VECTOR GAMES WITH POTENTIAL FUNCTION

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Abstract. *In this paper many classes of multiobjective games with potential functions are studied. The notions of generalized, best-reply and Pareto potential games are introduced in a multicriteria set. Some properties and Pareto equilibria are investigated*

Key words: multiobjective games; Pareto equilibria; generalized potential; best reply potential; Pareto potential.

1 Introduction

In general potential games in the scalar case have in common an attractive feature: every maximizer of the potential function, a real valued function on the set of strategy profile is an equilibrium (NE for short) for the game. It is natural to ask if the same is valid for multiobjective games, also called vector games, with the suitable changes, for example by considering Pareto equilibria instead of Nash equilibria and defining suitable Best-reply.
The problem was partially investigated in [PPT, P]. In this paper we consider other classes of games.
In a potential game, the potential function is similar to a payoff function of one agent who chooses the strategies for all players.
Rosenthal in 1973 ([R]) introduced the class of congestion games which have an equilibrium in pure strtegie, if finite ones. Some years late, in 1996, Monderer and Shapley ([MS]), introduced potential games (exact, ordinal and generalized). They proved that the exact potential games have interesting relations with the games introduced by Rosenthal and all potential games have at least an equilibrium in pure strategies: the maximum of a a potential function corresponds to an equilibrium of the potential game.
In previous papers ([PPT, P]) were studied exact and ordinal potential games in the multicriteria setting. The goal of studying multicriteria games

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is to have more applications in the real life, infact the decision makers have not one but several objectives "to maximize" and often not comparable.
Many other classes of potential games were considered in literature as generalized, best-reply potential, Nash potential (see [V] and references in this). We study these classes in a multicriteria setting, we investigate about the finite improvement property (FIP for short), a cycle of best reply property and the relations between the equilibria of a potential game and those of the coordination game (where the payoff functions are equal to the potential function for each player). The paper is so organized: section 2 is a background about results, definitions and notations known; section 3 is about generalized potential games, section 4 is about best-reply potential games, section 5 is about Pareto potential games and we conclude studying the relations among these. Section 6 concludes with some suggestions for further research. Many examples illustrate the proven properties.

2 Background

Given a vector $x = (x_1, ..., x_n) \in \prod_{i=1}^n X_i$
we write $X_{-i} = \prod_{j \neq i} X_j$, $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i}$ and
for all $y_i \in X_i$ and $x_{-i} \in X_{-i}$ $(y_i, x_{-i}) = (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)$, $(x_i, x_{-i}) = x = (x_1, ..., x_n)$.
Given $x, y \in \mathbb{R}^n$ we consider the following inequalities on $\mathbb{R}^n$:
$x \geqq y \Leftrightarrow x_i \geq y_i \quad \forall i = 1, ..., n;$
$x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$;
$x > y \Leftrightarrow x_i > y_i \quad \forall i = 1, ..., n.$
Analogously we define $\leqq, \ \leq, \ <$.
We write $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x_i > 0 \ \forall i = 1, ..., n\}$ and
$\mathbb{R}^m_{+} = \{x \in \mathbb{R}^m : \ x_i \geq 0 \ \forall i = 1, ..., n\}$

We say that $U \subset \mathbb{R}^n$ is upper bounded (u.b. for short) if there exists $b \in \mathbb{R}^n$ such that $x \leq b \ \forall x \in U$.

For a function $F : V \subset \mathbb{R}^n \to \mathbb{R}^m$ a point $\hat{x} \in V$ is *strongly Pareto optimal* (sPE(F) for short) if there is no other feasible point x for which $F(x)$ is larger than $F(\hat{x})$ in at least one coordinate and not smaller in all other coordinates, i.e.
$\nexists x \in V$ s.t. $F(x) \geq F(\hat{x})$.
A feasible point $\hat{x} \in \mathbb{R}^m$ is *weakly Pareto-optimal* if there is no other feasible point x such that $F(x)$ is larger than $F(\hat{x})$ in each coordinate, i.e.

$\nexists\, x \in V$ s.t. $F(x) > F(\hat{x})$.

