

Nonlocal scaling operators with entanglement renormalization

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The multiscale entanglement renormalization ansatz (MERA) can be used, in its scale invariant version, to describe the ground state of a lattice system at a quantum-critical point. From the scale invariant MERA one can determine the local scaling operators of the model. Here we show that, in the presence of a global symmetry \mathcal{G} , it is also possible to determine a class of nonlocal scaling operators. Each operator consists, for a given group element $g \in \mathcal{G}$, of a semi-infinite string Γ_g^∞ with a local operator φ attached to its open end. In the case of the quantum Ising model, $\mathcal{G}=\mathbb{Z}_2$, they correspond to the disorder operator μ , the fermionic operators ψ and $\bar{\psi}$, and all their descendants. Together with the local scaling operators identity \mathbb{I} , spin σ , and energy ϵ , the fermionic and disorder scaling operators ψ , $\bar{\psi}$, and μ are the complete list of primary fields of the Ising CFT. Therefore the scale invariant MERA allows us to characterize all the conformal towers of this CFT.

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I. INTRODUCTION

The multiscale entanglement renormalization ansatz (MERA) (Refs. 1 and 2) is a tensor network introduced to efficiently represent ground states and low-energy subspaces of quantum many-body systems on a lattice. It is based on a real-space renormalization-group (RG) technique known as *entanglement renormalization*¹ that employs unitary tensors (*disentangler*s) to remove short-range entanglement from the system at each RG iteration. The removal of entanglement is a key difference with other real-space RG techniques, such as Wilson's ground breaking numerical RG for the Kondo problem³ or White's extremely successful density matrix RG (Ref. 4) for arbitrary one-dimensional systems. At a fixed point of the RG flow, it produces a representation that is explicitly scale invariant: the *scale invariant* MERA. This ansatz is characterized by only a small number of tensors and can be used to describe systems with topological order (at a noncritical RG fixed point)⁵ as well as continuous quantum phase transitions (corresponding to a critical RG fixed point).^{1,2,6-10}

Here we shall be concerned with the characterization of a (scale invariant) quantum critical point with the scale invariant MERA.^{1,2,6-10} Evidence that the scale-invariant MERA is capable of describing critical ground states was presented in Ref. 1 for the quantum Ising model and in Ref. 6 for noninteracting systems of fermions and bosons. On the other hand it was argued that this ansatz naturally reproduces two important aspects of critical ground states: the logarithmic scaling for the entanglement entropy of a block of contiguous sites (in one-dimensional systems) and the power-law decay of correlations.² The latter was seen to follow from the fact that a two-point correlator $C(s_1, s_2)$ between two points separated a distance $r=|s_1-s_2|$ is obtained after $O[\log(r)]$ applications of a fixed superoperator that introduces a constant factor $z < 1$ after each application, and therefore

$$C_2(s_1, s_2) \approx z^{\log(r)} = r^{-q}, \quad q \equiv -\log z. \quad (1)$$

This result was formalized in Ref. 7 by relating the possible values of the factor z with the eigenvalues of that superoperator and by identifying the *scaling operators* of the theory

with the corresponding eigenoperators. A connection between the scale invariant MERA and the conformal field theory (CFT) underlying a quantum-critical point was then established in Ref. 8, including a way to extract the conformal data: central charge, primary fields, and their scaling dimensions and operator product expansion (OPE).¹¹ However, the analysis of Refs. 7 and 8 was only concerned with *local* scaling operators. For instance, for the quantum Ising model, Ref. 8 identified the primary fields identity \mathbb{I} , energy ϵ , and spin σ , as well as some of their descendants, all of which were expressed as an operator acting on two contiguous sites of a coarse-grained lattice. Instead, *nonlocal* scaling operators were not considered. One reason is that entanglement renormalization, being based on locally coarse graining the system, has a computational cost that grows exponentially with the size of the support of the operators under consideration.

