

Drag Force on an Impurity below the Superfluid Critical Velocity in a Quasi-One-Dimensional Bose-Einstein Condensate

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The existence of frictionless flow below a critical velocity for obstacles moving in a superfluid is well established in the context of the mean-field Gross-Pitaevskii theory. We calculate the next order correction due to quantum and thermal fluctuations and find a nonzero force acting on a delta-function impurity moving through a quasi-one-dimensional Bose-Einstein condensate at all subcritical velocities and at all temperatures. The force occurs due to an imbalance in the Doppler shifts of reflected quantum fluctuations from either side of the impurity. Our calculation is based on a consistent extension of Bogoliubov theory to second order in the interaction strength, and finds new analytical solutions to the Bogoliubov-de Gennes equations for a gray soliton. Our results raise questions regarding the quantum dynamics in the formation of persistent currents in superfluids.

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Quantum fluids and their macroscopic manifestations have long intrigued physicists. Recent rapid advances in the experimental manipulation of dilute ultracold gases [1–6] have opened a new window to the understanding of macroscopic quantum phenomena. Such experiments provide an exciting opportunity to advance our theoretical understanding of quantum fluids as they probe regimes where calculations are more tractable than many traditional condensed matter systems.

One of the most remarkable features is the emergence of superfluidity and the resulting critical velocity, below which an obstacle is able to move without experiencing any friction [7]. The presence of a Bose-Einstein condensate (BEC) is commonly believed to be intrinsically related to superfluidity. In this Letter we probe the superfluidity of a quasi-1D BEC with a delta-function impurity moving through it at constant velocity. We introduce a complete set of zero eigenvalue solutions to the Bogoliubov-de Gennes equations for the excitation spectrum of the BEC and impurity. Our calculation is consistent to second order in the interaction strength. We find a drag force is present at *subcritical* speeds for all temperatures. We emphasize that this result is not in conflict with Landau's argument, as the drag force in this situation arises from the scattering of fluctuations (either quantum or thermal) rather than the creation of quasiparticles [7]. This mechanism was first suggested as a source of dissipation at zero temperature via a perturbative calculation for a uniform three-dimensional BEC using the Born approximation [8]. The current work goes beyond the perturbative regime and is applicable to repulsive delta-function impurities of any strength. Furthermore, we calculate the force for systems at finite temperature and find a clear distinction between the quantum and thermal regime. These results could be tested experimentally using foreseeable technology [9].

The second-quantized Hamiltonian of the system in the frame moving with the impurity at velocity v is

$$\hat{H} = \int dx \hat{\psi}^\dagger \left(-\frac{\hbar^2}{2m} \partial_x^2 + i\hbar v \partial_x + \eta \delta(x) + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi} \right) \hat{\psi}, \quad (1)$$

where $\hat{\psi} = \hat{\psi}(x, t)$ is the bosonic field operator obeying the usual equal-time commutation relations, and $g > 0$ is related to the 3D s -wave scattering length a : $g \approx 2\hbar\omega_\perp a$ (we have assumed a sufficiently tight transverse trapping potential of frequency ω_\perp [10]). We use the following dimensionless coordinates for the system. Time, $\tau = (gn_\infty/\hbar)t$, and length, $z = x/\xi$, where $\xi = \hbar/\sqrt{mgn_\infty}$ is the healing length, and n_∞ is the total density, at $x = \pm\infty$ (far away from the impurity). The impurity strength and impurity velocity are parametrized by $\bar{\eta} = \eta/(v_s\hbar) > 0$ and $\bar{v} = v/v_s$, respectively, where $v_s = \sqrt{gn_\infty/m}$ is the speed of sound far from the impurity. Finally we rescale the field operator, $\hat{\psi} = \sqrt{n_\infty} \hat{\phi}$. In these units the commutation relations become $[\hat{\phi}(z, \tau), \hat{\phi}^\dagger(z', \tau)] = \sqrt{\gamma} \delta(z - z')$, where $\gamma = mg/(\hbar^2 n_\infty)$ is the ratio of interaction energy to kinetic energy. Our calculations are based on the assumption that $\epsilon \equiv \gamma^{1/4} \ll 1$, and this defines the small parameter in our perturbative expansion. The equation of motion for $\hat{\phi}$ is then

$$i\partial_\tau \hat{\phi} = \left(-\frac{1}{2} \partial_z^2 + i\bar{v} \partial_z + \bar{\eta} \delta(z) + \hat{\phi}^\dagger \hat{\phi} \right) \hat{\phi}. \quad (2)$$