Definition 2.1 *A strategic multiobjective game is a tuple*

$$\Gamma = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N} \rangle$$

where N is the set of players, X_i is the strategy space for player $i \in N$, X is the cartesian product $\prod_{i\in N} X_i$ of the strategy spaces $(X_i)_{i\in N}$ and each player has $m(i)$ objectives, i.e. the utility function for player i is a function $u_i : X \to \mathbb{R}^{m(i)}$.

In general in vector games each player i can have $m(i)$ different objectives to "optimize"; the existence of a potential requires that each player has the same number of objectives: $m(i) = m$.
In previous papers exact potential games ([PPT]) and ordinal potential one ([P]) in the multicriteria case were studied, so we recall them because there are some relations with other potential games which we are going to study.

Definition 2.2 *The strategic form of an exact potential game is a tuple $\Gamma = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N} \rangle$, $u_i : X \to \mathbb{R}^m$ and there exist a map $P : X \to \mathbb{R}^m$, such that for all $i \in N$, x_i, $y_i \in X_i$, $x_{-i} \in X_{-i}$, it holds*

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

Definition 2.3 $\Gamma = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N} \rangle$ *with $u_i : X \to \mathbb{R}^m$ is called an ordinal potential game if there exists a map $P : X \to \mathbb{R}^m$ such that for all $i \in N$, x_i, $y_i \in X_i$, $x_{-i} \in X_{-i}$ it holds*

$$u_i^j(x_i, x_{-i}) > u_i^j(y_i, x_{-i}) \Leftrightarrow P^j(x_i, x_{-i}) > P^j(y_i, x_{-i})$$

for all $j = 1, \ldots, m$.

Shapley [S] gave a generalization of the classical definition of Nash equilibrium, the so called Pareto equilibrium (weak and strong Pareto equilibrium) for a game. This is a generalization of Nash equilibrium (NE for short) to a multicriteria setting. We will use these definition for our multicriteria games.

Definition 2.4 *Given a strategy profile $\hat{x} = (\hat{x}_i, \hat{x}_{-i}) \in X$, it is called*
a) a weak Pareto equilibrium for the multiobjective strategic game Γ if for all $i \in N$ $\nexists\, x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) > u_i(\hat{x}_i, \hat{x}_{-i})$.
b) a strong Pareto equilibrium for the game G if for all $i \in N$
$\nexists\, x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) \geq u_i(\hat{x}_i, \hat{x}_{-i})$.

The set of all strong (weak) Pareto equilibria of G will be denoted by $sPE(G)$ ($wPE(G)$). We will write $PE(G)$ when we consider indifferently the strong or weak Pareto equilibria to our goals. In other words:
given a game $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, $\hat{a} \in \prod_{i \in N} X_i$ is called
a1) a weak Pareto equilibrium if
$\forall i \in N$, $\hat{a}_i \in wPB(\hat{a}_{-i}, u_i)$ *where we define the weak Pareto best reply (wPB for short) in the following way:*
$wPB(\hat{a}_{-i}, u_i) = \{a_i \in A_i : u_i(b_i, \hat{a}_{-i}) \notin u_i(a_i, \hat{a}_{-i} + \mathbb{R}^m_{++}$ *and*
b1) a strong Pareto equilibrium if
$\forall i \in N$, $\hat{a}_i \in sPB(\hat{a}_{-i}, u_i)$ *where we define the strong Pareto best reply (sPB for short) in the following way:*
$sPB(\hat{a}_{-i}, u_i) = \{a_i \in A_i : u_i(b_i, \hat{a}_{-i}) \notin u_i(a_i, \hat{a}_{-i} + \mathbb{R}^m_{+}$

3 Generalized potential games

The notion of generalized potential games in the scalar case has been given in [MS] and it can be extended to a multiobjective setting.

Definition 3.1 *The strategic form of a generalized potential game is a tuple $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, $u_i : X \to \mathbb{R}^m$ and there exists a function $P : X \to \mathbb{R}^m$ such that for all $i \in N$, $y_i \in Y_{-i}$, $\forall x, z \in Y_i$, it holds*

$$u_i(y_{-i}, x) - u_i(y_{-i}, z) > 0 \Rightarrow P(y_{-i}, x) - P(y_{-i}, z) > 0,$$

The symbol $>$ have to be intended componentwise.
In the following we will $\mathcal{G}$ the set of generalized potential games.

