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Relationship Between Text-Book Orientation and Mathematics Achievement and Attitude

BY
K. F. COLLIS

FACULTY OF EDUCATION

Volume I Number 8

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K. F. COLLIS

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RELATIONSHIP BETWEEN TEXT-BOOK ORIENTATION AND MATHEMATICS ACHIEVEMENT AND ATTITUDE

I. INTRODUCTION

Mathematics has a twofold significance in the modern curriculum. First of all, as Gnedenko (1965, p. 49) says, "The mathematical style of thinking and precise quantitative methods penetrate literally every sphere of human activitiy—economics and biology, technical science and medicine, physics and astronomy. Normal functioning of society requires an ever growing number of persons with a knowledge of mathematics."

Secondly, few writers on the subject in the last fifty years have omitted mention of the value of mathematics in a general education.

It seems a reasonable conjecture that this recognition of its twofold significance has been an important factor in making mathematics the centre of considerable educational concern in recent years. The large number of experimental programmes set up during the past twenty years bears witness to this concern. In general two problems are mentioned repeatedly: the unfavourable attitude that many students have for the subject (Beberman, 1959, pp. 35-36; Biggs, 1962, p. 107; Allendoerfer, 1965, p. 694), and the failure of courses to develop an understanding of principles among students. In the main the new mathematics programmes have been devised in an attempt to overcome these deficiencies. Russia, France, and Belgium are but three European countries reporting considerable experimentation in the area. Two of the best known programmes of an experimental nature to come out of England, the School Mathematics Project and the Midlands Mathematical Experiment, were both begun in the early 1960's. In the United States over a dozen major experimental programmes are reported to be in operation. Some were begun in the early 1950's and new ones were still being put into operation during the early 1960's.

In general, all of these programmes show evidence of the influence of two professional groups, the mathematicians and the classroom teachers. The collaboration between mathematician and teacher has been very fruitful. The influence of the mathematician can be seen in the emphasis placed on the careful use of language, the slackening of interest in social applications, and the awareness of structure in mathematics. The teacher's influence shows up in the attention given to the grading of programmes and in the classroom procedures suggested. However, the expertise which could have been contributed by another professional group, the psychologists specializing in learning and teaching theory, is lacking. This latter neglect has led to four rather serious consequences.

(a) Objectives of courses are quite often not stated at all, it being assumed, apparently, that topic headings, illustrative examples, and recommended procedures for teaching are a sufficient guide to the teacher responsible for implementing the programme.

(b) When objectives are stated they are presented using such general terms as understanding, insight, imagination, and basic skills—all of which require careful definition in behavioural terms before they become useful from the point of view of evaluation.

(c) Current thinking and research concerned with classroom learning finds inadequate expression in the curricula. James or Dewey would be familiar with the theories implied in many of the new programmes (Woodring, 1964, p. 300).

(d) Teaching theories have tended to be ignored or assumed to be inferable from learning theories. Williams' research (1966) shows that teachers tend to revert to previously learned patterns of teaching behaviour in a situation where a new approach is warranted. Gage (1964) seems to get to the heart of this matter when he expresses the view that, although learning theories give direction to the teacher by implication, they show the educative process from the standpoint of the pupil and thus many teachers see them as inadequate guides to specific teaching behaviour.

The purpose of this study was to contribute to the development and evaluation of a teaching theory in relation to mathematics at the first year secondary level.

II. THE TEACHING THEORY

Developing a teaching theory in this area requires that specific attention be given to three variables: (a) the child's stage of intellectual maturity; (b) the nature and structure of the content matter; and (c) the way in which pupils are likely to learn most effectively (Collis, 1967 and 1969). These three can be seen as somewhat independent threads which can be made to come together to contribute to a teaching "theory". It is appropriate at this point to examine the three variables separately and then to show how a theory of mathematics teaching emerges from them.

Stages of Intellectual Maturity

The Swiss psychologist, Piaget, has probably contributed most to our understanding of what the successive stages in thinking and cognitive development are (Piaget, 1952 and 1956; Inhelder and Piaget, 1958 and 1964; Peel, 1960; Flavell, 1964). He suggests that there are three recognizable successive periods in a child's cognitive development:—

I sensory-motor, up to 2 years;

II preparation for and organization of concrete operations, approximately 2-11 years;

III formal operations, beginning at approximately 12 years for most children.

These broad periods are sub-divided by Piaget into several stages and sub-periods. Period I need not be considered here. Period II may be broken into two important sub-periods, preoperational (2-7 years) and concrete operational (7-11 years). The first of these sub-periods may be broken into two stages, one of pre-operational representation (2-4½ years), and one where intuitive thought becomes evident (4½-7 years). These latter stages are seen as preparatory to the kind of thinking which is typical of the concrete operational level (7-11 years), just as this last is seen as preparing the child for the highest level of cognitive development, formal operations (period III).

The preoperational representation stage which follows the period of sensory-motor intelligence marks a big advance in the child's thinking in that he is no longer tied to immediate sensory perception. Through his new ability to call forth, internally, a signifier, such as a word, which symbolizes an event which is not in his present perceptual field and from which the signifier is clearly differentiated, he is able to recall the past and anticipate the future. This is his first step towards freedom from concrete reality which reaches its final development in formal reasoning in adolescence.

During the stage of intuitive thought the child's reasoning is dominated by the perceptual context and he finds it difficult to see relationships and to make consistent judgments. He tends to be egocentric, that is, to see his own point of view as the only one, to centre on some striking feature of a problem, and to neglect other important accompanying circumstances. He is also unable to relate present to past in such a way as to be freed from the idea that a new configuration must mean a new set of circumstances. Perhaps the most often mentioned characteristic of this period, irreversibility of the thought processes, is also the most important. The child at this stage is incapable of thinking back to the beginning of his line of reasoning from some end point which he has reached.

The stage of intuitive thought is still preoperational[1] in the sense in which Piaget uses the term. It is during the stage of development achieved by the child between the ages 7-11 years that thinking becomes "operational" in Piaget's sense. After he has attained the notions of reversibility and conservation in concrete situations the child gradually elaborates a systematic logical mode of thinking which enables him "to classify material, to break down groups into constituent parts, to place a series in order, to pair corresponding elements and to substitute equivalent elements" (Peel, 1960, p. 76). True concepts can be formed at this level but they must be directly related to the child's current or very recent empirical experience as he is still unable to manipulate relations between abstractions. In addition, the specificity of this link to empirical reality severely handicaps the child in deducing valid logical implications from given data.

Period III, the period of formal operations, brings to fruition the developmental progress made during the previous periods. At this level the child's conception of the principles of conservation and reversibility is fully developed, his thinking is marked by hypothesis formation and the testing thereof, and he is no longer tied to concrete-empirical experiences for the matter of his thought processes—"the most distinctive property of formal thought is reversal of direction between reality and possibility" (Inhelder & Piaget, 1958, p. 6). The adolescent, when presented with a problem, is able to begin by envisaging all the possible relations involved and then, by a combination of logical analysis and experimentation, to deduce which of these relations is true. The logical model appropriate for this stage is that of formal propositions; the adolescent is able to see abstract relationships, and to use formal logical propositions (including definitions and axioms) as the raw material for his reasoning.

The age ranges for the periods discussed above are quite flexible, and, especially during the transition from one period to the next, considerable overlapping of

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[1]An "operation" is "any representational act which is an integral part of an organised network of related acts". (Flavell, 1964, p. 166.)

the different types of reasoning would be expected both from individual to individual and within the cognitive operations typical of the one person. Variation within the group of students studied in this experiment would be expected to range from completely concrete operational to completely formal operational with the majority somewhere on the continuum between these two extremes.