In this Brief Report we show that the scale invariant MERA can be used to characterize a whole class of nonlocal scaling operators of a critical lattice model. We consider a quantum spin chain whose Hamiltonian H is invariant under a symmetry group \mathcal{G}

$$\Gamma_g H \Gamma_g^\dagger = H, \quad \forall g \in \mathcal{G}, \quad (2)$$

where $\Gamma_g \equiv \cdots V_g \otimes V_g \otimes V_g \cdots$ is an infinite string of copies of a matrix V_g , with V_g a unitary representation of \mathcal{G} . We shall see that, by incorporating the symmetry \mathcal{G} into the MERA, it is possible to study nonlocal operators that have a semi-infinite string of V_g 's. Important examples of such nonlocal operators are the disorder operator μ and the fermionic operators ψ and $\bar{\psi}$ of the quantum Ising model, which are associated to the low energy spectrum of a chain with antiperiodic boundary conditions.

II. LOCAL SCALING OPERATORS

Recall that the scale invariant MERA is made of copies of a unique pair of bulk tensors, namely a disentangler u and an isometry w , distributed in layers according to Fig. 1(i). In the presence of the symmetry in Eq. (2), we choose these tensors to be invariant under \mathcal{G} (Ref. 12)

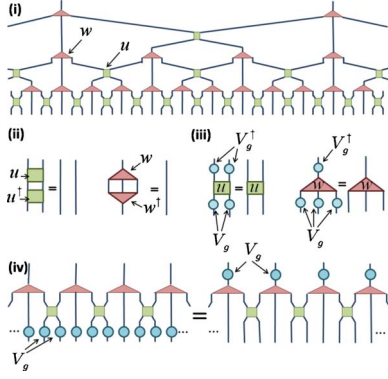


FIG. 1. (Color online) (i) Scale invariant MERA, characterized by a disentangler u and an isometry w that are copied throughout the ansatz. (ii) These tensors are isometric, meaning that $u^\dagger u = I$, $w^\dagger w = I$. (iii) We choose u and w to be symmetric, Eq. (3). (iv) As a result, the infinite string Γ_g^{\leftarrow} commutes with a layer of disentanglers and isometries. In other words, Γ_g^{\leftarrow} is invariant under coarse graining.

$$(V_g \otimes V_g)u(V_g \otimes V_g)^\dagger = u,$$

$$(V_g \otimes V_g \otimes V_g)w(V_g)^\dagger = w, \quad (3)$$

where V_g acting on different indices may actually denote different (in general, reducible) representations of \mathcal{G} . The layers of disentanglers and isometries define a real space RG transformation and a sequence of increasingly coarse-grained lattices $\{\mathcal{L}, \mathcal{L}', \mathcal{L}'', \dots\}$. Under coarse graining, a local operator o transforms according to the scaling superoperator \mathcal{S} of Fig. 2(v) for $g=\mathbb{I}$

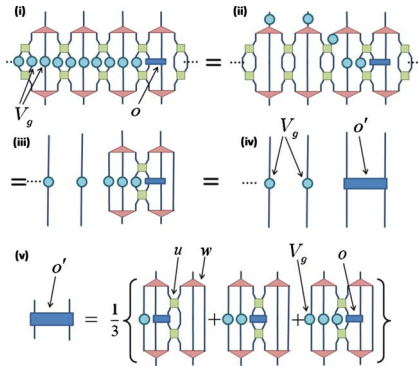


FIG. 2. (Color online) (i) Coarse graining of a nonlocal operator $o^\leftarrow = \Gamma_g^{\leftarrow} \otimes o$. (ii) Most of the string Γ_g^{\leftarrow} of V_g 's commutes with the coarse graining thanks to Eq. (3). (iii) Then we can remove most of disentanglers and isometries using Fig. 1(ii). (iv) o' is defined in term of o , u , w , and V_g . (v) Scaling superoperator \mathcal{S}_g , $o' = \mathcal{S}_g(o)$, for the local part of a nonlocal operator o_g^\leftarrow in Eq. (6), see Eqs. (7) and (8). Notice the average over the three possible ways in which o can be coarse grained. In the case of $g=\mathbb{I}$, we have $V_g=\mathbb{I}$, so that o_1^\leftarrow is simply a local operator and we recover the “usual” scaling superoperator \mathcal{S} of Eqs. (4) and (5) (see Fig. 1 of Ref. 8 for further details).

