NETWORK SYNTHESIS OF LINEAR DYNAMICAL QUANTUM STOCHASTIC SYSTEMS*

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Abstract. The purpose of this paper is to develop a synthesis theory for linear dynamical quantum stochastic systems that are encountered in linear quantum optics and in phenomenological models of linear quantum circuits. In particular, such a theory will enable the systematic realization of coherent/fully quantum linear stochastic controllers for quantum control, amongst other potential applications. We show how general linear dynamical quantum stochastic systems can be constructed by assembling an appropriate interconnection of one degree of freedom open quantum harmonic oscillators and, in the quantum optics setting, discuss how such a network of oscillators can be approximately synthesized or implemented in a systematic way from some linear and nonlinear quantum optical elements. An example is also provided to illustrate the theory.

Key words. quantum networks, quantum network synthesis, quantum control, linear quantum stochastic systems, linear quantum circuit theory

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1. Background and motivation. In recent years there has been an explosion of interest in exploitation of quantum mechanical systems as a basis for new quantum technologies, giving birth to the field of quantum information science. To develop quantum technologies, it has been recognized from early on that quantum control systems will play a crucial role for tasks such as manipulating a quantum mechanical system to perform a desired function or to protect it from external disturbances [1, 2]. Moreover, recent advances in quantum and nanotechnology have provided a great impetus for research in the area of quantum feedback control systems; e.g., see [3, 4, 5, 6, 7, 8].

Perhaps just about the simplest and most tractable controller to design would be the linear quantum controllers, and this makes them an especially attractive class of controllers. In this class, one can have classical linear quantum controllers that process only classical signals which are obtained from a quantum plant by measurement of some plant output signals (e.g., [5, 9, 10]), but more recently there has also been interest in fully quantum and mixed quantum-classical linear controllers that are able to manipulate quantum signals [11, 12, 13, 14, 15]. In fact, an experimental realization of a fully quantum controller in quantum optics has been successfully demonstrated in [8]. As noted in that paper, the class of fully quantum controllers or *coherent-feedback controllers*, as they are often known in the physics literature, presents genuinely new control-theoretic challenges for quantum controller design. An important open problem raised in the works [12, 13, 14, 15] is how one would systematically build or implement a general, arbitrarily complex, linear quantum controller, at least approximately, from basic quantum devices, such as quantum optical devices. This prob-

lem can be viewed as a quantum analogue of the synthesis problem of classical electrical networks (in this paper the qualifier "classical" refers broadly to systems that are not quantum mechanical) that asks the question of how to build arbitrarily complex linear electrical circuits from elementary passive and active electrical components such as resistors, capacitors, inductors, transistors, op-amps, etc. Therefore, the quantum synthesis problem is not only of interest for the construction of linear quantum stochastic controllers, but also more broadly as a fundamental aspect of linear quantum circuit theory that arises, for example, in quantum optics and when working with phenomenological models of quantum RLC circuits such as described in [16], as well as in relatively new fields such as nanomechanical circuit quantum electrodynamics [17, 18].

A key result of this paper is a new synthesis theorem (Theorem 5.1) that prescribes how an arbitrarily complex linear quantum stochastic system can be decomposed into an interconnection of basic building blocks of one degree of freedom open quantum harmonic oscillators and thus be systematically constructed from these building blocks. In the context of quantum optics, we then propose physical schemes for "wiring up" one degree of freedom open quantum harmonic oscillators and the interconnections between them that are required to build a desired linear quantum stochastic system, using basic quantum optical components such as optical cavities, beam splitters, squeezers, etc. An explicit yet simple example that illustrates the application of the theorem to the synthesis of a two degrees of freedom open quantum harmonic oscillator is provided.

1.1. Elements of linear electrical network synthesis. To motivate synthesis theory in the context of linear dynamical quantum systems, we start with a brief overview of aspects of linear electrical network synthesis that are relevant for the current work.

As is well known, a classical (continuous time, causal, linear time invariant) electrical network described by a set of (coupled) ordinary differential equations can be analyzed using various representations, for example, with a frequency domain or transfer function representation, with a modern state space representation and, more recently, with a behavioral representation. It is well known that the transfer function and state space representation are equivalent in the sense that one can switch between one representation to the other for any given network. However, although one can associate a unique transfer function representation to a state space representation, the converse is not true: for a given transfer function there are infinitely many state space representations. The state space representation can be made to be unique (up to a similarity transformation of the state space matrices) by requiring that the representation be of minimal order (i.e., the representation is both controllable and observable). The synthesis question in linear electrical networks theory deals with the inverse scenario, where one is presented with a transfer function or state space description of a linear system and would like to synthesize or build such a system from various linear electrical components such as resistors, capacitors, inductors, op-amps, etc. A particularly advantageous feature of the state space representation, since it is given by a set of first order ordinary differential equations, is that it can be inferred directly from the representation how the system can be *systematically* synthesized. For example, consider the system below, given in a state space representation:

(1.1)
$$\frac{dx(t)}{dt} = \begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} u(t),$$
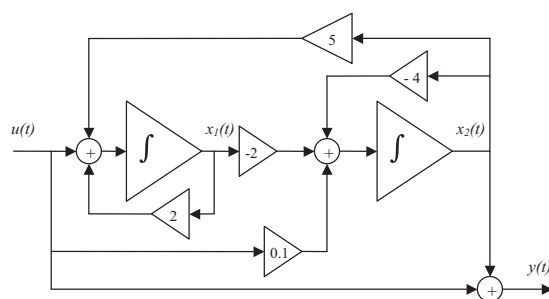$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + u(t),$$

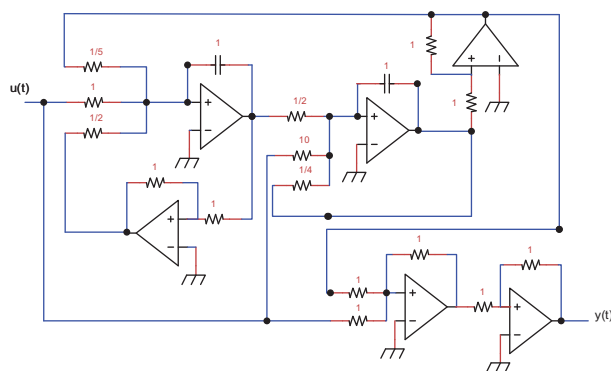FIG. 1.1. *Schematic for the implementation of the classical system* (1.1).