There are also certain stage-free notions within Piaget's developmental theory which have important implications for this study. One of these is the concept of equilibrium of which Flavell (1964) gives an excellent summary. This concept is stage-free in that it is concerned with the elements that mark the periods and stages rather than the stages themselves. Flavell (1964, pp. 245-46) takes conservation as an example and shows how Piaget distinguishes four steps in "la marche à équilibre". In the early stages of the development of this concept the child centres on one only of the opposing properties x or y; in the next step he tends to alternate between choosing one of the two properties. This disjunction in step 2 after many concrete experiences leads to a conjunction in step 3 as the child sees x and y in relationship to each other. By step 4 the concept has reached equilibrium and the child has a permanent and adequate concept of conservation in the particular field concerned, e.g. conservation of number, quantity, etc. This equilibrium model applies equally well to the transition between the logic typical of the concrete operational child and that of the later period. After many experiences involving one step, the child, in seeking the most adequate and consistent explanation, moves up to the next step. This move results in fewer conflicting results than he had had previously (Lawrence, 1967, p. 105) but he will not have the most satisfactory, self-consistent results until he has taken step 4. As the child reaches each step in the hierarchy and continues to perform at this new level the probability of his achieving the next step becomes greater (Inhelder and Piaget, 1964, p. 288).

Although the ages at which the characteristics of the various types of thought emerge may vary and the levels of operation may vary within the individual's various subject-matter fields (Piaget, 1956, p. 17), there is evidence to show invariance in the order in which the various types of cognition emerge. Keats (1955) is responsible for one of these experiments. His experiment is of particular interest here because it deals with this problem in relation to the formal and concrete thought processes. Keats constructed three reasoning tests, arithmetic, probabilities, and inequalities and tested children in the age group 9-15 years. Each set of problems was constructed at two levels—the first could be solved concretely while the second required formal reasoning. Keats compared the performance on pairs of equivalent items, concrete and formal, by using simple twofold frequency tables. His results tended to confirm Piaget's hypothesis that concrete operations *necessarily* precede formal ones, and that the ability to solve problems at the formal level presupposes that children have mastered the problems at the concrete level. In addition he was able to show that the change from concrete to formal thinking occurred in its most marked form between the ages of 12 and 13 years.

Piaget's view of the child's cognitive development may be looked at as a gradual developing and maturing of logical constructs on which he bases his reasoning—the logic forming a structure behind the more obvious structures built up by the child in his various subject matter areas. Unless this structure within a structure is capable of absorbing and useful for manipulating the material presented, then the material, although possibly retained on a rote basis, will be little used by the child in his subsequent reasoning. In fact, his rote retention of the topic will enable him to respond to the original stimulus and, apart from some stimulus generalization, little else; the skill or technique will be of use only in the specific area in which it was first presented.

If one accepts Piaget's view of the development of the mental processes it is clear that, in the primary school, mathematics should be presented so that the child can attain concepts at the concrete operational level. In addition, at about the age of

12 to 13 years teachers can expect that, although some pupils will be capable of formal operational thinking, many will still require considerable work at the concrete operational level of thinking before they can hope to attain the higher level of reasoning. Thus the nature and structure of the subject-matter must next come up for consideration in order to study its suitability for students at this transitional level of development.

The Nature and Structure of the Content Matter

From the view-point of the teacher of mathematics at this level two mathematical streams may be discerned: the "pure" stream which is concerned with the manipulation of numerals and pronumerals and the elementary interrelationships inhering in the mathematical structure; and the "applied" stream where the student is expected to apply his developed cognitive structure in mathematics to measurement and prediction in relation to situations in the physical environment.

In general pupils in primary and early secondary schools work entirely in the field of "real" numbers (all the numbers which can be represented on a number line). As Allendoerfer and Oakley (1963, p. 57) state, the formal properties of these, upon which all manipulations and algorisms are based, can be encompassed within relatively few postulates. Although this set of postulates appears highly formal, the critical factor is probably that each postulate can be understood and used by the pupils without recourse to formal propositional reasoning. All that is necessary is the ability to classify and reclassify material, to separate groups into their constituent parts, to pair corresponding elements and to replace elements in a given set by equivalent elements; in short, the establishment of notions of reversibility and conservation in simple concrete situations. The essential requirement in the teaching situation is to provide the pupil with a concrete referent by means of which he can readily make valid deductions. This can no doubt be achieved in a variety of ways but the use of set arrays has been found useful.

As illustrations of the ideas outlined above one can consider a concrete-operational approach to two of the axioms of the real number field (Collis, 1969), the Commutative Law of Multiplication [$a \times b = b \times a$, where a and b stand for arbitrary real numbers] and the Distributive Law of Multiplication over Addition [$a \times (b + c) = (a \times b) + (a \times c)$, where a, b, and c stand for arbitrary real numbers].

The Commutative Law of Multiplication is easily seen by children at the concrete-operational level of reasoning because it involves simply the considering of the same pattern of elements from two differing viewpoints. The set array, useful for demonstrating this law, reinforces and extends the previously established Closure Principle ($a \times b$ is a unique real number). A suitable set array for demonstrating $3 \times 2 = 2 \times 3$, a specific representation of the law in question, would be the following pattern of crosses:

X X X
X X X

By considering the columns as equivalent sets of elements, the total number of crosses in the arrays may be expressed as 3×2; by reorienting one's viewpoint and considering the rows as equivalent sets, the expression can be seen as 2×3. Clearly the number of elements has not changed and thus it is appropriate to write $3 \times 2 = 2 \times 3$.

The Distributive Law of Multiplication over Addition can be handled in a like manner. If children are asked to write down several expressions to account for the total number of crosses in the following array many will come upon this principle of their own accord.

X X X X   X X X
X X X X   X X X
$(2 \times 4) + (2 \times 3) = 2 \times (4 + 3) = 2 \times 7$

After many experiences with arrays and the natural numbers associated with them the children become familiar enough with the operations to deduce of their own accord the various laws stated in symbolic form, e.g. $a + b = b + a$ (where a and b are counting numbers or zero). It must be noted, however, that it cannot be presumed that pupils of this age group are able to regard these laws as a set of formal axioms which define an axiomatic system. However, they are capable of understanding them and are able to see them as a set of standard and consistent results which lie behind all their manipulations in arithmetic and ordinary algebra.

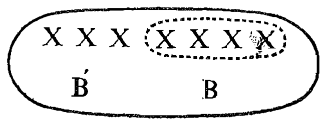
In addition, the various operations signs used in algebra and arithmetic, i.e. $+$, $-$, $\times$, $\div$, can take on a meaning at the level of concrete-operational thinking. For example, the idea of subtraction embodied in an expression such as $7 - 4$ can be represented in terms of two models, the first representing an excess comparison and the second the complement of a set. The following models illustrate these points:

Excess Comparison

A X X X X X X X
      | | | |
B X X X X

Set A has 3 more elements than B,
i.e. $n(A) - n(B) = 7 - 4 = 3$.

Complement
A



Set A has 7 elements, i.e. $n(A) = 7$
Set B has 4 elements, i.e. $n(B) = 4$
Set B' has 3 elements, i.e. $n(A) - n(B) = 7 - 4 = 3$.

The elementary symbolism embodied in these models presents no difficulty to children as they are directly related to concrete empirical data in the form of set arrays. Again it should be emphasized that the meaning attached to the symbols by the child should not be taken to have the generalized meaning which mathematicians would read into them.