An understanding of the relevant length scales in a 1D Bose gas is important [11]. The ground state of an *infinitely* extended, weakly interacting 1D Bose gas is that of a quasicondensate [12], with phase coherence length $l_\phi = \xi \exp(\sqrt{\pi^2/\gamma})$. Hence phase coherence is present across

regions of size $l_\phi \gg \xi$, but not over all space. We define the size of our system, L , to be the same order as the phase coherence length. That is $L \gg \xi$ and $L \lesssim l_\phi$, such that we have a true condensate which can be considered homogeneous [13]. We emphasize that we are not considering regimes of 1D systems for which there does not exist a true BEC [14].

We ultimately wish to calculate the force on the impurity, given by the gradient of the potential, $F = -\langle \hat{\phi}^\dagger \partial_z [\bar{\eta} \delta(z)] \hat{\phi} \rangle$. First we find the solution to Eq. (2) from Bogoliubov's perturbation expansion $\hat{\phi} \simeq (\varphi_0 + \epsilon \hat{\phi}_1 + \epsilon^2 \varphi_2) e^{-i(\mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2)}$. The expansion is conceptually summarized as (i) φ_0 is the condensate wave function in the absence of all fluctuations, (ii) $\hat{\phi}_1$ describes the fluctuations of the field in the presence of a condensate φ_0 , and (iii) φ_2 is the modification of the condensate wave function due to the presence of fluctuations.

(i) $\mathcal{O}(\epsilon^0)$: At zeroth order the commutators vanish and a c -number field describes the system. The condensate wave function is given by the solution to the Gross-Pitaevskii equation

$$L[\varphi_0]\varphi_0 = 0, \quad (3)$$

where $L[\varphi_0] \equiv -\frac{1}{2} \partial_z^2 + i\bar{v} \partial_z + \bar{\eta} \delta(z) + |\varphi_0|^2 - 1$ ($\mu_0 = 1$). This can be solved analytically [15] for impurity velocities less than the critical velocity $\bar{v} < \bar{v}_c$, where \bar{v}_c depends on $\bar{\eta}$ by $\bar{v}_c^2 = 1 - \frac{2\bar{\eta}^2}{3} + \frac{\bar{\eta}}{\sqrt{2}}(\frac{3}{R} - R) + \bar{\eta}^2(\frac{R}{9} + \frac{1}{R^2})$, where $R^3 \equiv 2\sqrt{2}\bar{\eta}/(\sqrt{1 + 8\bar{\eta}^2/27} - 1)$ [16]. To do so, the term $\bar{\eta} \delta(z)$ in Eq. (3) is replaced with a *derivative jump*: $\partial_z \varphi_0(z = 0^+) - \partial_z \varphi_0(z = 0^-) = 2\bar{\eta}$. The dark soliton solutions [17] are used to give

$$\varphi_0(z) = \begin{cases} e^{i\sigma_-} \{\cos(\theta) \tanh[z_c^-] + i \sin(\theta)\} & (z < 0) \\ e^{i\sigma_+} \{\cos(\theta) \tanh[z_c^+] + i \sin(\theta)\} & (z > 0), \end{cases} \quad (4)$$