Here an example of a generalized potential bicriteria game:

Example 3.1 Let us consider the following game

G_1:

	L	R
T	(0,1) (0,0)	(1,1) (0,2)
B	(0,0) (1,2)	(2,3) (2,2)

where $wPE(G_1) = \{(T, L), (B, L), (B, R)\}$. $sPE(G_1) = \{(B, R)\}$. A generalized potential is equal to

P :

	L	R
T	(0,0)	(0,2)
B	(1,−1)	(2,4)

Another generalized potential function is the following:

P_1 :

	L	R
T	$(0,0)$	$(0,2)$
B	$(-1,-2)$	$(2,3)$

We call G the set of all generalized potential game so $G_1 \in G$
Note that $sPE(P) = wPE(P) = \{(B,R)\} = sPE(P_1) = wPE(P_1)$.

Studying Γ^{P_1} and Γ^P which are the pure coordination game having the potential function as utility function for the players, we can note that $wPE(\Gamma^P) = wPE(\Gamma)$ and $wPE(\Gamma^{P_1}) \subset wPE(\Gamma)$. and this is not a case, in general it holds:

Proposition 3.1 *If Γ is a generalized potential finite game the following relations are valid:*
1) $wPE(\Gamma) \neq \emptyset$
2) $wPE(P) \subseteq wPE(\Gamma)$
3) $wPE(\Gamma^P) \subseteq wPE(\Gamma)$
The same relations are valid for sPE.

Proof We prove the result only for the weak Pareto equilibria, the proof of the strong is similar.
$\diamond$ Let $\hat{x} \in wPE(P)$ then for all $i \in N$
$\nexists x_i \in X_i$ s.t. $P(x_i, \hat{x}_{-i}) > P(\hat{x}_i, \hat{x}_{-i})$
and this inequality implies that
$\nexists x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) \geq u_i(\hat{x}_i, \hat{x}_{-i})$
so $\hat{x} \in sPE(\Gamma) \subset wPE(\Gamma)$ so we have proved the inclusion 2) and the relation in 1) being the games finite.
$\diamond$ let $\hat{x} \in wPE(\Gamma^P)$ so by definition $\nexists x_i \in X_i$ s.t. $P(x_i, \hat{x}_{-i}) > P(\hat{x}_i, \hat{x}_{-i})$
and this implies that $\nexists x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) \geq u_i(\hat{x}_i, \hat{x}_{-i})$
and so the 3). $\square$

Remark 3.1 *In the definition 3.1 of generalized potential game we have to note that:*
a) If $u_i^j(y_{-i}, x) - u_i^j(y_{-i}, z) = 0$ for some $j = 1, ..., m$ then nothing we can say about the corresponding relations on P;
b) If the relations about u_i are non confrontable then the corresponding relations about P are not confrontable and in the same sense (because the intuitive idea is that the strict preferences are preserved going from u_i to P.)

Theorem 3.1 *Let Γ be a game with n players and the strategy sets be intervals in $\mathbb{R}$. Let us suppose that the utility functions are continuously differentiable. If Γ is a generalized potential game then the following relation is valid:*

$$\frac{\partial u_i^k}{\partial x_i} > 0 \Rightarrow \frac{\partial P^k}{\partial x_i} > 0$$

$\forall k = 1, ..., m$ and $\forall i = 1, ..., n$.

Proof: Starting from the definition of a generalized potential game and fixing the objective k, dividing the two members with $x_i - y_i$ and passing to the limit for $x_i \to y_i$, we obtain the relations between the first order derivative of u_i and P. □

Definition 3.2 *A finite path $\ell = (x_1, ..., x_m)$ in the strategy space X is a finite sequence of elements $x_k \in X$ such that $\forall k$, the strategy combination x_k and x_{k+1} differs in the $i(k)$-th coordinate. It is called closed or cycle if $x_1 = x_m$. It is a simple cycle if it is closed and all strategy combinations are different except the initial and final point. A finite path $(x_1 ... x_m)$ is called a weak improvement cycle if*
$x_1 = x_m$
$u_{i(k)}(x_k) \leq u_{i(k)}(x_{k+1})$ for some $k \in 1, 2, ..., m$.
A multiobjective game has the finite improvement property, (FIP for short), if every improvement path is finite

Proposition 3.2 *Let Γ be a finite game. The following properties are equivalent:*
a) Γ has the FIP
b) Γ has no strict improvement cycle
c) Γ is a generalized potential game.