Practice at this level where the child can reason successfully using the concrete-operational logical model leads him into a transitional stage where he does not need the empirical sets of arrays to make expressions meaningful for him. At this level he is able to use numerals and elementary operations (not involving inverses or reciprocals) with understanding. Finally, the child progresses to the formal stage where the economy and generality of algebraic techniques becomes clear to him.

As was mentioned earlier, the "pure" aspects of the arithmetic-algebra area are not the only ones which concern the teacher at the level of mathematics teaching of major concern in this study. From one point of view they may be regarded as the tools with which to manipulate the symbols derived from the mathematical model of some real situation. The ability to apply a mathematical model to the real situation, for the purpose of *measurement* or *prediction,* is a function of "applied" mathematics and requires at least two further skills. First, the child has to be able to make sense of the real life situation involved. This assumes some extensive and well integrated experiences which will allow him to focus on the problem as such and not regard it merely as a puzzle; he must have sufficient experience so that he can discard data that is irrelevant as far as the appropriate mathematical model is concerned. Secondly, the selection of a mathematical model by a student implies the availability of such a model in the student's cognitive structure.

One does not have to go far to find clear illustrations of the first point. The social implications of applied topics such as Profit and Loss, Bank Interest, Commission and Discount are very complex, and problems in these areas must be almost meaningless to students with no experience in the life situations involved. A short lecture on the allied social and business concomitants of these topics followed by a setting down of standard definitions and rules to be followed clearly implies formal level reasoning.

Piagetian-oriented research takes up the second point of the availability of a *meaningful* model. "Proportion" has been taught in the latter stages of primary school for generations; in most cases with singular lack of success. Inhelder and Piaget (1958, p. 314) explain the difficulty when they point out that, despite the fact that mathematical proportions consist simply of the equality of two ratios, difficulty arises because the psychological sub-strata upon which their formation depends does not occur during the stage of concrete operations.

The illustrations given above are by no means exhaustive (Collis, 1967) but are meant to be indicative of the general principle that, because the child's level of cognitive development is the factor which determines whether the mathematical experiences with which he is presented can be assimilated into his existing cognitive structure, Piagetian developmental theory has a twofold significance for mathematics teaching at the level of concern in this study. Firstly, it provides teachers with a *means* of presenting appropriate mathematical experiences at the various stages of logical development. Secondly, it sounds a warning against emphasizing, at the primary and early secondary level, applied topics which can only be adequately handled by formal reasoning unless the learner has had a good deal of empirical experience in the areas concerned.

A Theory of Learning

As the concern here is with learning potentially meaningful material we turn to Ausubel, who appears to have devised a most useful theory concerned with the *manner* in which educational experiences should be presented. Ausubel (1963) proposes that the organization of a particular subject matter discipline in an individual's own mind is hierarchical in nature. "The most inclusive concepts occupy a position at the apex of the structure and subsume progressively less inclusive and more highly-differentiated sub-concepts and factual data." (Ausubel, 1963, p. 79.) The theory assumes a basic distinction between logical ordering of subject matter and its psychological ordering. Both clearly involve arrangement of the subject material but the former strives for the interrelatedness of topics and homogeneity of the parts comprising the system whereas the latter strives for a sequential arrangement by which the learner can most effectively grasp a new body of knowledge. In psychological ordering there is progressive differentiation in terms of degree of generality and inclusiveness. Each part of the structure is linked to the next higher step in the hierarchy by a process of subsumption.

According to this theory, the human nervous system is a data processing and storing mechanism, involving in the first instance directional, relational, and cataloguing operations. These preliminary operations are necessary for meaningful learning and retention because the incorporation of the new material into the existing cognitive structure forms the basis for the emergence of new meaning and also must conform to the prevailing cognitive organization. In other words, the new learning is related in a particular way to existing ideas and thus is stored with these ideas. In this way the new idea exists as an identifiable entity anchored to subsuming concepts and, for varying periods of time, is discriminable from its subsumers. However, sooner or later, depending upon the idea and its relationship to its subsumers, the new identifiable idea becomes incorporated or submerged within the subsuming entity.

Ausubel is not alone in using a hierarchical model for meaningful learning and retention. Bruner (1965), Dienes (1963), and Skemp (1963), all subscribe to this model, the last two with particular respect to mathematics learning. Moreover, Ausubel, along with Bruner (1965) and Gagné (1965), proposes that the existing cognitive structure is the primary independent variable influencing the meaningful learning and retention of new subject-matter material. The existing cognitive structure affects the attainment of meaningful verbal learning in three principal ways:
(a) it determines the availability of relevant subsuming concepts at an appropriately proximate level of inclusiveness for optimal anchorage to the established structure; (b) it affects the extent to which the new material is discriminable from the established conceptual system that subsumes it; and (c) the stability and clarity of the subsumers are determinants of the strength of the anchorage provided for new materials related to present knowledge.

It appears to be generally agreed by the theorists mentioned that in a form of learning where there is an organized structure of knowledge, such as in the area of mathematics, there must be available a schema to enable meaningful learning to take place or the child is forced to resort to rote learning methods in which the learned material is isolated from the remainder of his knowledge. The latter alternative leads to difficulties for the learner; not being related to the rest of the structure, rotely-learned material is not readily retained nor is it usually adaptable to solving problems even in a closely associated area.

Perhaps the basic theme to come out of Ausubel's theory is that it behoves teachers deliberately to order the organization and sequence of lessons along the lines indicated by this naturally occurring subsumption process. Out of this there arise several implications for teaching practice, three of which are set out below.

Firstly, the most general and inclusive ideas should be introduced at the outset and these should be gradually differentiated towards less inclusive ideas, specific data and acquired facts. Secondly, some care must be taken to link new data in a meaningful way to the child's existing cognitive structure in the area. Finally, there is a serious need for what Ausubel calls integrative reconciliation of presented material.

The application of these teaching principles in the field of mathematics teaching is readily inferred. From what has been said above about the nature and structure of mathematics at the primary to early secondary level, the implications of the principle of progressive differentiation and of the linking of subsequent experiences with earlier knowledge of the subject matter area are clear: a few postulates represent the basic axioms on which all the manipulations of symbols in arithmetic and elementary algebra are based—specific techniques are derived from these; a broad general notion of variation (for example) becomes differentiated into more specific areas of ratio, proportion, rates, graphical representation, and algebraic variables. With respect to the integrative reconciliation principle, Ausubel (1963, p. 80) points out that it is directly opposed, both in spirit and practice, to the text-book writer's predilection for compartmentalizing and segregating particular areas and topics within their respective chapters and sections. He points out that overlapping topics are treated as though self-contained, that students are left with the task of cross-referencing and explaining ideas and seeing relationships between ideas. Contrasts, similarities, and inconsistencies are not brought out and multiple terms are used to represent ideas which are intrinsically equivalent. The end result of this state of affairs is that artificial barriers are erected between related topics and adequate use is not made of relevant previously learned ideas for subsuming new ideas into the existing structure.

It should be clear that a child at the concrete-operational level would find the task of reconciling and integrating this material, unaided, quite beyond his cognitive capabilities. The child at the formal level of reasoning, on the other hand, can, and very often does, interrelate the topics for himself. In practice, mathematics teaching lends itself readily to the use of this principle of integrative reconciliation. For example, the teacher can establish a close relationship between vulgar fractions, decimal fractions and percentages and yet, in many syllabuses and text books, these three topics are treated as completely separate entities; the obvious interrelationships are not drawn out at any stage. As a further illustration, most texts treat percentage type problems as completely separate topics, one chapter being given over to "Profit and Loss", another to "Commission and Discount", another to various types of "Interest" problems and still another to "Growth and Decay" type problems. The examples of this kind of practice could be extended almost indefinitely but sufficient illustrations have been given to make the point. Very infrequently have teachers consciously set out to break down these artificial topic boundaries, and rarely has the student been told that the mathematics is virtually the same; it is only the technical terms used, and perhaps the social situations involved, which differ.