where the impurity velocity is parametrized by $\bar{v} = \sin(\theta) < \bar{v}_c$, and $z_c^\pm = (z \pm z_0) \cos(\theta)$. The phases $\sigma_+ = -\theta$, and $\sigma_- = \theta - \pi + 2 \arctan\{\sin(2\theta)/[\exp(2z_0 \cos\theta) - \cos(2\theta)]\}$ are fixed, although φ_0 has an arbitrary global phase factor. Here, we work in the limit of a large system, where the effects of fluctuations in the total number of particles are negligible [18]. The quantity z_0 is determined by the derivative jump condition, which reduces to numerically solving $\bar{\eta} = \cos^3(\theta) \times \tanh[\cos(\theta)z_0]/(\sin^2(\theta) + \sinh^2[\cos(\theta)z_0])$ for a given value of $\bar{\eta}$ and $\bar{v} < \bar{v}_c$. Two solutions exist for z_0 , but only one is physical [15]. Figure 1 shows a plot of $\varphi_0(z)$ for a specific $\bar{\eta}$ and \bar{v} . As expected, at this level of approximation the drag force $F = 0$. For numerical solutions to Eq. (3) in higher dimensions, see Ref. [19].

(ii) $\mathcal{O}(\epsilon^1)$: The quantum and thermal depletion of the condensate is accounted for at first order. Linearization of Eq. (2) with respect to ϵ results in

$$i\partial_\tau \hat{\phi}_1 = L[\sqrt{2}\varphi_0]\hat{\phi}_1 + \varphi_0^2 \varphi_1^\dagger. \quad (5)$$

The density is unaffected at order ϵ (as $\langle \hat{\phi}_1 \rangle = 0$) and

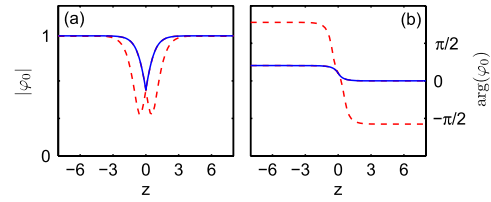


FIG. 1 (color online). Mean field solutions for (a) the condensate amplitude $|\varphi_0(z)|$ and (b) the condensate phase from Eq. (4) for $\bar{\eta} = 1$ and $v/v_s = 0.35$. The dashed line shows the continuation of each of the underlying dark soliton solutions.

therefore $\mu_1 = 0$. A Bogoliubov transformation is then performed, $\hat{\phi}_1 = \int dk [u_k(z) e^{-iE_k \tau} \hat{\alpha}_k + v_k^*(z) e^{iE_k \tau} \hat{\alpha}_k^\dagger]$, where $\hat{\alpha}_k$ ($\hat{\alpha}_k^\dagger$) are the annihilation (creation) operators of the excitations (note that k is a continuous index). They obey the usual commutation relations, $[\hat{\alpha}_k, \hat{\alpha}_{k'}^\dagger] = \delta(k - k')$ and $[\hat{\alpha}_k, \hat{\alpha}_{k'}] = [\hat{\alpha}_k^\dagger, \hat{\alpha}_{k'}^\dagger] = 0$. The quantum state of the system is defined such that the occupation number of each excitation is $n_k \equiv \langle \hat{\alpha}_k^\dagger \hat{\alpha}_k \rangle = 1/(e^{E_k/T} - 1)$. The amplitudes, u_k and v_k , are determined by

$$\begin{bmatrix} L[\sqrt{2}\varphi_0] & \varphi_0^2 \\ -(\varphi_0^*)^2 & -L^*[\sqrt{2}\varphi_0] \end{bmatrix} \vec{u}_k = E_k \vec{u}_k, \quad (6)$$

where $\vec{u}_k = (u_k, v_k)^T$. The normalization which preserves the bosonic commutation relations is given by $\int dz \vec{u}_k^\dagger \sigma_z \vec{u}_{k'} = \delta(k - k')$, where σ_z is a Pauli matrix.

