigma\bar{\nu}} \partial_{\bar{j}} H_{\gamma\bar{\tau}}. \tag{4.8}$$

After applying the interior L^p theory of the linear equation in U_δ, we have for any ς, τ, $H_{\varsigma\bar{\tau}} \in W^{2,p}_{\mathbf{s}}(\omega_B)(K)$ for any $K \subset\subset U_\delta$. And Proposition 3.16 for weaker Sobolev spaces tells us $H_{\varsigma\bar{\tau}} \in W^{2,p}(\omega_B)(M)$. Thus F_{h_δ} is bounded in $L^p(M)$ norm for any $1 \leq p < \infty$. Note that $W^{2,p}_{\mathbf{s}}$ is the strong Sobolev spaces and $W^{2,p}$ is the weaker one, see Definitions 3.3 and 3.2.

Step: Long time existence. In each U_δ, we can see deduce from the estimates above that the solution to the approximation equation (4.3) has long time existence. For any compact subset $K \subset\subset B \setminus D$, we could choose a small enough δ_K such that for any $\delta < \delta_K$, $K \subset\subset U_\delta$. Since the metrics $h_\delta(x,t)$ have uniform $W^{2,p}(\omega_B)(M)$ estimates for any p, and independently of δ, h_δ converges to a limit flow $h = \lim_{\delta\to 0} h_\delta$ in $W^{2,p}(\omega_B)(M)$-norm for any $p \geq 1$ and furthermore h solves

$$\begin{cases} \dot{h}h^{-1} = -(\Lambda_{\omega_B} F_h - \lambda I) \text{ in } B \setminus D, \\ h(x,0) = h_0, \quad x \in B \setminus D. \end{cases} \tag{4.9}$$

Step: Convergence. We need a subspace of the space of Hermitian metrics,

$$\mathcal{H}_{bounded}(E) = \Big\{ h \text{ is a Hermitian metric on } E \text{ over } B \setminus D \text{ such that}$$
$$\sup_{B\setminus D} |h| < +\infty \text{ and } \sup_{B\setminus D} |\Lambda_{\omega_B} F_h|_h < 2 \sup_{B\setminus D} |\Lambda_{\omega_B} F_{h_0}|_{h_0} \Big\}.$$

Obviously, $h_0 \in \mathcal{H}_{bounded}(E)$. Furthermore, for any $|h - h_0|_{C^{2,\alpha,\beta}} \leq \epsilon$, we still have $\sup_{B\setminus D} |\Lambda_{\omega_B} F_h|_h < 2 \sup_{B\setminus D} |\Lambda_{\omega_B} F_{h_0}|_{h_0}$, provided that ϵ is small enough. Under the topology induced by the $C^{2,\alpha,\beta}$-topology, we consider the path-connected branch of h_0, denoted by $\mathcal{H}_{bounded,h_0}(E)$, i.e. the set of metrics that be connected to h_0 by a path

$$\{h_s, 0 \leq s \leq 1\} \subset \mathcal{H}_{bounded}(E).$$

Donaldson's functional for the cone version is well-defined on the space of Hermitian metrics $\mathcal{H}_{bounded,h_0}(E)$ with suitable asymptotic behavior near the divisor,

$$\begin{aligned} M_D(h_0, h) &= \int_0^1 ds \int_B \operatorname{tr}(\dot{h}_s h_s^{-1} \cdot F_{h_s}) \frac{\omega_B^{n-1}}{(n-1)!} \\ &\quad - \lambda \int_B \log\det(h_0 h^{-1}) \frac{\omega_B^n}{n!}, \\ &= \int_0^1 ds \int_B \operatorname{tr}(\dot{h}_s h_s^{-1} \cdot \Lambda_{\omega_B} F_{h_s}) \frac{\omega_B^n}{n!} \end{aligned}$$

$$- \lambda \int_B \log\det(h_0 h^{-1}) \frac{\omega_B^n}{n!},$$

where $h_s \in \mathcal{H}_{bounded,h_0}(E)$ is path connecting h_0 and h. The definition is independent of the choice of the path. Actually, one can adapt to our setting the proof of the classical smooth case. The proof consists in showing that the variation of $M_D(h_0, .)$ is a closed 1-form. It requires to study the term $\phi_h := \mathrm{tr}(h^{-1}\tilde{d}h \cdot F_h)$ where

$$h : \{(t,s), a \leq t \leq b, 0 \leq s \leq 1\} = \Delta \to \mathcal{H}_{bounded}(E)$$

is a smooth map and $\tilde{d} = (\partial_s)ds + (\partial_t)dt$ is the exterior differentiation on the domain Δ. But the 1-form ϕ_h is well defined from our assumption on $\mathcal{H}_{bounded}(E)$ and one can apply Stokes theorem $\int_{\partial\Delta} \phi_h = \int_\Delta \tilde{d}\phi_h$. Then one can follow word by word the proof of [31, Lemma 3.6]. Alternatively, one can show that the curvature of h is a moment map for the action of the Gauge group on the space of Chern connections associated to $\mathcal{H}_{bounded}(E)$, see for instance [19] and also [47, Lemma 7.2]. Then, as a classical result of the moment map theory, M_D is the associated integral to this moment map and is consequently independent of the choice of the path.

We have by the arguments of [47, Proposition 5.3], that there are two constants C_1 and C_2 such that for any $h \in \mathcal{H}_{bounded}(E)$

$$\|\log \mathrm{tr} H\|^2_{L^1(\omega_B)} \leq C_1 + C_2 M_D(h_0, h). \tag{4.10}$$

Since $\Lambda_{\omega_B} F_h$ and $h_0 h^{-1}$ are both bounded, the following functional is well-defined along the flow h_t,

$$\begin{aligned} M_D(h_0, h_t) = & \int_0^t d\tau \int_B \mathrm{tr}(\dot{h}_\tau h_\tau^{-1} \cdot \Lambda_{\omega_B} F_{h_\tau}) \frac{\omega_B^n}{n!} \\ & - \lambda \int_B \log\det(h_0 h_t^{-1}) \frac{\omega_B^n}{n!}. \end{aligned}$$

We need its first variation formula along the flow,

$$\begin{aligned} \frac{d}{dt} M_D(h_0, h_t) &= \int_B \mathrm{tr}(\dot{h}_t h_t^{-1} \cdot \Lambda_{\omega_B} F_{h_t}) \frac{\omega_B^n}{n!} - \lambda \int_B tr(\dot{h}_t h_t^{-1}) \frac{\omega_B^n}{n!}, \\ &= - \int_B |\Lambda_{\omega_B} F_{h_t} - \lambda Id_E|^2 \frac{\omega_B^n}{n!}. \end{aligned} \tag{4.11}$$

Thus the functional M_D is non-increasing along the flow. This leads to a uniform upper bound to $\|\log \mathrm{tr} H\|_{L^1(\omega_B)}$ from (4.10).

Now, we wish to apply De Giorgi-Nash-Moser iteration method for Kähler cone metrics of [41, Section 4] to the bounded $\log \mathrm{tr} H$ in the following functional inequality

$$\triangle_{\omega_B} \log \mathrm{tr} H \geq -(|\Lambda_{\omega_B} F_{h_0}| + |\Lambda_{\omega_B} F_h|) := -f. \tag{4.12}$$

In order to do so, we need to examine the conditions of Proposition 4.8 in [41]. Firstly, the following Sobolev inequality with respect to

ω_B holds, i.e. for any $w \in W^{1,2}(\omega_B)$, there is a Sobolev constant $C_S(\omega_B) < +\infty$ such that

$$\|w\|^2_{L^{2^*}(\omega_B)} \leq C_S(\omega_B)(\|\nabla w\|^2_{L^2(\omega_B)} + \|w\|^2_{L^2(\omega_B)}),$$

where $2^* = \frac{2n}{n+1}$. Secondly, we need to rewrite (4.12) to the following form by using integration by parts, i.e. $v = \log \operatorname{tr} H$ is a $W^{1,2}$ subsolution of the linear equation in the weak sense, i.e. for any $\eta \in C^{2,\alpha,\beta}$,

$$\int_B (\partial v, \partial \eta)_{\omega_B} \omega_B^n \leq - \int_B f \eta \omega_B^n. \tag{4.13}$$

This is achieved by using the approximation sequence and v vanishes on the exhaustion domains. So, outside the δ neighbourhood of the divisor D,

$$\int_{B \setminus D_\delta} (\partial v, \partial \eta)_{\omega_B} \omega_B^n = \int_{B \setminus D_\delta} -\triangle v \eta \omega_B^n + \int_{\partial D_\delta} v \cdot \partial \eta d\nu \ \ \leq - \int_B f \eta \omega_B^n. \tag{4.14}$$

The boundary $\int_{\partial D_\delta} v \cdot \partial \eta d\nu \to 0$, as $\delta \to 0$.