At the outset three variables were selected as comprising the necessary conditions for shaping the child's cognitive structure in this area: the cognitive level of the child, the nature of the subject matter, and a theory of learning potentially meaningful material. It remains but to draw the threads together in such a way as to give a succinct statement of the view taken here of the theory of teaching mathematics to children of the late primary to early secondary school levels.

The child's cognitive developmental level determines the kinds of experiences which can have meaning for him and the theoretical orientation here postulates that the child's mathematical development is dependent upon a substrate logical development suggested by Piaget. This development is shown by the child's progress along the concrete-operational to formal-operational level of thinking continuum. Moreover, it is suggested that an understanding of the child's development in the whole area is most successful when Ausubel's theoretical constructs of progressive differentiation and integrative reconciliation are integrated with Piaget's developmental theory. The two theoretical positions merge into a coherent theory in the area of mathematics teaching at the point where the teacher can see how the most highly inclusive concepts can be interpreted at the concrete-operational level of thinking, for it is then that the teacher is able to ensure that the child builds up a structure of his own out of his own experiences.

III. THE STUDIES

Two investigations were conducted on the basis of the theoretical outline summarized above. The first was concerned with the evaluation of two sets of texts, the second with tracing the development of the conceptual structures in mathematics being built up during the course of one year by children using a text which made its prime aim the development of these structures.

The First Study

If the method of presenting mathematics to students in first year secondary classes followed the theoretical outline of teaching set out above it would be expected that two major outcomes would result:—

(a) the students using this method would attain a higher standard of achievement in mathematics than would those following a conventional presentation;

(b) the attitude to mathematics would be more favourable among students using the recommended approach than among those taught by conventional methods.

To test these hypotheses, under classroom conditions, it was necessary to have some children taught by a method consistent with the theoretical outline above and others taught the same course by a conventional approach. Following the introduction of a new mathematical curriculum to the lower secondary schools in Queensland three texts were written to cover the course set out. While two (Olsen and Jones, 1964; Young and Radcliffe, 1964) followed the traditional form of presentation, the third (Hubbard, 1964) approximated the structure suggested above.

As Nelson (1965) and Begle (1965) have argued, there are sound reasons for believing that the approach of the text-book will be a main factor in determining the teacher's presentation of the subject-matter to his pupils. This is especially so in Queensland where mathematics courses have been, traditionally, curriculum dominated and text-book oriented.

At this point, it is necessary to elaborate a little upon the differences between the text-books. The experimental text treats the curriculum-dominated sections of arithmetic, mensuration, and algebra as different aspects of the same system and every effort is made by the author to integrate them thoroughly. This process is by no means accidental as the author writes in his introduction to the teaching notes, "Instead of two subjects, Algebra and Arithmetic, we find ONE unified subject—Mathematics." (Hubbard, 1965.)

This integrating process has four clear features:—

1. the rationalization and restructuring of known arithmetic techniques by appealing to familiar everyday concrete situations and using concrete and semi-concrete operational level reasoning;
2. the use of (1) to introduce the generalized strucure of algebra;
3. using developments of (1) and (2) to enable the student to undertake simple mathematical modelling for use in applied situations;
4. the attempt throughout to induce the child to see the value of formal approaches for efficiency but allowing for the use of concrete and semi-concrete operational processes in doing exercises.

The threads of progressive differentiation of subject matter and its integrative reconciliation run through this text.

The control texts, on the other hand, conform to the stereotype set up by Ausubel (1963, p. 79) when he describes the typical practice among text-book writers of segregating topically homogeneous materials into separate chapters and presenting them throughout at a uniform level of conceptualization in accordance with an apparent logical outline of subject-matter organization. These texts have separate chapters for algebra, arithmetic, and mensuration and make little attempt to integrate the topics and no attempt to bring out the inherent logical structure of the mathematical subject-matter.

As the aim of this first study was to compare the achievement of two sets of students under normal classroom conditions, a Groups within Treatments Design, used in conjunction with covariance techniques, offered several advantages. The former avoids possible contamination effects and enables the use of intact classes; the latter helps to offset the disadvantage of lack of precision in the Groups within Treatments Design.[2]

The sampling unit decided upon was the school, and each school selected was represented by one class. A complete list of all Brisbane secondary schools was obtained and schools were classified according to the texts they used. Nine schools were chosen at random from each group so that the total sample consisted of eighteen schools. The total number of students involved was 665; the proportion of boys to girls in each set of nine schools was approximately the same. The mean age of the children was 12.7 years at the beginning of the year.

Five tests were given. Three were administered during the first term of 1965 (26/1/65–11/2/65), and the remaining two during the last four weeks of school in

[2]For full technical details see Collis (1967).

1965 (2/11/65–29/11/65). The first set of tests consisted of an Intelligence Test, a Mathematics Attitude Scale (Thurstone Type), and Mathematics Test I. The final criterion tests were the same Mathematics Attitude Scale as that given at the beginning of the year and Mathematics Test II.

A preliminary examination of the results of the testing shows a trend favouring the experimental group. Table 1 records the general means on the tests given, calculated from the group means. As would be expected from the method of sampling used there was no significant difference between the means of the two groups on any of the three initial tests, and the differences throughout the table suggest the effectiveness of the experimental method. From the attitude results, it can be seen that the experimental group began the year with a less favourable attitude than the control group, but by the end of the year the positions were reversed. A similar trend is shown in the differences in the means on the mathematics tests, although in this case they are very much smaller.

TABLE 1
General Means Calculated from Group Means

Variable	Experimental	Control	Difference
I.Q.	47.3012	45.2041	2.0971 in favour of Exp. Group
Initial Attitude	69.8088	73.4574	3.6486 in favour of Control Group
Maths. Test I	13.2445	13.3294	.0849 in favour of Control Group
Final Attitude	70.2093	68.0177	2.1916 in favour of Exp. Group
Maths. Test II	37.2273	36.3305	.8968 in favour of Exp. Group

The results of the covariance analysis carried out on the data confirmed that the experimental group had indeed improved their attitude to the subject; the difference between the two groups on scores on this variable could not reasonably be attributed to chance. On the other hand, in the case of the achievement variable, the difference in the criterion scores was not large enough to rule out the possibility of chance being a causal factor.

While this latter finding may be a little disappointing it was not altogether unexpected, as a number of uncontrolled factors were operating against the success of the experimental group. First, as Bloom (1963) states, ". . . the research worker must not expect major modification of teaching practices in a brief period of time. Nor should he expect to secure significant evidence of growth toward new objectives in a single study carried on over a one-year period". Secondly, Bruner, Goodnow, and Austin (1960) point out that once a cognitive structure has been shaped it is difficult to alter. This point has application to the present discussion at both the teacher's and the child's level. Because of the method of selection of the samples, the teachers taking both sets of classes were a random group. However, the experimental approach was unfamiliar to the teachers, who furthermore had all been familiar with the conventional approach at school, in training, and in previous classroom practice. As a direct result of this, method reversion undoubtedly occurred on occasions. Moreover, the children in both sets had been trained in the conventional approach for seven years, during which time they would have developed quite complex, even if, in many cases, inadequate, structures and strategies associated with mathematical material. While the control students continued along the same lines, with teachers versed in the method, the experimental group was obliged to adopt a new approach to the subject-matter and to restructure the mathematical knowledge which had already been learnt at the primary school level. Thirdly, any text-book, particularly one not deliberately based upon a theory, is unlikely to be a completely adequate representation of the theory of teaching which might be seen to lie behind it.