FIG. 1.2. *Hardware implementation of the schematic diagram shown in Figure* 1.1.

where $x(t)$ is the state, $u(t)$ is the input signal, and $y(t)$ is the output signal. In an electrical circuit, $u(t)$ could be the voltage at certain input ports of the circuit and $y(t)$ could be the voltage at another set of ports of the circuit, different from the input ports. This system can be implemented according to the schematic shown in Figure 1.1. This schematic can then be used to to implement the system at the hardware level as shown in Figure 1.2 [19, Chapter 13]. However, linear electrical network synthesis is a mature subject that deals with much more than just how one can obtain *some* realization of a particular system. For instance, it also addresses fundamental issues such as how a passive network, a network that does not require an external source of energy, can also be synthesized using only passive electrical components, and how to synthesize a given circuit with a minimal number of circuit elements or with a minimal number of certain types of elements (such as active elements). In this paper our primary objective is to develop an analogously systematic method for synthesizing arbitrarily complex linear dynamical *quantum* stochastic systems that are given in an abstract description that is similar in form to (1.1). These linear dynamical quantum stochastic systems are ubiquitous in linear quantum optics, where they arise as idealized models for linear open quantum systems. However, since there is currently no comprehensive synthesis theory available for linear dynamical quantum systems (as opposed to *static* linear quantum systems in linear quantum optics that have been studied in, e.g., [20]) and related notions such as passivity have not been extensively studied and developed, here we focus our attention solely on the development of a *general* synthesis method that applies to *arbitrary* linear dynamical quantum systems which does not exploit specific physical properties or characteristics that a particular

system may possess (say, for instance, passivity). Although the latter will be an important issue to be dealt with in further development of the general theory, it is beyond the scope of the present paper (which simply demonstrates the existence of *some* physical realization).

1.2. Open quantum systems and quantum Markov models. A quantum system is never completely isolated from its environment and can thus interact with it. Such quantum systems are said to be *open quantum systems* and are important in modeling various important physical phenomena such as the decay of the energy of an atom. The environment is modeled as a separate quantum system in itself and can be viewed as a *heat bath* to which an open quantum system can dissipate energy or from which it can gain energy (see [21, Chapters 3 and 7]). An idealization often employed in modeling the interaction between an open quantum system and an external heat bath is the introduction of a *Markovian* assumption: the dynamics of the coupled system and bath is essentially "memoryless" in the sense that future evolution of the dynamics of the coupled system depends only on its present state and not at all on its past states. Open quantum systems with such a property are said to be *Markov*. The Markov assumption is approximately valid under some physical assumptions made on the system and bath, such as that the heat bath is so much "larger" than the system (in the sense that it has many more degrees of freedom than the system) and is weakly coupled to the system that its interaction with the latter has little effect on its own dynamics and can thus be neglected; for details on the physical basis for this Markovian assumption, see [21, Chapters 3 and 5]. Markov open quantum systems are important, as they are often employed as very good approximations to various practically relevant open quantum systems, particularly those that are encountered in the field of quantum optics, yet at the same time are relatively more tractable to analyze as their dynamics can be written in terms of first order operator differential equations.

In Markov open quantum systems, heat baths can be idealistically modeled as a collection of a continuum of harmonic oscillators oscillating at frequencies in a continuum of values. An important consequence of the Markov approximation in this model is that the heat bath can be effectively treated in a quantum statistical sense as quantum noise [21, section 3.3], and thus Markov open quantum systems have inherently stochastic quantum dynamics that are most appropriately described by quantum stochastic differential equations (QSDE) [21, 22, 23, 24]. To be concrete, a single heat bath in the Markov approximation is formally modeled as an operator-valued *quantum white noise* process $\eta(t)$, where $t \geq 0$ denotes time, that satisfies the singular commutation relation $[\eta(t), \eta(t)^*] = \delta(t - t')$, where $*$ denotes the adjoint of an operator, $\delta(t)$ is the Dirac delta function, and the commutator bracket $[\cdot, \cdot]$ acts on operators A and B as $[A, B] = AB - BA$. Examples of heat baths that have been effectively modeled in such a way include vacuum noise, squeezed and laser fields in quantum optics [21], and infinitely long bosonic transmission lines [16]. See also [25] for a brief intuitive overview of the modeling of a free-traveling quantized electromagnetic wave as quantum white noise. The formal treatment with quantum white noises can be made mathematically rigorous by considering the bosonic annihilation process $A(t)$ (on a Fock space) that can be formally defined as the "integral" of $\eta(t)$: $A(t) = \int_0^t \eta(s)ds$ and its adjoint process $A^*(t) = A(t)^*$. We shall refer to the operator process $A(t)$ simply as a *bosonic field*. The celebrated Hudson–Parthasarathy (H-P) stochastic calculus provides a framework for working with differential equations involving the processes $A(t)$ and $A^*(t)$, as well as another fundamental process on a Fock space called the gauge process, denoted by $\Lambda(t)$, that models scattering of the photons

of the bosonic heat bath (at a formal level, one could write $\Lambda(t) = \int_0^t \eta(s)^* \eta(s) dt$). More generally, a quantum system can be coupled to several independent bosonic fields $A_1(t), \ldots, A_n(t)$, with $A_j(t) = \int_0^t \eta_j(s) ds$, and in this case there can be scattering between different fields modeled by interfield gauge processes $\Lambda_{jk}(t) = \int_0^t \eta_j(s)^* \eta_k(s) ds$ (in interfield scattering, a photon is annihilated in one field and then created in another).