Let

$$\tilde{v} = v - \frac{1}{V} \int_M v \, \omega_B^n,$$

then there exists from [41, Proposition 4.8] a constant C depending on the Sobolev constant C_S with respect to ω_B such that for $p^* = \frac{2np}{2n+p}$,

$$\sup_B \tilde{v} \leq C(\|f\|_{L^{p^*}(\omega_B)} + \|\tilde{v}\|_{L^1(\omega_B)}). \tag{4.15}$$

We thus obtain the L^∞ bound of H,

$$\sup_B |\log \operatorname{tr} H| \leq C(\|f\|_\infty + \|\log \operatorname{tr} H\|_{L^1(\omega_B)}). \tag{4.16}$$

From the monotonicity of the energy along the flow (4.11), the right hand side is uniformly bounded. Letting $\eta = v$ in (4.13), we have

$$\|\log \operatorname{tr} H\|_{W^{1,2}(\omega_B)} \leq C. \tag{4.17}$$

Thus we are able to prove that H_t converges in C^0 norm to some H_∞ and then get C^1 norm of H_t which is independent of t, as the gradient estimate above in *Step: Gradient estimate*. After applying the L^p theory of the linear equation with respect to Kähler cone metrics developed in [13] to the curvature equation (4.8), we have $H_{\varsigma\bar{\tau}} \in W^{2,p}(\omega_B)$. Furthermore, the $W^{2,p}$ norm of $H(t)$ is independent of t, as the proof in *Step: $W^{2,p}(\omega_B)$ estimate.*

Step: $W^{1,2}(\omega_B)$ weak solution. Now we have a Hermitian-Einstein metric on the regular part, but we still need to verify that the limit metric satisfies the Hermitian-Einstein equation in $W^{1,2}(\omega_B)$ sense,

$$Const \cdot \delta_{\varsigma\bar{\tau}} = -\triangle_{\omega_B} H_{\varsigma\bar{\tau}} + g_B^{k\bar{j}} \sum_{\gamma,\nu=1}^{r} H^{\gamma\bar{\nu}} \partial_k H_{\varsigma\bar{\nu}} \partial_{\bar{j}} H_{\gamma\bar{\tau}}. \tag{4.18}$$

We fix $H = H_{\varsigma\bar{\tau}}$ and denote the nonlinear term

$$\mathsf{N} = g_B^{k\bar{j}} \sum_{\gamma,\nu=1}^{r} H^{\gamma\bar{\nu}} \partial_k H_{\varsigma\bar{\nu}} \partial_{\bar{j}} H_{\gamma\bar{\tau}}.$$

We need

$$C \int_B \mathrm{Id}\, \eta\, \omega_B^n = \int_B (\partial H, \partial\eta)_{\omega_B} \omega_B^n + \int_B \mathsf{N}\eta\, \omega_B^n. \tag{4.19}$$

It suffices to use the approximation sequence again and prove the boundary term $\int_{\partial D_\delta} \partial H \cdot \eta \to 0$, as $\delta \to 0$. Thus is true, since H has uniform C^1 norm.

Step: $C^{4,\alpha,\beta}(\omega_B)$ estimate. Once we have $H_{\varsigma\bar{\tau}}$ is a $W^{1,2}(\omega_B)$ weak solution and lies in $W^{2,p}(\omega_B)$, we can apply the Sobolev embedding theorem [13] to obtain $H_{\varsigma\bar{\tau}} \in C^{1,\alpha,\beta}$, thus returning to (4.18), the nonlinear term N is $C^{,\alpha,\beta}$. Then we have $H_{\varsigma\bar{\tau}} \in C^{2,\alpha,\beta}$ by Donaldson's Schauder estimate and bootstrap to $H_{\varsigma\bar{\tau}} \in C^{4,\alpha,\beta}(\omega_B)$ similar to the proof of Proposition 3.22 in Section 3.4.

□

Remark 4.9. *From [38], one gets that δ_0 depends on the (difference of the) weights of the parabolic structure of E. So a priori, δ_0 is fixed and we don't know its size, it can be $>$ or $<1/2$. But we consider here the theorem only for angle $\beta < \min(1/2, \delta_0)$. In general, we believe the asymptotic behaviour of Hermitian-Einstein metric h_E could be well-understood with the method in [52] and our angle restriction could be removed.*

4.4. **Parabolic stability and holomorphic vector fields.** It is well-known that Mumford stable vector bundles are simple. In the parabolic setting, we have the following result.

Lemma 4.10. *Assume E is parabolic stable with parabolic structure along a simple normal crossings divisor $D = \sum_{i=1}^m D_i$. Then the holomorphic endomorphism are the homotheties, i.e*

$$H^0(End(E)) = \mathbb{C}.$$

Proof. The proof is similar to the non parabolic case. Let f be a holomorphic endomorphism which is not zero or an isomorphism. Then by holomorphicity of f, $\ker(f), \mathrm{Im}(f)$ have a coherent subsheaf of E with quotient torsion free. One can obtain a parabolic structure for $F = \ker(f), \mathrm{Im}(f)$ by intersection $F_{|D_i}$ with the elements of the flag of $E_{|D_i}$, discarding the subspaces of $F_{|D_i}$ that coincide with another one, and considering the associated largest parabolic weights. Thanks to the parabolic stability of E, we have now the inequalities

$$\frac{\mathrm{par}\deg(\ker(f))}{\mathrm{rk}(\ker(f))} < \mathrm{par}\mu(E), \tag{4.20}$$

$$\frac{\operatorname{par}\deg(\operatorname{Im}(f))}{\operatorname{rk}(\operatorname{Im}(f))} < \operatorname{par}\mu(E). \tag{4.21}$$

But, the parabolic weights of $\ker(f)$ and $\operatorname{Im}(f)$ satisfy also

$$\operatorname{par}\mu(E) = \frac{\operatorname{par}\deg(\ker(f)) + \operatorname{par}\deg(\operatorname{Im}(f))}{\operatorname{rk}(\ker(f)) + \operatorname{rk}(\operatorname{Im}(f))}. \tag{4.22}$$

Using inequalities (4.20), (4.21) and Equation (4.22), one gets a contradiction: actually $\operatorname{par}\mu(\ker(f)) < \operatorname{par}\mu(\operatorname{Im}(f))$ and $\operatorname{par}\mu(\operatorname{Im}(f)) < \operatorname{par}\mu(\ker(f))$. Thus, f is an isomorphism or trivial. In the first case, fix $x \in B$ and consider any eigenvalue of $f : E_x \to E_x$. Let's call λ_0 this eigenvalue. Then by the reasoning as above, $f - \lambda_0 Id_E$ is zero and consequently f is an homothety. $\square$

We need the following classical definition of logarithmic tangent bundle.

Definition 4.11. *Consider B a complex manifold of complex dimension n and D a divisor with simple normal crossings singularities. In local coordinates $D = \{z \ s.t \ \prod_{i=1}^d z^i = 0\}$. Then the logarithmic tangent bundle $TB(-\log D)$ is the locally free sheaf generated by the vector fields $z^i \frac{\partial}{\partial z^i}$ where $1 \leq i \leq d$ and the vector fields $\frac{\partial}{\partial z^i}$ for $d < i \leq n$. Let us consider the stratification of B given by $B_0 = B \setminus D$, $B_1 = D \setminus Sing(D)$ and recursively B_k is the non-singular part of $Sing(B_{k-1})$. $T_B(-\log D)$ can be seen as the sheaf of holomorphic vector fields v on B such that for every $k \geq 0$, every $x \in B_k$, v_x is tangent to B_k.*

We denote X the projectivised bundle and π the associated projection map to B,

$$X := \mathbb{P}E^*, \qquad \pi : X \to B.$$

Set $\mathcal{D} := \pi^{-1}(D)$ where D is a divisor with simple normal crossings singularities of B. Next, we derive some information on the holomorphic automorphisms of X when E is parabolic stable and B has no nontrivial holomorphic vector field.

Corollary 4.12. *Assume E is a parabolic stable vector bundle with respect to the Kähler cone metric ω_B and the base B has trivial Lie algebra $Lie(Aut_D(X, [\omega_B]))$, so there is no non trivial holomorphic vector field tangent to D. Then $Lie(Aut_{\mathcal{D}}(X, [k\pi^*\omega_B + \hat{\omega}_E]))$ is actually trivial.*

Proof. Consider $TFibre(-\log \mathcal{D})$ the sheaf of logarithmic tangent vectors to the fibre of π with respect to $\mathcal{D}$. There is an exact sequence

$$0 \to TFibre(-\log \mathcal{D}) \to TX(-\log \mathcal{D}) \to \pi^*(TB(-\log D)) \to 0$$

which provides a long exact sequence

$$0 \to H^0(X, TFibre(-\log \mathcal{D})) \to H^0(X, TX(-\log \mathcal{D})) \\ \to H^0(X, \pi^*(TB(-\log D))) \to \ldots$$

Now, $H^0(X, \pi^*(TB(-\log D))) = 0$ by assumption, and thus we obtain that the space $H^0(X, TFibre(-\log \mathcal{D}))$ is isomorphic to $H^0(X, TX(-\log \mathcal{D}))$. On another hand, $H^0(B, \pi_* TFibre(-\log \mathcal{D}))$ can be identified with the holomorphic parabolic endomorphisms along D that are trace free. By parabolic endomorphisms, we mean endomorphisms of E that preserve the parabolic structure of E. Hence,

$$H^0(X, TFibre(-\log \mathcal{D})) \simeq H^0(B, \pi_* TFibre(-\log \mathcal{D})).$$

Eventually, using Lemma 4.10, we obtain $H^0(X, TFibre(-\log D)) = 0$ and thus,

$$H^0(X, TX(-\log \mathcal{D})) = 0.$$

This means that there is no nontrivial holomorphic vector fields tangent to D. $\square$

5. Construction of cscK cone metrics over projective bundles

Given a (hermitian) vector space Ξ, there is an isomorphism between Ξ and the space $H^0(\mathbb{P}\Xi^*, \mathcal{O}_{\mathbb{P}\Xi^*}(1))$. This leads to define a metric on $\mathcal{O}_{\mathbb{P}\Xi^*}(1)$ by the following construction. For $v \in \Xi$, the element $\hat{v} \in H^0(\mathbb{P}\Xi^*, \mathcal{O}_{\mathbb{P}\Xi^*}(1))$ is such that $\hat{v}(\xi) = \xi(v)$ for $\xi \in \Xi^*$. Then from any metric h on Ξ, we get a metric h^* on Ξ^* and a Fubini-Study metric $\hat{h}$ on the line bundle $\mathcal{O}_{\mathbb{P}\Xi^*}(1)$ by the formula

$$\hat{h}(\hat{v}, \hat{w})(\xi) = \frac{\xi(v)\overline{\xi(w)}}{|\xi|^2_{h^*}} \tag{5.1}$$

for $v, w \in \Xi$, and $\xi \in \Xi^*$.
Consequently, from the Hermitian metric h_E on the holomorphic vector bundle E, we get a Hermitian metric $\hat{h}_E$ on the line bundle $\mathcal{O}_{\mathbb{P}E^*}(1)$.