The solutions of Eq. (6) for $z \in \mathbb{R}^\pm$ are [20,21]

$$\vec{\xi}_k^\pm \equiv e^{ikz} \begin{bmatrix} e^{i\sigma_\pm [\frac{k}{2} + \frac{E_k}{k} + i \cos\theta \tanh(z_c^\pm)]^2} \\ e^{-i\sigma_\pm [\frac{k}{2} - \frac{E_k}{k} + i \cos\theta \tanh(z_c^\pm)]^2} \end{bmatrix}, \quad (7)$$

where E_k has two distinct branches: $E_k = E_k^\pm \equiv \pm k[\mp \sin(\theta) + \sqrt{k^2/4 + 1}]$, corresponding to the left and right propagating excitations and hence $E_k > 0$ (special attention must be paid to the $E_k = 0$ mode). For a fixed excitation energy E , there are four possible wave numbers, given by the four roots of $(1/4)k^4 + \cos^2(\theta)k^2 + 2E \sin(\theta)k - E^2 = 0$. Two of the roots will be in \mathbb{R} , denoted k and k_r with one positive and one negative. k_r is a *reflected mode*, and $k_r = -k$ only when $v = 0$. For $v \neq 0$ a Doppler shift will occur. The other two roots will be complex conjugates (with nonzero imaginary parts) denoted k_e and k_e^* . On either side of the impurity, one must select the exponentially *decreasing* mode to satisfy the boundary conditions of the problem. Each wave number corresponds to a particular solution of Eq. (6), which is ultimately a 4th order differential equation. The normalization of the eigenvectors in Eq. (7) is $N_k^2 \int dz \vec{\xi}_k^\dagger \sigma_z \vec{\xi}_k = \delta(k - k')$, where $N_k = \frac{1}{4} \sqrt{k/\pi} (k^2/4 + 1)^{-1/4} E_k^{-1}$. Thus the solutions in Eq. (7) correspond exactly to the usual Bogoliubov excitations of a uniform BEC traveling at velocity v for $z \rightarrow \pm\infty$ [22].

To incorporate the impurity at $z = 0$, we solve two independent scattering problems: One in which the incom-

ing mode approaches from $z = -\infty$ and the other in which it approaches from $z = +\infty$. The boundary conditions we impose are (i) the amplitude of the incoming wave is given by N_k in accordance with the assumption that, away from the impurity, the BEC has no knowledge of the impurity. Implicit in this condition is a time scale over which it is assumed the scattered waves have not reached edge of the system. (ii) Scattered waves cannot exponentially increase as they move away from the impurity. (iii) Scattered waves must be *causally* related to the incoming wave. For the case where the incoming wave approaches from $z = -\infty$ the solution is

$$\vec{u}_k = \begin{cases} N_k \vec{\xi}_k^- + A_r \vec{\xi}_{k_r}^- + A_{er} \vec{\xi}_{k_e}^- & (z < 0) \\ A_l \vec{\xi}_k^+ + A_{el} \vec{\xi}_{k_e}^+ & (z > 0), \end{cases} \quad (8)$$

where $k > 0$ and the complex constants A_r, A_l, A_{el} , and A_{er} are completely determined by $\vec{u}_k|_{z=0^+} = \vec{u}_k|_{z=0^-}$ and $\partial_z \vec{u}_k|_{z=0^+} - \partial_z \vec{u}_k|_{z=0^-} = 2\bar{\eta} \vec{u}_k|_{z=0}$, which arise from the $\bar{\eta} \delta(z)$ term in Eq. (6). Solving the scattering problem where the incoming wave approaches from $z = +\infty$ follows the same logic, except now $k < 0$ and the $z > 0$ and $z < 0$ conditions in Eq. (8) have to be swapped.

At this stage the problem has been reduced to a set of linear algebraic equations for each value of k . Analytic progression is hindered by the fourth order polynomial determining k_e and k_r . However, quantities such as $\langle \hat{\varphi}_1^\dagger(z) \hat{\varphi}_1(z) \rangle = \int dk [|u_k|^2 n_k + |v_k|^2 (1 + n_k)]$ and $\langle \hat{\varphi}_1(z) \hat{\varphi}_1(z) \rangle = \int dk [u_k v_k^* (1 + 2n_k)]$ are easily attained numerically to very high accuracy once one fixes $\bar{\eta}$, \bar{v} , and T . One must include a small k cutoff (of order $\sim 1/L$) to prevent long wavelength fluctuations destroying the long range order in the system. Results are logarithmically dependent on the choice of this cutoff [23].

