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Classes of multiojectives games possessing Pareto equilibria

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A B S T R A C T

In this paper we study non cooperative games with potential as introduced by Monderer and Shapley in 1996. We extend the notions of weighted and ordinal potential games to a multicriteria setting and study their Pareto equilibria. The importance of these games is the existence of Pareto equilibria in pure strategies and in the finite case and the approximate equilibria for some classes of infinite potential games. Some applications are studied via potential games: a water resource problem, a voluntary contribution model, peering games for telecommunication models.

1. Introduction

Potential games were introduced by Monderer and Shapley [6] and then extensively studied by many authors, among them [4,8,9,11,19]. This class of games has many interesting properties such as the existence of equilibria in pure strategies for finite or upper bounded games. Furthermore it has been proved that exact potential games are isomorphic to congestion ones introduced by Rosenthal [13], and these have important applications.

The aim of this paper is to discuss the extension of two classes of potential games namely weighted potential and ordinal potential, to games with vector payoffs and to investigate their exact and approximate equilibria.

Since the utility functions have values in $\mathbb{R}^m$ also the potential functions are vector valued.

We will use indifferently the words multiobjective, multicriteria or vector games.

Frequently in real life applications many objectives have to be “optimized” and often these are not comparable (see [1,5,12]). The generalization of the concept of Nash equilibrium in this setting is neither easy nor unique. Here we consider the (weak and strong) Pareto equilibria following the idea given by Shapley in [16].

We study both weighted and ordinal potential games and we generalize to these classes some results obtained in [9]. The existence of a potential function helps us to understand how the game will be played. In general a game with n players has n utility functions (n could be very large), but if the game has a potential, some of the equilibria can be obtained by studying this function only.

The paper is organized as follows: in Section 2 we give the basic definitions and notations. In Section 3 we give some results about weighted potential games and in Section 4 about ordinal potential ones.

In Section 5 we study examples and applications to a water resource problem, to voluntary contribution games, to peering games for telecommunication models. In Section 6 we investigate approximate equilibria studied in [7,9,10,17,18] and generalized in [12]. In Section 7 we discuss other possible generalizations of ordinal potential games to the multicriteria setting. The last section is about the conclusions. Many examples complete the paper.

2. Background

Given a vector $x=(x_1,\dots,x_n)\in\prod_{i=1}^n X_i$ we write $X_{-i}=\prod_{j\neq i}X_j$, $x_{-i}=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in X_{-i}$ and for all $y_i\in X_i$ and $x_{-i}\in X_{-i}(y_i,x_{-i})=(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_n)$, $(x_i,x_{-i})=x=(x_1,\dots,x_n)$.

Given $x,y\in\mathbb{R}^n$ we consider the following inequalities on $\mathbb{R}^n$:

$x\geqq y \Leftrightarrow x_i\geq y_i\ \forall i=1,\dots,n;$

$x\geq y \Leftrightarrow x\geqq y$ and $x\neq y;$

$x>y \Leftrightarrow x_i>y_i\ \forall i=1,\dots,n.$

Analogously we define $\leqq$, $\leq$, $<$.

We say that $U\subset\mathbb{R}^n$ is upper bounded (u.b. for short) if there exists $b\in\mathbb{R}^n$ such that $x\leq b\,\forall x\in U$.

We write $\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x_i\geq 0\ \forall i\}$ and $\mathbb{R}^n_{++}=\{x\in\mathbb{R}^n : x_i>0\ \forall i\}$.

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Given $\epsilon > 0$, for a scalar function $f : X \to \mathbb{R}$, we define:

$$\text{argsup}^{\epsilon}_{x\in X} f(x) = \left\{ y \in X : \; f(y) \geq \sup_{x\in X} f(x) - \epsilon \right\}$$

For a function $F : V \subset \mathbb{R}^n \to \mathbb{R}^m$ a point $\hat{x} \in V$ is *strongly Pareto optimal* ($sPE(F)$ for short) if there is no other feasible point x for which $F(x)$ is larger than $F(\hat{x})$ in at least one coordinate and not smaller in all other coordinates, i.e. $\nexists x \in V$ s.t. $F(x) \geq F(\hat{x})$.

A feasible point $\hat{x} \in \mathbb{R}^m$ is *weakly Pareto-optimal* if there is no other feasible point x such that $F(x)$ is larger than $F(\hat{x})$ in each coordinate, i.e. $\nexists x \in V$ s.t. $F(x) > F(\hat{x})$.

A feasible point $\hat{x} \in \mathbb{R}^m$ is called approximate Pareto optimal for F, ($\epsilon PE(F)$ for short) if given $\epsilon \in \mathbb{R}^m_+$, $\nexists x \in V$ s.t. $F(x) > F(\hat{x}) + \epsilon$.

Definition 2.1. A strategic multiobjective game is a tuple $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where N is the set of players, X_i is the strategy space for player $i \in N$, X is the cartesian product $\prod_{i\in N} X_i$ of the strategy spaces $(X_i)_{i\in N}$ and each player has $m(i)$ objectives, i.e. the utility function for player i is a function $u_i : X \to \mathbb{R}^{m(i)}$.

In general in vector games each player i can have $m(i)$ different objectives to "optimize"; the existence of a potential requires that each player has the same number of objectives i.e. $m(i) = m$.

Given k scalar games $G_1, \ldots, G_k$ (with the same number of players and strategy spaces) with abuse of notation we will write:

$G = (G_1, \ldots, G_k)$ to indicate the multiobjective game with k components.