5.1. Construction of background metrics.

Over X, we consider the Kähler metric $\hat{\omega}_E \in c_1(\mathcal{O}_{\mathbb{P}E^*}(1))$ outside $\mathcal{D}$ induced by the metric h_E on E and given by the formula

$$\hat{\omega}_E = \hat{\omega}_E(h_E) := i\bar{\partial}\partial \log \hat{h}_E. \tag{5.2}$$

Proposition 5.1. *Let h_E the Hermitian-Einstein cone metric obtained in Theorem 4.7. Then $\hat{\omega}_E$ is a (1,1)-form on $X = \mathbb{P}E^*$ with conical singularity along $\mathcal{D}$ in $C^{2,\alpha,\beta}$ topology.*

Proof. The local computation of $\hat{\omega}_E$ using (5.1) involves only terms of the form $\partial_k H_{\varsigma\bar{\tau}}$, $\partial_j \partial_{\bar{k}} H_{\varsigma\bar{\tau}}$ where H is the matrix representing h in a local frame close to D. Thus, in order to have a cone metric we only need to have the entries of H to be $C^{4,\alpha,\beta}$, and $h_E \in C^{4,\alpha,\beta}(\mathcal{H}^+(E); \omega_B)$, which is the case from Theorem 4.7. $\square$

Thus, from the metric on the base B, we obtain a reference Kähler cone metric in $[k\pi^*\omega_B + \hat{\omega}_E]$, that we denote by

$$\omega_k = k\pi^*\omega_B + \hat{\omega}_E. \tag{5.3}$$

Lemma 5.2. *Let h_E the Hermitian-Einstein cone metric obtained in Theorem 4.7 and assume that condition (C) holds. Then we have $\omega_k \in C^{2,\alpha,\beta}$ and $S(\omega_k) \in C^{,\alpha,\beta}$.*

Proof. This is an application of Theorem 2.5.1. The cscK cone metric ω_B on the base is $C^{2,\alpha,\beta}$, while $\hat{\omega}_E$ is also $C^{2,\alpha,\beta}$ from the previous result. □

5.2. **Expansion of scalar curvature.** Let us remember that we know from Hong's techniques see [28, Section II] or [43, Theorem 3.1].

Lemma 5.3. *On the regular part of $\mathbb{P}E^*$, we know the scalar curvature of the metric ω_k,*

$$\begin{aligned} S(\omega_k)([v]) =& r(r-1) + \frac{1}{k}\left(\pi^* S(\omega_B) + 2r\frac{\sqrt{-1}}{2\pi}\Lambda_{\omega_B}\mathrm{tr}\left([F_{h_E}]^0 \frac{v\otimes v^{*h_E}}{\|v\|^2}\right)\right) \\ & + O\left(\frac{1}{k^2}\right) \end{aligned} \tag{5.4}$$

where $r = \mathrm{rk}(E)$, $[v] \in \mathbb{P}E^*$ and $[.]^0$ denotes the trace free part.

5.3. **Approximate cscK cone metrics.** In view of Theorem 1.4, we shall deform the metric ω_k by deforming the metrics ω_B and h_E in order to obtain, after another deformation, a sequence of Kähler cone metrics that have almost constant scalar curvature. Then we will apply the contraction mapping theorem, following the main idea of [28, 29].

Proposition 5.4. *Assume ω_B is cscK with conical singularities along D with Hölder exponent α and angle $2\pi\beta$ satisfying Condition (C) and such that $Lie(Aut_D(B, [\omega_B]))$ is trivial. Assume the holomorphic vector bundle E is parabolic stable and equipped with h_E Hermitian-Einstein cone metric obtained via Theorem 4.7. Fix $p > 0$. There exist deformations of*

- *the form ω_B as $\omega_B + \sqrt{-1}\partial\bar{\partial}\eta_{k,p}$, with*
$$\eta_{k,p} = \eta_0 + \eta_1 k^{-1} + ... + \eta_{p-2}k^{-p+2} \in C^{4,\alpha,\beta}(B, \omega_B),$$
- *the Hermitian-Einstein metric h_E as $h_E(Id_E + \Phi_{k,p})$ with*
$$\Phi_{k,p} = \Phi_0 k^{-1} + ... + \Phi_{p-2}k^{-p+1} \in C^{2,\alpha,\beta}(Herm(E, h_E)),$$

such that the induced Kähler form

$$\tilde{\omega}_k = k\pi^*(\omega_B + \sqrt{-1}\partial\bar{\partial}\eta_{k,p}) + \hat{\omega}_E\left(h_E(Id_E + \Phi_{k,p})\right)$$

on X can be deformed to obtain an almost cscK cone metric outside $\mathcal{D}$ of order $p+1$, i.e there exists real valued functions on X,

$$\phi_{k,p} = \phi_0 k^{-2} + ... + \phi_{p-2}k^{-p} \in C^{4,\alpha,\beta}(X),$$

such that over X, we have in $C^{,\alpha,\beta}$ topology

$$S(\tilde{\omega}_k + \sqrt{-1}\partial\bar{\partial}\phi_{k,p}) = \underline{S}_\beta + O(k^{-p-1})$$

where $\underline{S}_\beta$ is the topological constant. The metric $\tilde{\omega}_k + \sqrt{-1}\partial\bar{\partial}\phi_{k,p}$ belongs to the Kähler class $[k\pi^*\omega_B + \hat{\omega}_E]$.

Remark 5.5. *Note that given $k, p > 0$, the $C^{4,\alpha,\beta}(\tilde{\omega}_k)$ topology and the $C^{4,\alpha,\beta}(\omega_k)$ topology are actually equivalent, according to the construction provided by Proposition 5.4 and Corollary 2.8.*

We construct the approximation solution by the implicit function theorem inductively, using Lemma 5.6, Lemma 5.8 and Proposition 5.14. The proof is given at the end of this section, page 45, and requires several preliminary results. The deformation of the scalar curvature is divided into three parts, the function on B, the section $\Gamma(B, W)$ and the function on $\mathbb{P}E^*$. Actually, we need to deform all the metrics ω_B, h_E and $\tilde{\omega}_k$.

Firstly, we need to understand the deformation of the cscK equation on X with respect to ω_B and h_E, where $\omega_B \in \mathcal{H}_\beta(B, D)$ is a Kähler cone metric on B and h_E a hermitian metric on E compatible to the parabolic structure with respect to ω. In order to do so, we are going to study the maps

$$A_1(\omega_B, h_E) = S(\omega_B)Id_E + \frac{\sqrt{-1}}{2\pi}\Lambda_{\omega_B}[F_{h_E}]^0 \in C^{,\alpha,\beta}(End(E))$$
$$S_1(\omega_B, h_E) = \mathrm{tr}\left(A_1(\omega_B, h_E)\frac{v \otimes v^{*h_E}}{\|v\|^2}\right) \in C^{,\alpha,\beta}.$$

When ω_B has constant scalar curvature and h_E is Hermitian-Einstein, then the linearization of A_1 at (ω_B, h_E) is given by

(5.5)
$$DA_1(\eta, \Phi) = (\mathbb{L}\mathrm{ic}_{\omega_B}\eta)Id_E + \frac{\sqrt{-1}}{2\pi}[\Lambda_{\omega_B}\bar{\partial}\partial\Phi + 2\Lambda^2_{\omega_B}(F_{h_E} \wedge \sqrt{-1}\partial\bar{\partial}\eta)]^0$$

where $\eta \in C^{4,\alpha,\beta}$ and Φ is a hermitian endomorphism with respect to h_E.

Lemma 5.6. *Suppose that $Lie(Aut_D(B, [\omega_B]))$ is trivial and E is parabolic stable with respect to ω_B as in Proposition 5.4. Set $C^\infty(End(E))^0$ the space of trace-free endomorphisms of E and set $C_0^{4,\alpha,\beta}(B)$ the space of $C^{4,\alpha,\beta}(\omega_B)$ real functions on B with vanishing integral with respect to ω_B. Then the map*

$$\begin{aligned} DA_1 \ :\ C_0^{4,\alpha,\beta}(B) \oplus C^{2,\alpha,\beta}(B, End(E))^0 &\to C_0^{,\alpha,\beta}(B) \oplus C^{,\alpha,\beta}(B, End(E))^0 \\ (\eta, \Phi) &\mapsto DA_1(\eta, \Phi) \end{aligned}$$

defined by (5.5), is an isomorphism.

Proof. If $DA_1(\eta,\Phi)=0$, then we get the system of decoupled equations on $B\setminus D$ by considering the trace part,

$$\mathbb{L}\mathrm{ic}_{\omega_B}\eta = 0, \tag{5.6}$$

$$\frac{\sqrt{-1}}{2\pi}[\Lambda_{\omega_B}\bar\partial\partial\Phi + 2\Lambda^2_{\omega_B}(F_{h_E}\wedge\sqrt{-1}\partial\bar\partial\eta)]^0 = 0. \tag{5.7}$$

Solutions of this system live on B. Using Theorem 1.1, the first equation has for solution only the constants by assumption on $Lie(Aut_D(B,[\omega_B]))$. But then $\eta=0$ since η has vanishing integral. From this fact, this leads from the second equation of the system, to

$$[\Lambda_{\omega_B}\bar\partial\partial\Phi]^0=0$$

and since Φ is trace-free and $C^{2,\alpha,\beta}(B,End(E))^0$,

$$\bar\partial^*\bar\partial\Phi=\sqrt{-1}\Lambda_{\omega_B}\bar\partial\partial\Phi=0 \tag{5.8}$$

and then we get $\int_B|\bar\partial\Phi|^2\frac{\omega_B^n}{n!}=0$ and so $\Phi\in H^0(B,End(E))$, and thus constant from Lemma 4.10. Now, this constant vanishes since Φ is trace-free. Eventually we apply Fredholm alternative. From the first equation, we are able to find η via the theory of the Lichnerowicz equation (Theorem 1.1). Putting it into the second equation, we could solve each component of Φ, i.e. (5.7), by the theory of second order elliptic equations with conical singularities (see Section 2.4). ☐

Secondly, we deform the cscK equation on X with respect to $\tilde\omega_k$.