1.3. Linear dynamical quantum stochastic systems. Linear dynamical quantum stochastic systems (e.g., see [10, 12]) arise in practice as idealized models of *open* quantum harmonic oscillators whose canonical position and momentum operators are *linearly* coupled to one or more external (quantum) heat baths (the mathematical modeling involved is discussed in section 2). Here a quantum harmonic oscillator is a quantized version of a classical harmonic oscillator in which the classical position and momentum variables q_c and p_c, respectively, are replaced by operators q, p on an appropriate Hilbert space (in this case the space $L^2(\mathbb{R})$) satisfying the canonical commutation relations (CCR) $[q, p] = 2i$. It is said to be open if it is interacting with elements of its environment. For instance, consider the scenario in [5] of an atom trapped in an optical cavity. The light in the cavity is strongly coupled to the atomic dipole, and as the atom absorbs and emits light, there are random mechanical forces on the atom. In an appropriate parameter regime, the details of the optical and atomic dipole dynamics are unimportant, and the optical field can be modeled as an environment for the atomic motion. Under the assumptions of [5] the "motional observables" of the trapped atom (its position and momentum operators) can then be treated like those of an open quantum harmonic oscillator. Linear Markov open quantum models are extensively employed in various branches of physics in which the Markov type of arguments and approximations such as discussed in the preceding subsection can be justified. They are particularly prominent in quantum optics, but have also been used, among others, in phenomenological modeling of quantum RLC circuits [16], in which the dissipative heat baths are realized by infinitely long transmission lines attached to a circuit. For this reason, the general synthesis results developed herein (cf. Theorem 5.1) are anticipated to be be relevant in various branches of quantum physics that employ linear Markov models. For example, it has the potential of playing an important role in the systematic and practical design of complex linear photonic circuits as the technology becomes feasible.

A general linear dynamical quantum stochastic system is simply a many degrees of freedom open quantum harmonic oscillator with several pairs of canonical position and momentum operators q_k, p_k, with k ranging from 1 to n, where n is the number of degrees of freedom of the system, satisfying the (many degrees of freedom) CCR $[q_j, p_k] = 2i\delta_{jk}$ and $[q_j, q_k] = [p_j, p_k] = 0$, where δ_{jk} is the Kronecker delta which takes on the value 0 unless $j = k$, in which case it takes on the value 1, that is linearly coupled to a number of external bosonic fields $A_1, \ldots, A_m$. In the interaction picture with respect to the field and oscillator dynamics, the operators q_j, p_j evolve unitarily in time as $q_j(t), p_j(t)$ while preserving the CCR $[q_j(t), p_k(t)] = 2i\delta_{jk}$ and $[q_j(t), q_k(t)] = [p_j(t), p_k(t)] = 0 \ \forall t \geq 0$, and the dynamics of the oscillator is given by (here $x(t) = (q_1(t), p_1(t), \ldots, q_n(t), p_n(t))^T$ and $A(t) = (A_1(t), \ldots, A_m(t))^T$)

$$dx(t) = Ax(t)dt + B \left[\begin{array}{c} dA(t) \\ dA(t)^* \end{array} \right],$$
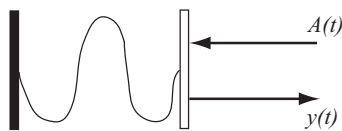
(1.2)
$$dy(t) = Cx(t)dt + DdA(t),$$

FIG. 1.3. *A Fabry–Perot optical cavity. The black rectangle denotes a mirror fully reflecting at the cavity resonance frequency, while the white rectangle denotes a mirror partially transmitting at that frequency.*

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{C}^{n \times 2m}$, $C \in \mathbb{C}^{m \times n}$, and $D \in \mathbb{C}^{m \times m}$. Here the variable $y(t)$ acts as the output of the system due to interaction of the bosonic fields with the oscillator; a component $y_j(t)$ of $y(t)$ is the transformed version of the field $A_j(t)$ that results *after* it interacts with the oscillator. Hence, $A_j(t)$ can be viewed as an *incoming* or *input* field, while $y_j(t)$ is the corresponding *outgoing* or *output* field. To make the discussion more concrete, let us consider a well-known example of a linear quantum stochastic system in quantum optics: an optical cavity (see section 6.1.1 for further details of this device), shown in Figure 1.3. The cavity depicted in the picture is known as a standing wave or Fabry–Perot cavity and consists of one fully reflecting mirror at the cavity resonance frequency and one partially transmitting mirror. Light that is trapped inside the cavity forms a standing wave with an oscillation frequency of ω_{cav}, while parts of it leak through the partially transmitting mirror. The loss of light through this mirror is modeled as an interaction between the cavity with an incoming bosonic field $A(t)$ in the vacuum state (i.e., a field with zero photons or a zero-point field) incident on the mirror. The dynamics for a cavity is linear and given by

$$dx(t) = -\frac{\gamma}{2}x(t)dt - \sqrt{\gamma}dA(t),$$
$$dy(t) = \sqrt{\gamma}x(t)dtdt + dA(t),$$

where $\gamma > 0$ is the coupling coefficient of the mirror, $x(t) = (q(t), p(t))^T$ are the interaction picture position and momentum operators of the standing wave inside the cavity, and $y(t)$ is the outgoing bosonic field that leaks out of the cavity. A crucial point to notice about (1.2) is that it is in a similar form to the classical deterministic state space representation such as given in (1.1), with the critical exception that (1.2) is a (quantum) stochastic system (due to the quantum statistical interpretation of $A(t)$) and involves quantities which are operator-valued rather than real/complex-valued. Furthermore, the system matrices A, B, C, D in (1.2) cannot take on arbitrary values for (1.2) to represent the dynamics of a physically meaningful system (see [12] and [13, Chapter 7] for further details). For instance, for arbitrary choices of A, B, C, D the many degrees of freedom CCR may not be satisfied for all $t \geq 0$ as required by quantum mechanics; hence these matrices cannot represent a physically feasible system. In [12, 13], the notion of *physically realizable* linear quantum stochastic systems has been introduced that corresponds to open quantum harmonic oscillators (hence are physically meaningful), which do not include scattering processes among the bosonic fields. In particular, necessary and sufficient conditions have been derived on the matrices A, B, C, D for a system of the form (1.2) to be physically realizable. More generally, however, are linear quantum stochastic systems that are completely described and parameterized by three (operator-valued) parameters: its Hamiltonian $H = \frac{1}{2}x^T R x$ ($R \in \mathbb{R}^{n \times n}$, $R = R^T$), its linear coupling operator to the external bosonic fields $L = Kx$ ($K \in \mathbb{C}^{m \times n}$), and its *unitary* scattering matrix

$S \in \mathbb{C}^{m \times m}$. In particular, when there is no scattering involved ($S = I$), then it has been shown in [12] that (S, L, H) can be recovered from (A, B, C, D) (since $S = I$, here necessarily $D = I$) and vice-versa. Although [12] does not consider the scattering processes, the methods and results therein can be adapted accordingly to account for these processes (this is developed in section 4 of this paper).

