

SOLVABLE 3-LIE ALGEBRAS WITH A MAXIMAL HYPO-NILPOTENT IDEAL N^*

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Abstract. This paper obtains all solvable 3-Lie algebras with the m -dimensional filiform 3-Lie algebra N ($m \geq 5$) as a maximal hypo-nilpotent ideal, and proves that the m -dimensional filiform 3-Lie algebra N can't be as the nilradical of solvable non-nilpotent 3-Lie algebras. By means of one dimensional extension of Lie algebras to the 3-Lie algebras, we get some classes of solvable Lie algebras directly.

Key words. 3-Lie algebra, hypo-nilpotent ideal, filiform n -Lie algebra.

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1. Introduction. The concept of n -Lie algebras appeared in two different contexts [1, 2]. In [1], Nambu introduced n -ary multilinear operations in his description of simultaneous classical dynamics of n particles, and extended the Poisson bracket to the n -ary multilinear bracket. In [2], Filippov formulated a theory of n -Lie algebras based on his proposed $(2n - 1)$ -fold Jacobi type identity and gave a classification for n -Lie algebras of lower ($\leq n + 1$) dimensions. The connection between the Nambu mechanics and the Filippov's theory of n -Lie algebras was established in 1994 by Takhtajan [3]. Recently n -Lie algebras have found important applications in string and membrane theories. For instance, in [4, 5] Bagger and Lambert proposed a supersymmetric field theory model for multiple M2-branes based on the metric 3-Lie algebras. More application of n -Lie algebras can be found in e.g., [6, 7, 8, 9, 10, 11, 12, 13].

In recent years, the structure of n -Lie algebras has been widely studied. Kasymov [14] developed the structure and representation theory of n -Lie algebras. Ling [15] proved that there is a unique $(n + 1)$ -dimensional simple n -Lie algebra for $n > 2$ over an algebraically closed field of characteristic zero. The first author of the current paper and her collaborators showed in [16] that there exist only $\lfloor \frac{n}{2} \rfloor + 1$ classes of $(n + 1)$ -dimensional simple n -Lie algebras over a complete field of characteristic 2 and

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gave a complete classification in [17] for six dimensional 4-Lie algebras. There are other results on structures and representations of n -Lie algebras.

The structure of n -Lie algebras is very different from that of Lie algebras, due to the n -ary multilinear operations involved. In particular, it turns out that the fundamental identity for an n -Lie algebra is much more restrictive than the Jacobi identity for a Lie algebra. One consequence is that higher, finite dimensional n -Lie algebras may be rare and are difficult to find. So it is very important to construct new examples of n -Lie algebras.

Filiform n -Lie algebras, i.e., nilpotent n -Lie algebras L satisfying $\dim L^i = \dim L - n - i$, are important class of nilpotent n -Lie algebras. In [18] we introduced the concept of hypo-nilpotent ideals of n -Lie algebras, and proved that an m -dimensional *simplest* filiform 3-Lie algebra N_0 can't be a nilradical of solvable non-nilpotent 3-Lie algebras. By m -dimensional *simplest* filiform 3-Lie algebra, we mean an m -dimensional filiform 3-Lie algebra with the following multiplication table in the basis e_1, e_2, \dots, e_m ,

$$(1.1) \quad [e_1, e_2, e_j] = e_{j-1}, 4 \leq j \leq m.$$

Moreover, it was shown that there are only four classes of $(m + 1)$ -dimensional and one class of $(m + 2)$ -dimensional solvable non-nilpotent 3-Lie algebras with N_0 as their maximal hypo-nilpotent ideal.

In this paper we generalize the results of [18]. Namely we consider a more complicated m -dimensional filiform 3-Lie algebra N ($m \geq 5$) defined by the multiplication table (3.1) below (c.f. (1.1)). We obtain all solvable 3-Lie algebras with such an N as a maximal hypo-nilpotent ideal and prove that N can't be a nilradical of solvable non-nilpotent 3-Lie algebras.

The organization for the rest of this paper is as follows. Section 2 introduces some basic notions. Section 3 describes the structure of solvable 3-Lie algebras with the maximal hypo-nilpotent ideal N . Section 4 studies the solvable 3-Lie algebras with nilradical N . Section 5 gives an application of one dimensional extension of Lie algebras.

Throughout this paper we consider 3-Lie algebras over a field F of characteristic zero.

2. Fundamental notions. First we introduce some notions of n -Lie algebras (see [2, 14, 18]). A vector space A over a field F is an n -Lie algebra if there is an n -ary multilinear operation $[\cdot, \dots, \cdot]$ satisfying the following identities

$$(2.1) \quad [x_1, \dots, x_n] = (-1)^{\tau(\sigma)} [x_{\sigma(1)}, \dots, x_{\sigma(n)}],$$

and

$$(2.2) \quad [[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n],$$

where σ runs over the symmetric group S_n and the number $\tau(\sigma)$ is equal to 0 or 1 depending on the parity of the permutation σ .