5.4. **Decomposition of the holomorphic tangent bundle.** In the sequel of this section, we assume that we are under the setting of Proposition 5.4, and h_E is a Hermitian-Einstein cone metric. Given $\hat\omega_E$ a form induced by the h_E on E, one can define the following operator Δ_V on $C^{2,\alpha,\beta}$ functions on the ruled manifold

$$\Delta_V f\hat\omega_E^{r-1}\wedge\pi^*\omega_B^n=(r-1)\sqrt{-1}\partial\bar\partial f\wedge\hat\omega_E^{r-2}\wedge\pi^*\omega_B^n$$

where n is the dimension of the base B and $\pi:X\to B$ is the projection onto B.

Remark 5.7. *It is not a Laplacian. Nevertheless, once restricted to the fibers, it is a Laplacian with respect to the Fubini-Study metric.*

Now, we know that for a general Kähler form ω, the linearization of the scalar curvature is given by

$$\mathcal{L}(\phi)=(\Delta^2-S(\omega)\Delta)\phi+n(n-1)\frac{\sqrt{-1}\partial\bar\partial\phi\wedge Ric(\omega)\wedge\omega^{n-2}}{\omega^n}.$$

Thus for a smooth function ϕ the linearization of the operator the scalar curvature operator at the metric ω_k is given by

$$\mathcal{L}_{\omega_k}=\Delta_V(\Delta_V-r)+O(k^{-1}),$$

when k tends to $+\infty$ using (5.4). We notice that when k tends to $+\infty$ we have by construction of the metric $\tilde{\omega}_k$,

$$\mathcal{L}_{\tilde{\omega}_k} = \Delta_V(\Delta_V - r) + O(k^{-1}).$$

We deform $\tilde{\omega}_k$ and obtain

$$\begin{aligned} S(\tilde{\omega}_k + \sqrt{-1}\partial\bar{\partial}\phi_{k,1}) =& S(\tilde{\omega}_k) + k^{-2}\Delta_V(\Delta_V - r)\phi_0 + O(k^{-3}), \\ =& r(r-1) + k^{-1}S_1(\omega_B, h_E) + k^{-2}DS_1(\eta_0, \Phi_0) \qquad (5.9) \\ &+ k^{-2}\Delta_V(\Delta_V - r)\phi_0 + O(k^{-3}). \qquad (5.10) \end{aligned}$$

Consider the metric $\hat{\omega}_E$ and its Laplacian acting on functions on $\mathbb{P}E^*_x$. We denote W_x as the space of all eigenfunctions associated to the first nonzero eigenvalue of this Laplacian, which is $r = \mathrm{rk}(E)$.

This defines a vector bundle W over B. We have the following result.

Lemma 5.8. *To any trace-free hermitian endomorphism*

$$\Phi \in C^{,\alpha,\beta}(End(E))^0 \cap Herm(E, h_E),$$

one can associate

$$\mathrm{tr}(\Phi \frac{v \otimes v^{*h_E}}{\|v\|^2}) \in C^{,\alpha,\beta}(B)$$

such that its restriction over $B \setminus D$ belongs to $C^{,\alpha,\beta}(B \setminus D, W)$ (i.e is a eigenfunction). The converse is also true, i.e given a eigenfunction that belongs to $C^{,\alpha,\beta}(B \setminus D, W) \cap C^{,\alpha,\beta}(B)$, we obtain a trace free hermitian endomorphism in $C^{,\alpha,\beta}(B, End(E))^0$.

Proof. It is known the eigenfunctions of the Laplacian on $\mathbb{CP}^{r-1}$ with respect to the Fubini-Study metric that are associated to the first non zero eigenvalue of the Laplacian. These eigenfunctions are given by harmonic polynomials on $\mathbb{C}^r$ that correspond to certain hermitian endomorphisms, see [24], [46, Proposition 2.4]. We apply this correspondence over $B \setminus D$. Consequently, we obtain that given Φ as above, $\mathrm{tr}(\Phi \frac{v \otimes v^{*h_E}}{\|v\|^2}) \in C^{,\alpha,\beta}(B)$ is an eigenfunction over $B \setminus D$ and the regularity is clear. Now, for the converse, starting with an eigenfunction, we get an endomorphism Φ_1 over $B \setminus D$ as before. We extend the subbundle $Im(\Phi_1)$ as a subbundle of $C^{,\alpha,\beta}(B, End(E))$. For doing that, we apply [47, Lemma 10.6]. We just need to see that the curvature of $End(E)$ is L^1, but this is the case as $F_{End(E)} = F_{E,h_E} \otimes Id_{E^*} + Id_E \otimes F_{E^*,h^*_E}$ and h_E is Hermitian-Einstein. □

Remark 5.9. *With the lemma in hands, we see why it is not sufficient to deform the metrics ω_B, h_E and that we need to also deform the metric $\tilde{\omega}_k$.*

One can consider the orthogonal space to $C^{4,\alpha,\beta}(B \setminus D, W)$ of functions in $C^{4,\alpha,\beta}(X; \omega_k)$ with vanishing integral i.e

$$\mathcal{Z}_0^{p,\alpha,\beta} = \{\phi \in C^{4,\alpha,\beta}(X; \omega_k), \int_{\mathbb{P}E^*_x} \phi \, \hat{\omega}_E^{r-1} = 0$$

$$\text{and} \int_{\mathbb{P}E_x^*} \phi\theta\, \hat{\omega}_E^{r-1} = 0, \forall x \in B \setminus D, \theta \in C^{p,\alpha,\beta}(B \setminus D, W)\}.$$

Similarly, the space $\mathcal{Z}_0^{p,\alpha,\beta}$ can be defined when one is considering the functions in $C^{,\alpha,\beta}(X)$.
We consider the map

$$\begin{aligned} L_V = \Delta_V(\Delta_V - r) : \mathcal{Z}_0^{4,\alpha,\beta} &\longrightarrow \mathcal{Z}_0^{,\alpha,\beta} \\ \phi &\longmapsto \Delta_V(\Delta_V - r)(\phi). \end{aligned}$$

Lemma 5.10. *L_V is an injective.*

Proof. For the injectivity, assume that $\Delta_V(\Delta_V - r)\phi = 0$. Then over each fiber above $x \in B \setminus D$, $\Delta_V(\Delta_V - r)\phi_x = 0$. By compactness of the fiber, it implies that $(\Delta_V - r)\phi$ is constant and since ϕ has vanishing integral, we get $(\Delta_V - r)\phi = 0$ over $B \setminus D$. But then ϕ is an eigenfunction associated to the r-th eigenvalue, and thus by orthogonality, $\phi = 0$.

□

Next we prove that the map L_V is also surjective. We adapt an idea from the theory of differential families of strongly elliptic differential operators, see Section 7 in Kodaira [32] for instance.

We first cut out a small open neighbourhood D_δ containing D with radius δ. On each fibre at $x \in B \setminus D_\delta$, the ellipticity of the operator $\Delta_V(\Delta_V - r)(\phi)$ tells us that there is a complete orthonormal set of eigenfunctions $\{e_i(x)\}$. Each corresponding eigenvalues $\lambda_i(x)$ is a continuous function of $x \in B \setminus D_\delta$.

Given $\psi \in \mathcal{Z}_0^{,\alpha,\beta}$ we construct a solution ϕ_x for all $x \in B \setminus D$ of

$$L_{V,x}\phi_x = \Delta_{V,x}(\Delta_{V,x} - r)\phi_x = \psi_x$$

over $\mathbb{P}E_x^*$. Along each fibre, we are able to solve the equation $L_{V,x}\phi_x = \psi_x$, since the operator is strongly elliptic and self-adjoint with respect to the inner product $\int_{\mathbb{P}E_x^*}(\cdot, \cdot)\hat{\omega}_E^{r-1}$.

We define the spaces $C_V^{k,\alpha}$, $k \geq 0$ as the Hölder spaces defined on the fibre $\mathbb{P}E_x^*$. Along each fibre, we then prove a priori estimates, with the help of the injectivity of the operator L_V.

Lemma 5.11. *Suppose that $\phi \in \mathcal{Z}_0^{4,\alpha,\beta}$ in $\mathbb{P}E^*$. There exists a constant $C > 0$ depending on $\mathbb{P}E_x^*$ and the coefficients of $L_{V,x}$ such that*

$$|\phi_x|_{C_V^{4,\alpha}} \leq C |L_{V,x}\phi_x|_{C_V^{\alpha}}.$$

Proof. Since $L_{V,x}$ is strongly elliptic, we have

$$|\phi_x|_{C_V^{4,\alpha}} \leq C |L_{V,x}\phi_x|_{C_V^{\alpha}} + |\phi_x|_{C_V^{,\alpha}}.$$

We now assume the conclusion is not true. Then there exists a sequence of $\phi_x(k) \in \mathcal{Z}_0^{4,\alpha,\beta}$, $k \geq 1$ such that

$$|\phi_x(k)|_{C_V^{,\alpha}} = 1, \forall k \geq 1 \text{ and } |L_{V,x}\phi_x(k)|_{C_V^{\alpha}} \to 0, \text{ as } k \to \infty.$$

Thus from compactness of $C^{4,\alpha}_V$, $|\phi_x(\infty)|_{C^\alpha_V} = 1$ and $L_{V,x}\phi_x(\infty) = 0$. But $\mathcal{Z}^{4,\alpha,\beta}_0$ is closed and then we use Lemma 5.10, there is no nontrivial kernel in $\mathcal{Z}^{4,\alpha,\beta}_0$. Thus $\phi_x(\infty) = 0$, which is a contradiction. $\square$

Let x, y be two points in $B \setminus D_\delta$. We let ϕ_x, ϕ_y solve the equations $L_{V,x}\phi_x = \psi_x$ and $L_{V,y}\phi_y = \psi_y$ respectively. Now we are ready to prove in the next two lemma, the asymptotic behaviors of the solution ϕ_x with respect to the point x in the base manifold $B \setminus D$.