Viewed against the background of militating factors listed above, the significant difference in attitude and the tendency for pupils being taught from the experimental text to do better in achievement provide some support for the teaching principles advocated earlier. It is possible, for example, that the significant difference in attitude was only beginning to influence the achievement scores, and that a long-term study, even with the existing handicaps, would provide clear evidence of the superiority of an experimental-type text.

The Second Study

As indicated earlier, the intention in this investigation was to trace the mathematical structure presumably being formed by the students as they passed through the first year of a course specifically designed to reorganize their existing cognitive structure in mathematics. It was thought that evidence might be obtained which would support or deny the teaching theory outlined earlier.

There is obviously a vast array of stimulus data presented to the child in mathematics, all of which, it is assumed, is capable of being subsumed under general categories of various kinds. One of the aims in teaching mathematics, along the lines suggested in the experimental text, is to form general categories so that the students may use them to solve specific problems. This involves, in Bruner's terms (1957) "going beyond the information given". The ability of the child to do mathematics is assumed to depend on the concepts he forms, and on how he goes about forming them. One problem which arises immediately is how to gather data on the thought processes. Verbal reports from the child as to how he goes about organizing the stimulus field and how he reduces the complexity of the vast number of stimuli by forming categories and extracting generalizations are likely to be of doubtful validity. Some kind of representation of the child's thoughts and thinking was required. It was decided (a) to select a group of students, all of one sex, following the experimental text exclusively and in no way connected with the students used in the first study; and (b) to prepare a set of tests, both group and individual, designed to isolate the mathematical concepts inhering in the text, and to administer these *same tests to the same students* at regular intervals throughout the school year while keeping a detailed record of both individual and class results.

Four convent girls' schools permitted a monthly testing of their first year secondary classes. The mean age of these students was 12.7 years at the time of the initial testing. All girls in the four classes were given the group tests, and a sub-set of five girls, chosen at random from each class, was given individual tests as well.

Three tests were devised, each of which was focused on a different facet of the problem. The items in Mathematics Test I, mentioned above, were analysed and placed into one of four pre-established concept categories[3] by competent judges, the test was given to the girls on successive occasions and a record kept of the development of the individual girls on each concept category as the administrations went by. A set of Flash Cards was developed which presented the testee with appropriate mathematical stimuli to which the girls recorded an immediate written response—the data were collected with a view to measuring the degree to which the children moved towards a generalized category for each response. The individual test, given to the set of five girls selected from each class, consisted of a Card Sorting Task. In this last test the girls were presented with a pack of fifty-four cards with mathematical stimuli on them; the cards could be grouped in various ways and each girl was asked simply to sort her pack into piles of cards which seemed to her to go together—she was not asked to give a reason or to label the results of her sorting.

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[3]These categories were devised by a committee set up by Educational Testing Service (1957).

Because this study was, in the main, exploratory, no formal hypotheses were set up. However, with respect to the readministrations of the Mathematics Test I, it was expected that, from administration to administration, there would be a general increase in scores showing the achievement in each concept category; this improvement would be manifested by an early rapid improvement in the categories with which the children were familiar from their previous work in the primary school and a slower improvement where the children were introduced to concepts for the first time during the course of the testings. Concept Categories I and III (number and operation, measurement and geometry respectively) were in the former group, while Concept Categories II and IV (symbolism and function and relation respectively) formed the latter group. The results obtained from an analysis of the data confirmed that there was certainly an overall improvement in achievement, but showed also that the rate of improvement did not depend upon whether the child had had previous experience with the concept in the primary school. The rates of improvement in scores on the categories were approximately the same (Figure 1).

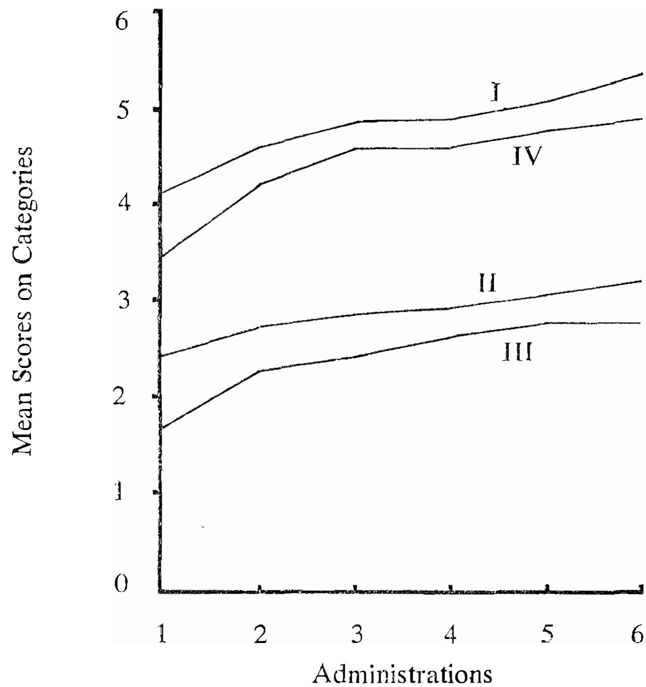

FIG. 1.—Maths. Test I, Overall Mean Scores on Concept Categories.

The relationships shown on the graph in Figure 1 warrant closer consideration as they are suggestive of interrelationships between certain pairs of categories, categories I and IV forming one pair and II and III the other.

Category I (number and operation) covers a concept area with which the children were familiar from primary school, while category IV (function and relation) introduced a concept area which was largely new to the students. Although both categories were represented in the test by seven items each, achievement on the former always remained ahead of achievement on the latter and their rates of improvement remained parallel throughout. No causal connection can be shown to exist between the two categories by the methods of analysis adopted in this study, but it is perhaps significant that the experimental text considered many aspects of number and operation as the concrete operational model upon which to base the higher level concepts of function and relation. For example, solution of equations, and algebraic processes generally, were directly and specifically related to earlier work on number and operation.

Concept categories II (symbolism) and III (measurement and geometry) were represented in the test by four items and five items respectively, and yet achievement on category II always remained ahead of achievement in category III. In the light of the theoretical underpinning for this project, perhaps this phenomenon is to be expected also.

Category II was concerned with a new area for these children, symbolism. The specific areas in which this symbolism was applied in the experimental text at this level were those concerned with interpretation of signs of operation and algebraic symbolism—two areas where it can be readily presented, understood and manipulated through concrete operational thought. On the other hand, concept category III is an area which includes, *inter alia,* inverse relationships between size of unit and number of units, converting units, areas and volume measurement. All of these topics had been covered by students undergoing the test series in the primary school and thus the experimental text had the task of restructuring the children's knowledge in these areas. However, according to Piaget and Piagetian-oriented research, these are areas where the necessary substrate concepts (proportion, area, volume) are not developed by the child until he or she is at a stage of development where formal operational level reasoning has begun to appear, and thus most of the students' prior knowledge in these areas would have been acquired by rote learning. It is reasonable to deduce that the task of restructuring their knowledge in this area would have been made more difficult than it was in category I concepts, for example, because the girls, in the main, would not, at this stage, have fully developed the necessary substrate concepts.