$$\overset{\mathcal{S}}{o} \rightarrow \overset{\mathcal{S}}{o'} \rightarrow \overset{\mathcal{S}}{o''} \dots \quad (4)$$

The scaling operators ϕ_α and scaling dimensions Δ_α are obtained from the eigenvalue decomposition of the scaling superoperator \mathcal{S} (Refs. 7 and 8)

$$\mathcal{S}(\phi_\alpha) = \lambda_\alpha \phi_\alpha, \quad \Delta_\alpha \equiv -\log_3 \lambda_\alpha. \quad (5)$$

III. NONLOCAL SCALING OPERATORS

In this work we consider the coarse graining of nonlocal operators o_g^\leftarrow of the form

$$o_g^\leftarrow = \Gamma_g^{\leftarrow} \otimes o, \quad \Gamma_g^{\leftarrow} \equiv \underbrace{\dots V_g \otimes V_g \otimes V_g}_{\infty} \quad (6)$$

where Γ_g^{\leftarrow} is a semi-infinite string made of copies of V_g and o is a local operator attached to the open end of Γ_g^{\leftarrow} . Notice that, under coarse graining, o_g^\leftarrow is mapped into another nonlocal operator $o_g^{\leftarrow'}$ of the same type

$$o_g^\leftarrow = \Gamma_g^{\leftarrow} \otimes o \rightarrow o_g^{\leftarrow'} = \Gamma_g^{\leftarrow} \otimes o' \quad (7)$$

since the semi-infinite string Γ_g^{\leftarrow} commutes with the coarse graining everywhere except at its open end, as illustrated in Fig. 2, where we exploit that the disentangler u and isometry w have been chosen to be symmetric, Eq. (3). In other words, we can study the sequence of coarse-grained nonlocal operators $o_g^\leftarrow \rightarrow o_g^{\leftarrow'} \rightarrow o_g^{\leftarrow''} \dots$ by just coarse graining the operator o with the modified scaling superoperator \mathcal{S}_g of Fig. 2

$$\overset{\mathcal{S}_g}{o} \rightarrow \overset{\mathcal{S}_g}{o'} \rightarrow \overset{\mathcal{S}_g}{o''} \dots \quad (8)$$

In particular, by diagonalizing this scaling superoperator

$$\mathcal{S}_g(\phi_{g,\alpha}) = \lambda_{g,\alpha} \phi_{g,\alpha}, \quad \Delta_{g,\alpha} \equiv -\log_3 \lambda_{g,\alpha} \quad (9)$$

we obtain nonlocal scaling operators $\phi_{g,\alpha}^\leftarrow$ of the form

$$\phi_{g,\alpha}^\leftarrow = \Gamma_g^{\leftarrow} \otimes \phi_{g,\alpha}. \quad (10)$$

Notice that for $g=\mathbb{I}$ we recover the local scaling operators ϕ_α of Refs. 7 and 8.

IV. QUANTUM ISING MODEL

As an example, we use the above formalism to identify the nonlocal operator content of the Ising CFT starting from the Ising quantum spin chain, as described by the Hamiltonian

$$H_{\text{Ising}} \equiv \sum_{r=-\infty}^{\infty} [X(r)X(r+1) + Z(r+1)], \quad (11)$$

where X and Z are Pauli matrices. This model preserves parity, $\mathcal{G}=\mathbb{Z}_2$, so that $g \in \{+1, -1\}$, with $V_{+1}=\mathbb{I}$ and $V_{-1}=Z$, and

$$\Gamma_{-1} H_{\text{Ising}} \Gamma_{-1}^\dagger = H_{\text{Ising}}, \quad \Gamma_{-1} \equiv \bigotimes_{m=-\infty}^{\infty} Z. \quad (12)$$

Each index i of tensors u and w decomposes as $i=(p, \alpha_p)$, where p labels the parity ($p=1$ for even parity and $p=-1$ for

TABLE I. Scaling dimensions of a few nonlocal operators of the quantum Ising model. The conformal towers of ψ and $\bar{\psi}$ have identical scaling dimensions.