Lemma 5.12. *The family of solution ϕ_x is continuous in x.*

Proof. We have

$$L_{V,x}(\phi_y - \phi_x) + (L_{V,y} - L_{V,x})\phi_y = \psi_y - \psi_x.$$

From Lemma 5.11, we have

$$|\phi_y - \phi_x| \le C|(L_{V,y} - L_{V,x})\phi_y|_{C^\alpha_V} + |\psi_y - \psi_x|_{C^\alpha_V}.$$

Here C depends on the geometry of the fibre $\mathbb{P}E^*_x$ and the coefficients of L_V at x. Since the coefficients of L_V and ψ are both $C^{,\alpha,\beta}$, we see that $\phi_y \to \phi_x$ as $y \to x$. $\square$

Lemma 5.13. *The family of solution ϕ_x is $C^{1,\beta}$ in x.*

Proof. We denote ∇_x be the covariant derivative with respect to the metric $\pi^*\omega_B$. We let η_x solves the equation

$$L_{V,x}\eta_x = \nabla_x\psi_x - (\nabla_x L_{V,x})\phi_x.$$

We also denote distance δ between x, y are measured under the metric $\pi^*\omega_B$. It is sufficient to prove that as $y \to x$,

$$\frac{\phi_y - \phi_x}{\delta} \to \eta_x.$$

From Lemma 5.12, it suffices to prove that as $y \to x$,

$$L_{V,y}[\frac{\phi_y - \phi_x}{\delta} - \eta_x] \to 0.$$

Using the formulas above, it becomes

$$\begin{aligned} L_{V,y}[\frac{\phi_y - \phi_x}{\delta} - \eta_x] =& \frac{\psi_y - \psi_x}{\delta} - \nabla_x\psi_x \\ &- [\frac{(L_{V,y} - L_{V,x})\phi_x}{\delta} - (\nabla_x L_{V,x})\phi_x] \\ &- (L_{V,y} - L_{V,x})\eta_x. \end{aligned}$$

The first two terms on the right hand side converge to zero by definition and the third term converges to zero, since the coefficients are in $C^{,\alpha,\beta}$. $\square$

Proposition 5.14. *L_V is bijective.*

Proof. Thanks to Lemma 5.10, we just need to prove that L_V is surjective. The problem now is how to glue the family of pointwise solutions together to produce a solution on the whole ruled manifold. Combining Lemma 5.12 and Lemma 5.13, we obtain $\phi \in C^{4,\alpha,\beta}(X;\omega_k)$ by using the induction argument and repeating the strategy of Lemma 5.13. □

Remark 5.15. *Note that the proofs of Lemmas 5.10, 5.11, 5.12, 5.13 and Proposition 5.14 do not require h_E to be Hermitian-Einstein but only to have the regularity obtained in Theorem 4.7.*

of Proposition 5.4. The first two terms of the expansion of the scalar curvature $S(\tilde{\omega}_k+\sqrt{-1}\partial\bar{\partial}\phi_{k,p})$ are constant since $S_1(\omega_B,h_E)$ is constant by definition of ω_B and h_E. Now, we wish to make constant the k^{-2} term of the expansion of $S(\tilde{\omega}_k+\sqrt{-1}\partial\bar{\partial}\phi_{k,p})$ constant. For that, writing the topological constant $\underline{S}_\beta = \underline{S}_\beta^0 + k^{-1}\underline{S}_\beta^1 + k^{-2}\underline{S}_\beta^2 + ...$ and considering (5.9), (5.10), we need to find (η_0, Φ_0, ϕ_0) solving the equation

$$DS_1(\eta_0,\Phi_0) + \Delta_V(\Delta_V - r)\phi_0 = \underline{S}_\beta^2$$

Since $\underline{S}_\beta^2$ is a constant, we are lead to solve both equations $\Delta_V(\Delta_V - r)\phi_0 = 0$ and $DS_1(\eta_0,\Phi_0) = \underline{S}_\beta^2$. Note that first equation has an obvious solution. Hence, we can apply Lemma 5.6 and Lemma 5.8 to obtain (η_0,Φ_0).
Next, we need to make constant the k^{-3} term. As there is a contribution coming from expansion of $S(\tilde{\omega}_k)$, we are lead this time to solve

$$DS_1(\eta_1,\Phi_1) + \Delta_V(\Delta_V - r)\phi_1 = \underline{S}_\beta^3 + \gamma_3$$

for a certain function γ_3 with real values and $C^{,\alpha,\beta}$ regularity. Here we use the fact that we found η_0 and ϕ_0 in $C^{4,\alpha,\beta}$ and $\Phi_0 \in C^{2,\alpha,\beta}$. The term γ_3 appears as a combination of 4th order derivatives of η_0,ϕ_0 and 2nd order derivatives in Φ_0. Consequently, from the fact that $C^{,\alpha,\beta}$ is an algebra, γ_3 is actually an element of $C^{,\alpha,\beta}(X,\mathbb{R})$. Again, Lemma 5.6, Lemma 5.8 and Proposition 5.14 ensure that a solution (η_1,Φ_1,ϕ_1) can be found with the right regularity.

Then Proposition 5.4 is obtained by induction using the same reasoning for higher order terms, thanks to the fact that we have surjectivity from the space $C_0^{4,\alpha,\beta}(B)\times C^{2,\alpha,\beta}(B,End(E))^0\times \mathcal{Z}_0^{p,\alpha,\beta}$ onto $C^{,\alpha,\beta}(X,\mathbb{R})$ and applying Lemmas 5.6 and 5.8 and Proposition 5.14.

□

5.5. **Proof of Theorem 1.4.**

of Theorem 1.4. Note that the proof is technically different to [43], as in the smooth case it is used Sobolev spaces while we use Hölder spaces. Since E is parabolic stable, we obtain a Hermitian-Einstein cone metric from Theorem 4.7. We can apply Proposition 5.4 to obtain an approximate cscK cone metric $\widetilde{\omega}_{k,p} := \tilde{\omega}_k+\sqrt{-1}\partial\bar{\partial}\phi_{k,p}$ with expected regularity. To obtain a genuine cscK cone metric, we need to apply the implicit function theorem. Indeed, we wish to find for $k >> 0$ a potential $\theta \in C^{4,\alpha,\beta}(\omega_D)$ such that $\widetilde{\omega}_{k,p} + \sqrt{-1}\partial\bar{\partial}\theta > 0$ and we have over X

$$S(\widetilde{\omega}_{k,p} + \sqrt{-1}\partial\bar{\partial}\theta) = Cst.$$

The linearization operator $\mathcal{L}$ of the previous equation is given by

$$\theta \mapsto \mathbb{L}\mathrm{ic}_{\widetilde{\omega}_{k,p}}\theta + \nabla_{\widetilde{\omega}_{k,p}}\theta.\nabla_{\widetilde{\omega}_{k,p}}S(\widetilde{\omega}_{k,p}) = \mathbb{L}\mathrm{ic}_{\widetilde{\omega}_{k,p}}\theta + \nabla_{\widetilde{\omega}_{k,p}}\theta.O\left(\frac{1}{k^{p+1}}\right).$$

Thanks to Corollary 4.12, the kernel of the Lichnerowicz operator $\mathbb{L}\mathrm{ic}_{\widetilde{\omega}_{k,p}}$ is trivial. Moreover, the techniques of [22, Section 6.2] and [8, Section 4.3.1] can be applied without change to get a rough lower bound on the lowest eigenvalue of this operator and this will allow us to see that $\mathcal{L}$ is a Banach isomorphism. Actually, there exists a constant $C_p > 0$ such that for any $\theta \in C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})$,

$$\|\mathbb{L}\mathrm{ic}_{\widetilde{\omega}_{k,p}}\theta\|_{L^2(\widetilde{\omega}_{k,p})} \geq C_p\frac{1}{k^3}\|\theta\|_{L^2(\widetilde{\omega}_{k,p})}. \tag{5.11}$$

This is proved by the following lemmas.

Consequently, the isomorphism of $\mathcal{L}$ will give us a solution $\theta \in C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})$ for $\mathcal{L}(\theta) = f$ for any $f \in C^{,\alpha,\beta}(\widetilde{\omega}_{k,p})$. Furthermore, we can deduce that $\theta \in C^{4,\alpha,\beta}(\omega_D)$ by Proposition 5.4, Proposition 5.14 and Corrollary 2.8.

Lemma 5.16. *For $k >> 0$, the L^2 norms $\|.\|_{L^2(\widetilde{\omega}_{k,p})}$ and $\|.\|_{L^2(k\pi^*\omega_B+\hat{\omega}_E)}$ are uniformly equivalent on a fixed bundle of tensors over X.*

Proof. In terms of tensors, the associated metrics $\widetilde{g}_{k,p}$ and g_k satisfy by construction

$$\|\widetilde{g}_{k,p} - g_k\|_{C^{,\alpha,\beta}(\omega_k)} \leq \frac{1}{2}$$

for $k >> 0$ and $1/2$ independent of k. The difference of the induced metrics on the cotangent space of X is bounded in $C^{,\alpha,\beta}$ norm from which the conclusion follows. □

Next lemma can be proved using exactly the same arguments as in [22, Lemma 6.5]. It is obtained by considering the first eigenvalue of the Laplacian for the metric $\tilde{\omega}_1$ and by using previous lemma.

