BIFURCATION SETS ARISING FROM NON-INTEGER BASE EXPANSIONS

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Abstract. Given a positive integer M and $q \in (1, M+1]$, let $\mathcal{U}_q$ be the set of $x \in [0, M/(q-1)]$ having a unique q-expansion: there exists a unique sequence $(x_i) = x_1x_2\ldots$ with each $x_i \in \{0, 1, \ldots, M\}$ such that

$$x = \frac{x_1}{q} + \frac{x_2}{q^2} + \frac{x_3}{q^3} + \cdots .$$

Denote by $\mathbf{U}_q$ the set of corresponding sequences of all points in $\mathcal{U}_q$. It is well-known that the function $H : q \mapsto h(\mathbf{U}_q)$ is a Devil's staircase, where $h(\mathbf{U}_q)$ denotes the topological entropy of $\mathbf{U}_q$. In this paper we give several characterizations of the bifurcation set

$$\mathscr{B} := \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

Note that $\mathscr{B}$ is contained in the set $\mathscr{U}$ of bases $q \in (1, M+1]$ such that $1 \in \mathcal{U}_q$. By using a transversality technique we also calculate the Hausdorff dimension of the difference $\mathscr{U} \setminus \mathscr{B}$. Interestingly this quantity is always strictly between 0 and 1. When $M = 1$ the Hausdorff dimension of $\mathscr{U} \setminus \mathscr{B}$ is $\frac{\log 2}{3 \log \lambda^*} \approx 0.368699$, where λ^* is the unique root in $(1, 2)$ of the equation $x^5 - x^4 - x^3 - 2x^2 + x + 1 = 0$.

1. Introduction

Fix a positive integer M. For $q \in (1, M+1]$, a sequence $(x_i) = x_1x_2\ldots$ with each $x_i \in \{0, 1, \ldots, M\}$ is called a *q-expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i} =: \pi_q((x_i)). \tag{1.1}$$

Here the *alphabet* $\{0, 1, \ldots, M\}$ will be fixed throughout the paper. Clearly, x has a q-expansion if and only if $x \in I_q := [0, M/(q-1)]$. When $q = M+1$ we know that each $x \in I_{M+1} = [0, 1]$ has a unique $(M+1)$-expansion except for countably many points, which have precisely two expansions. When $q \in (1, M+1)$ the set of expansions of an $x \in I_q$ can be much more complicated. Sidorov showed in [26] that Lebesgue almost every $x \in I_q$ has a continuum of q-expansions. Therefore, the set of $x \in I_q$ with a unique q-expansion is negligible in the sense of Lebesgue measure. On the other hand, the third author and his coauthors showed in [20] (see also Glendinning and Sidorov [13] for the case $M = 1$) that the set of $x \in I_q$ with a unique q-expansion has positive Hausdorff dimension when $q > q_{KL}$, where $q_{KL} = q_{KL}(M)$ is the *Komornik-Loreti constant* (see Section 2 for more details).

For $q \in (1, M+1]$ let $\mathcal{U}_q$ be the *univoque set* of $x \in I_q$ having a unique q-expansion. This means that for any $x \in \mathcal{U}_q$ there exists a unique sequence $(x_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ such that

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$x = \pi_q((x_i))$. Denote by $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q)$ the corresponding set of q-expansions. Note that π_q is a bijection from $\mathbf{U}_q$ to $\mathcal{U}_q$. So the study of the univoque set $\mathcal{U}_q$ is equivalent to the study of the *symbolic univoque set* $\mathbf{U}_q$.

De Vries and Komornik [8] discovered an intimate connection between $\mathcal{U}_q$ and the set

$$\mathscr{U} := \{q \in (1, M+1] : 1 \in \mathcal{U}_q\} \tag{1.2}$$

of bases for which the number 1 has a unique expansion. For $M = 1$, the set $\mathscr{U}$ was first studied by Erdős et al. [10, 11]. They showed that the set $\mathscr{U}$ is uncountable, of first category and of zero Lebesgue measure. Later, Daróczy and Kátai [7] proved that the set $\mathscr{U}$ has full Hausdorff dimension. Komornik and Loreti [18] showed that the topological closure $\overline{\mathscr{U}}$ is a *Cantor set*: a non-empty perfect set with no interior points. Indeed, for general $M \geq 1$, the above properties of $\mathscr{U}$ also hold (cf. [9, 16]). Some connections with dynamical systems, continued fractions and even the Mandelbrot set can be found in [6].

1.1. Set-valued bifurcation set $\hat{\mathscr{U}}$.

Let $\Omega := \{0, 1, \ldots, M\}^{\mathbb{N}}$ be the set of all sequences with each element from $\{0, 1, \ldots, M\}$. Then (Ω, ρ) is a compact metric space with respect to the metric ρ defined by

$$\rho((c_i), (d_i)) = (M+1)^{-\inf\{j \geq 1 : c_j \neq d_j\}}. \tag{1.3}$$

Under the metric ρ the Hausdorff dimension of any subset $E \subseteq \Omega$ is well-defined.

Note that the set-valued map $F : q \mapsto \mathbf{U}_q$ is increasing, i.e., $\mathbf{U}_p \subseteq \mathbf{U}_q$ for any $p, q \in (1, M+1]$ with $p < q$ (see Section 2 for more explanation). In [8] de Vries and Komornik showed that the map F is locally constant almost everywhere. On the other hand, the third author and his coauthors proved in [21] that there exist infinitely many $q \in (1, M+1]$ such that the difference between $\mathbf{U}_q$ and $\mathbf{U}_p$ for any $p \neq q$ is significant: $\mathbf{U}_q \triangle \mathbf{U}_p$ has positive Hausdorff dimension, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ stands for the symmetric difference of two sets A and B. Let $\hat{\mathscr{U}}$ be the *bifurcation set* of the set-valued map F, defined by

$$\hat{\mathscr{U}} = \hat{\mathscr{U}}(M) := \{q \in (1, M+1] : \dim_H(\mathbf{U}_p \triangle \mathbf{U}_q) > 0 \text{ for any } p \neq q\}.$$

Compared to the set $\mathscr{U}$ from (1.2), we know by [21, Theorems 1.1 and 1.2] that $\hat{\mathscr{U}} \subset \mathscr{U}$ and the difference $\mathscr{U} \setminus \hat{\mathscr{U}}$ is countably infinite. As a result, $\hat{\mathscr{U}}$ is a Lebesgue null set of full Hausdorff dimension. Furthermore,

$$(1, M+1] \setminus \hat{\mathscr{U}} = (1, q_{KL}] \cup \bigcup [q_0, q_0^*]. \tag{1.4}$$

The union on the right hand-side of (1.4) is pairwise disjoint and countable. By the definition of $\hat{\mathscr{U}}$ it follows that each connected component $[q_0, q_0^*]$ is a maximum interval such that the difference $\mathbf{U}_{q_0} \triangle \mathbf{U}_{q_0^*} = \mathbf{U}_{q_0^*} \setminus \mathbf{U}_{q_0}$ has zero Hausdorff dimension. So the closed interval $[q_0, q_0^*]$ is called a *plateau* of F. Indeed, for any $q \in (q_0, q_0^*)$ the difference $\mathbf{U}_q \setminus \mathbf{U}_{q_0}$ is at most countable, and for $q = q_0^*$ the difference $\mathbf{U}_{q_0^*} \setminus \mathbf{U}_{q_0}$ is uncountable but of zero Hausdorff dimension (cf. [21, Lemma 3.4]). Furthermore, each left endpoint q_0 is an algebraic integer, and each right endpoint q_0^*, called a *de Vries-Komornik number*, is a transcendental number (cf. [19]).

Instead of investigating the bifurcation set $\hat{\mathscr{U}}$ directly, we consider two modified bifurcation sets:

$$\mathscr{U}^L = \mathscr{U}^L(M) := \{q \in (1, M+1] : \dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) > 0 \text{ for any } p \in (1, q)\};$$
$$\mathscr{U}^R = \mathscr{U}^R(M) := \{q \in (1, M+1] : \dim_H(\mathbf{U}_r \setminus \mathbf{U}_q) > 0 \text{ for any } r \in (q, M+1]\}.$$

The sets $\mathscr{U}^L, \mathscr{U}^R$ are called the *left bifurcation set* and the *right bifurcation set* of F, respectively. In view of [21, Theorem 1.1], the right bifurcation set $\mathscr{U}^R$ is equal to the set of univoque bases such that 1 has a unique expansion, i.e., $\mathscr{U}^R = \mathscr{U}$. Clearly, $\hat{\mathscr{U}} \subset \mathscr{U}^L$ and $\hat{\mathscr{U}} \subset \mathscr{U}^R$. Furthermore,

$$\mathscr{U}^L \cap \mathscr{U}^R = \hat{\mathscr{U}} \quad \text{and} \quad \mathscr{U}^L \cup \mathscr{U}^R = \overline{\hat{\mathscr{U}}}.$$

By (1.4) it follows that the difference set $\mathscr{U}^L \setminus \hat{\mathscr{U}}$ consists of all left endpoints of the plateaus in $(q_{KL}, M+1]$ of F, and hence it is countable. Similarly, the difference set $\mathscr{U}^R \setminus \hat{\mathscr{U}}$ consists of all right endpoints of the plateaus of F. Therefore,

$$\begin{aligned} (1, M+1] \setminus \mathscr{U}^L &= (1, q_{KL}] \cup \bigcup (q_0, q_0^*], \\ (1, M+1] \setminus \mathscr{U}^R &= (1, q_{KL}) \cup \bigcup [q_0, q_0^*). \end{aligned} \tag{1.5}$$

Since the differences among $\hat{\mathscr{U}}, \mathscr{U}^L, \mathscr{U}^R = \mathscr{U}$ and $\overline{\mathscr{U}}$ are at most countable, the dimensional results obtained in this paper for $\mathscr{U} = \mathscr{U}^R$ also hold for $\hat{\mathscr{U}}, \mathscr{U}^L$ and $\overline{\mathscr{U}}$.

Now we recall from [21] the following characterizations of the left and right bifurcation sets $\mathscr{U}^L$ and $\mathscr{U}^R$ respectively.

Theorem 1.1 ([21]).

(i) *$q \in \mathscr{U}^L$ if and only if $\dim_H(\mathscr{U} \cap (p, q)) > 0$ for any $p \in (1, q)$.*
(ii) *$q \in \mathscr{U}^R$ if and only if $\dim_H(\mathscr{U} \cap (q, r)) > 0$ for any $r \in (q, M+1]$.*

Remark 1.2. Since $\hat{\mathscr{U}} = \mathscr{U}^L \cap \mathscr{U}^R$, Theorem 1.1 also gives an equivalent condition for the bifurcation set $\hat{\mathscr{U}}$, i.e., $q \in \hat{\mathscr{U}}$ if and only if

$$\dim_H(\mathscr{U} \cap (p, q)) > 0 \qquad \text{and} \qquad \dim_H(\mathscr{U} \cap (q, r)) > 0$$

for any $1 < p < q < r \leq M+1$.

1.2. **Entropy bifurcation set $\mathscr{B}$.** For a symbolic subset $X \subset \Omega$ its *topological entropy* is defined by

$$h(X) := \liminf_{n \to \infty} \frac{\log \# B_n(X)}{n},$$

where $B_n(X)$ denotes the set of all length n subwords occurring in elements of X, and $\#A$ denotes the cardinality of a set A. Here and throughout the paper we use base $M+1$ logarithms. Recently, Komornik et al. showed in [16] (see also Lemma 2.5 below) that the function

$$H : (1, M+1] \to [0, 1]; \qquad q \mapsto h(\mathbf{U}_q)$$

is a Devil's staircase:

- H is a continuous and non-decreasing function from $(1, M+1]$ onto $[0, 1]$.
- H is locally constant Lebesgue almost everywhere in $(1, M+1]$.

Let $\mathscr{B}$ be the *bifurcation set* of the entropy function H, defined by

$$\mathscr{B} = \mathscr{B}(M) := \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

In [1] Alcaraz Barrera with the second and third authors proved that $\mathscr{B} \subset \mathscr{U}$, and hence $\mathscr{B}$ is of zero Lebesgue measure. They also showed that $\mathscr{B}$ has full Hausdorff dimension. Furthermore, $\mathscr{B}$ has no isolated points and can be written as

$$(1, M+1] \setminus \mathscr{B} = (1, q_{KL}] \cup \bigcup [p_L, p_R], \tag{1.6}$$

where the union on the right hand side is countable and pairwise disjoint. By the definition of the bifurcation set $\mathscr{B}$ it follows that each connected component $[p_L, p_R]$ is a maximal interval on which H is constant. Thus each closed interval $[p_L, p_R]$ is called a *plateau* of H (or an *entropy plateau*). Furthermore, the left and right endpoints of each entropy plateau in $(q_{KL}, M+1]$ are both algebraic numbers (see also Lemma 3.1 below).