Definition 2.2. A strategy profile $\hat{x} = (\hat{x}_i, \hat{x}_{-i}) \in X$ is a weak Pareto equilibrium for the multiobjective strategic game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ if for all $i \in N$

$\nexists x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) > u_i(\hat{x}_i, \hat{x}_{-i})$.

It is a strong Pareto equilibrium for the game G if for all $i \in N$

$\nexists x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) \geq u_i(\hat{x}_i, \hat{x}_{-i})$.

The set of all strong (weak) Pareto equilibria of G will be denoted by $sPE(G)$ ($wPE(G)$). We will write $PE(G)$ when we consider indifferently the strong or weak Pareto equilibria.

A game G is a *pure coordination game* if $u_i = u$ for all $i \in N$.

For a given function P we will denote by G^P the pure coordination game where $u_i = P$ for all $i \in N$.

A game G is a *dummy game* if for all $i \in N$ and $x_{-i} \in X_{-i}$ it holds $u_i(x_i, x_{-i}) = u_i(y_i, x_{-i})$ for all $x_i, y_i \in X_i$.

3. Weighted potential games

The notion of weighted potential games in the scalar case has been given in [6] and it can be extended to multiobjective games.

Definition 3.1. The strategic form of a weighted potential game is a tuple $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, $u_i : X \to \mathbb{R}^m$ and there exist a map $P : X \to \mathbb{R}^m$ and vectors $w_i \in \mathbb{R}^m$, $w_i > 0$ for all i, such that for all $i \in N, x_i, y_i \in X_i$, $x_{-i} \in X_{-i}$, it holds

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = w_i \circ \{P(x_i, x_{-i}) - P(y_i, x_{-i})\},$$

where $\circ$ stands for the Hadamard (componentwise) product of vectors.

The function P is a w-potential of the game G.

It is easy to prove the following:

Proposition 3.1. *$G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a weighted potential game iff for all $k = 1, \ldots, m$ the scalar game*

$G_k = \langle N, (X_i)_{i\in N}, (\pi_k(u_i))_{i\in N}\rangle$ is a (scalar) weighted potential game. The function $\pi_k(P)$ is a weighted potential for G_k, (π_k is the projection on the kth component).

As for scalar games, we can prove that a weighted potential game can be written as the sum of a dummy and a weighted pure coordination game.

Theorem 3.1. *A game G is a weighted potential game iff for all $i \in N$ there exist $w_i \in \mathbb{R}^m$, $w_i > 0$ and $c, d_i : X \to \mathbb{R}^m$, such that $u_i = w_i \circ c + d_i$ $\forall i \in N$ and $D = \langle N, (X_i)_{i\in N}, (d_i)_{i\in N}\rangle$ is a dummy multicriteria game.*

Proof. Let G be a weighted potential game with w-potential P. Then if $d_i = u_i - w_i \circ P$, the game D is a dummy game and thus the thesis follows with $c = P$.

Conversely, if $u_i - w_i \circ c = d_i$ and $d_i(x_i, x_{-i}) - d_i(y_i, x_{-i}) = 0$ for any x_i, y_i and x_{-i} the function c is a weighted potential for G. □

Example of weighted potential bicriteria game:

Example 3.1. Let us consider the following game

G_1 :

	L	R
T	(4, 3) (4, 2)	(1, 0) (6, 0)
B	(8, 0) (1, 0)	(3, 3) (2, 2)

where $sPE(G) = wPE(G) = \{(T, L), (B, R)\}$. It is a weighted potential game, with w-potential equal to

P :

	L	R
T	(0, 1)	(2, 0)
B	(2, 0)	(3, 1)

and weights $w_1 = (2, 3)$ and $w_2 = (1, 2)$. Note that $PE(P) = \{(B, R)\}$

This game is the sum of the following weighted pure coordination game C and the dummy game D

C :

	(0, 3) (0, 2)	(4, 0) (2, 0)
	(4, 0) (2, 0)	(6, 3) (3, 2)

D :

	(4, 0) (4, 0)	(−3, 0) (4, 0)
	(4, 0) (−1, 0)	(−3, 0)(−1, 0)

We can note that $PE(G) = PE(G^P)$; this is not a case, in general it holds:

Proposition 3.2. *If G is a weighted potential finite game the following relations are valid:*

1) $PE(G) \neq \emptyset$

2) $PE(P) \subseteq PE(G) = PE(G^P)$

Proof. We prove the result only for the strong Pareto equilibria, leaving the weak Pareto equilibria to the reader.

◇ Let $\hat{x} \in sPE(P)$, then for all $i \in N$ $\nexists x_i \in X_i$ s.t. $P(x_i, \hat{x}_{-i}) \geq P(\hat{x}_i, \hat{x}_{-i})$; being $w_i > 0$ it is valid also:

$w_i \circ P(x_i, \hat{x}_{-i}) \geq w_i \circ P(\hat{x}_i, \hat{x}_{-i})$ and this inequality is equivalent to $\nexists x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) \geq u_i(\hat{x}_i, \hat{x}_{-i})$ so we have proved the first inclusion in 2). ◇ let $\hat{x} \in sPE(G)$ so by definition $\nexists x_i \in X_i$ s.t. $u_i(x_i, \hat{x}_{-i}) \geq u_i(\hat{x}_i, \hat{x}_{-i})$ if and only if $w_i \circ P(x_i, \hat{x}_{-i}) \geq w_i \circ P(\hat{x}_i, \hat{x}_{-i})$ if and only if $\nexists x_i \in X_i$ s.t. $P(x_i, \hat{x}_{-i}) \geq P(\hat{x}_i, \hat{x}_{-i})$ and so we have proven the equality in 2). □

The first inclusion in the previous proposition could be strict as the Example 3.1 proves.