Proof The equivalence between a) an b) is obvious in fact if Γ had strict improvement cycles, then it can be run infinite times against the FIP.
For the equivalence between a) and c), the proof is as in [Mi] adapted to multicriteria case. □

Remark 3.2 *If a game Γ has ordinal potential then it has generalized potential. The converse is not true as the following example proves.*

Example 3.2 Let us consider the following example:

G_2:

	L		R	
T	$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$
B	$(0,1)$	$(0,1)$	$(2,2)$	$(1,0)$

An ordinal potential is the following

P:

	L	R
T	$(0,0)$	$(0,2)$
B	$(0,4)$	$(1,3)$

and a generalized potential (not ordinal) is for example

P_2

	L	R
T	$(0,0)$	$(1,2)$
B	$(0,4)$	$(2,3)$

Calling O the set of ordinal potential games, it turns out $O \subset G$, and $G_2 \in G \cap O$.

Example 3.3 The following game is a generalized potential but not an ordinal one:

G_3:

	L		R	
T	$(0,0)$	$(0,0)$	$(0,1)$	$(0,0)$
B	$(0,0)$	$(0,0)$	$(1,0)$	$(0,0)$

A generalized potential is:

P_3:

	L	R
T	$(0,0)$	$(-1,2)$
B	$(0,0)$	$(0,0)$

$G_3 \in G \setminus O$.

Example 3.4 An application: a multicriteria Cournot model.
Let us suppose that there are two companies I and II, which make a duopoly about a certain product. Both companies decide to advertise their product through illustrative papers and television spots. The problem is that the printing of these panels has a bad impact in the environment of the zone because of the panel-factory is high polluting. It is well known by the two

companies that if the first invests in advertisement and II does not, then I increases the value of its products of 5 and decreases the environment value of 3. In the same time the company I decreases its value products of 5 but that of environment increases of 1. If both the companies invest in advertisement, the company I will have an advantage of 1 and a loss of 2 in the environment impact and company II an advantage of 1 for sailing and a loss of 1 in the environment impact. If both companies do not invest in advertisement, they obtain 0 for sailing and 0 for pollution. If the company I invest in advertising and II does not, I will loss 4 for the sailing but it gains 1 for the environment and company II gains 4 for the sailed products but is losses 1 for the pollution. We can modelize this situation with the following game:

(X, Y, F_1, F_2), $F_i : X \times Y \to \mathbb{R}^2$ where $X = Y = \{P, NP\}$ are the strategy sets and P is the strategy to invest in advertisement and NP is the strategy to not invest.
$(F_1(P, NP), F_2(P, NP))((3, -3), (-3, 1))$,
$(F_1(P, P), F_2(P, P))((1, -2), (1, -1))$,
$(F_1(NP, P), F_2(NP, P))((-1, 1), (3, -1))$,
$(F_1(NP, NP), F_2(NP, NP))((3, 0), (3, 0))$,
The strategic form of the game is:

G_4:

	P	NP
P	$(1, -2)$ $(1, -1)$	$(3, -3)$ $(-3, 1)$
NP	$(-1, 1)$ $(3, -1)$	$(3, 0)$ $(3, 0)$

This is a generalized potential game with two criteria, a generalized potential is the following:

P_4 :

	P	NP
P	$(1, -1)$	$(0, 2)$
NP	$(0, 2)$	$(1, 3)$

see [FP] to know more about this model in the exact potential case.

4 Best-reply potential games

Definition 4.1 $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *with* $u_i : X \to \mathbb{R}^m$ *is called a weak Best-Reply potential game (wBRP game for short) if there exists a* *map $P : X \to \mathbb{R}^m$ such that for all $i \in N$, $\hat{a_{-i}} \in X_{-i}$ it holds*

$$wPB(\hat{a_{-i}}, u_i) = wPB(\hat{a_{-i}}, P)$$

The function P is called weak Pareto Best-Reply potential of G.
In a similar way we can define the strong Pareto potential game.