In analogy with $\mathscr{U}^L$ and $\mathscr{U}^R$ we also define two one-sided bifurcation sets of H:

$$\mathscr{B}^L = \mathscr{B}^L(M) := \{q \in (1, M+1] : H(p) < H(q) \text{ for any } p \in (1, q)\};$$
$$\mathscr{B}^R = \mathscr{B}^R(M) := \{q \in (1, M+1] : H(r) > H(q) \text{ for any } r \in (q, M+1]\}.$$

We call $\mathscr{B}^L$ and $\mathscr{B}^R$ the *left bifurcation set* and the *right bifurcation set* of H, respectively. Comparing these sets with the bifurcation sets $\hat{\mathscr{U}}, \mathscr{U}^L$ and $\mathscr{U}^R$ of F, we have analogous properties for the bifurcation sets $\mathscr{B}, \mathscr{B}^L$ and $\mathscr{B}^R$. For example, $\mathscr{B} \subset \mathscr{B}^L$ and $\mathscr{B} \subset \mathscr{B}^R$. Furthermore,

$$\mathscr{B}^L \cap \mathscr{B}^R = \mathscr{B} \qquad \text{and} \qquad \mathscr{B}^L \cup \mathscr{B}^R = \overline{\mathscr{B}}.$$

The difference set $\mathscr{B}^L \setminus \mathscr{B}$ consists of all left endpoints of the plateaus in $(q_{KL}, M+1]$ of H. Similarly, $\mathscr{B}^R \setminus \mathscr{B}$ consists of all right endpoints of the plateaus of H. In other words, by (1.6) we have

$$\begin{aligned} (1, M+1] \setminus \mathscr{B}^L &= (1, q_{KL}] \cup \bigcup (p_L, p_R], \\ (1, M+1] \setminus \mathscr{B}^R &= (1, q_{KL}) \cup \bigcup [p_L, p_R). \end{aligned} \tag{1.7}$$

We emphasize that $M+1$ belongs to $\mathscr{B}, \mathscr{B}^L$ and $\mathscr{B}^R$. Since $\mathscr{B} \subset \hat{\mathscr{U}}$, by (1.5) and (1.7) we also have

$$\mathscr{B}^L \subset \mathscr{U}^L \qquad \text{and} \qquad \mathscr{B}^R \subset \mathscr{U}^R.$$

Now we state our main results. Inspired by the characterizations of $\mathscr{U}^L$ and $\mathscr{U}^R$ described in Theorem 1.1, we characterize the left and right bifurcation sets $\mathscr{B}^L$ and $\mathscr{B}^R$ respectively.

Theorem 1. *If $M = 1$ or M is even, the following statements are equivalent.*

(i) $q \in \mathscr{B}^L$.
(ii) $\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H \mathbf{U}_q > 0$ *for any* $p \in (1, q)$.
(iii) $\lim_{p \nearrow q} \dim_H(\mathscr{B} \cap (p, q)) = \dim_H \mathcal{U}_q > 0$.
(iv) $\lim_{p \nearrow q} \dim_H(\mathscr{U} \cap (p, q)) = \dim_H \mathcal{U}_q > 0$.

For odd $M \geq 3$ this theorem must be modified. This is due to the surprising presence of a single exceptional base $q_\star$ which is not an element of $\mathscr{B}^L$, but for which (ii) and (iv) of Theorem 1 nonetheless hold. Let

$$q_\star = q_\star(M) := \begin{cases} \frac{k+3+\sqrt{k^2+6k+1}}{2} & \text{if} \quad M = 2k+1, \\ \frac{k+3+\sqrt{k^2+6k-3}}{2} & \text{if} \quad M = 2k. \end{cases} \tag{1.8}$$

(We will have use for $q_\star(M)$ with M even later on.)

Theorem 1′. Suppose $M = 2k+1 \geq 3$.

(a) $q \in \mathscr{B}^L$ if and only if $\lim_{p\nearrow q} \dim_H(\mathscr{B} \cap (p,q)) = \dim_H \mathcal{U}_q > 0$.
(b) The following statements are equivalent:
  (i) $q \in \mathscr{B}^L \cup \{q_\star(M)\}$.
  (ii) $\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H \mathbf{U}_q > 0$ for any $p \in (1,q)$.
  (iii) $\lim_{p\nearrow q} \dim_H(\mathscr{U} \cap (p,q)) = \dim_H \mathcal{U}_q > 0$.

The characterization of $\mathscr{B}^R$ is more straightforward:

Theorem 2. *The following statements are equivalent for every $M \in \mathbb{N}$.*

(i) $q \in \mathscr{B}^R$.
(ii) $\dim_H(\mathbf{U}_r \setminus \mathbf{U}_q) = \dim_H \mathbf{U}_r > 0$ *for any* $r \in (q, M+1]$.
(iii) $\lim_{r\searrow q} \dim_H(\mathscr{B} \cap (q,r)) = \dim_H \mathcal{U}_q > 0$, *or* $q = q_{KL}$.
(iv) $\lim_{r\searrow q} \dim_H(\mathscr{U} \cap (q,r)) = \dim_H \mathcal{U}_q > 0$, *or* $q = q_{KL}$.

The asymmetry between the characterizations of $\mathscr{B}^L$ and $\mathscr{B}^R$ can be partially explained by the asymmetry of entropy plateaus. For instance, if $[p_L, p_R]$ is an entropy plateau, it follows from [1, Lemma 4.10] that $p_L \in \overline{\mathscr{U}} \setminus \mathscr{U}$, whereas $p_R \in \mathscr{U}$. Moreover, p_R is a left and right accumulation point of $\mathscr{U}$, but p_L is not a right accumulation point of $\mathscr{U}$. This helps explain why there is no counterpart in Theorem 2 to the special base $q_\star(M)$ of Theorem 1′.

Remark 1.3.

(1) Since $\mathscr{B} = \mathscr{B}^L \cap \mathscr{B}^R$ and $q_{KL} \notin \mathscr{B}$, Theorems 1, 1′ and 2 give equivalent conditions for the bifurcation set $\mathscr{B}$. For example, when $M = 1$, $q \in \mathscr{B}$ if and only if
$$\lim_{p\nearrow q} \dim_H(\mathscr{U} \cap (p,q)) = \lim_{r\searrow q} \dim_H(\mathscr{U} \cap (q,r)) = \dim_H \mathcal{U}_q > 0.$$
(2) In view of Lemma 3.12 below, we emphasize that the limits in statements (iii) and (iv) of Theorems 1 and 2 are at most equal to $\dim_H \mathcal{U}_q$ for every $q \in (1, M+1]$. So, the theorems characterize when this largest possible value is attained.

Since the sets $\mathscr{U}$ and $\mathscr{B}$ are of Lebesgue measure zero and nowhere dense, a natural measure of their distribution within the interval $(1, M+1]$ are the local dimension functions
$$\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) \qquad \text{and} \qquad \lim_{\delta\to 0} \dim_H(\mathscr{B} \cap (q-\delta, q+\delta)).$$
In [15, Theorem 2] it was shown that
$$q \in \overline{\mathscr{B}} \setminus \{q_{KL}\} \quad \Longleftrightarrow \quad \lim_{\delta\to 0} \dim_H(\mathscr{B} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0.$$
As for the set $\mathscr{U}$, we will show in Lemma 3.12 below that
$$\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) \leq \dim_H \mathcal{U}_q \qquad \text{for all} \quad q \in (1, M+1]. \tag{1.9}$$
Observe that $q_\star(M) \in \mathscr{B}^R$ for $M = 2k+1 \geq 3$. (See Lemma 3.1 below.) Thus Theorems 1, 1′ and 2 imply that the upper bound $\dim_H \mathcal{U}_q$ for the limit in (1.9) is attained if and only if $q \in \overline{\mathscr{B}}$. Precisely:

Corollary 3. *$q \in \overline{\mathscr{B}} \setminus \{q_{KL}\}$ if and only if*

$$\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0.$$

Clearly, $\lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) = 0$ when $q \notin \overline{\mathscr{U}}$. It is interesting to ask which values this limit can take for $q \in \overline{\mathscr{U}} \setminus \overline{\mathscr{B}}$. This may be the subject of a future paper.

1.3. **The difference set $\mathscr{U} \setminus \mathscr{B}$.** Note that $\mathscr{B} \subset \mathscr{U}$, and both are Lebesgue null sets of full Hausdorff dimension. Furthermore, $\mathscr{U} \setminus \mathscr{B}$ is a dense subset of $\mathscr{U}$. So the box dimension of $\mathscr{U} \setminus \mathscr{B}$ is given by

$$\dim_B(\mathscr{U} \setminus \mathscr{B}) = \dim_B(\overline{\mathscr{U} \setminus \mathscr{B}}) = \dim_B \overline{\mathscr{U}} = 1.$$

On the other hand, our next result shows that the Hausdorff dimension of $\mathscr{U} \setminus \mathscr{B}$ is significantly smaller than one.

Theorem 4.

(i) *If $M = 1$, then*

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \frac{\log 2}{3 \log \lambda^*} \approx 0.368699,$$

where $\lambda^* \approx 1.87135$ is the unique root in $(1, 2)$ of the equation $x^5 - x^4 - x^3 - 2x^2 + x + 1 = 0$.

(ii) *If $M = 2$, then*

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \frac{\log 2}{2 \log \gamma^*} \approx 0.339607,$$

where $\gamma^* \approx 2.77462$ is the unique root in $(2, 3)$ of the equation $x^4 - 2x^3 - 3x^2 + 2x + 1 = 0$.

(iii) *If $M \geq 3$, then*

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \frac{\log 2}{\log q_\star(M)},$$

where $q_\star(M)$ is given by (1.8).

Table 1 below lists the values of $\dim_H(\mathscr{U} \setminus \mathscr{B})$ for $1 \leq M \leq 8$. For large M we have by Theorem 4 (iii) the simple approximation $\dim_H(\mathscr{U} \setminus \mathscr{B}) \approx \log 2 / \log(k+3)$, where k is the greatest integer less than or equal to $M/2$. This systematically underestimates the true value, with an error slowly tending to zero. Observe also that $\dim_H(\mathscr{U} \setminus \mathscr{B}) \to 0$ as $M \to \infty$.

M	1	2	3	4	5	6	7	8
$\dim_H(\mathscr{U} \setminus \mathscr{B})$	0.3687	0.3396	0.5645	0.4750	0.4567	0.4088	0.4005	0.3091

TABLE 1. The numerical calculation of $\dim_H(\mathscr{U} \setminus \mathscr{B})$ for $M = 1, \ldots, 8$.

In [15], Kalle et al. showed that $\dim_H(\mathscr{U} \cap (1, t]) = \max_{q \leq t} \dim_H \mathcal{U}_q$ for all $t > 1$, and they asked whether more generally it is possible to calculate $\dim_H(\mathscr{U} \cap [t_1, t_2])$ for any interval $[t_1, t_2]$. In the process of proving Theorem 4, we give a partial answer to their question by computing the Hausdorff dimension of the intersection of $\mathscr{U}$ with any entropy plateau $[p_L, p_R]$ (see Theorem 4.1).

The rest of the paper is arranged as follows. In Section 2 we recall some results from unique q-expansions, and give the Hausdorff dimension of the symbolic univoque set $\mathbf{U}_q$ (see Lemma 2.8). Based on these observations we characterize the left and right bifurcation sets $\mathscr{B}^L$ and $\mathscr{B}^R$ in Section 3, by proving Theorems 1, 1′ and 2. In Section 4 we prove Theorem 4.

2. Unique expansions

In this section we will describe the symbolic univoque set $\mathbf{U}_q$ and calculate its Hausdorff dimension. Recall that $\Omega = \{0, 1, \ldots, M\}^{\mathbb{N}}$. Let σ be the *left shift* on Ω defined by $\sigma((c_i)) = (c_{i+1})$. Then (Ω, σ) is a *full shift*. By a *word* $\mathbf{c}$ we mean a finite string of digits $\mathbf{c} = c_1 \ldots c_n$ with each digit $c_i \in \{0, 1, \ldots, M\}$. For two words $\mathbf{c} = c_1 \ldots c_m$ and $\mathbf{d} = d_1 \ldots d_n$ we denote by $\mathbf{cd} = c_1 \ldots c_m d_1 \ldots d_n$ their concatenation. For a positive integer n we write $\mathbf{c}^n = \mathbf{c} \cdots \mathbf{c}$ for the n-fold concatenation of $\mathbf{c}$ with itself. Furthermore, we write $\mathbf{c}^\infty = \mathbf{cc} \cdots$ for the infinite periodic sequence with period block $\mathbf{c}$. For a word $\mathbf{c} = c_1 \ldots c_m$ we set $\mathbf{c}^+ := c_1 \ldots c_{m-1}(c_m+1)$ if $c_m < M$, and set $\mathbf{c}^- := c_1 \ldots c_{m-1}(c_m - 1)$ if $c_m > 0$. Furthermore, we define the *reflection* of the word $\mathbf{c}$ by $\overline{\mathbf{c}} := (M - c_1)(M - c_2) \cdots (M - c_m)$. Clearly, $\mathbf{c}^+, \mathbf{c}^-$ and $\overline{\mathbf{c}}$ are all words with digits from $\{0, 1, \ldots, M\}$. For a sequence $(c_i) \in \Omega$ its reflection is also a sequence in Ω defined by $\overline{(c_i)} = (M - c_1)(M - c_2) \cdots$.

Throughout the paper we will use the *lexicographical ordering* $\prec, \preccurlyeq, \succ$ and $\succcurlyeq$ between sequences and words. More precisely, for two sequences $(c_i), (d_i) \in \Omega$ we say $(c_i) \prec (d_i)$ or $(d_i) \succ (c_i)$ if there exists an integer $n \geq 1$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Furthermore, we say $(c_i) \preccurlyeq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. Similarly, for two words $\mathbf{c}$ and $\mathbf{d}$ we say $\mathbf{c} \prec \mathbf{d}$ or $\mathbf{d} \succ \mathbf{c}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$.