Theorem 3.2. *Let G be a game with n players and let the strategy sets be intervals in $\mathbb{R}$. Let us suppose that the utility functions are twice continuously differentiable. It is a weighted potential game if and only if the following relation is valid:*

$$\frac{\partial^2 u_i^k}{\partial x_j \partial x_i} = \left(\frac{w_i^k}{w_j^k}\right) \frac{\partial^2 u_j^k}{\partial x_i \partial x_j}$$

$\forall k = 1, \ldots, m$ and $\forall i, j = 1, \ldots, n$.

Proof. Starting from the definition of a weighted potential game and fixing the objective k, we obtain

$\frac{\partial u_i^k}{\partial x_i} = w_i^k \frac{\partial P^k}{\partial x_i}$, using the smoothness of the involved functions, the above condition is equivalent to the existence of a weighted potential, this in turn is equivalent to the requirement that:

$$\left(\frac{\partial u_1^k / w_1^k}{\partial x_1^k}, \ldots \ldots, \frac{\partial u_n^k / w_n^k}{\partial x_n^k}\right)$$

is a conservative vector field, which under the hypotheses is equivalent to the thesis. □

Remark 3.1. If $\left(\frac{w_i^k}{w_j^k}\right) = 1\ \forall k = 1, \ldots, m$ and $\forall i, j = 1, \ldots, n$, then the weighted potential games are exact potential ones:

G_2 :

	L	R
T	(1, 2) (0, 1)	(1, 0) (0, 0)
B	(2, 2) (0, 0)	(2, 1) (0, 0)

4. Ordinal potential games

Definition 4.1. $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N}\rangle$ with $u_i : X \to \mathbb{R}^m$ is called an ordinal potential game if there exists a map $P : X \to \mathbb{R}^m$ such that for all $i \in N$, $x_i, y_i \in X_i$, $x_{-i} \in X_{-i}$ it holds

$$u_i^j(x_i, x_{-i}) > u_i^j(y_i, x_{-i}) \Leftrightarrow P^j(x_i, x_{-i}) > P^j(y_i, x_{-i})$$

for all $j = 1, \ldots, m$.

Intuitively a potential game with m objectives is ordinal iff each component game is an ordinal potential one.

Example 4.1. Let us consider a generalization to the bicriteria setting of the oligopoly game studied in [6]:

$G_3 = ([0, a]^n, \pi_i)$, $i = 1, \ldots, n$. We add the hypotheses that the utility functions are vectors and the inverse demand function F is in each component twice continuous differentiable on an interval. The strategy space is $X_i = [0, a]$, $a \in \mathbb{R}$. The utility functions are:

$\pi_i(q_1, q_2, \ldots, q_n) = F(Q)q_i - cq_i, \quad i = 1, \ldots n, \quad Q = q_1 + \ldots q_n,$ $\pi_i : \prod_i X_i \to \mathbb{R}^m$.

This is an ordinal multicriteria potential game with ordinal potential function $P : \prod_i X_i \to \mathbb{R}^m$, $P(q_1, \ldots, q_n) = q_1 q_2 \ldots q_n (F(Q) - c)$

This is not a weighted potential game if we choose F such that:

$$\frac{\partial^2 u_i^k}{\partial x_i \partial x_j} \neq \left(\frac{w_i^k}{w_j^k}\right) \frac{\partial^2 u_j^k}{\partial x_i \partial x_j}$$

$\forall k = 1, \ldots, m$ and $\forall i, j = 1, \ldots, n$, $w_i^k \in \mathbb{R}_+$.

So the set of weighted potential games is a proper subset of the ordinal potential ones also for muticriteria games.

Furthermore this example shows that the equilibria set of the Cournot game coincides with the pure strategy equilibrium set of the game in which every firm's profit is the potential function.

The existence of an ordinal potential allows to study the Pareto equilibria of the game through the potential P in the following way:

Proposition 4.1. *Let G be a finite multicriteria ordinal potential game, the following relations are valid:*

(1) $PE(G) \neq \emptyset$
(2) $PE(P) \subseteq PE(G) = PE(G^P)$

Proof. The proof is similar to that of Proposition 3.2. □

Corollary 4.1. *Any finite multicriteria ordinal potential game has at least a weak Pareto equilibrium.*

Let us define a weak improvement cycle to illustrate some interesting properties of ordinal potential games.

Definition 4.2. A finite path $\ell = (x_1, \ldots, x_m)$ in the strategy space X is a finite sequence of elements $x_k \in X$ such that $\forall k$, the strategy combination x_k and x_{k+1} differs in the $i(k)$th coordinate. It is called closed or cycle if $x_1 = x_m$. It is a simple cycle if it is closed and all strategy combinations are different except the initial and final point. The number of different strategy combinations in a simple closed path is called the length of the path.

A finite path $(x_1 \ldots x_m)$ is called a weak improvement cycle if
$x_1 = x_m$
$u_{i(k)}(x_k) \leq u_{i(k)}(x_{k+1})$ for some $k = 1, 2, ..n$.