A derivation of an n -Lie algebra A is a linear map $D : A \rightarrow A$, such that for any elements x_1, \dots, x_n of A

$$D([x_1, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, D(x_i), \dots, x_n].$$

The set of all derivations of A is a subalgebra of Lie algebra $\text{gl}(A)$. This subalgebra is called the derivation algebra of A , and is denoted by $\text{Der}A$. The map $\text{ad}(x_1, \dots, x_{n-1}) : A \rightarrow A$ defined by $\text{ad}(x_1, \dots, x_{n-1})(x_n) = [x_1, \dots, x_n]$ for $x_1, \dots, x_n \in A$ is called a left multiplication. It follows from (2.2) that $\text{ad}(x_1, \dots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $\text{Der}A$ and is denoted by $\text{ad}(A)$. Every element in $\text{ad}(A)$ is by definition an inner derivation, and for all $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$ of A ,

$$(2.3) \quad [\text{ad}(x_1, \dots, x_{n-1}), \text{ad}(y_1, \dots, y_{n-1})]$$

$$= \text{ad}(x_1, \dots, x_{n-1})\text{ad}(y_1, \dots, y_{n-1}) - \text{ad}(y_1, \dots, y_{n-1})\text{ad}(x_1, \dots, x_{n-1})$$

$$= \sum_{i=1}^{n-1} \text{ad}(y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_{n-1}).$$

Let A_1, A_2, \dots, A_n be subalgebras of n -Lie algebra A and let $[A_1, A_2, \dots, A_n]$ denote the subspace of A generated by all vectors $[x_1, x_2, \dots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \dots, n$. The subalgebra $[A, A, \dots, A]$ is called the derived algebra of A , and is denoted by A^1 . If $A^1 = 0$, then A is called an abelian n -Lie algebra.

An ideal of an n -Lie algebra A is a subspace I such that $[I, A, \dots, A] \subseteq I$. If $A^1 \neq 0$ and A has no ideals except for 0 and itself, then A is by definition a simple n -Lie algebra.

An ideal I of an n -Lie algebra A is called a solvable ideal, if $I^{(r)} = 0$ for some $r \geq 0$, where $I^{(0)} = I$ and $I^{(s)}$ is defined by induction,

$$I^{(s+1)} = [I^{(s)}, I^{(s)}, A, \dots, A]$$

for $s \geq 0$. When $A = I$, A is a solvable n -Lie algebra.

An ideal I of an n -Lie algebra A is called a nilpotent ideal, if I satisfies $I^r = 0$ for some $r \geq 0$, where $I^0 = I$ and I^r is defined by induction, $I^{r+1} = [I^r, I, A, \dots, A]$ for $r \geq 0$. If $I = A$, A is called a nilpotent n -Lie algebra.

The sum of two nilpotent ideals of A is nilpotent, and the largest nilpotent ideal of A is called the nilradical of A , and is denoted by $NR(A)$.

Denote by A^* an associative algebra generated by all operators $\text{ad}(x)$, where $x = (x_1, \dots, x_{n-1}) \in A^{(n-1)}$. If I is an ideal of A , denote by I^* , $K(I)$ and $\text{ad}(I, A)$ respectively the subalgebra of A^* , the ideal of A^* and the subalgebra of $\text{ad}(A)$ generated by the operators of the form $\text{ad}(c, x_1, \dots, x_{n-2})$, $c \in I, x_i \in A, i = 1, \dots, n-2$. It follows at once from (2.3) that $K(I) = I^* \cdot A^* = A^* \cdot I^*$, and $\text{ad}(I, A)$ is an ideal of $\text{ad}(A)$.

LEMMA 2.1. [14] *An ideal I of an n -Lie algebra A is a nilpotent ideal if and only if $K(I)$ is a nilpotent ideal of the associative algebra A^* .*

An ideal I of an n -Lie algebra A may not be a nilpotent ideal although it is a nilpotent subalgebra. This property is different from that of Lie algebras. In the following, we concern such types of ideals of n -Lie algebras.

DEFINITION 2.2. *Let A be an n -Lie algebra and I be an ideal of A . If I is a nilpotent subalgebra but is not a nilpotent ideal, then I is called a hypo-nilpotent ideal of A . If I is not properly contained in any hypo-nilpotent ideals, then I is called a maximal hypo-nilpotent ideal of A .*

From (2.2), a hypo-nilpotent ideal of A is a proper ideal, and the nilradical $NR(A)$ is properly contained in every maximal hypo-nilpotent ideals. But the sum of two hypo-nilpotent ideals of A may not be hypo-nilpotent.

In the following, any brackets of basis vectors not listed in the multiplication table of n -Lie algebras are assumed to be zero.

3. 3-Lie algebras with maximal hypo-nilpotent ideal N . In the following we suppose that N is an m -dimensional filiform 3-Lie algebra with the multiplication table

$$(3.1) \quad \begin{cases} [e_1, e_2, e_j] = e_{j-1}, & 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2}, & 5 \leq j \leq m-1, \end{cases}$$

where e_1, \dots, e_m is a basis of N .

LEMMA 3.1. *Let N be an m -dimensional 3-Lie algebra with a basis e_1, \dots, e_m satisfying (3.1). Then the inner derivation algebra $\text{ad}(N)$ has a basis $\text{ad}(e_1, e_2)$,*

$ad(e_1, e_j), ad(e_2, e_j), j = 4, 5, \dots, m$. And with respect to the basis e_1, \dots, e_m , $ad(e_k, e_l)$ is represented by the following matrix form

$$ad(e_1, e_2) = \sum_{j=4}^m E_{jj-1}, ad(e_1, e_m) = \sum_{j=5}^{m-1} E_{jj-2} + E_{2m-1},$$

$$ad(e_1, e_i) = E_{2i-1} + E_{mi-2}, ad(e_2, e_i) = E_{1i-1} \text{ for } 5 \leq i \leq m-1,$$

$$ad(e_1, e_4) = E_{23}, ad(e_2, e_4) = E_{13}, ad(e_2, e_m) = E_{1m-1},$$

where E_{ij} is the $(m \times m)$ matrix unit.