The relations have meaning componentwise.
Intuitively a game $\Gamma = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a Pareto best reply potential game if there is a a pure coordination game $\Gamma^P = \langle N, (X_i)_{i\in N}, P\rangle$ where the payoff of each player is given by function P such that the best reply correspondance of each player i in Γ coincides with his (her) best response correspondence in the game where the payoff functions are the potential one for both players.
In the following we will write BR for the set of Best-Reply potential games.

Example 4.1 Let us see some examples:

G_5:

	C	D
A	$(2,1)$ $(2,0)$	$(0,0)$ $(0,2)$
B	$(0,0)$ $(1,1)$	$(1,0)$ $(0,2)$

A Pareto Best-Reply potential is:

P_5

	C	D
A	$(3,0)$	$(0,2)$
B	$(2,-1)$	$(1,2)$

It turns out $wPE(G_5) = \{(A,C);(A,D);(B,D)\} = wPE(G_5^P)$; $sPE(G) = \{(A,C);(B,D)\}$; and analougously for the strong equilibria.
This games is a BR potential game and an ordinal game too, for short we will write $G_5 \in BR\bigcap O$.
An ordinal potential is:

P

	C	D
A	$(2;0)$	$(0;2)$
B	$(1,5;-1)$	$(1;2)$

Example 4.2 The following is a BR-potential game but it is not an ordinal potential one:

G_6:

	D		E	
A	(2, 2)	(2, 2)	(0, 0)	(0, 0)
B	(0, 0)	(1, 1)	(1, 1)	(0, 0)
C	(1, 1)	(0, 0)	(0, 0)	(1, 1)

A best reply potential is:

P_6

	D	E
A	(2; 2)	(0; 0)
B	(1, 5; 1, 5)	(1; 1)
C	(−1; −1)	(0; 0)

So $G_5 \in BR \setminus O$

Example 4.3 The following is a generalized potential game but not a best reply one.

G_7:

	C		D	
A	(0, 0)	(0, 0)	(0, 1)	(0, 0)
B	(0, 1)	(0, 0)	(1, 0)	(0, 0)

$G_7 \in G \setminus BR$ in fact it is a generalized potential game because it has no strict improvement cycles but it is a Best-Reply one because it ha a best reply cycle.

The following proposition proves some relations between the equilibria of the game and of potential function:

Proposition 4.1 *If* Γ *is a best reply potential game and it is finite, the following relations are valid:*
1) $wPE(\Gamma) \neq \emptyset$
2) $wPE(P) \subseteq wPE(\Gamma)$
3) $wPE(\Gamma^P) = wPE(\Gamma)$ *We can prove the same result for strong Pareto equilibria.*

Proof. The proof is similar to that in Proposition 3.1.

Let us define a best reply cycle to illustrate some interesting properties of potential games.

Definition 4.2 *A finite path $\ell = (x_1, ..., x_m)$ in the strategy space X is a finite sequence of elements $x_k \in X$ such that $\forall k$, the strategy combination x_k and x_{k+1} differs in the $i(k) - th$ coordinate. It is called closed or cycle if $x_1 = x_m$. It is a simple cycle if it is closed and all strategy combinations are different except the initial and final point. A path $(x_1 ... x_m)$ is best reply compatible if the deviating player moves to a best response:*
$\forall k \ u_{i_{(k)}}(x^{k+1}) = a \in PBRu_{i_{(k)}}(y_i, x^k_{-i(k)})$
A finite path $(x_1, x_2, ..., x_m)$ is called a best reply cycle if it is a best reply compatible and $x_1 = x_m$ and for some $k \in \{1, ... m-1\}$ and $u_{i(k)}(x_k) \leq u_{i(k)}(x_{k+1})$

Intuitively a cycle of *weak Pareto best reply* (or strong Pareto best reply) is a cyclic path where in every side the final vertex is the weak (strong respectively) Pareto best reply of the deviating player to the other's strategy.