Theorem 4.1. *If G is a multicriteria ordinal potential game, then it has no weak improvement cycles.*

Proof. Let P be an ordinal potential for the game G and by absurdum let $(x_1, x_2, \ldots, x_m)$ be a weak improvement cycle, then

$u_{i(k)}(x_k) \leqq u_{i(k)}(x_{k+1})$ for $k = 1, 2, ..n-1$ and there is at least $\bar{k}$ s.t.

$u_{i(\bar{k})}(x_{\bar{k}}) < u_{i(\bar{k})}(x_{\bar{k}+1})$.

Being an ordinal potential game $P(x_k) \leqq P(x_{k+1})$ and for $\bar{k}$ it turns out in at least one component j, $P^j(x_{\bar{k}}) < P^j(x_{\bar{k}+1})$; by the hypothesis on the cycle, on at least a component j it turns out:

$P^j(x_1) \leq P^j(x_2) \leq \ldots P^j(x_{\bar{k}}) < P^j(x_{\bar{k}+1}) \leq \ldots P^j(x_1)$ and this is absurd. □

The converse is not true, even for finite games, as the following example shows:

Example 4.2.

(0, 0) (0, 0)	(0, 1) (0, 0)
(0, 0) (0, 0)	(1, 0)(0, 0)

The game has no weak improvement cycle, but it has not an ordinal potential.

Remark 4.1. The absence of weak improvement cycles involving only 4 deviations is not sufficient for G to be an ordinal potential game, (differently from what happens for exact potential ones) as the following example shows: G:

G :

	D	E	F
A	(0, 0) (1, 2)	(1, 2) (2, 3)	(0, 2) (0, 1)
B	(1 2) (1, 3)	(0, 1) (0, 1)	(0, 2) (0, 1)
C	(0 0) (0, 1)	(0, 1) (0, 1)	(1, 2) (1, 3)

The path $\gamma = ((C, D), (C, E), (B, E), (B, F), (A, F), (A, D))$ is a weak improvement cycle but it does not involve four deviations.

Definition 4.3. *A multiobjective game has the finite improvement property, (FIP for short), if every improvement path is finite.*

Proposition 4.2. *Every finite ordinal potential game has the FIP.*

The following drawing well illustrates the inclusions among the potential multicriteria games studied (Fig. 1).

The set F denotes the set of exact potential games, the set W that of weighted potential games and the set O that of ordinal potential ones.

In fact considering the previous examples: $G_1 \in W \backslash F$, $G_2 \in F$, $G_3 \in O \backslash W$.

Among the properties of potential games that contribute to their importance in applications, the following are perhaps the most relevant: (1) the existence of pure Pareto equilibria and (2) the FIP (Finite Improvement Path) property. The first depends on the observation that the set of equilibria of a potential game is strictly related to that of a game where players have the common

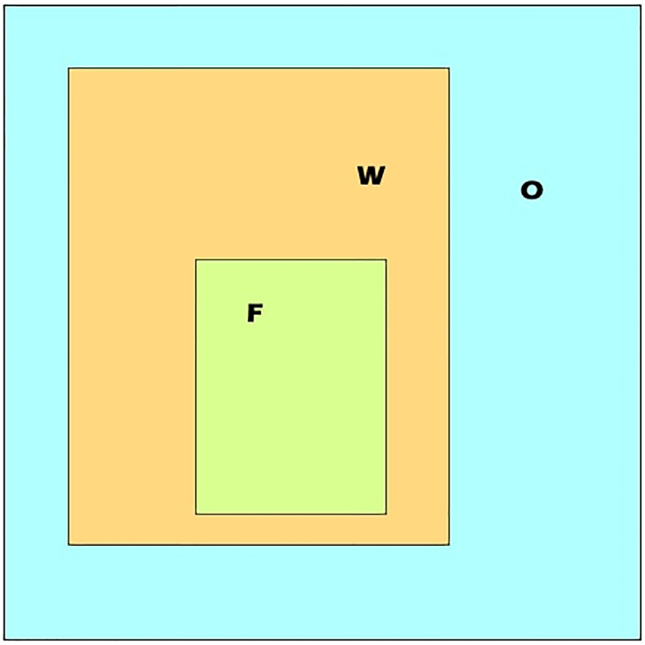

Fig. 1. Relations among potential games classes.

objective of optimising the potential. The second allows to create algorithms for the search of equilibria as the terminal points of finite improvement paths.

In the next two sections we present some applications of multicriteria potential games.

5. Examples and applications

Example 5.1 Application to a water resource problem. Players I and II are two firms that can collect water both from a common international water body and a private local source (different for each player) to be used for domestic and agricultural needs. Let us call $(x_1, x_2) \in X = X_1 \times X_2$ the quantities collected from the international water body, and $(y_1, y_2) \in Y = Y_1 \times Y_2$ those drilled out of the local source. We call $z_i = \binom{x_i}{y_i}$ $i = 1, 2$ the quantity of water used by player i.

Players have two objectives: the first is to maximize the benefits obtained by transforming water from the local sources, in terms of better hygienical conditions, improved local agricultural productions and increased health. The second is to maximize the profits obtained by selling the water taken from the international water body.