Proof. The result follows from a direct computation. \square

Let A be an $(m+1)$ -dimensional 3-Lie algebra with the ideal N , and x, e_1, \dots, e_m be a basis of A . Then the multiplication table of A in the basis x, e_1, \dots, e_m is given by

$$(3.2) \quad \begin{cases} [e_1, e_2, e_j] = e_{j-1}, & 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2}, & 5 \leq j \leq m-1, \\ [x, e_i, e_j] = \sum_{k=1}^m a_{ij}^k e_k, & 1 \leq i, j \leq m, \end{cases}$$

where $a_{ij}^k \in F, a_{ij}^k = -a_{ji}^k, 1 \leq i, j \leq m$. Therefore, the following $(\frac{m(m-1)}{2} \times m)$ matrix M determines the structure of A

$$(3.3) \quad M = \begin{pmatrix} a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{12}^4 & \cdot & a_{12}^{m-1} & a_{12}^m \\ a_{13}^1 & a_{13}^2 & a_{13}^3 & a_{13}^4 & \cdot & a_{13}^{m-1} & a_{13}^m \\ a_{14}^1 & a_{14}^2 & a_{14}^3 & a_{14}^4 & \cdot & a_{14}^{m-1} & a_{14}^m \\ a_{15}^1 & a_{15}^2 & a_{15}^3 & a_{15}^4 & \cdot & a_{15}^{m-1} & a_{15}^m \\ \vdots & \vdots & \vdots & \vdots & \cdot & \vdots & \vdots \\ a_{1m-2}^1 & a_{1m-2}^2 & a_{1m-2}^3 & a_{1m-2}^4 & \cdot & a_{1m-2}^{m-1} & a_{1m-2}^m \\ a_{1m-1}^1 & a_{1m-1}^2 & a_{1m-1}^3 & a_{1m-1}^4 & \cdot & a_{1m-1}^{m-1} & a_{1m-1}^m \\ a_{1m}^1 & a_{1m}^2 & a_{1m}^3 & a_{1m}^4 & \cdot & a_{1m}^{m-1} & a_{1m}^m \\ a_{23}^1 & a_{23}^2 & a_{23}^3 & a_{23}^4 & \cdot & a_{23}^{m-1} & a_{23}^m \\ a_{24}^1 & a_{24}^2 & a_{24}^3 & a_{24}^4 & \cdot & a_{24}^{m-1} & a_{24}^m \\ a_{25}^1 & a_{25}^2 & a_{25}^3 & a_{25}^4 & \cdot & a_{25}^{m-1} & a_{25}^m \\ \vdots & \vdots & \vdots & \vdots & \cdot & \vdots & \vdots \\ a_{2m-2}^1 & a_{2m-2}^2 & a_{2m-2}^3 & a_{2m-2}^4 & \cdot & a_{2m-2}^{m-1} & a_{2m-2}^m \\ a_{2m-1}^1 & a_{2m-1}^2 & a_{2m-1}^3 & a_{2m-1}^4 & \cdot & a_{2m-1}^{m-1} & a_{2m-1}^m \\ a_{2m}^1 & a_{2m}^2 & a_{2m}^3 & a_{2m}^4 & \cdot & a_{2m}^{m-1} & a_{2m}^m \\ a_{34}^1 & a_{34}^2 & a_{34}^3 & a_{34}^4 & \cdot & a_{34}^{m-1} & a_{34}^m \\ \vdots & \vdots & \vdots & \vdots & \cdot & \vdots & \vdots \\ a_{m-1m}^1 & a_{m-1m}^2 & a_{m-1m}^3 & a_{m-1m}^4 & \cdot & a_{m-1m}^{m-1} & a_{m-1m}^m \end{pmatrix}.$$

The matrix M is called the structure matrix of A with respect to the basis x, e_1, \dots, e_m .

By the above notations we have the following result.

THEOREM 3.2. *Let A be an $(m + 1)$ -dimensional 3-Lie algebra with a maximal hypo-nilpotent ideal N . Then A is solvable, and up to isomorphism the following is the only possibility for the structural matrix M of A :*

$$(3.4) \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & m-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m-2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m-3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, the multiplication table of A in a basis x, e_1, \dots, e_m ($m \geq 5$) is as follows

$$(3.5) \quad \begin{cases} [e_1, e_2, e_j] = e_{j-1} & \text{if } 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2} & \text{if } 5 \leq j \leq m-1, \\ [x, e_1, e_2] = e_2, \\ [x, e_1, e_k] = (m-k+2)e_k & \text{if } 3 \leq k \leq m. \end{cases}$$

Proof. Since N is an ideal of A and $\dim A = m + 1$, we have $A^1 = [A, A, A] = [A, A, N] \subseteq N$. Then the structural matrix M is of the form (3.3) with respect to a basis x, e_1, \dots, e_m .

Firstly, imposing the Jacobi identities

$$[[x, e_1, e_2], e_1, e_j] = [[x, e_1, e_j], e_1, e_2] + [x, e_1, [e_2, e_1, e_j]] \text{ for } 3 \leq j \leq m-1,$$

and

$$[[x, e_1, e_2], e_1, e_m] = [[x, e_1, e_m], e_1, e_2] + [x, e_1, [e_2, e_1, e_m]],$$

and using (3.2), we obtain

$$a_{1j}^1 = a_{1j}^2 = 0, a_{1j}^{j+1} = a_{1j}^{j+2} = \dots = a_{1j}^m = 0 \text{ for } 3 \leq j \leq m-2;$$

$$a_{1j-1}^{j-1} = a_{1j}^j + a_{12}^2, 4 \leq j \leq m-1;$$

$$a_{12}^m = a_{1j}^{j-1} - a_{1j-1}^{j-2}, a_{1j}^k = a_{1j-1}^{k-1}, k \neq j-1, \text{ for } 4 \leq k \leq m, 5 \leq j \leq m-1,$$

and

$$a_{12}^2 + a_{1m}^m - a_{1m-1}^{m-1} = 0, a_{1m-1}^1 = a_{1m-1}^2 = a_{1m-1}^m = 0,$$

$$a_{12}^{i+1} + a_{1m}^i - a_{1m-1}^{i-1} = 0, 4 \leq i \leq m-2, a_{1m}^{m-1} = a_{1m-1}^{m-2}$$

respectively.