Theorem 4.1 *If Γ is a finite and a weak Pareto best reply potential game then Γ has no weak Pareto best reply cycles.*

Proof. Let P be a wBR potential for G and suppose that $(x_1, ..., x_m)$ is a wBR compatible. By the best reply compatibility $P(x_k) < P(x_{k+1})$ so it turns out that there is j such that $P^j(x_1) < .... < P^j(x_m) = P^j(x_1)$ and this is a contradiction so X does not contain best reply cycles. □ We note that the converse is true if on X we define a preorder. The potential games with a preorder on the strategy space will be an argument of a next paper, for now see [V] for this topic in the scalar case.

5 Pareto potential games

Definition 5.1 *Given a game $\Gamma = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $u_i : X \to \mathbb{R}^m$, it is called game with Pareto potential if there is $P : X \to \mathbb{R}^m$ such that*

$$wPE(\Gamma) = wPE(\Gamma^P)$$

and

$$sPE(\Gamma) = sPE(\Gamma^P)$$

Note that all the potential games seen until now, except for generalized potential ones, are Pareto potential games.

Example 5.1 Let us consider the following bicriteria game:

G_8:

	D		E		F	
A	$(1,3)$	$(1,3)$	$(0,2)$	$(0,2)$	$(0,2)$	$(0,2)$
B	$(0,2)$	$(0,2)$	$(1,3)$	$(0,2)$	$(0,2)$	$(1,3)$
C	$(0,2)$	$(0,2)$	$(0,2)$	$(1,3)$	$(1,3)$	$(0,2)$

$wPE(\Gamma) = sPE(\Gamma) = \{(A, D)\}$.

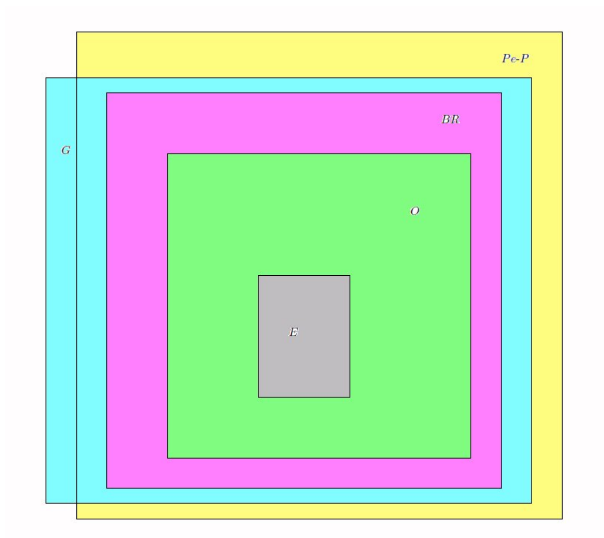
A Pareto potential is the following:

P_8:

	D	E	F
A	$(4,4)$	$(0,0)$	$(0,0)$
B	$(0,0)$	$(0,0)$	$(0,0)$
C	$(3,3)$	$(1,1)$	$(1,1)$

Note that G_8 is a Pareto potential game but no other type of potential one.

The following picture shows the studied inclusions among potential multi-objective games.



6 Conclusion and open problems

In this paper some classes of potential games have been studied: generalized, best reply and Pareto potential.
The importance of these games is that they have at least an equilibrium in pure strategy and it corresponds to a Pareto equilibrium of the potential function (generalized, best Pareto, Pareto ptential respectively). Furthermore these have interesting application in the real life: network models, environment problems, telecommunication models. See for application in the scalar case: [LCS], [MT], [MPT].
We have studied some properties of these classes but much more may be investigated, for example:
1) The study of approximate equilibria for infinite games (see [Mo], [PPT] for different concepts of approximate equilibria).
2) The study of the properties of equilibrium with improvement set as introduced in [PT], notion which captures contemporary the idea of exact and approximate equilibrium. 3) The FIP has relation with Pareto equilibria and approximate FIP ($aFIP$ for short) could be defined for a multicriteria setting and we could study the relations with approximate Pareto equilibria.
4) It could be interesting to investigate some well posedeness properties of the potential game G via the well posedness of the potential function ([MP], for exact potential games in the scalar case and [Mo] in the multicriteria setting).
5) Perhaps other classes of potential games can be defined and investigated.
6) Some applications to network models and telecommunication problems could be investigated via potential games ([LCS], [**?**]).
7) The potential games could be defined via a preorder on the strategy set and interesting properties could be found (see [V])
Some of these issues are work in progress.

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