We call B_i the benefit of player i from transforming the quantity of local water from his source, $B_1(z_1) = b_1 y_1$ with $b_1 \in \mathbb{R}_+$, $B_2(z_2) = b_2 y_2$ with $b_2 \in \mathbb{R}_+$. The costs incurred in this transforming work are

$C_1 = c.(x_1 + x_2 + y_1 + y_2)$ for the first player and for the second $C_2 = c.(x_1 + x_2 + +y_1 + y_2)$ with $c \in \mathbb{R}_+$.

So the first component of the utility function of player I is $b_1 y_1 - c.(x_1 + x_2 + y_1 + y_2)$.

The second component of the utility function of player I is the market profit, which is proportional to the quantity of water and to the inverse demand function E, $E > 0$. Taking into account the cost functions too, the payoffs for the players are:

$$u_1(x_1, y_1, x_2, y_2) = \begin{pmatrix} b_1 y_1 - c.(x_1 + x_2 + y_1 + y_2) \\ (x_1 + y_1)(E(x_1 + x_2 + y_1 + y_2) - k \end{pmatrix}$$

and analogously for player II:

$$u_2(x_1, y_1, x_2, y_2) = \begin{pmatrix} b_2 y_2 - c.(x_1 + x_2 + y_1 + y_2) \\ (x_2 + y_2)(E(x_1 + x_2 + y_1 + y_2) - k) \end{pmatrix}$$

This is an ordinal potential game with potential

$$P(x_1, y_1, x_2, y_2) = \begin{pmatrix} b_1 y_1 + b_2 y_2 - c.(x_1 + y_1 + x_2 + y_2) \\ (x_1 + y_1)(x_2 + y_2)(E(x_1 + x_2 + y_1 + y_2) - k \end{pmatrix}.$$

with $b_1, b_2, c, k \in \mathbb{R}+$

To prove in our model that P is an ordinal potential, the following result is useful:

Proposition 5.1. *Let G be an ordinal potential game as in Definition 4.1 and let u_i, P differentiable functions. Then the following is valid:*
$\forall i = 1, \dots, n; \forall j = 1, \dots, m;$

$$\frac{\partial u_i^j}{\partial x_i} > 0 \Leftrightarrow \frac{\partial P^j}{\partial x_i} > 0.$$

Proof. is similar to Theorem 3.2 □

To know more about a water resource problem in the model of partial cooperation and scalar case see [4] and references therein.

Example 5.2 Application to voluntary contribution. Now we study a class of games where players give private contributions to finance facilities or projects that are considered public goods, once they are established, all players can use them.

We consider a finite number of players and we suppose that each of them is interested in a finite number of facilities. Each player receives a benefit from the project s/he is interested in, if this is realized. In our context we are interested in the non-cooperative case.

Players present a contribution independently from the others, then the realization scheme determines which projects to finance and the players utility functions. This interactive problem is called a “contribution game”.

The strategy space of each player is the set of possible contributions: $[0, \alpha_i]$ where the $\alpha_i \in \mathbb{R}_+$ is the reward for player i, so that each player's contribution is strictly less than the compensation. The utility function depends on whether the projects s/he is interested in are realized or not and on the net benefit s/he gets from them. The social welfare function is defined as the sum of the utility functions of all players.

The realization scheme takes into account that each contributor is paying only for the project of interest and the funding for one project is never greater than its cost; excess contributions will be lost. Only projects completely financed are carried out. The players must decide how much they want to offer and after individual contributions are made an external person, independently from contribution makers, will decide which facilities will be funded. A contribution profile can be associated to many admissible facility sets and the decision maker use a *realization scheme* to select the best option.

A realization scheme specifies for each profile of contributions, which set of projects will be realized.

A realization problem is a tuple:

$$\mathcal{R} = \left\langle N, M, m \in \left(2^M\right)^N, \alpha \in \mathbb{R}^N_{++}, c \in \mathbb{R}^M_{++} \right\rangle$$

where

N is a finite, non empty set of n players,

M is a finite, non empty set of facilities or public goods,

$m = (m_i)_{i \in N} \in (2^M)^N$ is the vector of the facility set requested by each player. The vector $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}^N_{++}$ specifies the prize each player $i \in N$ gets if all projects in m_i are realized;
$c = (c_j)_{j \in M} \in \mathbb{R}^M_{++}$ specifies the cost c_j to realize the project $j \in M$.

A contribution problem is denoted by:

$$\mathcal{C} = \left\langle N, M, m \in \left(2^M\right)^N, \alpha \in \mathbb{R}^N_{++}, c \in \mathbb{R}^M_{++}, \mathcal{R} \right\rangle$$

Before the players decide their contribution, the arbiter's realization scheme is publicly announced.

A realization scheme follows two important rules:

(1) a player's contribution will be used for the projects of his interest

(2) players who get higher prizes are encouraged to contribute more.

The contribution problem can be seen as a non cooperative game where the strategy space of player i is $X_i = [0, \alpha_i)$.

The realization scheme R: $\prod_{i\in N} X_i \to 2^M$ selects the projects to be funded.

The payoff function of player i for each profile $x = (x_i)_{i\in N} \in X$ is:

$$u_i(x) = \begin{cases} -x_i, & \text{if } m_i \not\subseteq R(x), \\ \alpha_i - x_i & \text{if } m_i \subseteq R(x). \end{cases} \quad (1)$$

Intuitively, player i has a profit only if the facilities he is interested in are financed, otherwise he has a loss equal to his contribution. We have supposed that each player does not contribute with a quantity greater then the prize.

It is known that if $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a scalar contribution game, the social welfare function is

$U : X \to \mathbb{R}^m$ defined as $U = \sum_{i\in N} u_i$ and it is an ordinal potential function of the game.