Lemma 5.17. *There exists a constant $c_1 > 0$ such that for all functions $\phi \in C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})$, with $\int_M \phi\,\widetilde{\omega}_{k,p}^n = 0$ and for $k >> 0$,*

$$\|d\phi\|^2_{L^2(\widetilde{\omega}_{k,p})} \geq \frac{c_1}{k}\|\phi\|^2_{L^2(\widetilde{\omega}_{k,p})}.$$

Again, considering the $\bar{\partial}$-Laplacian determined by $\tilde{\omega}_1$, one obtains similarly to [22, Lemma 6.6] next lemma.

Lemma 5.18. *There exists a constant $c_2 > 0$ such that for all $\zeta \in C^{1,\alpha,\beta}(TX)$ and for $k >> 0$ we have*

$$\|\bar{\partial}\zeta\|^2_{L^2(\widetilde{\omega}_{k,p})} \geq \frac{c_2}{k^2}\|\zeta\|^2_{L^2(\widetilde{\omega}_{k,p})}.$$

In the following, ∇ is computed with respect to the metric induced by $\widetilde{\omega}_{k,p}$. An easy corollary of previous results is the next lemma.

Lemma 5.19. *There exists a constant $c_3 > 0$ such that for all $\phi \in C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})$, with $\int_M \phi\,\widetilde{\omega}^n_{k,p} = 0$ and $k >> 0$,*

$$\|\bar{\partial}\nabla\phi\|^2_{L^2(\widetilde{\omega}_{k,p})} \geq \frac{c_3}{k^3}\|\phi\|^2_{L^2(\widetilde{\omega}_{k,p})}.$$

The inequality (5.11) is obtained from the fact that $\mathbb{L}\mathrm{ic} = (\bar{\partial}\nabla)^*\bar{\partial}\nabla$ and Cauchy-Schwarz inequality. The operator $\mathbb{L}\mathrm{ic}_{\widetilde{\omega}_{k,p}} : C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p}) \to C^{,\alpha,\beta}$ is a Banach space isomorphism.

Lemma 5.20. *The inverse operator $\mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}$ satisfies for a certain constant $c_5 > 0$ and for $k >> 0$,*

$$\|\mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}\psi\|_{C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})} \leq c_5 k^3\|\psi\|_{C^{,\alpha,\beta}},$$

for any $\psi \in C^{,\alpha,\beta}$.

Proof. We apply (5.11) to $\theta = \mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}(\psi)$ and get

$$\|\mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}(\psi)\|_{L^2(\widetilde{\omega}_{k,p})} \leq C_p^{-1}k^3\|\psi\|_{L^2(\widetilde{\omega}_{k,p})}. \tag{5.12}$$

Then we use Claim 3.26 with $t = 1$, $K = 0$ and $u = \mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}(\psi)$ to deduce that

$$\|\mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}(\psi)\|_{C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})} \leq C_4\|\psi\|_{C^{,\alpha,\beta}} + \|\mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}(\psi)\|_{L^2(\widetilde{\omega}_{k,p})}$$

This gives with (5.12) and since $k \geq 1$,

$$\|\mathbb{L}\mathrm{ic}^{-1}_{\widetilde{\omega}_{k,p}}(\psi)\|_{C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})} \leq C_4\|\psi\|_{C^{,\alpha,\beta}} + C_p^{-1}k^3\|\psi\|_{L^2(\widetilde{\omega}_{k,p})} \leq (C_4 + C_p^{-1})k^3\|\psi\|_{C^{,\alpha,\beta}}.$$

□

Using the operator norm computed with respect to the Hölder norms, we obtain similarly to [22, Theorem 6.1]

Proposition 5.21. *For all large k and $p > 2$, the linearization operator $\mathcal{L}$ is a Banach space isomorphism and its inverse operator $\mathcal{L}^{-1}$ satisfies for $c_6 > 0$*

$$\|\mathcal{L}^{-1}\psi\|_{C^{4,\alpha,\beta}(\widetilde{\omega}_{k,p})} \leq c_6 k^3\|\psi\|_{C^{,\alpha,\beta}},$$

for any $\psi \in C^{,\alpha,\beta}$.

To apply the implicit function theorem, one needs to control the nonlinear term

$$N(\phi) = S(\widetilde{\omega}_{k,p} + \sqrt{-1}\partial\bar{\partial}\phi) - \mathcal{L}(\phi),$$

and to show that it has Lipschitz behaviour. This requires a few preliminary results.

Lemma 5.22. *Consider two Kähler cone structures* $(J,\omega),(J',\omega')$ *and denote* $R(J',\omega')$ *is the curvature tensor associated to* (J',ω')*. Fix a constant* $c>0$*. There exists constants* $c',c''>0$ *such that if these two Kähler structures satisfy*

$$\begin{cases} \|(J,\omega)-(J',\omega')\|_{C^{2,\alpha,\beta}} \leq & c', \\ \|R(J',\omega')\|_{C^{,\alpha,\beta}} \leq & c, \end{cases}$$

then the linearisations operators $\mathcal{L},\mathcal{L}'$ *of the scalar curvature* $S(\omega),S(\omega')$ *satisfy*

$$\|(\mathcal{L}-\mathcal{L}')\phi\|_{C^{,\alpha,\beta}} \leq c''\|(J,\omega)-(J',\omega')\|_{C^{2,\alpha,\beta}}\|\phi\|_{C^{4,\alpha,\beta}}.$$

Note that the norms are computed with respect to the same Kähler cone structure (J',ω')*.*

Proof. This is similar to [22, Lemma 2.9] and it is a local computation. The main ingredients of the proof are the following facts:

- the scalar curvature is analytic in the metric and can be extented to a smooth map over the space of potentials in $C^{4,\alpha,\beta}$
- for any integer $0\leq r\leq 4$, there exist a constant $c_r>0$ such that for any tensors T,T' that belong to $C^{r,\alpha,\beta}$,

  $$\|T\cdot T'\|_{C^{r,\alpha,\beta}} \leq c_r\|T\|_{C^{r,\alpha,\beta}}\|T'\|_{C^{r,\alpha,\beta}},$$

  where $\cdot$ stands for any algebraic operation (tensor products or contractions).

□

Lemma 5.23. *Denote* $B(p_0,r_0)\subset B$ *a ball centered at* $p_0\in B$ *in the base manifold of* $X=\mathbb{P}(E^*)$*. Define the complex structure* $J' = J_{\mathbb{CP}^{\mathrm{rk}(E)-1}} \oplus J_{B(p_0,r_0)}$ *using the complex structure on* $\mathbb{CP}^{\mathrm{rk}(E)-1}$ *and the ball. Define similarly* $\omega'_k=\omega_{\mathbb{CP}^{\mathrm{rk}(E)-1}}\oplus k\omega_{B(p_0,r_0)}$*. For all* $\epsilon>0$*,* $p_0\in B$*, there exists a ball* $B(p_0,r_0)\subset B$ *centered at* p_0 *such that for all* $k>>0$*, we have over* $\mathbb{P}(E^*)_{B(p_0,r_0)}$*,*

$$\|(J',\omega'_k)-(J,\widetilde{\omega}_{k,p})\|_{C^{2,\alpha,\beta}}<\epsilon.$$

Proof. It is a local computation. See [22, Theorem 5.2] and [8, Proposition 29]. □

Proposition 5.24 ((Control of the nonlinear term))**.** *For* $k>>0$*, there exists constants* $c,C'>0$ *such that for all* $\phi,\psi\in C^{4,\alpha,\beta}$ *with* $\max(\|\phi\|_{C^{4,\alpha,\beta}},\|\psi\|_{C^{4,\alpha,\beta}})\leq c$*, one has*

$$\|N(\phi)-N(\psi)\|_{C^{,\alpha,\beta}} \leq C'c\|\psi-\phi\|_{C^{4,\alpha,\beta}}.$$

Proof. The mean value theorem provides the inequality

$$\|N(\phi)-N(\psi)\|_{C^{,\alpha,\beta}} \leq \sup_{\chi\in[\phi,\psi]}\|(DN)_\chi\|\|\psi-\phi\|_{C^{4,\alpha,\beta}},$$

for $(DN)_\chi$ the derivative of N at χ. Here we denoted

$$[\phi,\psi]=\{\chi\in C^{4,\alpha,\beta},\chi=\phi+t(\psi-\phi),\ \text{for a certain } t\in[0,1]\}.$$

But $(DN)=\mathcal{L}_\chi-\mathcal{L}$ where $\mathcal{L}_\chi$ is the linearisation of the scalar curvature at the point χ. One applies Lemma 5.22 thanks to the fact that the norm of the curvature of $\widetilde{\omega}_{k,p}$ is bounded. This comes from the construction of $\widetilde{\omega}_{k,p}$ (Proposition 5.4) and Lemma 5.23 which implies a control of the curvature tensor similarly to [22, Lemma 2.7]. $\square$

We are ready to apply the following quantitative implicit function theorem.