Secondly, imposing the Jacobi identities on $\{[x, e_1, e_2], e_2, e_j\}$ for $3 \leq j \leq m$, we get

$$a_{2j}^1 = a_{2j}^2 = 0, a_{2j}^{j+1} = a_{2j}^{j+2} = \dots = a_{2j}^m = 0 \text{ for } 3 \leq j \leq m-1;$$

$$a_{2j-1}^{j-1} = a_{2j}^j - a_{12}^1, 4 \leq j \leq m; a_{2j}^k = a_{2j-1}^{k-1}, \text{ for } 4 \leq k < j \leq m.$$

From

$$[[x, e_1, e_i], e_2, e_j] = [[x, e_2, e_j], e_1, e_i] + [x, [e_1, e_2, e_j], e_i] = 0, \text{ for } 3 \leq i, j \leq m-1,$$

$$[[x, e_1, e_4], e_1, e_m] = [[x, e_1, e_m], e_1, e_4],$$

and

$$[[x, e_1, e_i], e_1, e_m] = [[x, e_1, e_m], e_1, e_i] - [x, e_1, e_{i-2}] \text{ for } 5 \leq i \leq m-1,$$

we get

$$a_{ij}^k = 0, 3 \leq i, j \leq m-1, 1 \leq k \leq m, a_{1m}^2 = 0,$$

50

R.-P. Bai, C.-H. Shen, and Y.-Z. Zhang

and

$$a_{1i}^i + a_{1m}^m - a_{1i-2}^{i-2} = 0, a_{1i}^k = a_{1i-2}^{k-2} \text{ for } k \neq i, 5 \leq k \leq m-1,$$

$$a_{1i-2}^1 = a_{1i-2}^2 = a_{1i-2}^{m-2} = a_{1i-2}^{m-1} = a_{1i-2}^m = 0.$$

Then we have $a_{1m}^m = 2a_{12}^2, a_{12}^m = 0$.

Again from

$$[[x, e_2, e_4], e_1, e_m] = [[x, e_1, e_m], e_2, e_4],$$

$$[[x, e_1, e_m], e_2, e_4] = [[x, e_2, e_4], e_1, e_m] + [x, e_3, e_m],$$

$$[[x, e_2, e_i], e_1, e_m] = [[x, e_1, e_m], e_2, e_i] - [x, e_2, e_{i-2}], 5 \leq i \leq m-1,$$

$$[[x, e_1, e_4], e_2, e_m] = [[x, e_2, e_m], e_1, e_4],$$

and

$$[[x, e_1, e_i], e_2, e_m] = [[x, e_2, e_m], e_1, e_i],$$

we get

$$a_{1m}^1 = 0, a_{3m}^k = 0, 1 \leq k \leq m, a_{2i}^k = a_{2i-2}^{k-2} \text{ for } 5 \leq k \leq m-1,$$

$$a_{2i-2}^1 = a_{2i-2}^2 = a_{2i-2}^{m-2} = a_{2i-2}^{m-1} = a_{2i-2}^m = 0, a_{12}^1 = 0, a_{2m}^2 = 0 \text{ and } a_{2m}^m = 0.$$

By

$$[[x, e_2, e_m], e_1, e_i] = [[x, e_1, e_i], e_2, e_m] - [x, e_{i-1}, e_m] + [x, e_2, e_{i-2}], 5 \leq i \leq m-1,$$

$$[[x, e_2, e_4], e_2, e_m] = [[x, e_2, e_m], e_2, e_4],$$

and

$$[[x, e_1, e_m], e_2, e_m] = [[x, e_2, e_m], e_1, e_m] + [x, e_{m-1}, e_m],$$

we obtain

$$a_{i-1m}^j = a_{2i-2}^j, 1 \leq j \leq m, a_{2m}^1 = 0, a_{2m}^i = a_{m-1m}^{i-2} \text{ for } 5 \leq i \leq m-1,$$

$$a_{m-1m}^1 = a_{m-1m}^2 = a_{m-1m}^{m-2} = a_{m-1m}^{m-1} = a_{m-1m}^m = 0.$$

Therefore, we get

$$\text{ad}(x, e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & a_{12}^3 & a_{12}^4 & a_{12}^5 & \cdot & a_{12}^{m-3} & a_{12}^{m-2} & a_{12}^{m-1} & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-1} & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m-1}^{m-3} & a_{1m}^{m-1} & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m-1}^{m-4} & a_{1m-1}^{m-3} & a_{1m}^{m-1} & \cdot & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_{1m-1}^4 & a_{1m-1}^5 & a_{1m-1}^6 & \cdot & a_{1m}^{m-1} & r_{m-4} a_{12}^2 & 0 & 0 \\ 0 & 0 & a_{1m-1}^3 & a_{1m-1}^4 & a_{1m-1}^5 & \cdot & a_{1m-1}^{m-3} & a_{1m}^{m-1} & r_{m-3} a_{12}^2 & 0 \\ 0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdot & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & r_{m-2} a_{12}^2 \end{pmatrix},$$

where $r_j = m - j$ for $1 \leq j \leq m - 2$, $a_{1m-1}^{i-1} = a_{12}^{i+1} + a_{1m}^i$ for $4 \leq i \leq m - 2$;

$$\text{ad}(x, e_2)|_N = \begin{pmatrix} 0 & -a_{12}^2 & -a_{12}^3 & -a_{12}^4 & -a_{12}^5 & \cdots & -a_{12}^{m-3} & -a_{12}^{m-2} & -a_{12}^{m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & a_{2m}^{m-1} & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 \end{pmatrix},$$