Let $q \in (1, M+1]$. Recall that $\mathbf{U}_q$ is the symbolic univoque set which contains all sequences $(x_i) \in \Omega$ such that (x_i) is the unique q-expansion of $\pi_q((x_i))$. Here π_q is the projection map defined in (1.1). The description of $\mathbf{U}_q$ is based on the *quasi-greedy* q-expansion of 1, denoted by $\alpha(q) = \alpha_1(q)\alpha_2(q) \ldots$, which is the lexicographically largest q-expansion of 1 not ending with 0^∞ (cf. [7]). The following characterization of $\alpha(q)$ was given in [4, Theorem 2.2] (see also [9, Proposition 2.3]).

Lemma 2.1. *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $(1, M+1]$ onto the set of all sequences $(a_i) \in \Omega$ not ending with 0^∞ and satisfying*

$$a_{n+1}a_{n+2} \ldots \preccurlyeq a_1 a_2 \ldots \qquad \text{for all} \quad n \geq 0.$$

Furthermore, the map $q \mapsto \alpha(q)$ is left-continuous.

Remark 2.2. Let $\mathbf{A} := \{\alpha(q) : q \in (1, M+1]\}$. Then Lemma 2.1 implies that the inverse map

$$\alpha^{-1} : \mathbf{A} \to (1, M+1]; \qquad (a_i) \mapsto \alpha^{-1}((a_i))$$

is bijective and strictly increasing. Furthermore, we can even show that α^{-1} is continuous; see the proof of Lemma 3.7 below.

Based on the quasi-greedy expansion $\alpha(q)$ we give the lexicographic characterization of the symbolic univoque set $\mathbf{U}_q$, which was essentially established by Parry [24] (see also [16]).

Lemma 2.3. *Let $q \in (1, M+1]$. Then $(x_i) \in \mathbf{U}_q$ if and only if*

$$\begin{cases} x_{n+1}x_{n+2} \ldots \prec \alpha(q) & \text{whenever} \quad x_n < M, \\ x_{n+1}x_{n+2} \ldots \succ \overline{\alpha(q)} & \text{whenever} \quad x_n > 0. \end{cases}$$

Note by Lemma 2.1 that when q is increasing the quasi-greedy expansion $\alpha(q)$ is also increasing in the lexicographical ordering. By Lemma 2.3 it follows that the set-valued map $q \mapsto \mathbf{U}_q$ is also increasing, i.e., $\mathbf{U}_p \subseteq \mathbf{U}_q$ when $p < q$.

Recall from [17] that the Komornik-Loreti constant $q_{KL} = q_{KL}(M)$ is the smallest element of $\mathscr{U}^R$, and satisfies

$$\alpha(q_{KL}) = \lambda_1\lambda_2\ldots, \tag{2.1}$$

where for each $i \geq 1$,

$$\lambda_i = \lambda_i(M) := \begin{cases} k + \tau_i - \tau_{i-1} & \text{if} \quad M = 2k, \\ k + \tau_i & \text{if} \quad M = 2k+1. \end{cases} \tag{2.2}$$

Here $(\tau_i)_{i=0}^\infty = 0110100110010110\ldots$ is the classical *Thue-Morse sequence* (cf. [3]). We emphasize that the sequence (λ_i) depends on M. The following recursive relation of (λ_i) was established in [17] (see also [19]):

$$\lambda_{2^n+1}\ldots\lambda_{2^{n+1}} = \overline{\lambda_1\ldots\lambda_{2^n}}^+ \qquad \text{for all} \quad n \geq 0. \tag{2.3}$$

By (2.1) and (2.2) it follows that $q_{KL}(M) \geq (M+2)/2$ for all $M \geq 1$ (see also [5]), and the map $M \mapsto q_{KL}(M)$ is strictly increasing.

Example 2.4. The following values of $q_{KL}(M)$ will be needed in the proof of Theorem 4 in Section 4.

(1) Let $M = 1$. Then by (2.2) we have $\lambda_1 = 1$. By (2.1) and (2.3) it follows that
$$\alpha(q_{KL}(1)) = 1101\,0011\,00101101\ldots = (\tau_i)_{i=1}^\infty.$$
This gives $q_{KL}(1) \approx 1.78723$.

(2) Let $M = 2$. Then by (2.2) we have $\lambda_1 = 2$, and by (2.1) and (2.3) that
$$\alpha(q_{KL}(2)) = 2102\,0121\,01202102\ldots.$$
So $q_{KL}(2) \approx 2.53595$.

(3) Let $M = 3$. Then by (2.2) we have $\lambda_1 = 2$, and by (2.1) and (2.3) that
$$\alpha(q_{KL}(3)) = 2212\,1122\,11212212\ldots.$$
Hence, $q_{KL}(3) \approx 2.91002$.

Now we recall from [16] the following result for the Hausdorff dimension of the univoque set $\mathcal{U}_q$.

Lemma 2.5.

(i) *For any $q \in (1, M+1]$ we have*
$$\dim_H \mathcal{U}_q = \frac{h(\mathbf{U}_q)}{\log q}.$$

(ii) *The entropy function $H : q \mapsto h(\mathbf{U}_q)$ is a Devil's staircase in $(1, M+1]$:*
- *H is non-decreasing and continuous from $(1, M+1]$ onto $[0, 1]$;*
- *H is locally constant almost everywhere in $(1, M+1]$.*

(iii) *$H(q) > 0$ if and only if $q > q_{KL}$. Furthermore, $H(q) = \log(M+1)$ if and only if $q = M+1$.*

We also need the following lemma for the Hausdorff dimension under Hölder continuous maps (cf. [12]).

Lemma 2.6. *Let $f : (X, \rho_X) \to (Y, \rho_Y)$ be a Hölder map between two metric spaces, i.e., there exist two constants $C > 0$ and $\xi > 0$ such that*

$$\rho_Y(f(x), f(y)) \le C\rho_X(x, y)^\xi \quad \textit{for any } x, y \in X.$$

Then $\dim_H f(X) \le \frac{1}{\xi} \dim_H X$.

Recall the metric ρ from (1.3). It will be convenient to introduce a more general family of (mutually equivalent) metrics $\{\rho_q : q > 1\}$ on Ω defined by

$$\rho_q((c_i), (d_i)) := q^{-\inf\{i\ge 1: c_i \ne d_i\}}, \qquad q > 1.$$

Then (Ω, ρ_q) is a compact metric space. Let $\dim_H^{(q)}$ denote Hausdorff dimension on Ω with respect to the metric ρ_q, so

$$\dim_H^{(M+1)} E = \dim_H E$$

for any subset $E \subseteq \Omega$. For $p > 1$ and $q > 1$,

$$\rho_q((c_i), (d_i)) = \rho_p((c_i), (d_i))^{\log q/\log p},$$

and by Lemma 2.6 this gives the useful relationship

$$\dim_H^{(p)} E = \frac{\log q}{\log p} \dim_H^{(q)} E, \qquad E \subseteq \Omega. \tag{2.4}$$

The following result is well known (see [14, Lemma 2.7] or [2, Lemma 2.2]):

Lemma 2.7. *For each $q \in (1, M + 1)$, the map π_q is Lipschitz on (Ω, ρ_q), and the restriction*

$$\pi_q : (\mathbf{U}_q, \rho_q) \to (\mathcal{U}_q, |.|); \qquad \pi_q((x_i)) = \sum_{i=1}^{\infty} \frac{x_i}{q^i}$$

is bi-Lipschitz, where $|.|$ denotes the Euclidean metric on $\mathbb{R}$.

Observe that the Hausdorff dimension does not exceed the lower box dimension (cf. [12]). This implies that $\dim_H E \le h(E)$ for any set $E \subset \Omega$. Using Lemmas 2.5–2.7 we show that equality holds for $\mathbf{U}_q$.

Lemma 2.8. *Let $q \in (1, M + 1]$. Then*

$$\dim_H \mathbf{U}_q = h(\mathbf{U}_q).$$

Proof. For $q = M + 1$, one checks easily that

$$\dim_H \mathbf{U}_{M+1} = h(\mathbf{U}_{M+1}) = 1.$$

Let $q \in (1, M+1)$. By Lemmas 2.7 and 2.6, $\dim_H^{(q)} \mathbf{U}_q = \dim_H \mathcal{U}_q$. So (2.4), Lemmas 2.7 and 2.5 give

$$\dim_H \mathbf{U}_q = \dim_H^{(M+1)} \mathbf{U}_q = \frac{\log q}{\log(M + 1)} \dim_H^{(q)} \mathbf{U}_q = \log q \dim_H \mathcal{U}_q = h(\mathbf{U}_q),$$

as desired. We emphasize that the base for our logarithms is $M + 1$. $\square$

Note that the symbolic univoque set $\mathbf{U}_q$ is not always closed. Inspired by the works of de Vries and Komornik [8] and Komornik et al. [16] we introduce the set

$$\mathbf{V}_q := \left\{(x_i) \in \Omega : \overline{\alpha(q)} \preccurlyeq x_{n+1}x_{n+2}\ldots \preccurlyeq \alpha(q) \text{ for all } n \geq 0\right\}. \tag{2.5}$$

We have the following relationship between $\mathbf{V}_q$ and $\mathbf{U}_q$.

Lemma 2.9. *For any* $0 < p < q \leq M+1$ *we have*

$$\dim_H \mathbf{V}_q = \dim_H \mathbf{U}_q \qquad \textit{and} \qquad \dim_H(\mathbf{V}_q \setminus \mathbf{V}_p) = \dim_H(\mathbf{U}_q \setminus \mathbf{U}_p).$$

Proof. By Lemma 2.3 it follows that for each $q \in (1, 2]$ the set $\mathbf{U}_q$ is a countable union of affine copies of $\mathbf{V}_q$ up to a countable set (see also [15, Lemma 3.2]), i.e., there exists a sequence of affine maps $\{g_i\}_{i=1}^\infty$ on Ω of the form

$$x_1x_2\ldots \mapsto ax_1x_2\ldots, \qquad x_1x_2\ldots \mapsto M^m bx_1x_2\ldots \qquad \text{or} \qquad x_1x_2\ldots \mapsto 0^m cx_1x_2\ldots,$$

where $a \in \{1, 2, \ldots, M-1\}$, $b \in \{0, 1, \ldots, M-1\}$, $c \in \{1, 2, \ldots, M\}$ and $m = 1, 2, \ldots$, such that

$$\mathbf{U}_q \sim \bigcup_{i=1}^\infty g_i(\mathbf{V}_q), \tag{2.6}$$

where we write $A \sim B$ to mean that the symmetric difference $A \triangle B$ is at most countable. Since the Hausdorff dimension is stable under affine maps (cf. [12]), this implies $\dim_H \mathbf{V}_q = \dim_H \mathbf{U}_q$.

Furthermore, for any $1 < p < q \leq M+1$ we have $\mathbf{U}_p \subseteq \mathbf{U}_q$ and $\mathbf{V}_p \subseteq \mathbf{V}_q$, so $g_i(\mathbf{V}_p) \subseteq g_i(\mathbf{V}_q)$ for all $i \geq 1$. Note that for $i \neq j$ the intersection $g_i(\mathbf{V}_q) \cap g_j(\mathbf{V}_q) = \emptyset$. Then by (2.6) it follows that

$$\begin{aligned} \mathbf{U}_q \setminus \mathbf{U}_p &\sim \bigcup_{i=1}^\infty g_i(\mathbf{V}_q) \setminus \bigcup_{i=1}^\infty g_i(\mathbf{V}_p) \\ &= \bigcup_{i=1}^\infty \big(g_i(\mathbf{V}_q) \setminus g_i(\mathbf{V}_p)\big) = \bigcup_{i=1}^\infty g_i(\mathbf{V}_q \setminus \mathbf{V}_p). \end{aligned}$$

We conclude that $\dim_H(\mathbf{U}_q \setminus \mathbf{U}_p) = \dim_H(\mathbf{V}_q \setminus \mathbf{V}_p)$. $\square$

3. Characterizations of $\mathscr{B}^L$ and $\mathscr{B}^R$

Recall from (1.7) that $\mathscr{B}^L$ and $\mathscr{B}^R$ are the left and right bifurcation sets of H. In this section we will characterize the sets $\mathscr{B}^L$ and $\mathscr{B}^R$, and prove Theorems 1, 1′ and 2. Since the theorems are very similar, we will prove only Theorem 1 in full detail, and comment briefly on the proofs of Theorems 1′ and 2.

Recall the definition of $q_\star(M)$ from (1.8). Its significance derives from the fact that

$$\alpha(q_\star(M)) = \begin{cases} (k+2)k^\infty & \text{if} \quad M = 2k+1, \\ (k+2)(k-1)^\infty & \text{if} \quad M = 2k. \end{cases}$$

By (2.1) and Lemma 2.1 it follows in particular that $q_\star(M) > q_{KL}$.

Recall that a closed interval $[p_L, p_R] \subseteq (q_{KL}, M+1]$ is an entropy plateau if it is a maximal interval on which H is constant. The following lemma was implicitly proven in [1].