(iii) $\mathcal{O}(\epsilon^2)$: At second order, the generalized Gross-Pitaevskii equation [18] is

$$\mathcal{H} |\varphi_2(z)\rangle = |f(z)\rangle, \quad (9)$$

where $f(z) = 2\varphi_0 \int dk [|u_k|^2 n_k + |v_k|^2 (n_k + 1)] + \varphi_0^* \int dk [u_k v_k^* (1 + 2n_k)] - \mu_2 \varphi_0$ (this definite integral is performed numerically using the analytic results of the previous section), and

$$\mathcal{H} = \begin{bmatrix} L[\sqrt{2}\varphi_0] & \varphi_0^2 \\ (\varphi_0^*)^2 & L^*[\sqrt{2}\varphi_0] \end{bmatrix}. \quad (10)$$

Here, a ket always corresponds to a complex conjugate pair: $|\bullet\rangle \equiv (\bullet, \bullet^*)^T$. The shift in the chemical potential μ_2 is defined on either side of the impurity such that $f(z) \rightarrow 0$ as $z \rightarrow \pm\infty$, and contains the necessary terms to ensure orthogonality between the condensate mode and the excitations [18].

To solve Eq. (9) for $\varphi_2(z)$, we use four linearly independent solutions of the homogeneous equation, $\mathcal{H}|\omega\rangle = |0\rangle$, to construct a 2×2 Green's matrix, $\mathcal{G}(z, s)$. Finding these four linearly independent solutions constitutes a nontrivial

step, and we were unable to find expressions for them in the previous literature; we provide a detailed presentation of these solutions in [24]. Briefly, linear combinations of Eqs. (2)–(5) of [24] are used to generate four solutions, denoted $|\Lambda_j(s)\rangle$, appropriate to the presence of an impurity [the $\bar{\eta} \delta(z)$ term in Eq. (9)]. Green's matrix satisfies

$$\mathcal{H} \mathcal{G}(z, s) = \delta(z - s) \mathbb{1}_2, \quad (11)$$

where $\mathbb{1}_2$ is the 2×2 identity. This condition yields Eqs. (13)–(16) of Ref. [24]. The boundary conditions for Eq. (9) are (i) φ_2 does not exponentially increase away from the impurity, (ii) φ_2 is symmetric about the impurity for $v = 0$, (iii) any nonzero drag force on the impurity acts to decrease the relative motion between condensate and impurity. These conditions combined with the symmetry of Green's matrix [specifically $\mathcal{G}(z, s) = \mathcal{G}^\dagger(s, z)$ which arises from the operator \mathcal{H} being Hermitian] uniquely define the solution

$$|\varphi_2(z)\rangle = \int ds \mathcal{G}(z, s) |f(s)\rangle, \quad (12)$$

up to a global phase factor. We note that Eq. (12) is not a general formula for any function $f(s)$ —the fact that $f(s)$ decays to zero as $s \rightarrow \pm\infty$ compensates for the linear