$$-\text{ad}(x, e_m)|_N = \begin{pmatrix} 0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m}^m \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^6 & a_{2m}^7 & a_{2m}^8 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & a_{2m}^{m-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we replace x by $x - a_{2m}^{m-1}e_1 + a_{1m}^{m-1}e_2 - a_{12}^3e_4 - a_{12}^4e_5 - \cdots - a_{12}^{m-1}e_m$, the above

maps are reduced to

$$\begin{aligned} & \text{ad}(x, e_1)|_N \\ = & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-2} & 0 & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & b_{1m}^5 & b_{1m}^6 & b_{1m}^7 & \cdot & 0 & r_{m-4} a_{12}^2 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^4 & b_{1m}^5 & b_{1m}^6 & \cdot & b_{1m}^{m-2} & 0 & r_{m-3} a_{12}^2 & 0 & 0 \\ 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdot & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & r_{m-2} a_{12}^2 & 0 \end{pmatrix}, \end{aligned}$$

where $r_j = m - j$, for $1 \leq j \leq m - 2$,

$$\begin{aligned} & \text{ad}(x, e_2)|_N \\ = & \begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} & -\text{ad}(x, e_m)|_N \\ = & \begin{pmatrix} 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & 2a_{12}^2 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^6 & a_{2m}^7 & a_{2m}^8 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Again by the Jacobi identities for vectors $\{[x, e_1, e_2], x, e_i\}$ for $3 \leq i \leq m$, we get

$a_{12}^2 a_{2m}^i = 0$ for $i = 3, 4, \dots, m-2$. Since $a_{12}^2 \neq 0$ (if $a_{12}^2 = 0$, then A is nilpotent), we get $a_{2m}^i = 0$, for $i = 3, 4, \dots, m-2$.

Therefore,

$$\begin{aligned} & \text{ad}(x, e_1)|_N \\ = & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-2} & 0 & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdot & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & b_{1m}^5 & b_{1m}^6 & b_{1m}^7 & \cdot & 0 & r_{m-4} a_{12}^2 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^4 & b_{1m}^5 & b_{1m}^6 & \cdot & b_{1m}^{m-2} & 0 & r_{m-3} a_{12}^2 & 0 & 0 \\ 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdot & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & r_{m-2} a_{12}^2 & 0 \end{pmatrix}, \end{aligned}$$

where $r_j = m - j$ for $1 \leq j \leq m - 2$,

$$\begin{aligned} & \text{ad}(x, e_2)|_N \\ = & \begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} & -\text{ad}(x, e_m)|_N \\ = & \begin{pmatrix} 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & 2a_{12}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \end{aligned}$$

that is

$$[x, e_1, e_2] = a_{12}^2 e_2, [x, e_1, e_3] = (m-1)a_{12}^2 e_3, [x, e_1, e_4] = (m-2)a_{12}^2 e_4,$$

$$[x, e_1, e_k] = \sum_{j=3}^{k-2} b_{2m}^{m-k+j} e_j + (m-k+2)a_{12}^2 e_k, \text{ for } k = 5, 6, \dots, m,$$

and other brackets of the basis vectors are equal to zero.

For any l satisfying $3 \leq l \leq m-2$, we take a series of linear transformations defined by

$$\tilde{e}_k = e_k \text{ for } 1 \leq k \leq l+1 \text{ and } \tilde{e}_k = e_k - \frac{b_{1m}^{m-l+1}}{(l-1)a_{12}^2} e_{k-l+1} \text{ for } l+2 \leq k \leq m.$$

Then the basis vectors $\tilde{e}_1, \dots, \tilde{e}_m$ satisfy (3.1). After replacing x by $\frac{x}{a_{12}^2}$, we get the structural matrix M of A with respect to the basis vectors $x, \tilde{e}_1, \dots, \tilde{e}_m$ as follows

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & m-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m-2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m-3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the multiplication table of A is

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1} & \text{for } 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2} & \text{for } 5 \leq j \leq m-1, \\ [x, e_1, e_2] = e_2, \\ [x, e_1, e_k] = (m-k+2)e_k & \text{for } 3 \leq k \leq m. \quad \square \end{cases}$$

THEOREM 3.3. *Let A be a solvable $(m + k)$ -dimensional 3-Lie algebra with the maximal hypo-nilpotent ideal N . Then we have $k = 1$.*

Proof. If $k \geq 2$, let $x_1, \dots, x_k, e_1, \dots, e_m$ be a basis of A . Thanks to the solvability of A , we have $[A, A, A] \subseteq N$. By the discussions of the proof of Theorem 3.2, we might as well suppose

$$\text{ad}(x_1, e_1)|_N = \text{diag}(0, 1, m - 1, m - 2, \dots, 4, 3, 2),$$

$$\text{ad}(x_1, e_2)|_N = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{ad}(x_1, e_m)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} & \text{ad}(x_2, e_1)|_N \\ = & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-2} & 0 & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdot & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & b_{1m}^5 & b_{1m}^6 & b_{1m}^7 & \cdot & 0 & r_{m-4} a_{12}^2 & 0 & 0 \\ 0 & 0 & b_{1m}^4 & b_{1m}^5 & b_{1m}^6 & \cdot & b_{1m}^{m-2} & 0 & r_{m-3} a_{12}^2 & 0 \\ 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdot & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & r_{m-2} a_{12}^2 \end{pmatrix}, \end{aligned}$$

where $r_j = m - j$ for $1 \leq j \leq m - 2$

$$\text{ad}(x_2, e_2)|_N = \begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \end{pmatrix},$$

$$-\text{ad}(x_2, e_m)|_N = \begin{pmatrix} 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & 2a_{12}^2 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^6 & a_{2m}^7 & a_{2m}^8 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\text{ad}(x_2, e_1)|_N - a_{12}^2 \text{ad}(x_1, e_1)|_N$, $\text{ad}(x_2, e_2)|_N - a_{12}^2 \text{ad}(x_1, e_2)|_N$ and

$$\text{ad}(x_2, e_m)|_N - a_{12}^2 \text{ad}(x_1, e_m)|_N$$

are nilpotent. It follows that $I = F(x_2 - a_{12}^2 x_1) + N$ is an $(m + 1)$ -dimensional hypo-nilpotent ideal of A . This is a contradiction. Therefore, we have $k = 1$. \square

COROLLARY 3.4. *There are no $(m + k)$ -dimensional solvable 3-Lie algebras with a maximal hypo-nilpotent ideal N when $k \geq 2$.*

4. 3-Lie algebras with nilradical N . In this section we study the solvable 3-Lie algebras with the nilradical N .