Lemma 3.1. *Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau.*

(i) *Then there exists a word $a_1 \dots a_m$ satisfying $\overline{a_1} < a_1$ and*
$$\overline{a_1 \dots a_{m-i}} \preccurlyeq a_{i+1} \dots a_m \prec a_1 \dots a_{m-i} \qquad \text{for all} \quad 1 \le i < m,$$
such that
$$\alpha(p_L) = (a_1 \dots a_m)^\infty \qquad \text{and} \qquad \alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty.$$

(ii) *Let $m \ge 1$ be defined as in (i). Then*
$$h(\mathbf{U}_{p_L}) \ge \frac{\log 2}{m},$$
where equality holds if and only if $M = 2k+1 \ge 3$ and $[p_L, p_R] = [k+2, q_\star(M)]$.

Proof. Part (i) was established in [1, Theorem 2 and Lemma 4.1]. Part (ii) was implicitly given in the proofs of [1, Lemmas 5.1 and 5.5]. It is shown there that $h(\mathbf{U}_{p_L}) > \log 2/m$ when $m \ge 2$. If $m = 1$, then $\alpha(p_L) = a_1^\infty$ for some $a_1 \ge (M+1)/2$, and
$$h(\mathbf{U}_{p_L}) = \log(2a_1 - M + 1).$$
(See [1, Example 5.13].) It follows that $h(\mathbf{U}_{p_L}) = \log 2/m$ if and only if $m = 1$, $M = 2k+1 \ge 3$ and $a_1 = k+1$, in which case
$$\alpha(p_L) = (k+1)^\infty \qquad \text{and} \qquad \alpha(p_R) = (k+2)k^\infty,$$
or equivalently,
$$[p_L, p_R] = \left[k+2, \frac{k+3+\sqrt{k^2+6k+1}}{2}\right] = [k+2, q_\star(M)]$$
for $M = 2k+1 \ge 3$. □

Remark 3.2. We point out that the condition in Lemma 3.1 (i) is not a sufficient condition for $[p_L, p_R] \subset (q_{KL}, M+1]$ being an entropy plateau. For a complete characterization of entropy plateaus we refer to [1, Theorem 2]. However, if $[p_L, p_R]$ is an interval satisfying the conditions of Lemma 3.1, then $[p_L, p_R]$ is either an entropy plateau or else it is contained in some entropy plateau (see Example 3.3 below). We refer to [1] for more details.

Example 3.3. Take $M = 1$ and let $a_1 \dots a_m = 1^{m-1}0$ with $m \ge 3$. Then the word $a_1 \dots a_m$ satisfies the inequalities in Lemma 3.1 (i), and the interval $[p_L, p_R]$ is indeed an entropy plateau, where $\alpha(p_L) = (1^{m-1}0)^\infty$ and $\alpha(p_R) = 1^m(0^{m-1}1)^\infty$.

On the other hand, take the word $b_1 \dots b_{2m} = 1^m 0^m$. One can also check that $b_1 \dots b_{2m}$ satisfies the inequalities in Lemma 3.1 (i). However, the corresponding interval $[q_L, q_R]$ is a proper subset of $[p_L, p_R]$ and hence not an entropy plateau, where $\alpha(q_L) = (b_1 \dots b_{2m})^\infty$ and $\alpha(q_R) = b_1 \dots b_{2m}^+ (\overline{b_1 \dots b_{2m}})^\infty$.

Definition 3.4. If $[p_L, p_R]$ is an entropy plateau with $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty$, we shall call $[p_L, p_R]$ an *entropy plateau of period m*.

Recall that $\mathscr{U}$ is the set of univoque bases $q \in (1, M+1]$ such that 1 has a unique q-expansion. The following characterization of its topological closure $\overline{\mathscr{U}}$ was established in [18] (see also [9]).

Lemma 3.5. *$q \in \overline{\mathscr{U}}$ if and only if*

$$\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \preccurlyeq \alpha(q) \qquad \text{for all} \quad n \geq 1.$$

Lemma 2.1 states that the map $\alpha : q \mapsto \alpha(q)$ is left-continuous on $(1, M+1]$. The following lemma strengthens this result when α is restricted to $\overline{\mathscr{U}}$.

Lemma 3.6. *Let $I = [p, q] \subset (1, M+1)$. Then the map α is Lipschitz on $\overline{\mathscr{U}} \cap I$ with respect to the metric ρ_q.*

Proof. Fix $1 < p < q < M+1$. We will show something slightly stronger, namely that there is a constant $C = C(p, q)$ such that for any $p \leq p_1 < p_2 \leq q$ with $p_2 \in \overline{\mathscr{U}}$,

$$\rho_q(\alpha(p_1), \alpha(p_2)) \leq C|p_2 - p_1|.$$

Let $p \leq p_1 < p_2 \leq q$ and $p_2 \in \overline{\mathscr{U}}$. Then by Lemma 2.1 we have $\alpha(p_1) \prec \alpha(p_2)$. So there exists $n \geq 1$ such that $\alpha_1(p_1) \ldots \alpha_{n-1}(p_1) = \alpha_1(p_2) \ldots \alpha_{n-1}(p_2)$ and $\alpha_n(p_1) < \alpha_n(p_2)$. Since $q < M+1$, we have $\alpha(q) \prec M^\infty$. Hence there exists a large integer $N \geq 1$, depending only on q, such that $\alpha(p_2) \preccurlyeq \alpha(q) \preccurlyeq M^{N-1}0^\infty$. Since $p_2 \in \overline{\mathscr{U}}$, it follows by Lemma 3.5 that

$$\alpha_{n+1}(p_2)\alpha_{n+2}(p_2) \ldots \succ \overline{\alpha(p_2)} \succcurlyeq 0^{N-1}M^\infty.$$

This implies

$$1 = \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^i} > \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} + \frac{1}{p_2^{n+N}}.$$

Therefore,

$$\begin{aligned}
\frac{1}{p_2^{n+N}} \leq 1 - \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} &= \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^i} - \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} \\
&\leq \sum_{i=1}^{n} \left(\frac{\alpha_i(p_2)}{p_1^i} - \frac{\alpha_i(p_2)}{p_2^i} \right) \\
&\leq \sum_{i=1}^{\infty} \left(\frac{M}{p_1^i} - \frac{M}{p_2^i} \right) = \frac{M|p_2 - p_1|}{(p_1 - 1)(p_2 - 1)} \\
&\leq \frac{M|p_2 - p_1|}{(p-1)^2}.
\end{aligned}$$

Here the second inequality follows by using $\alpha_1(p_1) \ldots \alpha_{n-1}(p_1) = \alpha_1(p_2) \ldots \alpha_{n-1}(p_2)$, $\alpha_n(p_1) < \alpha_n(p_2)$ and the property of quasi-greedy expansion that $\sum_{i=1}^\infty \alpha_{n+i}(p_1)/p_1^i \leq 1$. Therefore, we obtain

$$\rho_q(\alpha(p_1), \alpha(p_2)) = q^{-n} \leq p_2^{-n} \leq \frac{Mq^N}{(p-1)^2}|p_2 - p_1|.$$

The proof is complete. □

The following dimension estimates will be very useful throughout the paper:

Lemma 3.7. *For any interval $I = [p, q] \subseteq (1, M+1)$,*

$$\dim_H \pi_q(\mathbf{U}_I) \leq \dim_H(\overline{\mathscr{U}} \cap I) \leq \frac{h(\mathbf{U}_I)}{\log p},$$

where $\mathbf{U}_I := \{\alpha(\ell) : \ell \in \overline{\mathscr{U}} \cap I\}$.

Proof. Fix an interval $I = [p, q] \subseteq (1, M+1)$. We may view the map $\pi_q \circ \alpha : \overline{\mathscr{U}} \cap I \to \mathbb{R}$ as the composition of the maps $\alpha : \overline{\mathscr{U}} \cap I \to (\mathbf{U}_I, \varphi_q)$ and $\pi_q : (\mathbf{U}_I, \varphi_q) \to \mathbb{R}$. The first map is Lipschitz by Lemma 3.6, and the second is Lipschitz by Lemma 2.7, since $\mathbf{U}_I \subset \mathbf{U}_q$. Therefore, the composition $\pi_q \circ \alpha$ is Lipschitz. Using Lemma 2.6, this implies the first inequality.

The second inequality is proved as follows. Let $p \le p_1 < p_2 \le q$. Then $\alpha(p_1) \prec \alpha(p_2)$ by Lemma 2.1, so there is a number $n \in \mathbb{N}$ such that $\alpha_1(p_1) \dots \alpha_{n-1}(p_1) = \alpha_1(p_2) \dots \alpha_{n-1}(p_2)$ and $\alpha_n(p_1) < \alpha_n(p_2)$. As in the proof of Lemma 4.3 in [15], we then have

$$
\begin{aligned}
p_2 - p_1 &= \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^{i-1}} - \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^{i-1}} \\
&\le \sum_{i=1}^{n-1} \left(\frac{\alpha_i(p_2)}{p_2^{i-1}} - \frac{\alpha_i(p_1)}{p_1^{i-1}} \right) + \sum_{i=n}^{\infty} \frac{\alpha_i(p_2)}{p_2^{i-1}} \\
&\le p_2^{2-n} \le (M+1)^2 p^{-n},
\end{aligned}
$$

where the second inequality follows by the property of the quasi-greedy expansion $\alpha(p_2)$ of 1. We conclude that

$$
\rho(\alpha(p_1), \alpha(p_2)) = (M+1)^{-n} = p^{-n/\log p} \ge \left(\frac{p_2 - p_1}{(M+1)^2} \right)^{1/\log p},
$$

in other words, the map α^{-1} is Hölder continuous with exponent $\log p$ on the set $\{\alpha(\ell) : p \le \ell \le q\}$. It follows using Lemma 2.6 that

$$
\dim_H(\overline{\mathscr{U}} \cap I) = \dim_H(\alpha^{-1}(\mathbf{U}_I)) \le \frac{\dim_H \mathbf{U}_I}{\log p} \le \frac{h(\mathbf{U}_I)}{\log p},
$$

completing the proof. $\square$

Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau such that $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty$. The proofs of the following two propositions use the sofic subshift $(X_{\mathscr{G}}, \sigma)$ represented by the labeled graph $\mathscr{G} = (G, \mathscr{L})$ in Figure 1 (cf. [22, Chapter 3]).

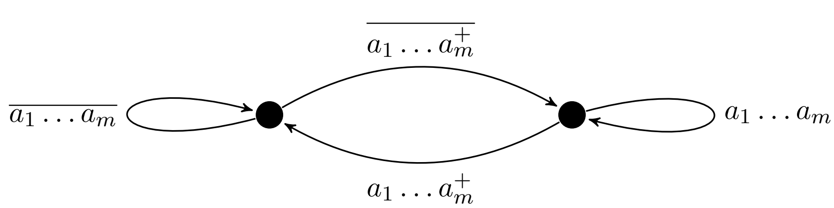

FIGURE 1. The picture of the labeled graph $\mathscr{G} = (G, \mathscr{L})$.

We emphasize that $(X_{\mathscr{G}}, \sigma)$ is in fact a subshift of finite type over the states

$$
a_1 \dots a_m, \qquad a_1 \dots a_m^+, \qquad \overline{a_1 \dots a_m} \qquad \text{and} \qquad \overline{a_1 \dots a_m^+}
$$

with adjacency matrix

$$A_{\mathscr{G}} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then it is easy to see (cf. [22, Theorem 4.3.3]) that

$$h(X_{\mathscr{G}}) = \frac{\log\lambda(A_{\mathscr{G}})}{m} = \frac{\log 2}{m}, \tag{3.1}$$

where $\lambda(A_{\mathscr{G}})$ denotes the spectral radius of $A_{\mathscr{G}}$.

Proposition 3.8. *Let $[p_L, p_R] \subseteq (q_{KL}, M+1)$ be an entropy plateau of period m. Then for any $p \in [p_L, p_R)$,*

$$\dim_H(\mathscr{U} \cap [p, p_R]) \geq \frac{\log 2}{m \log p_R}.$$

(We will show in Section 4 that this holds in fact with equality.)

Proof. We will construct a sequence of subsets $\{\Lambda_N\}$ of $\mathbf{U}_{[p,p_R]}$ such that the Hausdorff dimension of $\pi_{p_R}(\Lambda_N)$ tends to $\frac{\log 2}{m \log p_R}$ as $N \to \infty$, where $\mathbf{U}_{[p,p_R]} := \left\{\alpha(\ell) : \ell \in \overline{\mathscr{U}} \cap [p, p_R]\right\}$. This observation, when combined with Lemma 3.7 and the fact that the difference between $\mathscr{U}$ and $\overline{\mathscr{U}}$ is countable, will imply our lower bound.

Let $a_1 \dots a_m$ be the word such that $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^\infty$. Recall that $X_{\mathscr{G}}$ is a sofic subshift represented by the labeled graph $\mathscr{G}$ in Figure 1. For an integer $N \geq 2$ let Λ_N be the set of sequences $(c_i) \in X_{\mathscr{G}}$ beginning with

$$c_1 \dots c_{mN} = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}$$

and the tail sequence $c_{mN+1}c_{mN+2}\dots$ not containing the word $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}$ or $\overline{a_1 \dots a_m^+}(a_1 \dots a_m)^{N-1}$. Note that since $\alpha(p) \prec \alpha(p_R)$, we can choose N large enough so that $\alpha(p) \prec a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}0^\infty$. We claim that $\Lambda_N \subset \mathbf{U}_{[p,p_R]}$.