Further analyses of the data from this part of the study are summarized in Figures 2–5. As would be expected from what has already been said, the means of the categories show a general improvement throughout the test administrations with the curves adopting the typical shape of learning curves in general. The most significant feature of curves representing categories I, II, and IV, is the decrease in the amount of dispersion of the means. A close examination of the data from which these means were generated, other information obtained from the students concerned (intelligence and initial mathematics ability), and the graph lines themselves show that this narrowing of dispersion came about because the weaker schools (nos. 12 and 13) had a higher overall rate of improvement. Moreover, schools 12 and 13, with the lowest initial scores in these three categories as well as the lowest mean scores on intelligence, do not maintain these lowest positions throughout the administrations. The lines representing their achievement interact with each other and with those of schools 10 and 11. In the context of this study,[4] this result suggests that the experimental text was of most benefit to the weaker students. There are two likely, but not mutually exclusive, explanations for this phenomenon which fit the basic underpinning of the method of teaching suggested by the experimental text. Firstly, the weakest students were the ones most likely to benefit from the changed approach as they were the ones who had not built up a structure of mathematics which had given them success. Secondly, the continued appeal to concrete operational level reasoning gave these children a chance to become more successful as the experiences presented were able to be understood at their own level of logical development. On the other hand, the better students had built up a successful structure of their own and,

[4]The statements which follow leave out of consideration the teacher and other variables which must come up for consideration if the hypothesis implied was set up for formal testing.

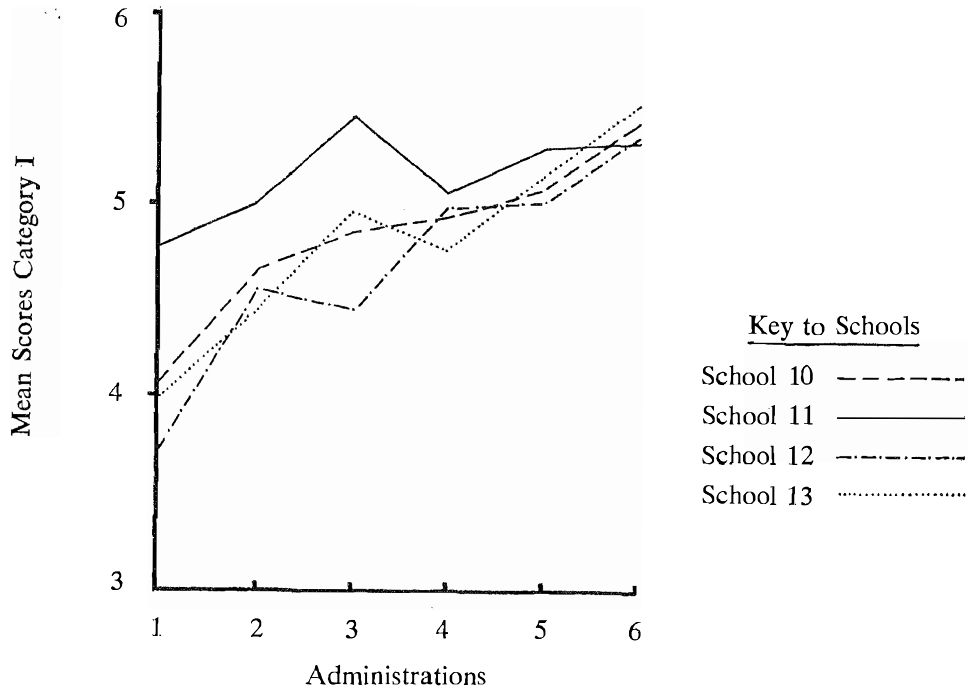

FIG. 2.—Maths. Test I, Mean Scores of Schools, Category I.

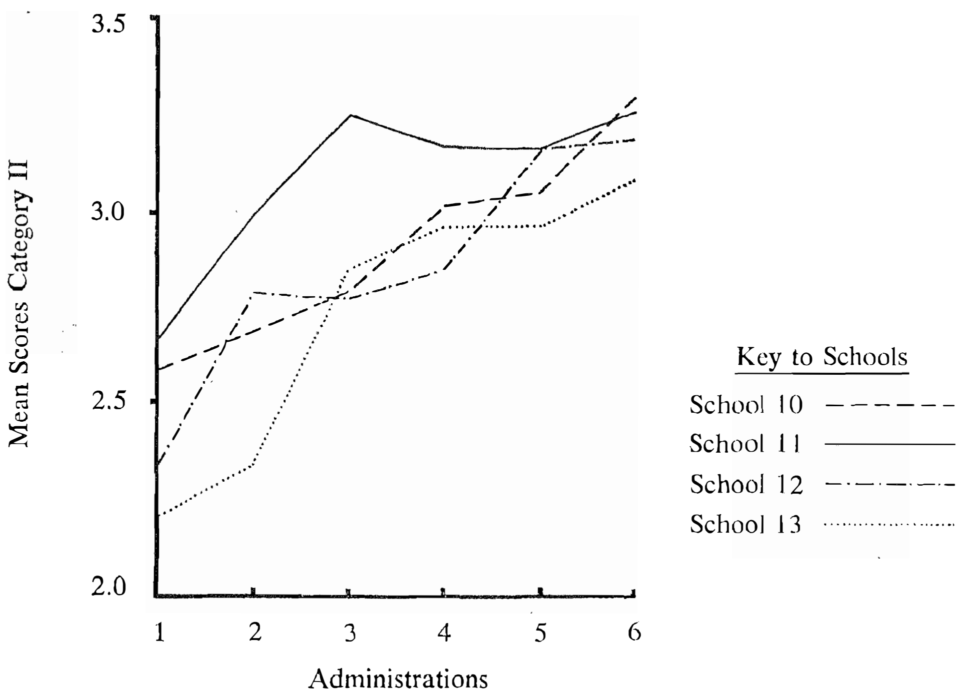

FIG. 3.—Maths. Test I, Mean Scores of Schools, Category II.

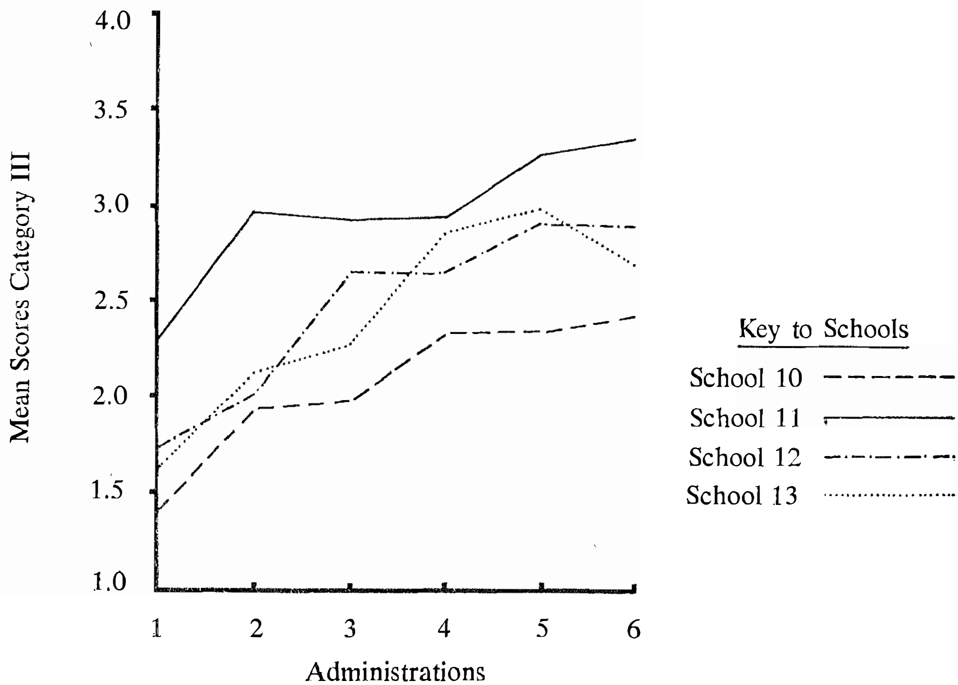

FIG. 4.—Maths. Test I, Mean Scores of Schools, Category III.