THEOREM 4.1. *There are no solvable non-nilpotent 3-Lie algebras with nilradical N .*

Proof. First let A be an $(m + k)$ -dimensional 3-Lie algebra with the nilpotent ideal N , where $1 \leq k \leq 2$. We will prove that A is nilpotent.

If $k = 1$, suppose x, e_1, \dots, e_m is a basis of A . Then the associative algebra A^* is generated by left multiplications $\text{ad}(x, e_i)$ and $\text{ad}(e_i, e_j)$, where $1 \leq i, j \leq m$. Therefore, we have $A^* = K(N, A)$. It follows from Lemma 2.1 that A is nilpotent.

If $k = 2$, let $x_1, x_2, e_1, \dots, e_m$ be a basis of A . Set $B = Fx_1 + Fe_1 + \dots + Fe_m$ and $C = Fx_2 + Fe_1 + \dots + Fe_m$. Then B and C are $(m+1)$ -dimensional subalgebras of A with the nilpotent ideal N . It follows from the result of the case $k = 1$, and Theorem 3.2 that the matrices of $\text{ad}(x_i, e_j)|_N$ ($i = 1, 2, 1 \leq j \leq m$) with respect to e_1, \dots, e_m are of the form

$$S = \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 & \cdots & a_{m-3} & a_{m-2} & 0 & 0 \\ 0 & 0 & b_3 & b_4 & b_5 & \cdots & b_{m-3} & b_{m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & c_{m-4} & c_{m-5} & c_{m-6} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{m-3} & c_{m-4} & c_{m-5} & \cdots & c_3 & 0 & 0 & 0 \\ 0 & 0 & c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_4 & c_3 & 0 & 0 \end{pmatrix},$$

where $a_l, b_l, c_l \in F, 3 \leq l \leq m - 2$. Therefore, $\text{ad}(x_i, e_j)$ are nilpotent maps of A for $i = 1, 2; j = 1, \dots, m$. Now suppose

$$[x_1, x_2, e_i] = \sum_{j=1}^m r_{ij} e_j, 1 \leq i \leq m.$$

With the help of the Jacobi identities for $\{[x_1, x_2, e_i], e_1, e_2\}, \{[x_1, x_2, e_i], e_1, e_4\}, i = 1, 2, \dots, m; \{[x_1, x_2, e_1], e_2, e_i\}$ for $4 \leq i \leq m$, we get that $\text{ad}(x_1, x_2)|_N$ has the form

$$\begin{pmatrix} 0 & 0 & r_{13} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{23} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{m3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\text{ad}(x_1, x_2)|_N$ is nilpotent, and $\text{ad}(x_1, x_2)$ is also nilpotent to A . This proves that A is nilpotent when $k = 2$.

Last we suppose that there is a solvable non-nilpotent $(m+k)$ -dimensional 3-Lie algebra with the nilradical N for $k \geq 3$. Let $x_1, \dots, x_k, e_1, \dots, e_m$ be a basis of A . Then there exist x_i, x_j such that $\text{ad}(x_i, x_j)|_N$ is not nilpotent. Set $T = Fx_i + Fx_j + Fe_1 + \dots + Fe_m$, then N is a nilpotent ideal of $(m+2)$ -dimensional subalgebra T . From the above discussions, T is a nilpotent subalgebra. Hence there exists an integer r such that $\text{ad}^r(x_i, x_j)(T) = 0$. Since A is solvable and N is the nilradical of A , we have $[A, \dots, A] \subseteq N$. Therefore,

$$\text{ad}^{r+1}(x_i, x_j)(A) \subseteq \text{ad}^r(x_i, x_j)(N) \subseteq \text{ad}^r(x_i, x_j)(T) = 0.$$

This is a contradiction. \square

REMARK 4.2. *The solvable condition in Theorem 4.1 is necessary. See the following example. Let A be an $(m+4)$ -dimensional 3-Lie algebra with the basis $x_1, x_2, x_3, x_4, e_1, \dots, e_m$, and the multiplication table*

$$\begin{cases} [x_1, x_2, x_4] = x_3, \\ [x_1, x_3, x_4] = x_2, \\ [x_2, x_3, x_4] = x_1, \\ [x_4, e_1, e_2] = e_3, \\ [e_1, e_2, e_j] = e_{j-1} \text{ for } 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2} \text{ for } 5 \leq j \leq m-1. \end{cases}$$

By a direct computation we get that N is the nilradical of A , and

$$A^{(1)} = Fx_1 + Fx_2 + Fx_3 + Fe_3 + \dots + Fe_{m-1},$$

$$A^{(s)} = Fx_1 + Fx_2 + Fx_3 \neq 0, s > 1.$$

It follows that A is an unsolvable 3-Lie algebra.

5. One dimensional extension of Lie algebras. In this section we describe the one dimensional extension of Lie algebras, first introduced in [6]. As an application of it we get all classes of solvable Lie algebras with the special nilradical given in [19, 20].