Observe that $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^\infty$ is the lexicographically largest sequence in $X_{\mathscr{G}}$, and $\overline{a_1 \dots a_m^+}(a_1 \dots a_m)^\infty$ is the lexicographically smallest sequence in $X_{\mathscr{G}}$. Take a sequence $(c_i) \in \Lambda_N$. Then (c_i) has a prefix $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}$, and the tail $c_{mN+1}c_{mN+2}\dots$ satisfies the inequalities

$$\overline{(c_i)} \preccurlyeq \overline{a_1 \dots a_m^+}(a_1 \dots a_m)^{N-1}M^\infty \prec \sigma^n((c_i)) \prec a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{N-1}0^\infty \preccurlyeq (c_i)$$

for all $n \geq mN$. By Lemma 3.5, to prove $(c_i) \in \mathbf{U}_{[p,p_R]}$ it suffices to prove $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $1 \leq n < mN$. Note by Lemma 3.1(i) that

$$\overline{a_1 \dots a_{m-i}} \preccurlyeq a_{i+1} \dots a_m \prec a_1 \dots a_{m-i} \quad \text{for all} \quad 1 \leq i < m. \tag{3.2}$$

This implies that

$$a_{i+1} \dots a_m^+ \overline{a_1 \dots a_i} \preccurlyeq a_1 \dots a_m \prec a_1 \dots a_m^+,$$

and

$$a_{i+1} \dots a_m^+ \succ a_{i+1} \dots a_m \succcurlyeq \overline{a_1 \dots a_{m-i}}$$

for all $1 \le i < m$. So $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $1 \le n < m$. Furthermore, by (3.2) it follows that

$$\overline{a_1 \dots a_m^+} \prec \overline{a_1 \dots a_m} \preccurlyeq a_{i+1} \dots a_m a_1 \dots a_i \prec a_1 \dots a_m^+$$

for all $0 \le i < m$. Taking the reflection we obtain

$$\overline{a_1 \dots a_m^+} \prec \overline{a_{i+1} \dots a_m a_1 \dots a_i} \prec a_1 \dots a_m^+ \tag{3.3}$$

for all $0 \le i < m$. Since $c_{m(N-1)+1} \dots c_{mN} = \overline{a_1 \dots a_m}$, we have $c_{mN+1} \dots c_{mN+m-1} = \overline{a_1 \dots a_{m-1}}$ (see Figure 1). Then by (3.3) it follows that $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $m \le n < mN$. Therefore, $\overline{(c_i)} \prec \sigma^n((c_i)) \prec (c_i)$ for all $n \ge 1$. So $(c_i) \in \mathbf{U}_{[p,p_R]}$, and hence $\Lambda_N \subset \mathbf{U}_{[p,p_R]}$.

Observe that $\pi_{p_R}(\Lambda_N)$ is the affine image of a graph-directed self-similar set whose Hausdorff dimension is arbitrarily close to the dimension of $\pi_{p_R}(X_{\mathscr{G}})$ as $N \to \infty$. Then

$$\lim_{N\to\infty} \dim_H \pi_{p_R}(\Lambda_N) = \dim_H \pi_{p_R}(X_{\mathscr{G}}) = \frac{\log 2}{m \log p_R}.$$

Therefore, by the first inequality in Lemma 3.7 and the claim we conclude that

$$\begin{aligned}\dim_H(\overline{\mathscr{U}} \cap [p, p_R]) &\ge \dim_H \pi_{p_R}(\mathbf{U}_{[p,p_R]}) \\ &\ge \lim_{N\to\infty} \dim_H \pi_{p_R}(\Lambda_N) = \frac{\log 2}{m \log p_R},\end{aligned}$$

completing the proof. □

Next, recall from (2.5) that $\mathbf{V}_q$ is the set of sequences $(x_i) \in \Omega$ satisfying the inequalities:

$$\overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q) \qquad \text{for all} \quad n \ge 0.$$

The next proposition shows that the set-valued map $q \mapsto \mathbf{V}_q$ does not vary too much inside an entropy plateau $[p_L, p_R]$, and gives a sharp estimate for the limit in Theorem 1(iv) when q lies inside an entropy plateau.

Proposition 3.9. *Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau of period m. Then*

(i) *For all p and q with $p_L \le p < q < p_R$,*

$$\dim_H(\mathbf{V}_q \backslash \mathbf{V}_p) < \dim_H(\mathbf{V}_{p_R} \backslash \mathbf{V}_p) = \frac{\log 2}{m}. \tag{3.4}$$

(ii) *For all $q \in (p_L, p_R]$,*

$$\lim_{p \nearrow q} \dim_H(\overline{\mathscr{U}} \cap (p, q)) \le \frac{\log 2}{m \log q}, \tag{3.5}$$

with equality if and only if $q = p_R$.

Proof. First we prove (i). By Lemma 3.1 there exists a word $a_1 \dots a_m$ such that

$$\alpha(p_L) = (a_1 \dots a_m)^\infty \qquad \text{and} \qquad \alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty. \tag{3.6}$$

Take a sequence $(c_i) \in \mathbf{V}_{p_R} \setminus \mathbf{V}_{p_L}$. Then there exists $j \ge 0$ such that

$$c_{j+1} \dots c_{j+m} = a_1 \dots a_m^+ \qquad \text{or} \qquad c_{j+1} \dots c_{j+m} = \overline{a_1 \dots a_m^+}.$$

We claim that the tail sequence $c_{j+1} c_{j+2} \dots \in X_{\mathscr{G}}$, where $X_{\mathscr{G}}$ is the sofic subshift determined by the labeled graph in Figure 1.

By symmetry we may assume $c_{j+1}\dots c_{j+m} = a_1\dots a_m^+$. Since $(c_i)\in \mathbf{V}_{p_R}$, by (3.6) the sequence (c_i) satisfies

$$\overline{a_1\dots a_m^+}(a_1\dots a_m)^\infty \preccurlyeq \sigma^n((c_i)) \preccurlyeq a_1\dots a_m^+(\overline{a_1\dots a_m})^\infty \tag{3.7}$$

for all $n\ge 0$. Taking $n=j$ in (3.7) it follows that $c_{j+m+1}\dots c_{j+2m}\preccurlyeq \overline{a_1\dots a_m}$. Again, by (3.7) with $n=j+m$ we obtain that $c_{j+m+1}\dots c_{j+2m}\succcurlyeq \overline{a_1\dots a_m^+}$. So, if $c_{j+1}\dots c_{j+m}=a_1\dots a_m^+$, then the next word $c_{j+m+1}\dots c_{j+2m}$ has only two choices: it either equals $\overline{a_1\dots a_m^+}$ or it equals $\overline{a_1\dots a_m}$.

- If $c_{j+m+1}\dots c_{j+2m}=\overline{a_1\dots a_m^+}$, then by symmetry and using (3.7) it follows that the next word $c_{j+2m+1}\dots c_{j+3m}$ equals either $a_1\dots a_m$ or $a_1\dots a_m^+$.
- If $c_{j+m+1}\dots c_{j+2m}=\overline{a_1\dots a_m}$, then $c_{j+1}\dots c_{j+2m}=a_1\dots a_m^+\overline{a_1\dots a_m}$. By using (3.7) with $k=j$ we have $c_{j+2m+1}\dots c_{j+3m}\preccurlyeq\overline{a_1\dots a_m}$. Again, by (3.7) with $k=j+2m$ it follows that the next word $c_{j+2m+1}\dots c_{j+3m}$ equals either $\overline{a_1\dots a_m^+}$ or $\overline{a_1\dots a_m}$.

By iteration of the above arguments we conclude that $c_{j+1}c_{j+2}\dots\in X_{\mathscr{G}}$. This proves the claim: any sequence in $\mathbf{V}_{p_R}\setminus\mathbf{V}_{p_L}$ eventually ends with an element of $X_{\mathscr{G}}$.

Using the claim and (3.1) it follows that

$$\dim_H(\mathbf{V}_{p_R}\backslash\mathbf{V}_p)\le\dim_H(\mathbf{V}_{p_R}\setminus\mathbf{V}_{p_L})\le\dim_H X_{\mathscr{G}}\le\ h(X_{\mathscr{G}})=\frac{\log 2}{m}. \tag{3.8}$$

On the other hand, since $p<p_R$ we have $\alpha(p)\prec\alpha(p_R)=a_1\dots a_m^+(\overline{a_1\dots a_m})^\infty$, so there exists $K\in\mathbb{N}$ such that $\alpha(p)\prec a_1\dots a_m^+(\overline{a_1\dots a_m})^K0^\infty$. Hence, the follower set

$$F_{X_{\mathscr{G}}}\big(a_1\dots a_m^+(\overline{a_1\dots a_m})^K\big):=\big\{(d_i)\in X_{\mathscr{G}}: d_1\dots d_{m(K+1)}=a_1\dots a_m^+(\overline{a_1\dots a_m})^K\big\}$$

is a subset of $\mathbf{V}_{p_R}\setminus\mathbf{V}_p$. By (3.1) this implies that

$$\dim_H(\mathbf{V}_{p_R}\setminus\mathbf{V}_p)\ge\dim_H F_{X_{\mathscr{G}}}\big(a_1\dots a_m^+(\overline{a_1\dots a_m})^K\big)=h(X_{\mathscr{G}})=\frac{\log 2}{m}, \tag{3.9}$$

where the first equality follows since, in view of the homogeneous structure of $X_{\mathscr{G}}$, there is no more efficient covering of this set than by cylinder sets of equal depth. Combining (3.8) and (3.9) gives

$$\dim_H(\mathbf{V}_{p_R}\setminus\mathbf{V}_p)=\frac{\log 2}{m}. \tag{3.10}$$

Next, observe that for $q\in(p_L,p_R)$ there exists $N\in\mathbb{N}$ such that

$$\alpha(q)\prec a_1\dots a_m^+(\overline{a_1\dots a_m})^N0^\infty.$$

Then the words $a_1\dots a_m^+(\overline{a_1\dots a_m})^N$ and $\overline{a_1\dots a_m^+}(a_1\dots a_m)^N$ are forbidden in $\mathbf{V}_q$. By the above argument it follows that any sequence in $\mathbf{V}_q\setminus\mathbf{V}_p$ eventually ends with an element of

$$\begin{aligned}X_{\mathscr{G},N}:=\big\{(d_i)\in X_{\mathscr{G}}: a_1\dots a_m^+&(\overline{a_1\dots a_m})^N\text{ and}\\ &\overline{a_1\dots a_m^+}(a_1\dots a_m)^N\text{ do not occur in }(d_i)\big\}.\end{aligned} \tag{3.11}$$

By (3.1) this implies that

$$\dim_H(\mathbf{V}_q\setminus\mathbf{V}_p)\le\dim_H X_{\mathscr{G},N}\le h(X_{\mathscr{G},N})<h(X_{\mathscr{G}})=\frac{\log 2}{m},$$

where the strict inequality holds by [22, Corollary 4.4.9], since $X_{\mathscr{G}}$ is a transitive sofic subshift and $X_{\mathscr{G},N} \subsetneq X_{\mathscr{G}}$. Later in Lemma 4.2 we will give an explicit formula for $h(X_{\mathscr{G},N})$. This completes the proof of (i).

To prove (ii), suppose first that $q \in (p_L, p_R)$. Let $a_1 \dots a_m$ be the word such that (3.6) holds. Take $p \in (p_L, q) \cap \overline{\mathscr{U}}$. By Lemma 2.1 it follows that for any $\ell \in (p, q)$ the quasi-greedy expansion $\alpha(\ell)$ begins with $a_1 \dots a_m^+$. As in the proof of (i), since $q < p_R$ it follows that there exists $N \in \mathbb{N}$ depending only on q such that

$$\mathbf{U}_{(p,q)} := \{\alpha(\ell) : \ell \in \overline{\mathscr{U}} \cap (p, q)\} \subseteq X_{\mathscr{G},N},$$

where $X_{\mathscr{G},N}$ was defined in (3.11). Therefore, by Lemma 3.7,

$$\begin{aligned}\lim_{p \nearrow q} \dim_H(\overline{\mathscr{U}} \cap (p, q)) &\le \lim_{p \nearrow q} \frac{h(\mathbf{U}_{(p,q)})}{\log p} \le \lim_{p \nearrow q} \frac{h(X_{\mathscr{G},N})}{\log p} \\ &= \frac{h(X_{\mathscr{G},N})}{\log q} < \frac{h(X_{\mathscr{G}})}{\log q} = \frac{\log 2}{m \log q}.\end{aligned}$$

For $q = p_R$ we have $h(\mathbf{U}_{(p,q)}) \le h(X_{\mathscr{G}})$, so as in the above calculation we obtain

$$\lim_{p \nearrow p_R} \dim_H(\overline{\mathscr{U}} \cap (p, p_R)) \le \frac{\log 2}{m \log p_R}.$$

The reverse inequality holds by Proposition 3.8, and hence we have equality in (3.5) for $q = p_R$. □

Corollary 3.10. *For any entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$ and any $q \in (p_L, p_R]$,*

$$\dim_H(\mathbf{V}_q \setminus \mathbf{V}_{p_L}) \le \dim_H \mathbf{V}_{p_L},$$

with equality if and only if $M = 2k + 1 \ge 3$ and $q = p_R = q_\star(M)$.