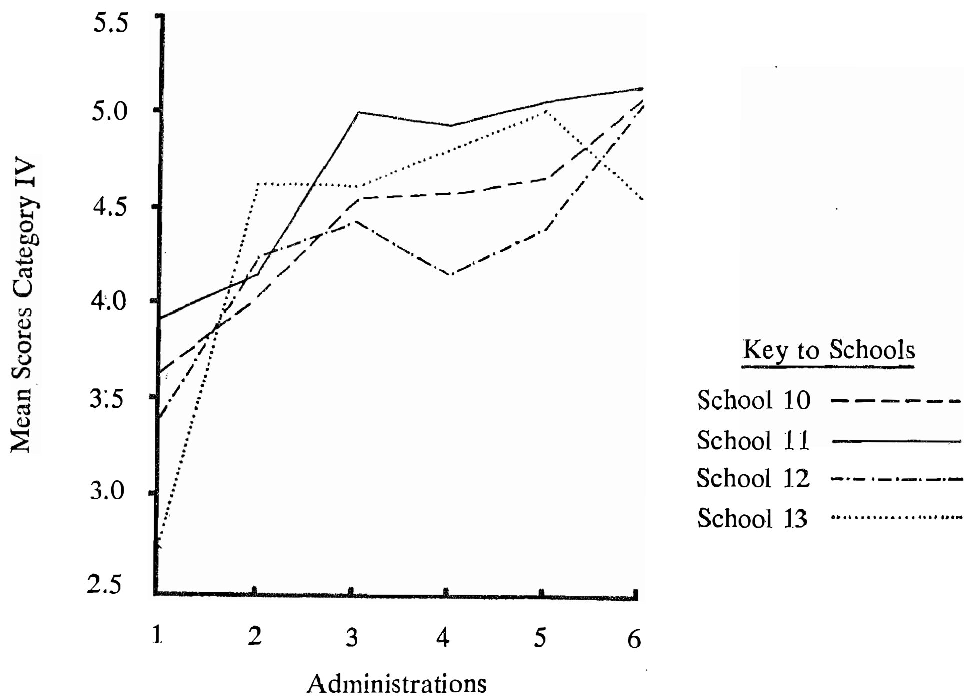

FIG. 5.—Maths. Test I, Mean Scores of Schools, Category IV.

although the concrete operational level approach was of use to them in the early stages, the continued emphasis on this approach throughout the course limited their ability to arrive at their full potential. These latter students, although finding it an advantage to have new concepts introduced at a concrete level, soon developed a need for a more formal approach to satisfy their new found ability to reason at a formal operational level.

The graph of achievement on concept category III, however, does not show exactly the same pattern as the others. Schools 12 and 13 interact in the typical pattern of the other graphs, but school 10, with the second rating on intelligence, remains throughout the lowest scorer on this category and this is not typical of this school in the other concept areas. School 11, with the highest initial score on intelligence and mathematics, maintains the highest average score on this concept category through all six administrations. In spite of the interaction in evidence between the scores in this category of schools 12 and 13, the four schools finish the sixth administration of the test in the same rank order as in the first administration. Furthermore, the dispersion of the means on this category widens slightly rather than contracts. All four schools make gains in this concept category as the administrations progress but it does appear that, in this concept area, an initial advantage in handling the concepts concerned is not readily over-taken. The points noted in this paragraph tend to offer further support to the Piagetian-oriented research which would show this as a problem area until the appropriate substrate concepts are well developed, and this will not have occurred before the child has reached the formal operational level of thinking.

It was mentioned earlier that one of the specific objectives of teaching mathematics is to ensure that the student becomes familiar with the general principles involved in the various parts of the mathematical structure. In addition to knowing and recognizing a general principle involved in a specific mathematical stimulus, the child normally has the task of directing his thoughts so that he recognizes a general principle in a situation where the relationships are highly structured by the problem itself. The Flash Card Test was constructed to study the children's development in these two areas.

Two sets of flash cards[5] were devised; one set consisted of ten single cards each with a different mathematical stimulus on it, e.g., $a + b$; the other set consisted of ten pairs of cards, each card in a pair relatable by comparison or contrast to its partner, e.g., $x + y = y + x$ and $p - q = q - p$ would be a suitable pair. Each card, or pair of cards, was shown to the class for fifteen seconds and then the child was to write down the best mathematical response to the card, or pair of cards, which occurred to her. The dimensions under consideration were concerned with (i) the child's ability to find a general principle per se, and (ii) her ability to find an appropriate general principle when the situation was pre-structured by the stimulus. The single (called unstructured) cards were designed and scored so that any general principle recognized in the stimulus pattern by the testee was appropriately scored. With the structured card set, however, the students were expected to find a general principle which was implied by relating the stimuli on the pair of cards to each other.

Analyses conducted on the data obtained from this series of tests revealed several interesting aspects of the children's approach to the learning of mathematics at this level. Both class and individual records showed that: (a) the children's ability to give satisfactory generalized responses to the stimulus presented (both unstructured and structured) improved as the year went by; (b) overall, this sample of students found the unstructured stimulus the easier of the two with which to make a satisfactory response; and (c) although within each school, the unstructured mean scores were always above the structured mean scores, there was a tendency for the schools to maintain the same rank order on one dimension as they did on the other.

[5]White cards with large black characters printed or written on them.

A close examination of the individual results showed that of the 118 students involved in the test series, 54 vacillated between a preference[6] for the unstructured dimension and a preference for the structured dimension, 49 began with a preference for the unstructured cards and maintained this preference throughout, 15 began with, and maintained, a preference for the structured cards.

With respect to these last results, several teachers of mathematics were asked by the writer which set of cards they felt that they themselves would score best on; all of them selected the structured set. Careful consideration of the children's results, together with the teachers' stated preference, suggested that the most successful students in mathematics might be those who had made their best scores on the structured dimension. To test this hypothesis each student's dimension preference was related to her achievement on each of three tests: Intelligence, Mathematics Test II, and the Attitude Test which were mentioned earlier in connection with the First Study described above. To accomplish this, three tables (2, 3, and 4) were set up. In each table, the rows represent the dimension preference shown by the child's scores throughout the test series, and the columns divide the groups into those whose scores on the relevant test place them above or below the mean of the whole group on the variable concerned.

Consideration of the three tables reveals that:

(a) a girl favouring the structured dimension is more likely to be above the mean on all three criteria than a girl whose scores place her in the unstructured dimension; in Table 2, of all the students in the structured set, .73 are above the mean intelligence compared with .51 in the unstructured group; Tables 3 and 4 show similar relation-

TABLE 2

Flash Card Test, Dimension Preference by Intelligence

	Intelligence		
	Above Mean	Below Mean	
Structured	11	4	15
Unstructured	25	24	49
Vacillating	15	16	31
Unstructured → Structured	4	5	9
Structured → Unstructured	8	6	14
	63	55	118

TABLE 3

Flash Card Test, Dimension Preference by Achievement on Mathematics Test II

	Mathematics Test II Achievement		
	Above Mean	Below Mean	
Structured	10	5	15
Unstructured	29	20	49
Vacillating	14	17	31
Unstructured → Structured	6	3	9
Structured → Unstructured	7	7	14
	66	52	118

[6]"Preference" as used here was judged by the student's score on a particular dimension.