The works [12, 13] were motivated by the problem of the design of robust fully quantum controllers and left open the question of how to systematically build arbitrary linear quantum stochastic controllers as a suitable network of basic quantum devices. This paper addresses this open problem by developing synthesis results for general linear quantum stochastic systems for applications that are anticipated to extend beyond fully quantum controller synthesis, and it also proposes how to implement the synthesis in quantum optics. The organization of the rest of this paper is as follows. Section 2 details the mathematical modeling of linear dynamical quantum stochastic systems and defines the notion of an open oscillator and a generalized open oscillator, section 3 gives an overview of the notions of the concatenation and series product for generalized open oscillators as well as the concept of a reducible quantum network with respect to the series product, and section 4 discusses the bijective correspondence between two descriptions of a linear dynamical quantum stochastic system. This is then followed by section 5 that develops the main synthesis theorem which shows how to decompose an arbitrarily complex linear dynamical quantum stochastic system as an interconnection of simpler one degree of freedom generalized open oscillators, section 6 that proposes the physical implementation of arbitrary one degree of freedom generalized open oscillators and direct interaction Hamiltonians between these oscillators, and section 7 that provides an explicit example of the application of the main synthesis theorem to the construction of a two degrees of freedom open oscillator. Finally, section 8 provides a summary of the contributions of the paper and conclusions.

2. Mathematical modeling of linear dynamical quantum stochastic systems. In the previous works [10, 12] linear dynamical quantum stochastic systems were essentially considered as open quantum harmonic oscillators. Here we shall consider a more general class of linear dynamical quantum stochastic systems consisting of the cascade of a static passive linear quantum network with an open quantum harmonic oscillator. However, in this paper we restrict our attention to synthesis of linear systems with purely quantum dynamics, whereas the earlier work [12] considers a more general scenario where a mixture of both quantum and classical dynamics is allowed (via the concept of an augmentation of a quantum linear stochastic system). The class of mixed classical and quantum controllers will be considered in a separate work. To this end, let us first recall the definition of an open quantum harmonic oscillator (for further details, see [10, 12, 13]).

In this paper we shall use the following notations: $i = \sqrt{-1}$, * will denote the adjoint of a linear operator as well as the conjugate of a complex number, if $A = [a_{jk}]$ is a matrix of linear operators or complex numbers, then $A^\# = [a_{jk}^*]$, and $A^\dagger$ is defined as $A^\dagger = (A^\#)^T$, where ^{T} denotes matrix transposition. We also define $\Re\{A\} = (A + A^\#)/2$ and $\Im\{A\} = (A - A^\#)/2i$ and denote the identity matrix by I whenever its size can be inferred from context and use $I_{n \times n}$ to denote an $n \times n$ identity matrix.

Let $q_1, p_1, q_2, p_2, \ldots, q_n, p_n$ be the canonical position and momentum operators, satisfying the canonical commutation relations $[q_j, p_k] = 2i\delta_{jk}$, $[q_j, q_k] = 0$, $[p_j, p_k] = 0$ of a *quantum harmonic oscillator* with a quadratic Hamiltonian $H = \frac{1}{2}x_0^T R x_0$

$(x_0 = (q_1, p_1, \ldots, q_n, p_n)^T)$, where $R = R^T \in \mathbb{R}$. The integer n will be referred to as the *degrees of freedom* of the oscillator. Digressing briefly from the main theme of this section, let us first discuss why the matrix R can be taken to be real (symmetric) rather than complex (Hermitian). Consider a general quadratic Hamiltonian H of the form $H = \frac{1}{2} \sum_{j=1}^{n} (\alpha_j q_j^2 + \beta_j q_j p_j + \gamma_j p_j q_j + \epsilon_j p_j^2) + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \kappa_{jk} q_j p_k$, with $\alpha_j, \epsilon_j, \kappa_{jk} \in \mathbb{R}$, $\beta_j, \gamma_j \in \mathbb{C}$, and $\beta_j^* = \gamma_j \; \forall \; j, k$, since H must be self-adjoint. Using the commutation relations for the canonical operators, we can then write $H = \frac{1}{2} \sum_{j=1}^{n} (\alpha_j q_j^2 + \Re\{\beta_j\}(q_j p_j + p_j q_j) + \epsilon_j p_j^2) + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \kappa_{jk} q_j p_k + c = \frac{1}{2} x_0^T R x_0 + c$ for some real symmetric matrix R and a real number $c = -2 \sum_{j=1}^{n} \Im\{\beta_j\}$. Since c contributes only a phase factor e^{ic} that has no effect on the dynamics of the oscillator, as $e^{i(H-c)t} x_0 e^{-i(H-c)t} = e^{iHt} x_0 e^{-iHt} \; \forall t \geq 0$, we may as well just discard c and take H to be $H = \frac{1}{2} x_0^T R x_0$ (i.e., the original H without the constant term). Returning to the main discussion, let $\eta_1, \ldots, \eta_m$ be independent vacuum quantum white noise processes satisfying the commutation relations $[\eta_j(t), \eta_k(t')^*] = \delta_{jk} \delta(t - t')$ and $[\eta_j(t), \eta_k(t')] = 0 \; \forall j, k$ and $\forall t, t' \geq 0$, and define $A_j(t) = \int_0^t \eta_j(s) ds \; (j = 1, \ldots, m)$ to be vacuum bosonic fields satisfying the quantum Ito multiplication rules [22, 23]

$$dA_j(t) dA_k^*(t) = \delta_{jk} dt, \quad dA_j^*(t) dA_k(t) = 0, \quad dA_j(t) dA_k(t) = 0, \quad dA_j^*(t) dA_k^*(t) = 0,$$

with all other remaining second order products between dA_j, dA_k^* and dt vanishing. An *open quantum harmonic oscillator*, or simply an *open oscillator*, is defined as a quantum harmonic oscillator coupled to $A(t)$ via the formal time-varying *idealized* interaction Hamiltonian [21, Chapter 11]

$$(2.1) \qquad H_{Int}(t) = i(L^T \eta(t)^* - L^\dagger \eta(t)),$$

where L is a *linear* coupling operator given by $L = K x_0$ with $K \in \mathbb{C}^{m \times n}$ and $\eta = (\eta_1, \ldots, \eta_m)^T$. Although the Hamiltonian is formal since the $\eta_j(t)$'s are singular quantum white noise processes, it can be given a rigorous interpretation in terms of Markov limits (e.g., [26], [21, Chapter 11]). The evolution of the open oscillator is then governed by the unitary process $\{U(t)\}_{t \geq 0}$ satisfying the QSDE [10, 12, 21, 27]

$$(2.2) \quad dU(t) = \left(-iH dt + dA(t)^\dagger L - L^\dagger dA(t) - \frac{1}{2} L^\dagger L dt \right) U(t); \qquad U(0) = I.$$