Proof. Immediate from Lemma 3.1(ii), Lemmas 2.8 and 2.9, and Proposition 3.9(i). □

As a final preparation for the proofs of Theorems 1, 1′ and 2, we need the following results about the local dimension of the bifurcation sets $\overline{\mathscr{B}}$ and $\overline{\mathscr{U}}$. We first recall from [15, Theorem 2] the local dimension of $\mathscr{B}$.

Lemma 3.11. *For any $q \in \overline{\mathscr{B}}$ we have*

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{B}} \cap (q - \delta, q + \delta)) = \dim_H \mathcal{U}_q.$$

For the local dimension of $\mathscr{U}$, we can prove the following:

Lemma 3.12. *For any $q \in (1, M+1]$ we have*

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{U}} \cap (q - \delta, q + \delta)) \le \dim_H \mathcal{U}_q.$$

Proof. Take $q \in (1, M+1]$. By Lemmas 2.1, 2.3 and 3.5 it follows that for each $\ell \in \overline{\mathscr{U}} \cap (q - \delta, q + \delta)$ the quasi-greedy expansion $\alpha(\ell)$ belongs to $\mathbf{U}_{q+\delta}$, where we set $\mathbf{U}_{q+\delta} = \Omega$ if $q + \delta > M + 1$. In other words, using the notation of Lemma 3.7,

$$\mathbf{U}_{(q-\delta, q+\delta)} \subseteq \mathbf{U}_{q+\delta}.$$

We now obtain by Lemma 3.7 and Lemma 2.5,

$$\begin{aligned}\dim_H\left(\overline{\mathscr{U}}\cap(q-\delta,q+\delta)\right)&\le\frac{h(\mathbf{U}_{(q-\delta,q+\delta)})}{\log(q-\delta)}\le\frac{h(\mathbf{U}_{q+\delta})}{\log(q-\delta)}\\&\le\frac{\log(q+\delta)}{\log(q-\delta)}\dim_H\mathcal{U}_{q+\delta}\to\dim_H\mathcal{U}_q\end{aligned}$$

as $\delta\to0$. This completes the proof. □

We are now ready to prove Theorems 1, 1′ and 2.

Proof of Theorem 1. Suppose $M=1$ or M is even. We prove (i) ⇔ (ii) and (i) ⇒ (iii) ⇒ (iv) ⇒ (i).

First we prove (i) ⇒ (ii). Let $q\in\mathscr{B}^L$, and take $p\in(1,q)$. Then $H(p)<H(q)$ by the definition of $\mathscr{B}^L$, so Lemma 2.8 implies

$$\dim_H\mathbf{U}_p=H(p)<H(q)=\dim_H\mathbf{U}_q.$$

Therefore,

$$\dim_H(\mathbf{U}_q\setminus\mathbf{U}_p)=\dim_H\mathbf{U}_q>\dim_H\mathbf{U}_p\ge0.$$

Next, we prove (ii) ⇒ (i). Let $q\in(1,M+1]\setminus\mathscr{B}^L$. By (1.7) we have $q\in(1,q_{KL}]$ or $q\in(p_L,p_R]$ for some entropy plateau $[p_L,p_R]\subset(q_{KL},M+1]$. If $q\in(1,q_{KL}]$, then by Lemma 2.5 we have

$$\dim_H(\mathbf{U}_q\setminus\mathbf{U}_p)=\dim_H\mathbf{U}_q=0$$

for any $p\in(1,q)$. Suppose $q\in(p_L,p_R]\subset(q_{KL},M+1]$, and take $p\in(p_L,q)$. By Corollary 3.10 and Lemma 2.9 it follows that

$$\begin{aligned}\dim_H(\mathbf{U}_q\setminus\mathbf{U}_p)\le\dim_H(\mathbf{U}_q\setminus\mathbf{U}_{p_L})&=\dim_H(\mathbf{V}_q\setminus\mathbf{V}_{p_L})\\&<\dim_H\mathbf{V}_{p_L}=\dim_H\mathbf{U}_{p_L}\le\dim_H\mathbf{U}_q.\end{aligned}$$

Thus, (ii) ⇒ (i).

We next prove (i) ⇒ (iii). Take $q\in\mathscr{B}^L$. Then $q>q_{KL}$ by (1.7), so Lemma 2.5 yields $\dim_H\mathcal{U}_q>0$. Thus, it remains to prove that $\lim_{p\nearrow q}\dim_H(\mathscr{B}\cap(p,q))=\dim_H\mathcal{U}_q$. Since $\mathscr{B}\subset\mathscr{U}$, by Lemma 3.12 it suffices to prove

$$\lim_{p\nearrow q}\dim_H(\mathscr{B}\cap(p,q))\ge\dim_H\mathcal{U}_q.\tag{3.12}$$

Fix $\varepsilon>0$. By Lemma 2.5 the function $q\mapsto\dim_H\mathcal{U}_q$ is continuous, so there exists $p_0:=p_0(\varepsilon)\in(1,q)$ such that

$$\dim_H\mathcal{U}_p\ge\dim_H\mathcal{U}_q-\varepsilon\qquad\text{for all}\quad p\in(p_0,q).\tag{3.13}$$

Since $q\in\mathscr{B}^L$, by the topological structure of the bifurcation set $\mathscr{B}^L$ there exists a sequence of entropy plateaus $\{[p_L(n),p_R(n)]\}$ such that $p_L(n)\nearrow q$ as $n\to\infty$. Fix $p\in(p_0,q)$. Then there exists a large integer N such that $p_L(N)\in(p,q)$. Observe that $p_L(N)\in\mathscr{B}^L\subset\overline{\mathscr{B}}$ and the difference $\overline{\mathscr{B}}\setminus\mathscr{B}$ is countable. By Lemma 3.11 there exists $\delta>0$ such that

$$(p_L(N)-\delta,p_L(N)+\delta)\subseteq(p,q),\tag{3.14}$$

and

$$\dim_H\left(\mathscr{B}\cap(p_L(N)-\delta,p_L(N)+\delta)\right)\ge\dim_H\mathcal{U}_{p_L(N)}-\varepsilon.\tag{3.15}$$

By (3.13), (3.14) and (3.15) it follows that

$$\begin{aligned}\dim_H(\mathscr{B}\cap(p,q)) &\geq \dim_H\big(\mathscr{B}\cap(p_L(N)-\delta,p_L(N)+\delta)\big)\\ &\geq \dim_H\mathcal{U}_{p_L(N)}-\varepsilon\geq\dim_H\mathcal{U}_q-2\varepsilon.\end{aligned}$$

Since this holds for all $p\in(p_0(\varepsilon),q)$, we obtain (3.12). This proves (i) $\Rightarrow$ (iii).

Note that (iii) $\Rightarrow$ (iv) follows directly from Lemma 3.12 since $\mathscr{B}\subset\mathscr{U}$.

It remains to prove (iv) $\Rightarrow$ (i). Let $q\in(1,M+1]\backslash\mathscr{B}^L$. By (1.7) it follows that $q\in(1,q_{KL}]$ or $q\in(p_L,p_R]$ for some entropy plateau $[p_L,p_R]\subset(q_{KL},M+1]$. If $q\in(1,q_{KL}]$, then $\dim_H\mathcal{U}_q=0$. Now we consider $q\in(p_L,p_R]\subset(q_{KL},M+1]$. If $q\notin\overline{\mathscr{U}}$, then $\lim_{p\nearrow q}\dim_H(\mathscr{U}\cap(p,q))=0$. So let $q\in\overline{\mathscr{U}}\cap(p_L,p_R]$. If $q<p_R$, then Proposition 3.9(ii), Lemma 3.1(ii) and Lemma 2.5 give

$$\lim_{p\nearrow q}\dim_H(\overline{\mathscr{U}}\cap(p,q))<\frac{\log 2}{m\log q}\leq\frac{h(\mathbf{U}_{p_L})}{\log q}=\frac{h(\mathbf{U}_q)}{\log q}=\dim_H\mathcal{U}_q. \tag{3.16}$$

Similarly, if $q=p_R$, then Lemma 3.1(ii) holds with strict inequality, and we obtain the same end result as in (3.16), but with the first inequality replaced by "$\leq$" and the second inequality replaced by "$<$". This proves (iv) $\Rightarrow$ (i), and completes the proof of Theorem 1. $\square$

Proof of Theorem 1′. The proof of Theorem 1′ is, for the most part, the same as the proof of Theorem 1. Asssume $M=2k+1\geq 3$. We need only check the following two facts for the entropy plateau $[p_L,p_R]=[k+2,q_\star]$, where $q_\star=q_\star(M)$:

$$\dim_H(\mathbf{U}_{q_\star}\backslash\mathbf{U}_p)=\dim_H(\mathbf{U}_{q_\star})\qquad\text{for any } p\in(1,q_\star), \tag{3.17}$$

and

$$\lim_{p\nearrow q_\star}\dim_H(\overline{\mathscr{U}}\cap(p,q_\star))=\dim_H\mathcal{U}_{q_\star}. \tag{3.18}$$

Here (3.17) is clear for $p\in(1,k+2)$, since $\dim_H\mathbf{U}_p<\dim_H\mathbf{U}_{q_\star}$. For $p\in[k+2,q_\star)$, (3.17) follows from Proposition 3.9(i) and the equality statement in Lemma 3.1(ii), noting that $[k+2,q_\star]$ is an entropy plateau of period $m=1$.

Similarly, (3.18) follows from the equality statements in Proposition 3.9(ii) and Lemma 3.1(ii). $\square$

Proof of Theorem 2. The proofs of (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are completely analogous to the proofs of the corresponding implications in Theorem 1.

Consider the implication (ii) $\Rightarrow$ (i). Suppose $q\in(1,M+1]\setminus\mathscr{B}^R$. By (1.7) we have $q\in(1,q_{KL})$ or $q\in[p_L,p_R)$ for some entropy plateau $[p_L,p_R]\subset(q_{KL},M+1]$. A similar argument as in the proof of Theorem 1 shows that either $\dim_H\mathbf{U}_q=0$ for $q\in(1,q_{KL})$, or $\dim_H(\mathbf{U}_r\setminus\mathbf{U}_q)<\dim_H\mathbf{U}_r$ for any $r\in(q,p_R)$. This proves (ii) $\Rightarrow$ (i).

Next, consider the implication (i) $\Rightarrow$ (iii). Take $q\in\mathscr{B}^R$. Then $q\geq q_{KL}$. If $q\neq q_{KL}$, then by Lemma 2.5 we have $\dim_H\mathcal{U}_q>0$. Since $q\in\mathscr{B}^R$, there exists a sequence of entropy plateaus $\{[\tilde{p}_L(n),\tilde{p}_R(n)]\}$ such that $\tilde{p}_L(n)\searrow q$ as $n\to\infty$. Using the continuity of the function $q\mapsto\dim_H\mathcal{U}_q$ and Lemma 3.11, we can show as in the proof of Theorem 1 that $\lim_{r\searrow q}\dim_H(\mathscr{B}\cap(q,r))=\dim_H\mathcal{U}_q$. This proves (i) $\Rightarrow$ (iii).

Finally, consider the implication (iv) $\Rightarrow$ (i). For $q\in(1,M+1]\backslash\mathscr{B}^R$ we have $q\in(1,q_{KL})$ or $q\in[p_L,p_R)$ for some entropy plateau $[p_L,p_R]\subset(q_{KL},M+1]$. By the same argument as in the proof of Theorem 1 we can prove that either $\dim_H \mathcal{U}_q = 0$ for $q < q_{KL}$, or $\lim_{r \searrow q} \dim_H(\overline{\mathscr{U}} \cap (q, r)) < \dim_H \mathcal{U}_q$ for $q \in [p_L, p_R)$. This establishes (iv) $\Rightarrow$ (i). $\square$

4. Hausdorff dimension of $\mathscr{U} \setminus \mathscr{B}$

In this section we will calculate the Hausdorff dimension of the difference set $\mathscr{U} \setminus \mathscr{B}$ and prove Theorem 4. First, we prove the following result for the local dimension of $\mathscr{U}$ inside any entropy plateau $[p_L, p_R]$.

Theorem 4.1. *Let $[p_L, p_R] \subset (q_{KL}, M+1)$ be an entropy plateau of period m. Then*

$$\dim_H(\mathscr{U} \cap [p_L, p_R]) = \frac{\log 2}{m \log p_R}.$$

Observe that the lower bound in Theorem 4.1, that is, the inequality

$$\dim_H(\mathscr{U} \cap [p_L, p_R]) \geq \frac{\log 2}{m \log p_R},$$

follows from Proposition 3.8 by setting $p = p_L$. The proof of the reverse inequality is more tedious, and we will give it in several steps.