Now let us consider the bicriteria game $H = \langle N, (X_i)_{i\in N}, (F_i)_{i\in N}\rangle$ where $F_i = (u_i, U)$ so the players have two objectives: to maximize the private payoff and the social welfare. H is an ordinal potential bicriteria game.

More about voluntary contribution games can be found in [20].

6. Application to peering games for telecommunication models.

Potential games are used to model and solve several problems in telecommunications, where resources have to be allocated in an efficient way among numerous players (see e.g. [3] for some specific examples of applications to wireless networks). We present here an example inspired by the peering games in [14,15]; we refer to these papers for the definitions and the previous results.

Let us define $N = \{1, \dots, n\}$ the set of neighbouring autonomous systems acting as players in the game, E the (finite) set of links that each player can use for inter-community communications and X_i the set of strategies of each player. If F is the number of possible outbound flows from the community of provider i, X_i is defined as the set of all vectors of length F whose components are in E. If for example four flows have to be routed to two links l_1, l_2, then the strategy $\sigma_i = (l_1, l_2, l_2, l_1)$ describes the decision of the ith autonomous system about the use of the peering links for the routing on the four inter-community flows. The routing choice of each player affects their egress and ingress costs at each router and the congestion of inter-peer links - the second aspect acquiring importance when many inter-community flows are considered. In the following multi-objective game we therefore define the payoff of player i as:

$$\pi_i(\sigma_i, \sigma_{-i}) = \left(\phi_{i,s}(\sigma_i) + \phi_{i,d}(\sigma_{-i}), \phi_{i,c}(\sigma_i, \sigma_{-i})\right),$$

where $\phi_{i,\,s}$ represents the egress costs of player i, $\phi_{i,\,d}$ its ingress costs and $\phi_{i,\,c}$ the congestion cost. The first only depends on the own strategy (selfish game), the second only on opponents' strategies (dummy game) and the last on the collective behaviour. For any choice of the functions $\phi_{i,\,s}$ and $\phi_{i,\,d}$ the scalar game $\langle N, (X_i)_{i\in N}, (\phi_{i,s} + \phi_{i,d})_{i\in N}\rangle$ is an exact potential game. For the game $\langle N, (X_i)_{i\in N}, (\phi_{i,\,c})_{i\in N}\rangle$ we refer to Harks et al. [2], where the conditions for the existence of a weighted potential for these games are established. Assuming that the functions $\phi_{i,\,c}$ satisfy the requirements in Section 3.2 of [2], the multi-objective game $G = \langle N, (X_i)_{i\in N}, (\pi_i)_{i\in N}\rangle$ is then a weighted potential game in the sense of Definition 3.1. By Proposition 3.2 the Pareto optimal points of its w-potential P provide Pareto equilibria for G and can be used as an equilibrium refinement tool. As in [14,15] the dummy game $\langle N, (X_i)_{i\in N}, (\phi_{i,\,d})_{i\in N}\rangle$, which has no influence on the potential P, can be used for the subsequent selection of an efficient equilibrium.

Example 6.1. For this numerical example we choose $n = 2$ (and name the players I and II), $E = \{\ell_1, \ell_2, \ell_3\}$ and assume that there is only one outbound flow from each system, so that strategies are couples of the form (ℓ_m, ℓ_n) with $m,n = 1,2,3$. We define the egress cost of player i on link m as c^i_m and the ingress cost of player i on link n as γ^i_n; the chosen numerical values are:

$$c^I_1 = 10, \quad c^I_2 = 7, \quad c^I_3 = 9$$
$$c^{II}_1 = 6, \quad c^{II}_2 = 8, \quad c^{II}_3 = 15$$
$$\gamma^I_1 = 5, \quad \gamma^I_2 = 8, \quad \gamma^I_3 = 10$$
$$\gamma^{II}_1 = 3, \quad \gamma^{II}_2 = 9, \quad \gamma^{II}_3 = 7$$

In this case

$$\phi_{I,s}(\ell_m, \ell_n) = c^I_m, \qquad \phi_{II,s}(\ell_m, \ell_n) = c^{II}_n$$
$$\phi_{I,d}(\ell_m, \ell_n) = \gamma^I_n, \qquad \phi_{II,d}(\ell_m, \ell_n) = \gamma^{II}_m$$

and the first element of the payoff of player i is given by the sum of the egress and ingress costs for player i when the strategy (ℓ_m, ℓ_n) is played.

For the congestion game, if d_i denotes the demand of player i and the load L_k of link ℓ_k in strategy (ℓ_m, ℓ_n) is:

$$L_k(\ell_m, \ell_n) = \delta_{km}\, d_I + \delta_{kn}\, d_{II}$$

it is:

$$\phi_{I,c}(\ell_m, \ell_n) = d_I\, g(L_m), \qquad \phi_{II,c}(\ell_m, \ell_n) = d_{II}\, g(L_n)$$

where g is a given cost function. If g is affine the congestion game is an exact potential game; choosing $g(x) = \exp(x)$ a weighted potential game is obtained with potential

$$P_c(\ell_m, \ell_n) = g(L_m) + g(L_n).$$

If $d_I = 5$ and $d_{II} = 7$ for the game:

G:

$I\backslash II$	ℓ_1	ℓ_2	ℓ_3
ℓ_1	$(15, 5e^{12})(9, 7e^{12})$	$(18, 5e^5)(11, 7e^7)$	$(20, 5e^5)(18, 7e^7)$
ℓ_2	$(12, 5e^5)(15, 7e^7)$	$(15, 5e^{12})(17, 7e^{12})$	$(17, 5e^5)(24, 7e^7)$
ℓ_3	$(14, 5e^5)(13, 7e^7)$	$(17, 5e^5)(15, 7e^7)$	$(19, 5e^{12})(22, 7e^{12})$

the potential P

	ℓ_1	ℓ_2	ℓ_3
ℓ_1	$(3, e^{12}+1)$	$(5, e^7+e^5)$	$(12, e^7+e^5)$
ℓ_2	$(0, e^7+e^5)$	$(2, e^{12}+1)$	$(9, e^7+e^5)$
ℓ_3	$(2, e^7+e^5)$	$(4, e^7+e^5)$	$(11, e^{12}+1)$

is a w-potential with weights $w_I = \left(1, \frac{7e^7}{e^7-1}\right)$ and $w_{II}\left(1, \frac{5e^5}{e^5-1}\right)$. The strategy profile (ℓ_2, ℓ_1) is a Pareto equilibrium of the game.

7. Approximate Pareto equilibria

In this section we recall the definitions of approximate Pareto equilibria (ϵPE for short) for multiobjective games as introduced in [9].

Definition 7.1. Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a multicriteria strategic game and $\epsilon \in \mathbb{R}^m$ with positive components. The strategy profile $\hat{x} \in \prod_{i\in N} X_i$ is called an ϵ-Pareto equilibrium of the game G if for each $i \in N$ it holds $\hat{x}_i \in \epsilon PB(\hat{x}_{-i})$, where $\epsilon PB(\hat{x}_{-i})$ is the set of ϵ-Pareto best answers to $\hat{x}_{-i}$, that is

$\epsilon PB(\hat{x}_{-i}) = \{x_i \in X_i \ s.t. \ \ u_i(y_i, \hat{x}_{-i}) \notin u_i(x_i, \hat{x}_{-i}) + \mathbb{R}^m_{+,\epsilon} \quad \forall y_i \in X_i\}$

with

$\mathbb{R}^m_{+,\epsilon} = \mathbb{R}^m_+ \setminus ([0, \epsilon]^m)$.

In the following we denote by $\epsilon PE(G)$ the set of ϵ-Pareto equilibria of the game G. For weighted potential games with vector payoff, the following result is valid:

Theorem 7.1. *Let G be a multicriteria weighted potential game. Let us suppose the weighted potential function P is upper bounded, then* $\epsilon PE(G) \neq \emptyset$

Proof. Let P be a weighted potential for G and $\epsilon > 0$. Take

$$\hat{x} \in \text{argsup}^{\epsilon}_{x\in X}\left(\sum_{i=1}^{n}\sum_{j=1}^{m} w_i^j \circ P^j(x)\right)$$

then $\forall x \in X$

$$\sum_{j=1}^{m} w_i^j \circ P^j(\hat{x}) \geq \sum_{j=1}^{m} w_i^j \circ P^j(x) - \epsilon$$

We want prove that $\hat{x}$ is an approximate equilibrium.

Suppose by absurd that $\hat{x} \notin \epsilon PE(G)$ then $\exists i \in N$ and $x_i \in X_i$ s.t.

$$u_i\big(x_i, \hat{x}_{-i}\big) \in u_i(\hat{x}) + \mathbb{R}^m_{+,\epsilon}.$$

since

$$w_i \circ \Big(P\big(x_i, \hat{x}_{-i}\big) - P(\hat{x})\Big) = u_i\big(x_i, \hat{x}_{-i}\big) - u_i(\hat{x}) \in \mathbb{R}^m_{+,\epsilon}$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_i^j . \Big(P^j(x_i, \hat{x}_{-i}) - P^j(\hat{x})\Big) > \epsilon$$

and this is a contradiction. So, for each $i \in N$, $\hat{x}_i \in \epsilon PB(\hat{x}_{-i})$ i.e. $\hat{x} \in \epsilon PE(G)$. □

Proposition 7.1. *Let G be a multicriteria weighted potential game. Let us suppose the weighted potential function P is upper bounded, then* $\epsilon PE(G) = \epsilon PE(G^P)$

Some games have no equilibria but approximate equilibria.

Example 7.1. Let us consider the following example with two players which have an infinite number of strategies and two objectives to reach:

$(1,0),(3-1,1)$	$(1,0),(3-1/2,1)$	$\cdots$	$(1,0),(3-1/n,1)$	$\cdots$
$(0,0),(0,1)$	$(0,0),(0,1)$	$\cdots$	$(0,0),(0,1)$	$\cdots$

In this game there are no strong Pareto equilibria but an infinite number of approximate Pareto equilibria.

This is an infinite bicriteria game with ordinal potential as follows:

$(3-1,1)$	$(3-1/2,1)$	$\cdots$	$(3-1/n,1)$	$\cdots$
$(0,1)$	$(0,1)$	$\cdots$	$(0,1)$	$\cdots$

8. Remarks about multicriteria ordinal potential games

Definition 4.1 of ordinal potential games is one of the possible ways of extending the classical notion to a multicriteria setting. We present here another viable alternative, which enlarges the class of admissible potentials, but subject to some criticisms which will be later discussed.