The time-evolved canonical operators are given by $x(t) = U(t)^* x_0 U(t)$ and satisfy the QSDE

$$dx(t) = 2\Theta(R + \Im\{K^\dagger K\}) x(t) dt + 2i\Theta [\begin{array}{cc} -K^\dagger & K^T \end{array}] \left[\begin{array}{c} dA(t) \\ dA(t)^\# \end{array} \right],$$

$$x(0) = x_0,$$

where Θ is a canonical commutation matrix of the form $\Theta = \mathrm{diag}(J, J, \ldots, J)$, with

$$J = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right],$$

while the output bosonic fields $Y(t) = (Y_1(t), \ldots, Y_n(t))^T$ that result from interaction of $A(t)$ with the harmonic oscillator are given by $Y(t) = U(t)^* A(t) U(t)$ and satisfy the QSDE

$$(2.3) \qquad dY(t) = K x(t) dt + dA(t).$$

Note that the dynamics of $x(t)$ and $Y(t)$ are linear.

Fig. 2.1. *A generalized open oscillator.*

The input $A(t)$ of an open oscillator can first be passed through a static passive linear (quantum) network (for example, a static passive linear optical network. See, e.g., [20, 28] for details) without affecting the linearity of the overall system dynamics; this is shown in Figure 2.1. Such an operation effects the transformation $A(t) \mapsto \tilde{A}(t) = SA(t)$, where $S \in \mathbb{C}^{m \times m}$ is a complex unitary matrix (i.e., $S^\dagger S = SS^\dagger = I$). Thus $\tilde{A}(t)$ will be a new set of vacuum noise fields satisfying the same Ito rule as $A(t)$.

Letting $S = [S_{jk}]_{j,k=1,\ldots,m}$, it can be shown by straightforward calculations using the quantum Ito stochastic calculus that the cascade is equivalent (in the sense that it produces the same dynamics for $x(t)$ and the output $y(t)$ of the system) to a linear quantum system whose dynamics is governed by a unitary process $\{\tilde{U}(t)\}_{t \geq 0}$ satisfying the QSDE (for a general treatment, see [29])

$$(2.4) \qquad d\tilde{U}(t) = \left(\sum_{j,k=1}^{m} (S_{jk} - \delta_{jk}) d\Lambda_{jk}(t) - iH dt + dA(t)^\dagger L - L^\dagger S dA(t) \right.$$
$$\left. - \frac{1}{2} L^\dagger L dt \right) \tilde{U}(t); \tilde{U}(0) = I,$$

where $\Lambda_{jk}(t)$ $(j, k = 1, \ldots, m)$ are fundamental processes, called the gauge processes, satisfying the quantum Ito rules

$$d\Lambda_{jk}(t) d\Lambda_{j'k'}(t) = \delta_{kj'} d\Lambda_{jk'}(t), \quad dA_j(t) d\Lambda_{kl}(t) = \delta_{jk} dA_l(t), \quad d\Lambda_{jk} dA_l(t)^* = \delta_{kl} dA_j^*(t),$$

with all other remaining second order products between $d\Lambda_{jl}(t)$ and $dA_{j'}(t), dA_{l'}^*(t), dt$ vanishing. This yields the following dynamics for $x(t) = \tilde{U}(t)^* x_0 \tilde{U}(t)$ and the system output $y(t) = \tilde{U}(t)^* A(t) \tilde{U}(t)$:

$$(2.5) \qquad dx(t) = Ax(t) dt + B \begin{bmatrix} dA(t) \\ dA(t)^\# \end{bmatrix},$$

$$(2.6) \qquad dy(t) = Cx(t) dt + D dA(t),$$

with

$$(2.7) \qquad A = 2\Theta(R + \Im\{K^\dagger K\}),$$
$$B = 2i\Theta[\ -K^\dagger S \quad K^T S^\# \ ],$$
$$C = SK,$$
$$D = S.$$

For convenience, in the remainder of the paper we shall refer to the cascade of a static passive linear quantum network with an open oscillator as a *generalized open oscillator*.

Let G be a generalized open oscillator that evolves according to the QSDE (2.4) with given parameters S, $L = Kx_0$, and $H = \frac{1}{2} x_0^T R x_0$. For compactness, we shall

use a shorthand notation of [29] and denote such a generalized open oscillator by $G = (S, L, H)$. In the next section we briefly recall the concatenation and series product developed in [29] that allows one to systematically obtain the parameters of a generalized open oscillator built up from an interconnection of generalized open oscillators of one degree of freedom.

3. The concatenation and series product of generalized open oscillators and reducible quantum networks. In this section we will recall the formalisms of concatenation product, series product, and reducible networks (with respect to the series product) developed in [29] for the manipulation of networks of generalized open oscillators as well as more general Markov open quantum systems.

Let $G_1 = (S_1, K_1 x_{1,0}, \frac{1}{2} x_{1,0}^T R_1 x_{1,0})$ and $G_2 = (S_2, K_2 x_{2,0}, \frac{1}{2} x_{2,0}^T R_2 x_{2,0})$ be two generalized open oscillators, where $x_{k,0} = x_k(0)$. The concatenation product $G_1 \boxplus G_2$ of G_1 and G_2 is defined as

$$G_1 \boxplus G_2 = \left(S_{1 \boxplus 2}, (K_1 x_{1,0}, K_2 x_{2,0})^T, \frac{1}{2} x_{1,0}^T R_1 x_{1,0} + \frac{1}{2} x_{2,0}^T R_2 x_{2,0} \right),$$

where

$$S_{1 \boxplus 2} = \left[\begin{array}{cc} S_1 & 0 \\ 0 & S_2 \end{array} \right].$$

It is important to note here that the possibility that $x_{1,0} = x_{2,0}$ or that some components of $x_{1,0}$ coincide with those of $x_{2,0}$ are allowed. If G_1 and G_2 are independent oscillators (i.e., the components of $x_{1,0}$ act on a distinct Hilbert space to that of the components of $x_{2,0}$), then the concatenation can be interpreted simply as the "stacking" or grouping of the variables of two noninteracting generalized open oscillators to form a larger generalized open oscillator.