For any given s -dimensional Lie algebra g with a basis y_1, \dots, y_s and the multiplication table

$$[y_i, y_j] = \sum_{k=1}^s a_{ij}^k y_k, 1 \leq i, j \leq s$$

where a_{ij}^k are structure constants, we can define a corresponding 3-Lie algebra as follows. Let y_0, y_1, \dots, y_s be the basis of the $(s+1)$ -dimensional vector space L_g . The

3-ary multiplication table of L_g is defined by

$$\begin{cases} [y_0, y_i, y_j] = \sum_{k=1}^s a_{ij}^k y_k, 1 \leq i, j \leq s, \\ [y_t, y_i, y_j] = 0, 1 \leq t, i, j \leq s. \end{cases}$$

It is not difficult to check that L_g is a 3-Lie algebra. L_g is called the one dimensional extension of the Lie algebra g . Then we have following results.

THEOREM 5.1. *Let I be a subalgebra of g , then I is an ideal of Lie algebra g if and only if I is an ideal of 3-Lie algebra L_g , and I is a solvable (nilpotent) ideal of g if and only if I is a solvable (nilpotent) ideal of L_g .*

Proof. Since $[L_g, L_g, I] = [y_0, g, I] = [g, I] \subseteq I$, we get the first result. Denote the derived series (descending central series) of I in 3-Lie algebra L_g by $I_g^{(s)}$ (I_g^s), that is $I_g^{(s+1)} = [I_g^{(s)}, I_g^{(s)}, L_g]$, ($I_g^{s+1} = [I_g^s, I, L_g]$) for $s \geq 0$, $I_g^{(0)} = I = I^{(0)}$, ($I^0 = I = I^0$). By induction on s we get

$$I^{(s+1)} = [I^{(s)}, I^{(s)}] = [y_0, I^{(s)}, I^{(s)}] = [L_g, I_g^{(s)}, I_g^{(s)}] = I_g^{(s+1)}, s \geq 0,$$

$$I^{s+1} = [I^s, I] = [y_0, I^s, I] = [L_g, I_g^s, I] = I_g^{s+1}, s \geq 0.$$

It follows that $I^{(s+1)} = 0$ if and only if $I_g^{(s+1)} = 0$, and $I^{s+1} = 0$ if and only if $I_g^{s+1} = 0$. \square

THEOREM 5.2. *Let I be an ideal of Lie algebra g . Then $J = I + Fy_0$ is an ideal of 3-Lie algebra L_g , and I is a solvable ideal of g if and only if J is a solvable ideal of L_g .*

Proof. It is evident that J is an ideal of L_g if I is an ideal of L . Since

$$J^{(1)} = [J, J, L_g] \subseteq I, J^{(2)} = [J^{(1)}, J^{(1)}, L_g] \subseteq [I, I, L_g] = [I, I] = I^{(1)},$$

by induction on s , we get

$$J^{(s+1)} = [J^{(s)}, J^{(s)}, L_g] \subseteq [I^{(s-1)}, I^{(s-1)}] = I^{(s)}.$$

Conversely,

$$I^{(1)} = [I, I] = [I, I, y_0] \subseteq [J, J, L_g] = J^{(1)},$$

by induction on s , we get

$$I^{(s+1)} = [I^{(s)}, I^{(s)}] = [I^{(s)}, I^{(s)}, y_0] \subseteq [J^{(s)}, J^{(s)}, L_g] = J^{(s+1)}.$$

Therefore, I is a solvable ideal of L if and only if J is a solvable ideal of L_g . \square

REMARK 5.3. *If I is a nilpotent ideal of g , then $J = I + Fy_0$ is a nilpotent subalgebra of L_g , but J may not be a nilpotent ideal of L_g . See the following example.*

EXAMPLE 5.4. *Suppose $g = Fx + Fe_1 + Fe_2 + Fe_3 + Fe_4$ is a 5-dimensional Lie algebra and the multiplication table of g in the basis x, e_1, e_2, e_3, e_4 is*

$$\left\{ \begin{array}{l} [e_1, e_3] = e_2, \\ [e_1, e_4] = e_3, \\ [x, e_1] = e_1, \\ [x, e_2] = 2e_2, \\ [x, e_3] = e_3. \end{array} \right.$$

$I = Fe_1 + Fe_2 + Fe_3 + Fe_4$ is a nilpotent ideal of g since $I^3 = 0$. Let $L_g = Fy_0 + g$ is the one dimensional extension of g . Then the multiplication table of L_g in the basis $y_0, x, e_1, e_2, e_3, e_4$ is as follows

$$\left\{ \begin{array}{l} [y_0, e_1, e_3] = e_2, \\ [y_0, e_1, e_4] = e_3, \\ [y_0, x, e_1] = e_1, \\ [y_0, x, e_2] = 2e_2, \\ [y_0, x, e_3] = e_3. \end{array} \right.$$

Then $J = Fy_0 + I = Fy_0 + Fe_1 + Fe_2 + Fe_3 + Fe_4$ is an ideal of L_g . Since $J^s = J^1 = Fe_1 + Fe_2 + Fe_3 \neq 0$ for $s \geq 1$, J is not a nilpotent ideal of L_g .