TABLE 4
Flash Card Test, Dimension Preference by Mathematics Attitude Test

	Attitude		
	More Favourable	Less Favourable	
Structured	11	4	15
Unstructured	22	27	49
Vacillating	16	15	31
Unstructured→Structured	5	4	9
Structured→Unstructured	6	8	14
	60	58	118

ships between structured and unstructured scores; on mathematics achievement, the corresponding proportions are .67 and .59 and on attitude to mathematics, .73 and .45;

(b) students who are either vacillating in their choice of a favoured dimension, or who change their orientation during the test series (last three rows), show almost a random distribution of scores in the cells concerned.

The results in this portion of the study tend to support the general theme. Quite sound mathematical generalizations can be obtained on the basis of purely concrete operational thought and the unstructured cards could certainly have been responded to at this level. On the other hand, the structured cards, given the brief exposure time, would need to be operated on by formal level reasoning for best results. This would account for the greater success with which the children handled the unstructured cards. It also helps to account for the preference shown by teachers, competent at mathematics, to favour the structured cards as they would have been long accustomed to looking at mathematical stimuli from the formal operational viewpoint. The attempts to link preferences for structured or unstructured cards with success on other variables showed an apparent trend for the most successful students to be those favouring the structured stimuli—statistical tests of significance, however, do not, in these cases, rule out the possibility that the differences noted were due to chance variation.

These results also highlight the necessity for a more detailed and specific study in this area, especially with respect to the effect which teacher preference for the structured stimulus as opposed to the child preference for unstructured would have on the teacher's communication with the child. Leaving aside general personality variables it would seem reasonable to suppose that lack of communication between a teacher and the majority of his class could be due to a basic difference in orientation towards the subject matter of the discipline concerned.

So far in this second study the tasks given to the children have depended either on the child's ability to read and extract relevant information in a standard test situation (Mathematics Test I) or on her ability to express in words or other symbols some mathematical principle (Flash Card Test); the Card Sorting Task was designed with a view to tracing the development of the child's conceptual structure in this area, with as little interference as possible from such intervening variables. Moreover, the first two sets of data were more likely to provide information regarding levels of operational thinking rather than information concerning the formation of cognitive structures with respect to mathematics in the child's mind.

The sorting procedure was the method by which the students categorized the stimulus patterns. Basically the process was one of putting together any two or more statements or patterns which involved, according to the sorter, the same area or principle of mathematics. In general, the card stimuli were capable of being sorted on the basis of quite superficial criteria, or on a basis of quite high level mathematical principles. For example, cards with the following stimuli on them, (356) and (35 gallons 6 pints), could be sorted together because the expressions on them use the numerals 3, 5, and 6 or they could be included in a group with cards having the following stimuli on them, ($2 \times 100 + 5 \times 10 + 6 \times 1$), ($2^3 + 2^2 + 2^1$) and (£2. 4s. 6d.), on the basis of the generalized principle of place value. The testees were given a minimum number of instructions but were told that they might like to establish a grouping in which they placed cards which did not seem to fit readily anywhere else. This group was called the unclassified group, and was indicated to the tester when the bundles were being returned. The children were allowed as much time for their sorting as they cared to use.

The writer kept a detailed record of the groupings into which each girl sorted her cards from administration to administration. These records were then analysed with a view to observing any patterns of development or strategies which were being used by the girls in order to set up their groups.

With the advantage of hind-sight and the record of all administrations in front of the investigator, several important phenomena appear which are clearly related to the process of concept formation in this area. First, the number of groups of cards forming "pure" categories (i.e. without misconceptions) increases rapidly after the second administration; secondly, the number of cards consigned to the "unclassified" category decreases steadily and constantly throughout the test series; finally, groupings occurred in most students' sortings which, after their first appearance, remained stable and were repeated unaltered in all the student's future sortings of these cards.

However, perhaps the most interesting feature of the results obtained with the card sorting task lies in the fact that the investigator was able to predict an individual child's future sorting after examining the patterns of development and strategies which the child was using and relating these to Ausubel's principles of progressive differentiation and integrative reconciliation. In general, in the initial sorting, the children formed two sets of groupings, a few large groupings and several smaller sets, both groups based on superficial criteria. For example, one large grouping, consisting of twenty-one cards, was set up apparently because all contained letters; another large grouping was set up with the only common aspect being "= s" signs; one small group consisted of two cards, the apparent reason being that both cards had the same numbers involved.

Subsequent sortings took place on the basis of card migrations, the net result being that large superficial groupings were broken down and reformed on the basis of more adequate mathematical criteria. At the same time smaller groups were amalgamated with one another or integrated into the larger groups. After the third sorting the investigator endeavoured to predict certain groupings which the child would attain in later applications of the task. Of these predictions 58 per cent emerged in the fourth administration, and by the sixth administration 95 per cent of the predictions had emerged.

One final phenomenon deserves comment at this stage. Although the movement of cards from one category to another was a necessary technique for re-forming categories, it was noticeable that certain cards, involving mathematical concepts which children of this age level frequently confuse, were continually shuffled about from one category (often inappropriate) to another, as though the child was vaguely aware that they must fit somewhere (rarely were they placed in the unclassified category) but was not quite sure where they should be placed. The cards most often treated in this fashion were the ones with the following stimuli on them, ($5 - 5$), (2×0), (0×7), ($\frac{0}{3}$), i.e. various replacements for zero; ($6 \div 6$), ($\frac{6}{0}$), i.e. stimuli commonly confused with zero; and (x is integral, $3x = 7$), ($a > b$, $b - a$), ($0 - 5$), i.e. "impossible" statements, the latter two being impossible only for students using the experimental text.

The most important result of the Card Sorting Task in this study was the independent evidence which was provided for the view put forward by Ausubel that in learning potentially meaningful material children do use naturally a process of progressive differentiation in order to proceed from broad highly inclusive areas of knowledge to regions of lesser inclusiveness, each step in the process being linked to the one above by a process of subsumption. In other words the more specific data and factual knowledge are linked to a broader more generalized principle and are, in fact, subsumed by that principle. At the same time a process of reconciling the present data or stimuli with past experiences in the field is carried on, and the child's process of integration of his subject-matter knowledge is, in large measure, due to this attempt at reconciliation.

IV. CONCLUSION

Of the studies reported here, the first set out to develop and evaluate a teaching theory in relation to the teaching of mathematics at the late primary-early secondary school level. Both the content area selected and the level of cognitive development of children concerned are highly significant variables in terms of current educational practice. Mathematics curricula and recommended methods in this area are undergoing rapid change—often without adequate scientific foundation; and assisting the child through from typical concrete operational level thinking to formal operational thinking can be seen as a major function of the teachers of children of this age group regardless of the subject-matter content with which the teacher is concerned. The results from this part of the investigation tended to favour the use of a text-book based on a theory of teaching which incorporates both Piaget's developmental concepts and Ausubel's view of cognitive structure.

In addition, as the teaching theory was concerned with the method of shaping a cognitive structure, certain features of the gradual development of a cognitive structure in the field of mathematics were explored with a view to exposing some of the variables underlying its development. The findings in this part of the investigation supported the theoretical orientation of the project as a whole and contained implications for both teaching practice and future research.

It is clear that theories of teaching can only be developed through a careful examination and integration of relevant psychological theories of learning and development, together with objective scientifically based investigations. It would be presumptuous to make extravagant claims for the present study but it is hoped that by adhering to these principles, the investigations reported here may make some contribution to the developing field of teaching technology.

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