It is also possible to feed the output of a system G_1 to the input of system G_2, with the proviso that G_1 and G_2 have the same number of input and output channels. This operation of cascading or loading of G_2 onto G_1 is represented by the series product $G_2 \lhd G_1$ defined by

$$G_2 \lhd G_1 = \left(S_2 S_1, K_2 x_{2,0} + S_2 K_1 x_{1,0}, \frac{1}{2} x_{1,0}^T R_1 x_{1,0} \right.$$
$$\left. + \frac{1}{2} x_{2,0}^T R_2 x_{2,0} + \frac{1}{2i} x_{2,0}^T (K_2^\dagger S_2 K_1 - K_2^T S_2^\# K_1^\#) x_{1,0} \right).$$

Note that $G_2 \lhd G_1$ is again a generalized open oscillator with a scattering matrix, coupling operator, and Hamiltonian as given by the above formula.

With concatenation and series products having been defined, we now come to the important notion of a *reducible network with respect to the series product* (which we shall henceforth refer to more simply as just a *reducible network*) of generalized open oscillators. This network consists of l generalized open oscillators $G_k = (S_k, L_k, H_k)$, with $L_k = K_k x_{k,0}$ and $H_k = \frac{1}{2} x_{k,0}^T R_k x_{k,0}$, $k = 1, \ldots, l$, along with the specification of a direct interaction Hamiltonian $H^d = \sum_j \sum_{k=j+1} x_{j,0}^T R_{jk} x_{k,0}$ $(R_{jk} \in \mathbb{R}^{2 \times 2})$ and a list $\mathcal{S} = \{ G_k \lhd G_j \}$ of series connections among generalized open oscillators G_j and G_k, $j \neq k$, with the condition that each input and each output has at most one connection, i.e., lists of connections such as $\{ G_2 \lhd G_1, G_3 \lhd G_2, G_1 \lhd G_3 \}$ are disallowed. Such a reducible network $\mathcal{N}$ again forms a generalized open oscillator and is denoted by $\mathcal{N} = \{ \{ G_k \}_{k=1,\ldots,l}, H^d, \mathcal{S} \}$. Note that if $\mathcal{N}_0$ is a reducible network

defined as $\mathcal{N}_0 = \{\{G_k\}_{k=1,\ldots,l}, 0, \mathcal{S}\} = (S_0, L_0, H_0)$, then $\mathcal{N}$, which is $\mathcal{N}_0$ equipped with the direct interaction Hamiltonian H^d, is simply given by $\mathcal{N} = \mathcal{N}_0 \boxplus (0, 0, H) = (S_0, L_0, H_0 + H^d)$.

The notion of a reducible network was introduced in [29] to study networks that are free of "algebraic loops" such as when connections like $\{G_2 \lhd G_1, G_3 \lhd G_2, G_1 \lhd G_3\}$ are present. The theory in [29] is not sufficiently general to treat networks with algebraic loops; they can be treated in the more general framework of quantum feedback networks developed in [30]. Since this work is based on [29], we also restrict our attention to reducible networks, but as we shall show in section 5 this is actually sufficient to develop a network synthesis theory of linear quantum stochastic systems. A network synthesis theory can indeed also be developed using the theory of quantum feedback networks of [30], and this has been pursued in a separate work [31].

Two important decompositions of a generalized open oscillator based on the series product that will be exploited in this paper are

$$(3.1) \qquad (S, L, H) = (I, L, H) \lhd (S, 0, 0),$$
$$(3.2) \qquad (S, L, H) = (S, 0, 0) \lhd (I, S^\dagger L, H),$$

where $(S, 0, 0)$ represents a static passive linear network implementing the unitary matrix S.

4. Correspondence between system matrices (A, B, C, D) and the parameters S, L, H. In [12] it has been shown that for $S = I$, then $D = I$, and there is a bijective correspondence between the system matrices (A, B, C, I) of a physically realizable linear quantum stochastic system [12, section III] and the parameters K, R of an open oscillator; see Theorem 3.4 therein (however, note that the B, C, and D matrices are defined slightly differently from here because [12] expresses all equations in terms of quadratures of the bosonic fields rather than their modes). Here we shall show that allowing for an arbitrary complex unitary scattering matrix S, a bijective correspondence between the system parameters (A, B, C, D) of an extended notion of a physically realizable linear quantum stochastic system and the parameters S, K, R of a generalized open oscillator (in particular, $D = S$) can be established. We begin by noting that we may write the dynamics (2.6) in the following way:

$$y(t) = Sy'(t),$$

with $y'(t)$ defined as

$$dy'(t) = S^\dagger K x(t)dt + dA(t).$$

Then by defining $K' = S^\dagger K$ and substituting $K = SK'$ in (2.7), we see that $x(t)$ in (2.5), and $y'(t)$ constitutes the dynamics for the open oscillator $(I, K'x_0, \frac{1}{2}x_0^T Rx_0)$ with system matrices given by $(A, B, S^\dagger C, I)$. Since $D = S$ and $(S, L, H) = (S, 0, 0) \lhd (I, K'x_0, \frac{1}{2}x_0^T Rx_0)$ (cf. (3.2)), from [12, Theorem 3.4] we see that there is a bijective correspondence between $(A, B, S^\dagger C)$ and the parameters (K', R) and that one set of parameters may be uniquely recovered from the other. Therefore, we may define a system of the form (1.2) to be physically realizable (extending the notion in [12]) if it represents the dynamics of a generalized open oscillator (this idea already appears in [13, Chapter 7]; see Remark 7.3.8 therein). This implies that a system (1.2) with matrices (A, B, C, D) is physically realizable if and only if D is a complex unitary matrix and $(A, B, D^\dagger C, I)$ are the system matrices of a physically realizable system

in the sense of [12] (i.e., $(A, B, D^\dagger C, I)$ are the system matrices of an open oscillator). Therefore, we may state the following theorem.