Observe that $\inf \mathscr{U} = q_{KL}$, and any entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$ satisfies $\alpha(q_{KL}) \prec \alpha(p_L) \prec \alpha(M+1)$. In the following we fix an arbitrary entropy plateau $[p_L, p_R] \subset (q_{KL}, M+1]$ of period m such that $\alpha(p_L) = (a_1 \dots a_m)^\infty$ and $\alpha(p_R) = a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^\infty$. Recall the definition of the generalized Thue-Morse sequence $(\lambda_i) = (\lambda_i(M))$ from (2.2), which has the property that $\alpha(q_{KL}) = (\lambda_i)$. If $M = 1$, then

$$1101 \ldots = \lambda_1 \lambda_2 \ldots \prec (a_1 \dots a_m)^\infty \prec 1^\infty,$$

so $m \geq 3$. Similarly, if $M = 2$, we have

$$210201 \ldots = \lambda_1 \lambda_2 \ldots \prec (a_1 \dots a_m)^\infty \prec 2^\infty,$$

so $m \geq 2$. But when $M \geq 3$, it is possible to have $m = 1$. In short, we have the inequality

$$M + m \geq 4. \tag{4.1}$$

We divide the interval (p_L, p_R) into a sequence of smaller subintervals by defining a sequence of bases $\{q_n\}_{n=1}^\infty$ in (p_L, p_R). Let $\hat{q} = \min(\overline{\mathscr{U}} \cap (p_L, p_R))$, and for $n \geq 1$ let $q_n \in (p_L, p_R)$ be defined by

$$\alpha(q_n) = \left(a_1 \dots a_m^+ (\overline{a_1 \dots a_m})^{n-1} \overline{a_1 \dots a_m^+} \right)^\infty. \tag{4.2}$$

Note that $\hat{q}$ is a de Vries-Komornik number which has a Thue-Morse type quasi-greedy expansion

$$\alpha(\hat{q}) = a_1 \dots a_m^+ \ \overline{a_1 \dots a_m} \ \overline{a_1 \dots a_m^+} \, a_1 \dots a_m^+ \cdots \tag{4.3}$$

That is, $\alpha(\hat{q})$ is the sequence $\alpha_1 \alpha_2 \ldots$ given by $\alpha_1 \ldots \alpha_m = a_1 \dots a_m^+$, and recursively, for $i \geq 0$, $\alpha_{2^i m+1} \ldots \alpha_{2^{i+1} m} = \overline{\alpha_1 \ldots \alpha_{2^i m}}^+$. Then $\alpha(q_1) \prec \alpha(\hat{q}) \prec \alpha(q_2) \prec \cdots \prec \alpha(p_R)$, and $\alpha(q_n) \nearrow \alpha(p_R)$ as $n \to \infty$. By Lemma 2.1 it follows that

$$q_1 < \hat{q} < q_2 < q_3 < \cdots < p_R, \quad \text{and} \quad q_n \nearrow p_R \quad \text{as } n \to \infty.$$

We will bound the dimension of $\overline{\mathscr{U}} \cap [q_n, q_{n+1}]$ for each $n \in \mathbb{N}$. In preparation for this, we first determine the entropy of the subshift $X_{\mathscr{G}, N}$ defined in (3.11).

Lemma 4.2. *The topological entropy of* $X_{\mathscr{G},N}$ *is given by*

$$h(X_{\mathscr{G},N}) = \frac{\log \varphi_N}{m},$$

where φ_N *is the unique root in* $(1,2)$ *of* $1 + x + \cdots + x^{N-1} = x^N$.

Proof. The m-block map Φ defined by

$$\Phi(a_1 \dots a_m^+) = \Phi(\overline{a_1 \dots a_m^+}) = 1, \quad \Phi(a_1 \dots a_m) = \Phi(\overline{a_1 \dots a_m}) = 0$$

induces a two-to-one map ϕ from $X_{\mathscr{G},N}$ into $\{0,1\}^{\mathbb{N}}$. Recall that $X_{\mathscr{G},N}$ is the subset of $X_{\mathscr{G}}$ with forbidden blocks $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^N$ and $\overline{a_1 \dots a_m^+}(a_1 \dots a_m)^N$. Then $Y := \phi(X_{\mathscr{G},N})$ is the subshift of finite type in $\{0,1\}^{\mathbb{N}}$ of sequences avoiding the word 10^N. It is well known that $h(Y) = \log \varphi_N$ (cf. [22, Exercise 4.3.7]); hence, $h(X_{\mathscr{G},N}) = (\log \varphi_N)/m$. $\square$

Lemma 4.3. *For any* $n \geq 1$*, we have*

$$\dim_H(\overline{\mathscr{U}} \cap [q_n, q_{n+1}]) \leq \frac{\log \varphi_{n+1}}{m \log q_n}.$$

Proof. Fix $n \geq 1$. Note by (4.2) and (4.3) that for any $p \in \overline{\mathscr{U}} \cap [q_n, q_{n+1}]$, $\alpha(p)$ begins with $a_1 \dots a_m^+$, and $\alpha(p) \in \mathbf{V}_p \subseteq \mathbf{V}_{q_{n+1}}$. By a similar argument as in the proof of Proposition 3.9 it follows that $\alpha(p) \in X_{\mathscr{G}}$, and $\alpha(p)$ does not contain the subwords $a_1 \dots a_m^+(\overline{a_1 \dots a_m})^{n+1}$ and $\overline{a_1 \dots a_m^+}(a_1 \dots a_m)^{n+1}$, where $X_{\mathscr{G}}$ is the sofic subshift represented by the labeled graph $\mathscr{G} = (G, \mathscr{L})$ in Figure 1. In other words, $\alpha(p) \in X_{\mathscr{G},n+1}$. By Lemma 4.2 this implies

$$h(\mathbf{U}_{[q_n, q_{n+1}]}) \leq h(X_{\mathscr{G},n+1}) = \frac{\log \varphi_{n+1}}{m}. \tag{4.4}$$

Applying Lemma 3.7 with $I = [q_n, q_{n+1}]$ completes the proof. $\square$

The next step is to prove that the upper bound in Lemma 4.3 is smaller than $\log 2/(m \log p_R)$. This requires us to show that q_n is sufficiently close to p_R, which we accomplish by applying a *transversality* technique (see [25, 27]) to certain polynomials associated with q_n and p_R. For this we need the estimation of the Komornik-Loreti constants $q_{KL}(M)$. Recall from Example 2.4 that

$$q_{KL}(1) \approx 1.78723, \qquad q_{KL}(2) \approx 2.53595 \qquad \text{and} \qquad q_{KL}(3) \approx 2.91002.$$

We emphasize that $q_{KL}(M) \geq (M+2)/2$ for each $M \geq 1$, and the map $M \mapsto q_{KL}(M)$ is strictly increasing.

Lemma 4.4. *Let* $[p_L, p_R] \subset (q_{KL}, M+1]$ *be an entropy plateau such that* $\alpha(p_L) = (a_1 \dots a_m)^\infty$ *and* $\alpha(p_R) = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^\infty$. *Define the polynomials*

$$\begin{aligned} P(x) := {} & a_1 x + \cdots + a_{m-1}x^{m-1} + (1 + a_m^+)x^m \\ & + (\overline{a_1} - a_1)x^{m+1} + \cdots + (\overline{a_{m-1}} - a_{m-1})x^{2m-1} + (\overline{a_m} - a_m^+)x^{2m} - 1 \end{aligned} \tag{4.5}$$

and

$$Q_n(x) := P(x) - x^{m(n+1)}(\overline{a_1}x + \cdots + \overline{a_m}x^m), \qquad n \in \mathbb{N}. \tag{4.6}$$

(i) *The number* $1/p_R$ *is the unique zero of* P *in* $[1/(M+1), 1]$.
(ii) *The number* $1/q_n$ *is the unique zero of* Q_n *in* $[1/(M+1), 1]$*, for all* $n \in \mathbb{N}$.
(iii) $P'(x) \geq a_1$ *for all* $x \in [1/p_R, 1/p_L]$.

Proof. (i) Since $\alpha(p_R) = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^\infty$, it follows that $1/p_R$ is the unique solution in $[1/(M+1), 1]$ of

$$\begin{aligned} 1 &= a_1 x + a_2 x^2 + \dots + a_{m-1}x^{m-1} + a_m^+ x^m \\ &\qquad + x^m(\overline{a_1}x + \dots + \overline{a_m}x^m) + x^{2m}(\overline{a_1}x + \dots + \overline{a_m}x^m) + \dots \\ &= a_1 x + a_2 x^2 + \dots + a_{m-1}x^{m-1} + a_m^+ x^m + \frac{x^m(\overline{a_1}x + \dots + \overline{a_m}x^m)}{1 - x^m}. \end{aligned}$$

Expanding and rearranging terms we see that $1/p_R$ is the unique zero in $[1/(M+1), 1]$ of P.

(ii) By (4.2), it follows that the greedy expansion of 1 in base q_n is

$$\beta(q_n) = a_1 \dots a_m^+(\overline{a_1 \dots a_m})^n 0^\infty,$$

so $1/q_n$ is the unique root in $[1/(M+1), 1]$ of the equation

$$1 = a_1 x + \dots + a_{m-1}x^{m-1} + a_m^+ x^m + \frac{x^m(\overline{a_1}x + \dots + \overline{a_m}x^m)(1 - x^{mn})}{1 - x^m}.$$

Expanding and rearranging gives that $1/q_n$ is the unique zero in $[1/(M+1), 1]$ of Q_n.

(iii) Consider first the case $m = 1$. In this case, the polynomial P should be interpreted as

$$P(x) = (1 + a_1^+)x + (\overline{a_1} - a_1^+)x^2 - 1.$$

Now observe that, since $\alpha(p_L) = a_1^\infty$, it follows that $p_L = a_1 + 1$. So for $x \in [1/p_R, 1/p_L]$, we have in particular that $x \le 1/(a_1 + 1)$. Therefore, since $a_1 \ge (M+1)/2$,

$$\begin{aligned} P'(x) &= 1 + a_1^+ + 2(\overline{a_1} - a_1^+)x = 2 + a_1 + 2(M - 2a_1 - 1)x \\ &\ge 2 + a_1 + \frac{2(M - 2a_1 - 1)}{a_1 + 1} = a_1 + \frac{2(M+1)}{a_1 + 1} - 2 \\ &\ge a_1, \end{aligned}$$

where the last inequality follows since $a_1 \le M$.

Assume next that $m \ge 2$. Here we use that the greedy expansion of 1 in base p_L is $\beta(p_L) = a_1 \dots a_m^+ 0^\infty$, so

$$a_1 p_L^{-1} + \dots + a_{m-1} p_L^{-(m-1)} + a_m^+ p_L^{-m} = 1. \tag{4.7}$$

Hence,

$$a_1 x + \dots + a_{m-1}x^{m-1} + a_m^+ x^m \le 1 \qquad \text{for } 0 \le x \le 1/p_L. \tag{4.8}$$

Now for $0 \leq x \leq 1/p_L$, writing $\overline{a_k} - a_k$ as $M - 2a_k$, we have

$$\begin{aligned} P'(x) &= a_1 + \sum_{k=2}^{m-1} k a_k x^{k-1} + m(1 + a_m^+)x^{m-1} \\ &\qquad + \sum_{k=1}^{m-1} (m+k)(M - 2a_k)x^{m+k-1} + 2m(M - 2a_m^+ + 1)x^{2m-1} \\ &\geq a_1 + \sum_{k=2}^{m-1} \left\{ k a_k x^{k-1} + \big(M(m+k) - 2(k-1)a_k\big)x^{m+k-1} \right\} \\ &\qquad + \left\{ m(1 + a_m^+) - 2(m+1) \right\} x^{m-1} + Mx^m\{m + 1 + 2mx^{m-1}\} \\ &\qquad + 2\{m - (m-1)a_m^+\}x^{2m-1}, \end{aligned}$$

where the inequality follows by multiplying both sides of (4.8) by $m + 1$ and some algebraic manipulation. Here, the terms in the summation over $k = 2, \ldots, m - 1$ are positive, since $a_k \leq M$ and so $M(m+k) - 2(k-1)a_k \geq M(m-k+2) > 0$. The sum of the remaining terms is increasing in a_m^+, since the coefficient of a_m^+ is

$$mx^{m-1} - 2(m-1)x^{2m-1} \geq mx^{m-1}(1 - 2x^m) \geq 0,$$

using that $m \geq 2$ and $x \leq 1/p_L \leq 1/q_{KL}(1) \leq 0.6$, which holds for all $M \geq 1$. Since $a_m^+ \geq 1$, it follows that

$$\begin{aligned} P'(x) &\geq a_1 - 2x^{m-1} + Mx^m\{m + 1 + 2mx^{m-1}\} + 2x^{2m-1} \\ &\geq a_1 - 2x^{m-1} + M(m+1)x^m = a_1 + x^{m-1}\{M(m+1)x - 2\}. \end{aligned}$$

At this point, we need that $x \geq 1/p_R \geq 1/(M+1)$. When $M \geq 2$, this implies

$$M(m+1)x - 2 \geq 3Mx - 2 \geq \frac{3M}{M+1} - 2 = \frac{M-2}{M+1} \geq 0,$$

recalling our assumption that $m \geq 2$. When $M = 1$, we have $m \geq 3$ by (4.1), and so $M(m+1)x - 2 \geq 4x - 2 \geq 0$, since $x \geq 1/2$. In both cases, it follows that $P'(x) \geq a_1$. $\square$

The following elementary lemma (an easy consequence of the mean value theorem) is the key to the proof of the next inequality, in Lemma 4.6 below.

Lemma 4.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function which has a zero x_0, and let $\gamma > 0$, $\delta > 0$. Suppose $|f'(x)| \geq \gamma$ for all $x \in (x_0 - \delta, x_0 + \delta)$. If g is a continuous function such that*

$$|g(x) - f(x)| \leq \gamma\delta \quad \text{for all} \quad x \in (x_0 - \delta, x_0 + \delta),$$

then g has at least one zero in $[x_0 - \delta, x_0 + \delta]$.