Theorem 5.5.1. *Let $\mathbb{S}:\mathfrak{B}_1\to\mathfrak{B}_2$ be a differentiable map of Banach spaces $\mathfrak{B}_1,\mathfrak{B}_2$ whose derivative at 0, $D\mathbb{S}_0$ is an isomorphism of Banach spaces. Let δ be the radius of the closed ball in $\mathfrak{B}_1$ centered at 0, on which $\mathbb{S}-D\mathbb{S}_0$ is Lipschitz, with Lipschitz constant $\frac{1}{2\|\mathbb{S}\|_{op}}$.*
Then, for any $y\in\mathfrak{B}_2$ that satisfies $\|y-\mathbb{S}(0)\|<\frac{\delta}{2\|\mathbb{S}\|_{op}}$, there exists $x\in\mathfrak{B}_1$ with

$$\mathbb{S}(x)=y.$$

Theorem 1.4 is now a consequence of Theorem 5.5.1 applied to $\theta\mapsto S(\widetilde{\omega}_{k,p}+\sqrt{-1}\partial\bar{\partial}\theta)-Cst$ and together with Proposition 5.4, Proposition 5.21, and Proposition 5.24 by choosing a sufficiently close almost cscK cone metric with $p\geq 6$.

$\square$

6. Kähler-Einstein cone metrics and Tangent bundle

It is well-known that if X is a compact Kähler manifold endowed with a smooth Kähler-Einstein metric (with positive, negative or zero curvature) then its tangent bundle TX admits a Hermitian-Einstein metric and thus is Mumford polystable (with respect to the anticanonical polarization, canonical polarization, or any polarization respectively). In this section, we study the case when X is a compact Kähler manifold and admits a Kähler-Einstein metric with conical singularities along a divisor. Let n be the complex dimension of X.
Let us consider ω_{KE} a Kähler-Einstein metric with conical singularities along D smooth divisor for which the Hölder exponent α and the angle $2\pi\beta$ satisfy Condition (C). Define ∇_{KE} the Chern connection associated to the induced hermitian metric h_{KE} on the tangent bundle TX and $F_{KE}=F_{\nabla_{KE}}\in\Omega^{1,1}(End(T^{1,0}X))$ its curvature. We obtain an operator

$$\widehat{\omega}_{KE}(\Lambda_{\omega_{KE}}F_{KE}):T^{1,0}X\to\Lambda^{0,1}X$$

by identifying $T^{1,0}X$ to $\Lambda^{0,1}X$ using ω_{KE}. Consequently, $\widehat{\omega}_{KE}(\Lambda_{\omega_{KE}}F_{KE})$ can be seen as an element in $\Omega^{1,1}(X)$. A local computation that remains valid outside of the divisor D, shows that

$$Ric(\omega_{KE})=\widehat{\omega}_{KE}(\Lambda_{\omega_{KE}}F_{KE}).$$

Using the Kähler-Einstein property, the previous equation provides a metric on TX which is Hermitian-Einstein metric outside D and has $C^{2,\alpha,\beta}$ regularity as $Ric(\omega_{KE})$ is $C^{,\alpha,\beta}$ from Theorem 2.5.1. We now check that we obtain furthermore a parabolic structure for which this metric is compatible. We consider the canonical section σ of D that vanishes precisely on D. Now, as the Kähler form ω_{KE} has conical singularities along D, it is quasi-isometric to

$$\frac{\sqrt{-1}}{2}a_1|\sigma|^{2(\beta-1)}dz^1\wedge d\bar{z}^1+\tilde{\omega} \tag{6.1}$$

using local cone chart coordinates. Here the z^1 is the local defining function of the hypersurface $D=\{\sigma=0\}$ where p locates, a_1 is a smooth function, $0<\beta<1/2$, and $\tilde{\omega}$ is a smooth form. Consequently, from (6.1), the curvature $|F_{KE}|_{h_{KE}}$ of the metric h_{KE} lies in $L^p(X)$ for $p<\frac{2}{1-\beta}$ as

$$|\sigma|^{1-\beta}|F_{KE}|_{h_{KE}}\leq C, \tag{6.2}$$

for a uniform constant $C>0$, see for instance the proof of [37, Lemma 5.2]. This bound implies the following statement.

Lemma 6.1. *With above assumptions, there exists $C>0$ such that*

$$\|F_{KE}\|_{L^p(\omega_{KE})}<C,$$

with $p>2$.

Theorem 6.2. *Assume (X,ω_{KE}) is a compact Kähler manifold endowed with a Kähler-Einstein cone metric along a smooth divisor D, with Hölder exponent α and angle $2\pi\beta$ satisfying Condition (C). Then its tangent bundle TX is parabolic polystable with respect to ω_{KE}.*

Proof. Firstly, note that in the case of a curve or a surface, we can use a strong result of Biquard that shows an equivalence between the category of hermitian bundles on $X\setminus D$ with L^p curvature and the category of holomorphic bundles on X with parabolic structure over D, see [5,6]. Applying Lemma 6.1, we obtain that the tangent bundle can be extended over D together with a parabolic structure along the divisor and such that the metric h_{KE} is compatible with this parabolic structure. Moreover this extension is essentially unique. Now, using the Hermitian-Einstein condition and [38, Theorem 6.3] or [47], we obtain the parabolic stability of each component of the tangent bundle if it is not indecomposable, i.e its polystability.

In general, we can adapt the construction of [38, Section 3] as the bundle we are interested in is already defined over the divisor. We choose local holomorphic coordinates in a neighbourhood $U=\{|z^i|<1,i=1,..,n\}$ of the point $p=(0,...,0)$ such that the intersection with the divisor can be written $D\cap U=\{z^1=0\}$. We may choose a holomorphic basis $\{e_i\}_{i=1,..,r}$ of $E_{|U}$ such that the matrix of the metric

h_{KE} in this basis is diagonal. We write $D_{h_{KE}}$ this matrix. The matrix $D_{h_{KE}}$ necessarily vanishes on $U \cap D$ as the curvature F_{KE} is singular and thus, fixing $\|.\|_D$ a norm on $\mathcal{O}(D)$, we can write $D_{h_{KE}}$ as

$$D_{h_{KE}} = \begin{pmatrix} \zeta_1 \|z^1\|_D^{2\gamma_1} & & \\ & \ddots & \\ & & \zeta_r \|z^1\|_D^{2\gamma_r} \end{pmatrix}.$$

Here ζ_j are positive smooth functions, the γ_j are non-negative real numbers and (6.2) ensures that $\gamma_j < 1$ for all j. Without loss of generality we can assume that the γ_j is an increasing sequence, by doing a permutation of $\{e_i\}$. We denote r_j the integers which count the numbers of equal γ_j, i.e that $r_1 = \mathrm{rk}(TM) = n$ and r_{j+1} is defined inductively by $\gamma_l = \gamma_{n-r_j+1}$ for $n - r_j + 1 \leq l \leq n - r_{j+1}$. Let l_E be the number of different integers r_i. We define $\mathcal{F}^i_{|D\cap U} = Vect(e_{n-r_i+1}, ..., e_n)$ for $i = 2, .., l_E$ and $\alpha_i = \gamma_{n-r_{i+1}}$. Clearly, the data $(\mathcal{F}^i, \alpha_i)$ defines a parabolic structure for $E_{|D\cap U}$. If we consider another neighbourhood U' that intersects $U \cap D$, then one can find as above a basis $\{e'_i\}_{i=1,..,r}$ of $E_{|U'}$ such that the matrix of h_{KE} is diagonal with diagonal entries $\zeta'_i \|\sigma\|_D^{2\gamma'_i}$ for $i = 1, .., r$ with γ'_i increasing sequence and σ the canonical section of D which vanishes precisely on D. The vanishing order must be the same on $U \cap U' \cap D \neq \emptyset$ which forces $\gamma_i = \gamma'_i$ and consequently $\mathcal{F}_{|D\cap U\cap U'}$ extends to $\mathcal{F}_{|D\cap U'}$. Consequently we have defined a filtration by subbundles of $E_{|D}$ and thus a parabolic structure for E along D. Then, following [38, Section 3], the metric h_{KE} is compatible with the structure and TM is endowed with a Hermitian-Einstein cone metric. We conclude as above using Remark (4.8). □

7. Further Applications and Remarks

7.1. **Simple normal crossings divisors.** We expect that Theorems 1.1, 1.4, 3.25, 4.7, 6.2 and Corollary 1.2 could be generalized to the case of simple normal crossings divisors $D = \sum_{i=1}^m D_i$ where D_i are irreducible. It is possible to define Hölder and Sobolev spaces for Kähler cone metrics ω with angles $2\pi\beta_i$ along D_i. The condition (C) would be replaced by the condition (C')

$$\text{(C')} \qquad 0 < \beta_i < \frac{1}{2}; \qquad \alpha\beta_i < 1 - 2\beta_i,\ \forall i = 1, ..., m.$$

The statements of our results would remain identical under above changes. The only missing step is a Schauder estimate that extends Proposition 3.22. This has been announced recently in the preliminary work of Guo-Song [25].

7.2. **Twisted conical path for cscK metric.** Donaldson introduced a continuity method for conical Kähler-Einstein metrics [20, Equation (27)] on Fano manifolds. A natural extension of this path for general polarizations is given by the following equation, that we call *scalar curvature twisted conical path.*

$$S(\omega_{\varphi(t)}) = c_t + 2\pi(1-t)\mathrm{tr}_{\omega_{\varphi(t)}}[D]. \tag{7.1}$$

where c_t is a constant that depends on the time given in terms of topological invariants, $[\omega_{\varphi(t)}] = 2\pi c_1(L)$ for L ample line bundle on the projective manifold X. Note that this is a variant of the continuity method introduced by Chen [12, Equation (2.16)]. Scalar curvature twisted conical path is expected to be helpful for the construction of smooth cscK metric as $t \to 1$. It is natural to ask if the set of times t for which (7.1) admits a solution is open and non empty. As far as we know, the existence of a solution at an initial time $t_0 > 0$ is not known. Nevertheless, [44, Corollary 5.10] shows the existence of an invariant $\beta(X, D)$ such for $0 < t_0 < \beta(X, D)$, (X, D) is log K-stable for angle t_0. If an initial solution does exist at time $t_0 < \min(1/2, \beta(X, D))$, Theorem 1.1 applies and provides the openness property if $Lie(Aut_D(X, L))$ is trivial. This is similar to the smooth case, see [12, Theorems 1.5 and 1.8].