THEOREM 4.1. *There is a bijective correspondence between the system matrices* (A, B, C, D) *and the parameters* (S, K, R) *of a generalized open oscillator. For given* (S, K, R), *the corresponding system matrices are uniquely given by* (2.7). *Conversely, for given* (A, B, C, D), *which are the system matrices of a generalized open oscillator* G *with parameters* (S, K, R), *then* D *is unitary, and* $S = D$ *and* $(A, B, D^\dagger C, I)$ *are the system matrices of some open oscillator* $G' = (I, K'x_0, \frac{1}{2}x_0^T R x_0)$. *The parameters* (K', R) *of the open oscillator* G' *is uniquely determined from* $(A, B, D^\dagger C)$ *by* [12, *Theorem* 3.4] *(by suitably adapting the matrices* B *and* $D^\dagger C$), *from which the parameter* K *of* G *is then uniquely determined as* $K = DK'$.

Due to this interchangeability of the description by (A, B, C, D) and by (S, K, R) for a generalized open oscillator, it does not matter with which set of parameters one works with. However, for convenience of analysis in the remainder of the paper we shall work exclusively with the parameters (S, K, R).

5. Main synthesis theorem. Suppose that there are two independent generalized open oscillators coupled to m independent bosonic fields, with m output channels: an n_1 degrees of freedom oscillator $G_1 = (S_1, L_1, H_1)$ with canonical operators $x_1 = (q_{1,1}, p_{1,1}, \ldots, q_{1,n_1}, p_{1,n_1})^T$, Hamiltonian operator $H_1 = \frac{1}{2}x^T R_1 x_1$, coupling operator $L_1 = K_1 x_1$, and scattering matrix S_1, and, similarly, an n_2 degrees of freedom oscillator $G_2 = (S_2, L_2, H_2)$ with canonical operators $x_2 = (q_{2,1}, p_{2,1}, \ldots, q_{2,n_2}, p_{2,n_2})^T$, Hamiltonian operator $H_2 = \frac{1}{2}x_2^T R_2 x_2$, coupling operator $L_2 = K_2 x_2$, and unitary scattering matrix S_2.

Consider now a reducible quantum network $\mathcal{N}_{12}$ constructed from G_1 and G_2 as $\mathcal{N}_{12} = \{\{G_1, G_2\}, H_{12}^d, G_2 \triangleleft G_1\}$, as shown in Figure 5.1, where H_{12}^d is a direct interaction Hamiltonian term between G_1 and G_2 given by

$$H_{12}^d = \frac{1}{2}x_1^T R_{12} x_2 + \frac{1}{2}x_2^T R_{12}^T x_1 - \frac{1}{2i}(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2); \ R_{12} \in \mathbb{R}^{2 \times 2}$$

$$= x_2^T R_{12}^T x_1 - \frac{1}{2i}(L_2^\dagger S_2 L_1 - L_2^T S_2^\# L_1^\#)$$

$$= x_2^T \left(R_{12}^T - \frac{1}{2i}(K_2^\dagger S_2 K_1 - K_2^T S_2^\# K_1^\#) \right) x_1,$$

where we recall that $A^\#$ denotes the elementwise adjoint of a matrix of operators A and the second equality holds, since elements of L_1 commute with those L_2. Also note that the matrix $\frac{1}{2i}(K_2^\dagger S_2 K_1 - K_2^T S_2^\# K_1^\#)$ is real. Some straightforward calculations (see [29] for details) then show that we may write

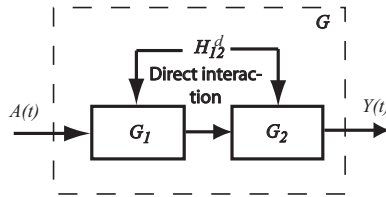$$\mathcal{N}_{12} = (S_2 S_1, S_2 L_1 + L_2, H_1 + H_2 + H_{12}^f + H_{12}^d),$$

FIG. 5.1. *Cascade connection of* G_1 *and* G_2 *with indirect interaction* H_{12}^d.

where $H_{12}^f = \frac{1}{2i}(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2)$. Now let us look closely at the Hamiltonian term of $\mathcal{N}_{12}$. Note that after plugging in the definition of H_1, H_2, H_{12}^d, and H_{12}^f, we may write

$$H_1 + H_2 + H_{12}^f + H_{12}^d = \frac{1}{2}\begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Letting $x = (x_1^T, x_2^T)^T$, $S_{2\leftarrow 1} = S_2 S_1$, and defining

$$(5.1) \qquad\qquad\qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix},$$

$$(5.2) \qquad\qquad\qquad K = \begin{bmatrix} S_2 K_1 & K_2 \end{bmatrix},$$

we see that

$$(5.3) \qquad\qquad\qquad \mathcal{N}_{12} = \left(S_{2\leftarrow 1}, Kx, \frac{1}{2}x^T R x \right).$$

Therefore, $\mathcal{N}_{12} = (S_{2\leftarrow 1}, L_{2\leftarrow 1}, H_{2\leftarrow 1})$, with $S_{2\leftarrow 1} = S_2 S_1$, $L_{2\leftarrow 1} = Kx$, and $H_{2\leftarrow 1} = \frac{1}{2}x^T R x$. In other words, a reducible network formed by a bilinear direct interaction and cascade connection of two generalized open oscillators having the same number of input and output fields results in another generalized open oscillator with a degrees of freedom which is the sum of the degrees of freedom of the two constituent oscillators and having the same number of inputs and outputs.

By repeated application of the above construction, we can prove the following synthesis theorem.