Δ^{exact}	$\Delta_{\chi=36}^{\text{MERA}}$	Error (%)	Δ^{exact}	$\Delta_{\chi=36}^{\text{MERA}}$	Error (%)
$(\mu)1/8$	0.1250002	0.0002	$(\psi)1/2$	0.5	$<10^{-8}$
$1+1/8$	1.124937	0.006	$1+1/2$	1.49999	$<10^{-5}$
$1+1/8$	1.124985	0.001	$2+1/2$	2.49931	0.028
$2+1/8$	2.123237	0.083	$2+1/2$	2.50118	0.047
$2+1/8$	2.124866	0.006			
$2+1/8$	2.125487	0.023			

odd parity) and α_p labels the distinct values of i with parity p . Then the tensors u, w are chosen to be parity preserving, e.g., $u_{i_1 i_2}^{j_1 j_2} = 0$ if $p(i_1)p(i_2)p(j_1)p(j_2) = -1$. An operator O acting on the spin chain has parity p if $(\Gamma_{-1})O(\Gamma_{-1})^\dagger = pO$.

We have used the algorithm of Refs. 8 and 13 to obtain a scale invariant MERA approximation for the ground state of H_{Ising} with a computational effort that scales as $O(\chi^4 \bar{\chi}^4)$ with the dimension χ (and $\bar{\chi}$) of the lower (and upper) indices of the disentangler u . The present results correspond to $\chi=36$ and $\bar{\chi}=20$, and required one week on a 3 GHz dual core desktop with 8 Gb of random access memory. The scaling superoperators \mathcal{S}_1 and \mathcal{S}_{-1} were diagonalized in each parity sector. The resulting nonlocal scaling operators are of the form

$$\phi_{-1,\alpha}^\square = \cdots Z \otimes Z \otimes Z \otimes \phi_{-1,\alpha}. \quad (13)$$

Table I contains a few scaling dimensions extracted from \mathcal{S}_{-1} . The second and fifth columns are for scaling operators with even and odd parity, respectively, and reproduce the exact results with several digits of accuracy. Figure 3 shows scaling dimensions for both local and nonlocal operators. Local scaling operators with even parity form the two conformal towers¹¹ of the primary fields identity \mathbb{I} and energy ϵ of the Ising CFT whereas those with odd parity form the conformal tower of the primary field spin σ . Nonlocal scaling operators with even parity form the conformal tower of the disorder operator μ and those with odd parity are organized according to two towers corresponding to the fermion operators ψ and $\bar{\psi}$.

We have also computed the coefficients $C_{\alpha\beta\gamma}$ of the OPE (Ref. 11) for all primary fields, by analyzing three-point cor-

relators as explained in Ref. 8. Notice that a three-point correlator $\langle \phi_{g_1\alpha_1}^\square \phi_{g_2\beta}^\square \phi_{g_3\gamma}^\square \rangle$ will vanish unless (i) the product of parities of the three operators is +1 (since the ground state is invariant under parity) and (ii) $g_1 g_2 g_3 = \mathbb{I} \in \mathcal{G}$ (since otherwise the product $\phi_{g_1\alpha_1}^\square \phi_{g_2\beta}^\square \phi_{g_3\gamma}^\square$ is a nonlocal operator ϕ^\square , which must decompose as a sum of nonlocal scaling operators ϕ^\square , and $\langle \phi^\square \rangle = 0$ since all nonlocal scaling dimensions $\Delta_{-1,\alpha}$ are larger than zero so that $\langle \phi^\square \rangle = 0$). Table II shows a numerical estimate of all nonvanishing OPE coefficients $C_{\alpha\beta\gamma}$. Again, the results match the exact solution with several digits of accuracy.