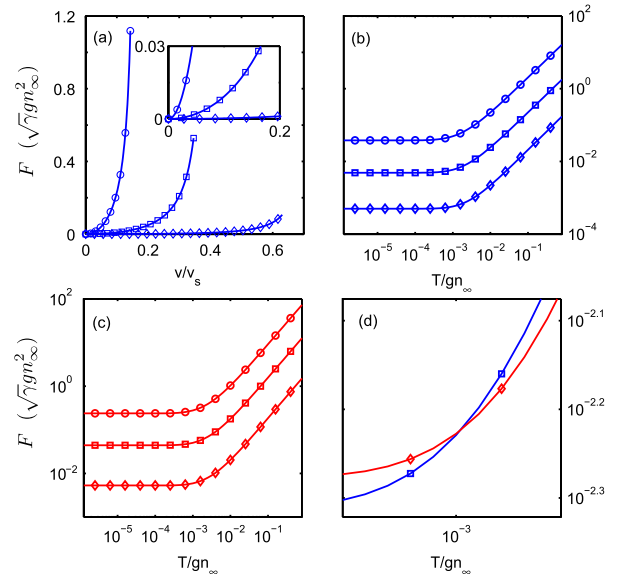


FIG. 2 (color online). The drag force on the impurity (in units $\sqrt{\gamma} g n_0^2$). (a) Drag force at $T = 0$ as a function of velocity. $\bar{\eta} = 0.3, 1, 3$ for diamond, square, circle, respectively, and each line terminates at their respective critical velocities $v_c/v_s = 0.64, 0.37, 0.16$. (b)–(d) Drag force as a function of temperature. (b) $\bar{\eta} = 0.3$ and velocities: blue diamond $\bar{v} = 0.14$, blue square $\bar{v} = 0.35$, blue circle $\bar{v} = 0.56$. (c) $\bar{\eta} = 1$ and velocities: red diamond $\bar{v} = 0.08$, red square $\bar{v} = 0.2$, and red circle $\bar{v} = 0.3$. Panel (d) is data taken from (b) and (c), and compares the drag on a strong impurity moving at slow speed with that on a weak impurity moving at high speed. The strong impurity can have a larger drag in the quantum regime while the weak impurity will have the larger drag in the thermal regime.

divergences in \mathcal{G} . The integral in Eq. (12) is also performed numerically.

We now evaluate the force to $\mathcal{O}(\epsilon^2)$

$$F \simeq \bar{\eta} \partial_z \left[\int dk [|u_k|^2 n_k + |v_k|^2 (1 + n_k)] + 2 \operatorname{Re} \{ \varphi_0 \varphi_2^* \} \right], \quad (13)$$

where all the derivatives are performed analytically using Eq. (8) and Eqs. (2)–(5) of [24]. The results are shown in Fig. 2(a) for the case of zero temperature. The fact that the force is *nonzero* below the critical velocity may be unexpected, although we note the magnitude is much smaller than the supercritical forces present above v_c [15]. We emphasize that quasiparticle creation is *not* the mechanism behind this drag force, and consequentially this is not a contradiction of Landau's statements [7]. At zero temperature fluctuations in the quantum vacuum scatter from each side of the impurity, and the asymmetric Doppler shift in the reflected waves results in an imbalance and a nonzero net force. A crucial assumption in this result is that the incoming waves have no knowledge of the impurity. In any finite system this assumption would *only* be valid up to a time $t_c \sim L/v_s$ after the initial acceleration of the impurity. It could be imagined that the force on the impurity will decay to zero over this time scale t_c , which can be thought of as the relaxation time of the quantum vacuum. We intend to test this conjecture by performing dynamical simulations that incorporate the effects of the quantum fluctuations; this project lies beyond the scope of this work.

The results at finite temperature are shown in Figs. 2(b)–2(d). The quantum and thermal regimes are clearly distinguished with a crossover near $T \sim 10^{-3} gn_\infty$. The force increases linearly with T in the thermal regime due to the scattering being most prominent for the low energy modes whose occupation is $n_k = 1/(e^{E_k/T} - 1) \sim T/E_k$ for $T \gg E_k$. The reflected modes for the high energy incoming waves have a negligible Doppler shift and only a small contribution to the force.

In conclusion, we have calculated the drag force acting on an impurity moving through a true BEC, in a quasi-1D geometry, using analytic solutions of the Bogoliubov perturbation expansion. We find that a nonzero force arises for zero and finite temperature at all velocities due to an imbalance in the Doppler shift in the scattering of quantum and thermal fluctuations. The crossover from the quantum to the thermal regime occurs at temperatures near $T \sim 10^{-3} gn_\infty$. We have proposed a mechanism that predicts a time scale of L/v_s for which it would be possible to observe the force in a finite system.

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