In this section we introduce the relation $\trianglerighteq$, where we mean, given $a, b, c, d \in \mathbb{R}^m \ : \ a \trianglerighteq b \Leftrightarrow c \trianglerighteq d$

$$\text{means}\begin{cases} a > b \Leftrightarrow c > d \\ a = b \Leftrightarrow c = d \\ \text{if } a,\ b \text{ are not comparable also} \\ \quad c,\ d \text{ are not comparable} \end{cases}$$

Definition 8.1. We call $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ an extended ordinal potential game if there exists $P : X \to \mathbb{R}^m$ such that

$\forall i \in N,\ \forall x_i, y_i \in X_i,\ \forall x_{-i} \in X_{-i}$

$u_i(x_i, x_{-i}) \trianglerighteq u_i(y_i, x_{-i}) \Leftrightarrow P(x_i, x_{-i}) \trianglerighteq P(y_i, x_{-i})$

If we call $\mathcal{G}$ the set of all strategic non cooperative games, $\mathcal{G}_0$ the set of ordinal potential games and $\mathcal{G}_E$ the set of ordinal extended ones, we illustrate by examples that:

$\mathcal{G}_E \setminus \mathcal{G}_0 \neq \emptyset$; $\mathcal{G} \neq \mathcal{G}_E$; $\mathcal{G}_0 \subset \mathcal{G}_E$.

Example 8.1. The following game G is not ordinal but extended ordinal:

G:

$(0,0)\ (0,2)$	$(1,1)\ (1,0)$
$(0,0)\ (1,1)$	$(2,0)\ (0,1)$

In this game there are no weak improvement cycles, but it is not an ordinal game, it is an extended ordinal game, in fact the following is an extended potential:

Q:

$(0,2)$	$(-3,3)$
$(0,2)$	$(-1,2)$

So we have shown that $\mathcal{G}_E \setminus \mathcal{G}_0 \neq \emptyset$.

Furthermore through this example we can see that the Proposition 4.1 can be not valid for extended ordinal games.

Example 8.2. The following example proves that $\mathcal{G} \neq \mathcal{G}_E$

G:

$(1,3)\ (2,1)$	$(0,1)\ (0,2)$
$(0,1)\ (2,3)$	$(1,2)\ (3,2)$

Example 8.3. This example shows that if G is an ordinal potential game, it is also extended potential one that is $\mathcal{G}_0 \subset \mathcal{G}_E$ The following games G_4 and G_5 are ordinal potential games:

G_4:

	C	D
A	$(0,0)$	$(1,1)$
B	$(1,2)$	$(1,2)$

G_5:

	C	D
A	$(1,1)$	$(1,0)$
B	$(0,0)$	$(1,1)$

with potentials

P_4:

	C	D
A	0	1
B	3	1

P_5:

	C	D
A	2	1
B	0	1

For the bicriteria game $G = (G_4, G_5)$

G:

	C	D
A	$(0,1)\ (0,1)$	$(1,1)\ (1,0)$
B	$(1,0)\ (2,0)$	$(1,1)\ (1,1)$

Obviously $P = (P_4, P_5)$ is an ordinal potential, but Q is an extended ordinal potential which is not ordinal.

$P:$

	C	D
A	(0, 2)	(1, 1)
B	(3, 0)	(1, 1)

$Q:$

	C	D
A	(−1, 2)	(0, 0)
B	(1, 1)	(0, 0)

Remark 8.1. It is well known that in Game Theory the preferences of players are very important and the numbers associated to utility functions are irrelevant. So in our opinion the generalization of an ordinal game to a multicriteria setting, given component by component, is more suitable to study applications because this preserves players preferences.

9. Conclusions and open problems

In this paper we have studied two classes of potential games: weighted potential and ordinal potential with multiobjective payoffs. Starting from the fact that often decision makers have not one but more objective "to maximize", we have generalized these classes of game to a multiobjective setting and we have studied how far is the theory of strategic games with potentials [6] can be extended to strategic games with vector payoffs.

We have investigated also the problem of the existence of Pareto equilibria for these classes of games (in the finite case) through the existence of Pareto optimal points of their potential functions. Other interesting properties as the decomposition of multicriteria weighted potential games into a coordination multicriteria and a dummy multicriteria one have been proven. We have studied the approximate equilibria too for some infinite games.

Then we have considered a new class of ordinal potential games: extended ordinal potential. We have made several examples comparing the properties. We conclude thinking that the given definition of ordinal potential game (componentwise) is the most suitable for applications in Game Theory, since it preserves players preferences.

Many examples have been presented to better illustrate the introduced concepts and to show the importance of potential games in applications: a water resource problem, a voluntary contribution model, peering games for telecommunication models.

Some open problems arise from our analysis:

1) It is well known that potential game and congestion ones are closely related [9,13], so a natural question is to investigate the connections between weighted congestion games and weighted potential games in the vector case.

2) We have generalized the FIP (finite improvement property) which is a very important property to design algorithms to find solutions. An open problem is to study for which potential games there is a relation between the FIP and the existence of Pareto equilibria and between the approximate FIP (ϵFIP for short) and ϵPE(G).

3) Interdomain peering links are the main bottleneck of Internet because of lack of coordination in the routing policies. As proposed in [14,15] modelling this problem as a non cooperative game can be effective in devising new solutions. A deeper investigation of the application of multiobjective games in this field could be a new and interesting research topic. A deeper investigation about internet peering settlement and telecommunication problem via multiobjective games could be a new and interesting research.

Some of these issues are work in progress.

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