Lemma 4.6. *For each $n \geq 1$,*

$$\frac{\log \varphi_{n+1}}{\log 2} < \frac{\log q_n}{\log p_R}.$$

Proof. Set $\mu_n := 1/q_n$ for $n \geq 1$, and set $\mu^* := 1/p_R$. Then $\mu_n > \mu^*$ for all $n \geq 1$. We will use Lemma 4.5 to show that μ_n is sufficiently close to μ^*.

By Lemma 4.4, μ^* is the unique zero in $[1/(M+1), 1]$ of the polynomial $P(x)$ from (4.5), and μ_n is the unique zero in $[1/(M+1), 1]$ of the polynomial $Q_n(x)$ from (4.6). Moreover,

$$P'(x) \geq a_1 \geq \frac{M+1}{2} \qquad \text{for all} \quad \mu^* \leq x \leq 1/p_L. \tag{4.9}$$

In order to estimate the difference $P(x) - Q_n(x)$, we show first that

$$\overline{a_1}x + \cdots + \overline{a_m}x^m < 1 \qquad \text{for all} \quad 0 \leq x \leq 1/p_L. \tag{4.10}$$

Observe that

$$\overline{a_1}x + \cdots + \overline{a_m}x^m = \frac{Mx(1-x^m)}{1-x} - (a_1 x + \cdots + a_m x^m).$$

Hence, recalling (4.7), we have for $0 \leq x \leq 1/p_L$,

$$\begin{aligned} \overline{a_1}x + \cdots + \overline{a_m}x^m \leq \overline{a_1}p_L^{-1} + \cdots + \overline{a_m}p_L^{-m} &= \frac{M(1-p_L^{-m})}{p_L - 1} - (1 - p_L^{-m}) \\ &= (1-p_L^{-m})\left(\frac{M}{p_L - 1} - 1\right) \\ &\leq 1 - p_L^{-m} < 1, \end{aligned}$$

where the next-to-last inequality follows since $p_L \geq q_{KL}(M) \geq (M+2)/2$. This proves (4.10).

Recall our convention that logarithms are taken with respect to base $M+1$. Below, we write $\ln x$ for the natural logarithm of x. Suppose we can show, for some number $\delta_n > 0$, that

$$\mu_n - \mu^* \leq \delta_n. \tag{4.11}$$

Using the inequality $\ln(1+x) \leq x$ for any $x > -1$, it then follows that

$$\ln \mu_n - \ln \mu^* = \ln\left(1 + \frac{\mu_n - \mu^*}{\mu^*}\right) \leq \frac{\mu_n - \mu^*}{\mu^*} \leq \frac{\delta_n}{\mu^*} = \delta_n p_R,$$

and so

$$\frac{\ln q_n}{\ln p_R} = 1 + \frac{\ln q_n - \ln p_R}{\ln p_R} = 1 - \frac{\ln \mu_n - \ln \mu^*}{\ln p_R} \geq 1 - \frac{\delta_n p_R}{\ln p_R}. \tag{4.12}$$

Next, observe that $\varphi_{n+1}^{n+1}(1-\varphi_{n+1}) = 1 - \varphi_{n+1}^{n+1}$, whence $\varphi_{n+1}^{n+1}(2-\varphi_{n+1}) = 1$. It follows that

$$2 - \varphi_{n+1} = \varphi_{n+1}^{-(n+1)} > 2^{-(n+1)},$$

and hence,

$$\ln \varphi_{n+1} - \ln 2 = \ln\left(1 + \frac{\varphi_{n+1} - 2}{2}\right) \leq \frac{\varphi_{n+1} - 2}{2} < -\frac{1}{2^{n+2}}.$$

This gives

$$\frac{\ln \varphi_{n+1}}{\ln 2} < 1 - \frac{1}{2^{n+2}\ln 2}. \tag{4.13}$$

In view of (4.12) and (4.13) and the change-of-base formula $\ln x = \ln(M+1) \cdot \log x$, it then remains to show that

$$\frac{\delta_n p_R}{\log p_R} < \frac{1}{2^{n+2}\log 2} \qquad \text{for each} \quad n \geq 1. \tag{4.14}$$

By (4.10) and (4.6) we have

$$0 \leq P(x) - Q_n(x) \leq p_L^{-m(n+1)} \leq q_{KL}^{-m(n+1)}, \qquad x \in [0, 1/p_L].$$

Since we know that $\mu_n \in [\mu^*, 1/p_L]$ and moreover, μ_n is the unique root of Q_n in $[1/(M+1), 1]$, it follows from (4.9) and Lemma 4.5 (with $\gamma = (M+1)/2$) that (4.11) holds with

$$\delta_n = \frac{2}{M+1} q_{KL}^{-m(n+1)}.$$

(i) Assume first that $m \geq 2$. Then we can estimate

$$\begin{aligned}\left(2^{n+2}\log 2\right)\frac{\delta_n p_R}{\log p_R} &\leq 2\log 2\cdot\frac{2}{M+1}\cdot\frac{M+1}{\log q_{KL}}\left(\frac{2}{q_{KL}^m}\right)^{n+1}\\ &= \frac{4\log 2}{\log q_{KL}}\left(\frac{2}{q_{KL}^m}\right)^{n+1},\end{aligned} \tag{4.15}$$

where the inequality follows since $p_R \leq M+1$ and $\log p_R \geq \log q_{KL}$. Now observe that $\log 2/\log q_{KL} \leq \log 2/\log q_{KL}(1) \leq \log 2/\log 1.787 < 1.2$. Furthermore, if $M \geq 2$ then $2/q_{KL}^m \leq 2/(q_{KL}(2))^2 \leq 2/(2.5)^2 < 0.33$; and if $M = 1$, then $m \geq 3$ by (4.1) and so $2/q_{KL}^m \leq 2/(1.787)^3 < 0.36$. In both cases, it follows that

$$\left(2^{n+2}\log 2\right)\frac{\delta_n p_R}{\log p_R} \leq (4.8)(0.36)^{n+1} \leq (4.8)(0.36)^2 < 1,$$

for all $n \geq 1$. Thus, we have proved (4.14) in the case $m \geq 2$.

(ii) Assume next that $m = 1$, so $M \geq 3$ by (4.1). In this case, the bound in (4.15) is just too large for $n = 1$. But we can use the easily verified fact that the function $x \mapsto x/\log x$ is increasing on $[e, \infty)$ and $p_R \geq q_{KL}(3) \geq 2.9 > e$, to replace the factor $\log q_{KL}$ in (4.15) with the sharper $\log(M+1)$. Since $\log(M+1) \geq \log 4 = 2\log 2$, this gives the estimate

$$\begin{aligned}\left(2^{n+2}\log 2\right)\frac{\delta_n p_R}{\log p_R} &\leq 2\log 2\cdot\frac{2}{M+1}\cdot\frac{M+1}{\log(M+1)}\left(\frac{2}{q_{KL}}\right)^{n+1}\\ &\leq 2\left(\frac{2}{q_{KL}}\right)^2 \leq 2\left(\frac{2}{2.9}\right)^2 \approx .9512 < 1.\end{aligned}$$

In both cases above, we have found a δ_n such that (4.11) holds, and proved (4.14). Therefore, the proof of the Lemma is complete. □

Proof of the upper bound in Theorem 4.1. By Lemmas 4.3 and 4.6, we have

$$\dim_H(\overline{\mathscr{U}} \cap [q_n, q_{n+1}]) < \frac{\log 2}{m\log p_R} \qquad \text{for each } n \geq 1.$$

Since $\overline{\mathscr{U}} \cap (p_L, p_R) \subseteq \bigcup_{n=1}^{\infty}(\overline{\mathscr{U}} \cap [q_n, q_{n+1}])$, it follows from the countable stability of Hausdorff dimension that

$$\dim_H(\overline{\mathscr{U}} \cap [p_L, p_R]) \leq \sup_{n\geq 1}\dim_H(\overline{\mathscr{U}} \cap [q_n, q_{n+1}]) \leq \frac{\log 2}{m\log p_R},$$

establishing the upper bound. □

Remark 4.7. The above method of proof shows that in fact, for any $\varepsilon > 0$ we have $\dim_H(\overline{\mathscr{U}} \cap [p_L, p_R - \varepsilon]) < \dim_H(\overline{\mathscr{U}} \cap [p_L, p_R])$ and therefore,

$$\dim_H(\overline{\mathscr{U}} \cap [p_R - \varepsilon, p_R]) = \dim_H(\overline{\mathscr{U}} \cap [p_L, p_R]) = \frac{\log 2}{m\log p_R}$$

for any $\varepsilon > 0$. Thus, one could say that within an entropy interval $[p_L, p_R]$, $\overline{\mathscr{U}}$ is "thickest" near the right endpoint p_R.

Proof of Theorem 4. Since $\mathscr{U} \setminus \mathscr{B} \subset [q_{KL}(M), M+1]$, by (1.6) we have $\mathscr{U} \setminus \mathscr{B} = \{q_{KL}\} \cup \bigcup(\mathscr{U} \cap [p_L, p_R])$, where the union is pairwise disjoint and countable. Then

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \dim_H \bigcup_{[p_L,p_R]} (\mathscr{U} \cap [p_L, p_R]) = \sup_{[p_L,p_R]} \dim_H(\mathscr{U} \cap [p_L, p_R]). \tag{4.16}$$

Here the supremum is taken over all entropy plateaus $[p_L, p_R] \subset (q_{KL}(M), M+1]$.

Assume first that $M = 1$. Recall that for any entropy plateau $[p_L, p_R] \subseteq (q_{KL}(1), 2]$ with $\alpha(p_L) = (a_1 \dots a_m)^\infty$, it holds that $m \geq 3$. Furthermore, $m = 3$ if and only if $[p_L, p_R] = [\lambda_*, \lambda^*] \approx [1.83928, 1.87135]$, where $\alpha(\lambda_*) = (110)^\infty$ and $\alpha(\lambda^*) = 111(001)^\infty$. Observe that $q_{KL}(1) \approx 1.78723$. By a direct calculation one can verify that for any $m \geq 4$ we have

$$\frac{\log 2}{m \log p_R} < \frac{\log 2}{4 \log q_{KL}} < \frac{\log 2}{3 \log \lambda^*}. \tag{4.17}$$

Therefore, by (4.16), (4.17) and Theorem 4.1 it follows that

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \dim_H(\mathscr{U} \cap [\lambda_*, \lambda^*]) = \frac{\log 2}{3 \log \lambda^*} \approx 0.368699.$$

Finally, since $\alpha(\lambda^*) = 111(001)^\infty$, λ^* is the unique root in $(1, 2]$ of the equation

$$1 = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^3(x^3-1)},$$

or equivalently, $x^5 - x^4 - x^3 - 2x^2 + x + 1 = 0$.

Consider next the case $M = 2$. Then $m \geq 2$, with equality if and only if $[p_L, p_R] = [\gamma_*, \gamma^*] \approx [2.73205, 2.77462]$, where $\alpha(\gamma_*) = (21)^\infty$ and $\alpha(\gamma^*) = 22(01)^\infty$. For any entropy plateau $[p_L, p_R]$ with period $m \geq 3$, we have $m \log p_R \geq 3 \log q_{KL}(2) \geq 3 \log 2.5 > 2 \log 3 > 2 \log \gamma^*$, so

$$\frac{\log 2}{m \log p_R} < \frac{\log 2}{2 \log \gamma^*}.$$

Hence, by (4.16) and Theorem 4.1,

$$\dim_H(\mathscr{U} \setminus \mathscr{B}) = \dim_H(\mathscr{U} \cap [\gamma_*, \gamma^*]) = \frac{\log 2}{2 \log \gamma^*} \approx 0.339607.$$

Furthermore, since $\alpha(\gamma^*) = 22(01)^\infty$, γ^* is the unique root in $(2, 3)$ of the equation

$$1 = \frac{2}{x} + \frac{2}{x^2} + \frac{1}{x^2(x^2-1)},$$

or equivalently, γ^* is the unique root in $(2, 3)$ of $x^4 - 2x^3 - 3x^2 + 2x + 1 = 0$.

Finally, let $M \geq 3$. The leftmost entropy plateau with period $m = 1$ is $[p_L, p_R]$, where

$$\begin{aligned}
M = 2k+1 \quad &\Rightarrow \quad \alpha(p_L) = (k+1)^\infty \quad \text{and} \quad \alpha(p_R) = (k+2)k^\infty, \\
M = 2k \quad &\Rightarrow \quad \alpha(p_L) = (k+1)^\infty \quad \text{and} \quad \alpha(p_R) = (k+2)(k-1)^\infty.
\end{aligned}$$

Note that for this entropy plateau, $p_R = q_\star(M)$, where $q_\star(M)$ was defined in (1.8). Now consider an arbitrary entropy plateau $[p_L, p_R]$ with period m. If $m = 1$, then $p_R \geq q_\star(M)$, so $m \log p_R \geq \log q_\star(M)$. And if $m \geq 2$, we have

$$\begin{aligned} m \log p_R \geq 2 \log q_{KL}(M) \geq 2 \log \left(\frac{M+2}{2} \right) &= \log(M^2 + 4M + 4) - \log 4 \\ &\geq \log(4M + 4) - \log 4 = \log(M + 1) > \log q_\star(M). \end{aligned}$$

In both cases, we obtain

$$\frac{\log 2}{m \log p_R} \leq \frac{\log 2}{\log q_\star(M)}.$$

Hence, by (4.16) and Theorem 4.1, $\dim_H(\mathscr{U} \setminus \mathscr{B}) = \log 2 / \log q_\star(M)$. This completes the proof. □

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