Thus, not only have we been able to identify the *entire* field content $\{\mathbb{I}, \epsilon, \sigma, \psi, \bar{\psi}, \mu\}$ of the Ising CFT from a simple and rather unexpensive analysis of a quantum spin chain but we can now also identify all possible subsets of primary fields that close a subalgebra by inspecting Table II. Indeed, it follows that we have the following fusion rules

$$\epsilon \times \epsilon = \mathbb{I}, \quad \sigma \times \sigma = \mathbb{I} + \epsilon, \quad \sigma \times \epsilon = \sigma, \quad (14)$$

$$\mu \times \mu = \mathbb{I} + \epsilon, \quad \mu \times \epsilon = \mu, \quad (15)$$

$$\psi \times \psi = \mathbb{I}, \quad \bar{\psi} \times \bar{\psi} = \mathbb{I}, \quad (16)$$

$$\psi \times \bar{\psi} = \epsilon, \quad \psi \times \epsilon = \bar{\psi}, \quad \bar{\psi} \times \epsilon = \psi \quad (17)$$

(as well as other, such as $\sigma \times \mu = \psi + \bar{\psi}$, etc.) from where we see that $\{\mathbb{I}, \epsilon\}$ and $\{\mathbb{I}, \epsilon, \sigma\}$ close subalgebras of local primary fields, whereas $\{\mathbb{I}, \epsilon, \mu\}$ and $\{\mathbb{I}, \epsilon, \psi, \bar{\psi}\}$ close subalgebras that

TABLE II. OPE coefficients for the local and nonlocal primary fields of the Ising CFT.

C^{exact}	$C_{\chi=36}^{\text{MERA}}$	Error (%)
$C_{\epsilon,\sigma,\sigma} = 1/2$	0.50008	0.016
$C_{\epsilon,\mu,\mu} = -1/2$	-0.49997	0.006
$C_{\psi,\mu,\sigma} = \frac{e^{-i\pi/4}}{\sqrt{2}}$	$\frac{1.00068e^{-i\pi/4}}{\sqrt{2}}$	0.068
$C_{\bar{\psi},\mu,\sigma} = \frac{e^{i\pi/4}}{\sqrt{2}}$	$\frac{1.00068e^{i\pi/4}}{\sqrt{2}}$	0.068
$C_{\epsilon,\psi,\bar{\psi}} = i$	1.0001i	0.010
$C_{\epsilon,\bar{\psi},\psi} = -i$	-1.0001i	0.010

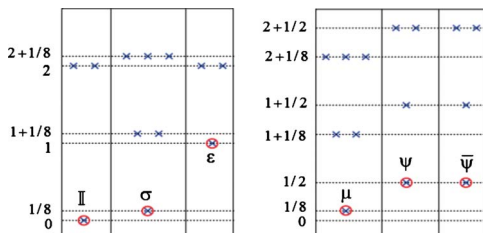


FIG. 3. (Color online) A few scaling dimensions of local (left) and nonlocal (right) scaling operators of the quantum Ising model, organized in its six conformal towers.

contain both local and nonlocal primary fields, where locality is relative to the spin variables.