7.3. **Other applications of Theorem 1.1.** The linear theory proved in this paper could be used to construct a large family of cscK cone metrics over blow-ups, extending the previous work of Arezzo-Pacard [2]. We also expect a generalization of the work of Fine (see [22] and subsequent works) for construction at the adiabatic limit of cscK cone metrics over a holomorphic submersion between compact Kähler manifolds $\pi : X \to B$ where the fibers and the base do not admit non trivial holomorphic vector fields and each fiber admits a cscK cone metric.

7.4. **Generalizations of Theorem 1.4.** In view of [1, 8, 29, 36, 43] it is natural to ask whether Theorem 1.4 admits a generalization in the case the base manifold B admits nontrivial holomorphic vector fields. As in the smooth case, this will not be automatically happen, and an extra condition on (B, D) will be required. A related question is about the existence of extremal Kähler cone metric when the bundle E splits as sum of parabolic stable bundles of different slopes. These problems will be investigated in a forthcoming paper.

7.5. **Log K-stability.** We expect that a purely algebraic version of Theorem 1.4 holds, i.e under the same assumptions, one gets also log-K-stability of $(X, \mathcal{D}, [\omega_k] = [k\pi^*\omega_B + \hat{\omega}_E])$ for large k. We refer to [20] and [34] for the notion of log-K-stability and the logarithmic version of Yau-Tian-Donaldson conjecture. In the particular case of vector bundles over a curve, by analogy to the smooth case, we expect the following to be true.

Conjecture 7.1. *Let C be a complex curve endowed with a cscK cone metric ω_C along a divisor D. Let E be a holomorphic vector bundle* *over C. Let $X = \mathbb{P}(E^*) \to C$ be its projectivisation and $\mathcal{D} = \pi^{-1}(D)$. The following three conditions are equivalent:*

(i) *X admits a cscK cone metric along $\mathcal{D}$ in any ample class $2\pi c_1(\mathcal{L})$ on X;*
(ii) *$(X, \mathcal{D})$ is log-K-polystable for any polarization $\mathcal{L}$ on X;*
(iii) *There exists a parabolic structure for E so that E is parabolic polystable with respect to ω_C, i.e it decomposes as the sum of stable parabolic bundles of same parabolic slopes.*

Before we provide some information on this conjecture, let us mention that from the work of Troyanov we have an algebraic characterization of complex curves that can be endowed with a cscK cone metric, see [50] and [34] for the relation with log K-stability.

(iii)⇒(i). At the boundary of the Kähler cone and when E is irreducible, on can invoke Theorem 1.4 which provides a more precise result than [30] in terms of regularity of the cscK cone metric. Nevertheless the implication (iii)⇒(i) is true in general as soon as condition (C) holds. Actually, starting from ω_B cscK cone metric and h_E Hermitian-Einstein metric on E, the Kähler cone metric ω_k given by (5.3) has constant scalar curvature outside $\mathcal{D}$, like in the smooth case (to check this fact one can also refine Lemma 5.3 by obtaining a complete expansion when the base has dimension 1). Moreover ω_k has $C^{2,\alpha,\beta}$ potential from Lemma 5.2 and thus is a genuine cscK cone metric. Note that for this reasoning we don't need to assume $k >> 0$ and that when the base is a curve, any Kähler class on X is of the form $[\omega_k]$.

(i)⇔(ii). Hashimoto [26] provides essentially the equivalence for a very particular bundle E and $C = \mathbb{P}^1$. Remark that the considered bundle E can be made parabolic polystable by using the techniques [30]. Moreover, Corollary 2.15 gives evidence that the implication (i)⇒(ii) holds as product test configurations correspond to holomorphic vector fields with a holomorphy potential.

(ii)⇒(iii). A weaker version is shown in terms of asymptotic Chow polystability with angle in [30].

In the special case of rank 2 bundles and rational weights for the parabolic structure, we also expect a relation between the cscK cone metrics and the smooth cscK cone metrics on the blow-ups in view of the recent work of Y. Rollin [45].

8. Appendix

8.1. **Local Hölder sapces.** We identify $\tilde{U}$ in the complex Euclidean space as the image of the cone chart $U \subset X$ under the cone chart, i.e a quasi-isometry $\rho : U \to \tilde{U} \subset \mathbb{C}^n$. We call $\tilde{U}$ an image cone chart. Then we have $C^{,\alpha,\beta}(\tilde{U})$ and $C^{2,\alpha,\beta}(\tilde{U})$ defined in $\tilde{U}$ as above with respect to ω_{cone}.

Definition 8.1. *The Hölder space $C^{3,\alpha,\beta}(\tilde{U};\omega_{cone})$ is defined as the set of function $u \in C^{2,\alpha,\beta}(\tilde{U})$ such that its 3rd order covariant derivatives with respect to ∇^{cone} associated to the metric ω_{cone} are $C^{,\alpha,\beta}(\tilde{U})$ in an image cone chart $\tilde{U}$. More precisely, written down with respect to the standard cone metric ω_{cone}, the following covariant derivatives are required to be $C^{,\alpha,\beta}(\tilde{U})$,*

$$
(8.1)\quad \begin{cases}
\nabla_i^{cone}\nabla_{\bar{l}}^{cone}\nabla_k^{cone}u, |z^1|^{3-3\beta}\nabla_1^{cone}\nabla_{\bar{1}}^{cone}\nabla_1^{cone}u \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{1-\beta}\nabla_i^{cone}\nabla_{\bar{1}}^{cone}\nabla_k^{cone}u, |z^1|^{1-\beta}\nabla_1^{cone}\nabla_{\bar{l}}^{cone}\nabla_k^{cone}u \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{2-2\beta}\nabla_i^{cone}\nabla_{\bar{1}}^{cone}\nabla_1^{cone}u, |z^1|^{2-2\beta}\nabla_1^{cone}\nabla_{\bar{l}}^{cone}\nabla_1^{cone}u \in C^{,\alpha,\beta}(\tilde{U}).
\end{cases}
$$

The $C^{4,\alpha,\beta}(\tilde{U};\omega_{cone})$ is defined in the similar way as follows.

Definition 8.2. *We say a $C^{3,\alpha,\beta}(\tilde{U};\omega_{cone})$ function u is $C^{4,\alpha,\beta}(\tilde{U};\omega_{cone})$ if it satisfies for any $2 \leq i,j,k,l \leq n$, in the image cone chart $\tilde{U}$,*

(8.2)

$$
\begin{cases}
\nabla_{\bar{l}}^{cone}\nabla_k^{cone}\nabla_{\bar{j}}^{cone}\nabla_i^{cone}u = \dfrac{\partial^4 u}{\partial z^{\bar{j}}\partial z^i\partial z^{\bar{l}}\partial z^k} \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{1-\beta}\nabla_{\bar{l}}^{cone}\nabla_1^{cone}\nabla_{\bar{j}}^{cone}\nabla_i^{cone}u = |z^1|^{1-\beta}\dfrac{\partial^4 u}{\partial z^{\bar{j}}\partial z^i\partial z^{\bar{l}}\partial z^1} \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{2-2\beta}\nabla_{\bar{1}}^{cone}\nabla_1^{cone}\nabla_{\bar{j}}^{cone}\nabla_i^{cone}u = |z^1|^{2-2\beta}\dfrac{\partial^4 u}{\partial z^{\bar{j}}\partial z^i\partial z^{\bar{1}}\partial z^1} \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{2-2\beta}\nabla_{\bar{l}}^{cone}\nabla_1^{cone}\nabla_{\bar{j}}^{cone}\nabla_1^{cone}u\\
\qquad = |z^1|^{2-2\beta}[\dfrac{\partial^4 u}{\partial z^{\bar{l}}\partial z^1\partial z^{\bar{j}}\partial z^1} + \dfrac{1-\beta}{z^1}\dfrac{\partial^3 u}{\partial z^{\bar{l}}\partial z^1\partial z^{\bar{j}}}] \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{3-3\beta}\nabla_{\bar{1}}^{cone}\nabla_1^{cone}\nabla_{\bar{1}}^{cone}\nabla_i^{cone}u\\
\qquad = |z^1|^{3-3\beta}[\dfrac{\partial^4 u}{\partial z^{\bar{1}}\partial z^1\partial z^{\bar{1}}\partial z^i} + \dfrac{1-\beta}{z^{\bar{1}}}\dfrac{\partial^3 u}{\partial z^1\partial z^{\bar{1}}\partial z^i}] \in C^{,\alpha,\beta}(\tilde{U}),\\
|z^1|^{4-4\beta}\nabla_{\bar{1}}^{cone}\nabla_1^{cone}\nabla_{\bar{1}}^{cone}\nabla_1^{cone}u\\
\quad = |z^1|^{4-4\beta}[\dfrac{\partial^4 u}{\partial z^{\bar{1}}\partial z^1\partial z^{\bar{1}}\partial z^1} + \dfrac{1-\beta}{z^{\bar{1}}}\dfrac{\partial^3 u}{\partial z^1\partial z^{\bar{1}}\partial z^1} + \dfrac{1-\beta}{z^1}\dfrac{\partial^3 u}{\partial z^1\partial z^{\bar{1}}\partial z^{\bar{1}}}\\
\qquad\qquad + \dfrac{(1-\beta)^2}{|z^1|^2}\dfrac{\partial^2 u}{\partial z^1\partial z^{\bar{1}}}] \in C^{,\alpha,\beta}(\tilde{U}).
\end{cases}
$$

We notice from the definitions above that when the same derivative ∂z^1 or $\partial z^{\bar{1}}$ appear twice, we need extra lower order derivatives to adjust the normal derivatives.

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