Suppose $N_1 = N_0 = Fe_1 + \dots + Fe_m$ (as vector spaces), with $m \geq 4$, the multiplication table of Lie algebra N_0 in the basis e_1, \dots, e_m is as follows

$$[e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m,$$

and the multiplication table of Lie algebra N_1 in the basis e_1, \dots, e_m is

$$\left\{ \begin{array}{l} [e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m, \\ [e_j, e_m] = e_{j-2} \text{ for } 4 \leq j \leq m - 1. \end{array} \right.$$

In [19, 20], authors constructed all solvable Lie algebras with the nilradical N_0 and N_1 respectively. By Theorem 3.1, and Theorem 4.1 and Theorem 4.2 in [18], we have

(1). Let A be an $(m + k)$ -dimensional solvable Lie algebra with the nilradical N_1 ($k \geq 1$). Then we have $k = 1$, and up to isomorphism the following is the only possibility:

$$\left\{ \begin{array}{l} [e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m, \\ [e_j, e_m] = e_{j-2} \text{ for } 4 \leq j \leq m - 1, \\ [x, e_1] = e_1, \\ [x, e_k] = (m - k + 2)e_k, \text{ for } 2 \leq k \leq m. \end{array} \right.$$

(2). Let A be an $(m+k)$ -dimensional solvable Lie algebra with the nilradical N_0 ($k \geq 1, m \geq 4$). Then we have $k = 1$ or $k = 2$. And in the case of $k = 1$, up to isomorphisms one and only one of the following possibilities holds:

$$\begin{aligned}
 (M_1) \cdot \left\{ \begin{array}{l} [e_1, e_j] = e_{j-1}, \\ [x, e_2] = e_2, \\ [x, e_3] = e_3, \\ [x, e_r] = \sum_{k=2}^{r-2} b_{r-k+1} e_k + e_r; \end{array} \right. & \quad (M_2) \left\{ \begin{array}{l} [e_1, e_j] = e_{j-1}, \\ [x, e_1] = e_1, \\ [x, e_t] = (m-t)e_t; \end{array} \right. \\
 (M_3) \left\{ \begin{array}{l} [e_1, e_j] = e_{j-1}, \\ [x, e_1] = e_1, \\ [x, e_t] = (m-t+\alpha)e_t; \end{array} \right. & \quad (M_4) \left\{ \begin{array}{l} [e_1, e_j] = e_{j-1}, \\ [x, e_1] = e_1 - e_m, \\ [x, e_t] = (m-t+1)e_t; \end{array} \right.
 \end{aligned}$$

where $3 \leq j \leq m$, $2 \leq t \leq m$, $4 \leq r \leq m$, $b_{ij}, \alpha \in F$, and $\alpha \neq 0$.

In the case of $k = 2$, up to isomorphism the only possibility is the following:

$$\left\{ \begin{array}{l} [e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m, \\ [x_1, e_2] = e_2, \\ [x_1, e_i] = (m-i)e_i \text{ for } 2 \leq i \leq m, \\ [x_2, e_i] = e_i \text{ for } 2 \leq i \leq m. \end{array} \right.$$

REFERENCES

- [1] Y. Nambu. Generalized Hamiltonian dynamics. *Physical Reviews*, D7:2405-2412, 1973.
- [2] V. Filippov. n -Lie algebras. *Sibirskii Matematicheskii Zhurnal*, 26:126-140, 1985.
- [3] L. Takhtajan. On foundation of the generalized Nambu mechanics. *Communications in Mathematical Physics*, 160:295-315, 1994.
- [4] J. Bagger and N. Lambert. Gauge symmetry and supersymmetry of multiple $M2$ -branes. *Physical Reviews*, D77:035008, 2008.
- [5] J. Bagger and N. Lambert. Modeling multiple $M2$'s. *Physical Reviews*, D75:045020, 2007.
- [6] P. Ho, R. Hou, and Y. Matsuo. Lie 3-algebra and multiple M_2 -branes. arXiv:0804.2110v2 [hep-th].
- [7] A. Vinogradov and M. Vinogradov. On multiple generalizations of Lie algebras and Poisson manifolds. *Contemporary Mathematics*, 219:273-287, 1998.
- [8] G. Marmo, G. Vilasi, and A. Vinogradov. The local structure of n -Poisson and n -Jacobi manifolds. *Journal of Geometry and Physics*, 25:141-182, 1998.
- [9] P. Gautheron. Some remarks concerning Nambu mechanics. *Letters in Mathematical Physics*, 37:103-116, 1996.
- [10] A. Panov. Multiple Poisson brackets. *Vestn. Samar. Gos. Univ. Mat. Mekh. Fiz. Khim. Biol.*, 2:33-42, 1996.
- [11] T.L. Curtright, C.K. Zachos. Branes, strings, and odd quantum Nambu brackets. arXiv:hep-th/0312048.
- [12] P. Ho, M. Chebotar and W. Ke. On skew-symmetric maps on Lie algebras. *Proceedings of the Royal Society Edinburgh*, 133A:1273-1281, 2003.

- [13] G. Papadopoulos. *M2-branes, 3-Lie algebras and Plucker relations*. arXiv:0804.2662 [hep-th].
- [14] S. Kasymov. On a theory of n -Lie algebras. *Algebra i Logika*, 26:277-297, 1987.
- [15] W. Ling. *On the structure of n -Lie algebras*. Dissertation, University-GHS-Siegen, Siegen, 1993.
- [16] R.-P. Bai, X. Wang, W. Xiao and H. An. Structure of low dimensional n -Lie algebras over a field of characteristic 2. *Linear Algebra and its Applications*, 428:1912-1920, 2008.
- [17] R.-P. Bai and G. Song. The classification of six-dimensional 4-Lie algebras. *Journal of Physics A: Mathematical and Theoretical*, 42:035207, 2009.
- [18] R.-P. Bai, C.-H. Shen and Y.-Z. Zhang. 3-Lie Algebras with an Ideal N . *Linear Algebra and its Applications*, 431:673-700, 2009.
- [19] L Snobl and P Winternitz. A class of solvable Lie algebras and their Casimir invariants. *Journal of Physics A: Mathematical and General*, 38:2687-2700, 2005.
- [20] L Snobl and P Winternitz. All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency $n - 1$. arXiv:0809.3259v1 [math-ph].