V. QUANTUM XX MODEL

As a second example we study the quantum spin chain with Hamiltonian

$$H_{XX} \equiv \sum_{r=-\infty}^{\infty} [X(r)X(r+1) + Y(r)Y(r+1)], \quad (18)$$

where X and Y are Pauli matrices. This model is invariant under rotations $V_{\theta} = e^{-i\theta Z/2}$ on all spins. Therefore $\mathcal{G} = U(1)$, group elements g can be labeled by an angle $\theta \in [0, 2\pi)$ and

$$\Gamma_{\theta} H_{XX} \Gamma_{\theta}^{\dagger} = H_{XX}, \quad \Gamma_{\theta} \equiv \bigotimes_{m=-\infty}^{\infty} V_{\theta}. \quad (19)$$

To simplify the analysis, we regard each site as containing two spins, so that the on-site \hat{z} component of the spin can take the values 0 and ± 1 . Then each index i of a tensor decomposes as $i = (n, \alpha_n)$, where $n \in \mathbb{Z}$ is the ‘‘particle number’’ (z spin component) and α_n labels the distinct values of i with particle number n . Tensors u and w are chosen to be invariant under $U(1)$, e.g., $u_{i_1 i_2}^{j_1 j_2} = 0$ if $n(i_1) + n(i_2) \neq n(j_1) + n(j_2)$. An operator O acting on the spin chain has particle number n if $(\Gamma_{\theta}) O (\Gamma_{\theta})^{\dagger} = e^{-in\theta}$.

The optimization of a scale invariant MERA with $\chi = 54$ and $\tilde{\chi} = 32$ took one week (by exploiting the block structure of the tensors¹²). For several values of $\theta \in [0, 2\pi)$, we diagonalized the scaling superoperator \mathcal{S}_{θ} in the particle number sectors $n = 0, \pm 1, \pm 2, \dots$. Figures 4(i) and 4(ii) show the resulting scaling dimensions $\Delta_{\theta, \alpha}$ for $\theta = 0$ and π that is for $V_0 = \mathbb{I}$ (local operators) and $V_{\pi} = Z$, which also appear clearly organized in conformal towers. Finally, Fig. 4(iii) shows the scaling dimensions as a function of θ . They are seen to accurately approximate the expression [denoted ‘‘exact’’ in Fig. 4(iii)]

$$\Delta_{\theta, \alpha} = \Delta_{0, \alpha} + \left(\frac{\theta}{2\pi} + q \right)^2 - q^2, \quad q = 0, \pm 1, \quad (20)$$

which is consistent with previous results.¹⁴ Up to a shift and a rescaling factor, the scaling dimensions $\Delta_{\theta, \alpha}$ reproduce the

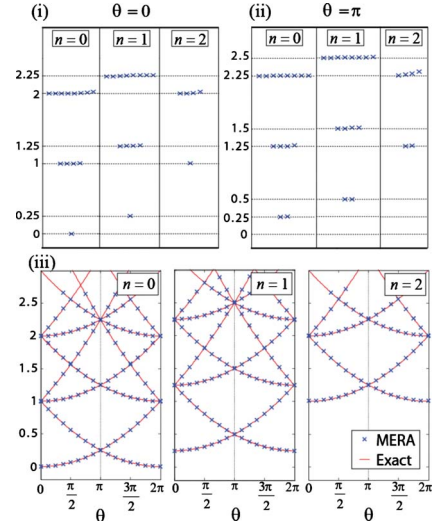


FIG. 4. (Color online) (i) Some scaling dimensions for *local* operators of the quantum XX model. (Sectors with particle numbers $+|n|$ and $-|n|$ yield the same scaling dimensions.) (ii) Some scaling dimensions for *nonlocal* operators with $V_{\theta} = Z$. (iii) Scaling dimensions $\Delta_{\theta, \alpha}$ as a function of θ , see Eq. (19). The scaling dimensions for $n = 0, 1, 2$ appear to only differ by a shift.

low energy spectrum of the XX chain with twisted boundary conditions with twisting angle θ .

In summary, we have explained how to use the scale invariant MERA to characterize nonlocal scaling operators of a critical quantum spin chain. For the quantum Ising model, we have identified all nonlocal primary fields and obtained remarkably accurate estimates of their scaling dimensions and OPE coefficients. For the quantum XX model, we have obtained continuous families of nonlocal operators associated to twisted boundary conditions.

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