THEOREM 5.1. *Let G be an n degrees of freedom generalized open oscillator with Hamiltonian matrix $R \in \mathbb{R}^{2n \times 2n}$, coupling matrix $K \in \mathbb{C}^{m \times 2n}$, and unitary scattering matrix $S \in \mathbb{C}^{m \times m}$. Let R be written in terms of blocks of 2×2 matrices as $R = [R_{jk}]_{j,k=1,\ldots,n}$, where the R_{jk}'s are real 2×2 matrices satisfying $R_{kj} = R_{jk}^T$ for all j, k, and let K be written as*

$$K = \begin{bmatrix} K_1 & K_2 & \ldots & K_n \end{bmatrix},$$

where, for each j, $K_j \in \mathbb{C}^{m \times 2}$. For $j = 1, \ldots, n$, let $G_j = (S_j, \tilde{K}_j x_j, \frac{1}{2}x_j^T R_{jj} x_j)$ be independent one degree of freedom generalized open oscillators with canonical operators $x_j = (q_j, p_j)^T$, m output fields, Hamiltonian matrix R_{jj}, coupling matrix $\tilde{K}_j$, and scattering matrix S_j. Also, define $S_{k\leftarrow j}$ for $j \leq k+1$ as $S_{k\leftarrow j} = \prod_{l=j}^{k} S_l = S_k \cdots S_{j+1} S_j$ for $j < k$, $S_{k\leftarrow k} = S_k$, and $S_{k\leftarrow k+1} = I_{m \times m}$, and let H^d be a direct interaction Hamiltonian given by

$$(5.4) \qquad H^d = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} x_k^T \left(R_{jk}^T - \frac{1}{2i}(\tilde{K}_k^\dagger S_{k\leftarrow j+1} \tilde{K}_j - \tilde{K}_k^T S_{k\leftarrow j+1}^\# \tilde{K}_j^\#) \right) x_j.$$

If $S_1, \ldots, S_n$ satisfies $S_n S_{n-1} \cdots S_1 = S$ and $\tilde{K}_k$ satisfies $\tilde{K}_k = S_{n\leftarrow k+1}^\dagger K_k$ for $k = 1, \ldots, n$, then the reducible network of harmonic oscillators $\mathcal{N}$ given by $\mathcal{N} = \{\{G_1, \ldots, G_n\}, H^d, \{G_2 \lhd G_1, G_3 \lhd G_2, \ldots, G_n \lhd G_{n-1}\}\}$ is equivalent to G. That is, G can be synthesized via a series connection $G_n \lhd \ldots \lhd G_2 \lhd G_1$ of n one degree of freedom generalized open oscillators, along with a suitable bilinear direct interaction Hamiltonian involving the canonical operators of these oscillators. In particular, if

$S = I_{m \times m}$ *(no scattering), then S_k can be chosen to be $S_k = I_{m \times m}$ and $\tilde{K}_k$ can be chosen to be $\tilde{K}_k = K_k$ for $k = 1, \ldots, n$.*

Proof. Let $H_j = \frac{1}{2} x_j^T R_{jj} x_j$, $L_j = \tilde{K}_j x_j$ and

$$H_k^f = \sum_{j=2}^{k} \left(L_j^\dagger \sum_{l=1}^{j-1} S_{j \leftarrow l+1} L_l - \sum_{l=1}^{j-1} L_l^\dagger S_{j \leftarrow l+1}^\dagger L_j \right), \quad k \geq 2.$$

Let us begin with the series connection $G_{12} = G_2 \triangleleft G_1$. By analogous calculations as given above for the two oscillator case, it is given by

$$G_{12} = (S_2 S_1, S_2 L_1 + L_2, H_1 + H_2 + H_2^f).$$

Repeating this calculation recursively for $G_{123} = G_3 \triangleleft G_{12}$, $G_{1234} = G_4 \triangleleft G_{123}, \ldots,$ $G_{12 \ldots n} = G_n \triangleleft G_{12 \ldots (n-1)}$, we obtain at the end that

$$G_{12 \ldots n} = \left(S_{n \leftarrow 1}, \sum_{k=1}^{n} S_{n \leftarrow k+1} L_k, \sum_{k=1}^{n} H_k + H_n^f \right).$$

Noting that H_n^f may be rewritten as

$$H_n^f = \frac{1}{2i} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (L_k^\dagger S_{k \leftarrow j+1} L_j - L_j^\dagger S_{k \leftarrow j+1}^\dagger L_k)$$

$$= \frac{1}{2i} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (L_k^\dagger S_{k \leftarrow j+1} L_j - L_k^T S_{k \leftarrow j+1}^\# L_j^\#)$$

$$= \frac{1}{2i} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} x_k^T (\tilde{K}_k^\dagger S_{k \leftarrow j+1} \tilde{K}_j - \tilde{K}_k^T S_{k \leftarrow j+1}^\# \tilde{K}_j^\#) x_j,$$

where the second equality holds since L_j commutes with L_k whenever $j \neq k$, we find that

$$\sum_{k=1}^{n} H_k + H_n^f + H^d = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j R_{jk} x_k = \frac{1}{2} x^T R x, \quad x = (x_1^T, x_2^T, \ldots, x_n^T)^T.$$

Therefore, if $S_1, \ldots, S_n$ and $\tilde{K}_1, \ldots, \tilde{K}_n$ satisfy the conditions stated in the theorem, we find that $\mathcal{N} = \{\{G_1, \ldots, G_n\}, H_d, \{G_2 \triangleleft G_1, G_3 \triangleleft G_2, \ldots, G_n \triangleleft G_{n-1}\}$ is given by

$$\mathcal{N} = \left(S, Kx, \frac{1}{2} x^T R x \right).$$

That is, $\mathcal{N}$ is a linear quantum stochastic system with Hamiltonian matrix R, coupling matrix K, and scattering matrix S, and is therefore equivalent to G. This completes the proof of the synthesis theorem. $\quad\square$

Therefore, according to the theorem, synthesis of an arbitrary n degrees of freedom linear dynamical quantum stochastic system is in principle possible if the following two requirements can be met:

1. Arbitrary one degree of freedom open oscillators $G = (I, L, H)$ with m input and output fields can be synthesized. In particular, it follows from this that one degree of freedom generalized open oscillators $G' = (S, L, H)$ can be synthesized as $G' = (I, L, H) \triangleleft (S, 0, 0)$.
2. The bilinear interaction Hamiltonian H^d as given by (5.4) can be synthesized.

One can observe certain parallels between the quantum synthesis described in the theorem with the active state-space synthesis method of linear electrical network synthesis theory (e.g., [19, Chapter 13]). To begin with, we may think of each oscillator G_j as a type of noisy quantum integrator, as the counterpart of a classical integrator (built from an operational amplifier, resistors, and capacitors) in an electrical network, and in both settings synthesis can be achieved by suitably cascading these basic integrating components. We may also view the direct interaction Hamiltonian between two oscillators as acting like a type of mutual "state feedback" between the oscillators, much like the state feedback employed in electrical network synthesis. However, because of the quite distinct nature of classical and quantum systems, of course the parallels should not be taken to be "exact" or "precise" in any way, the nature of these parallels are in spirit rather than detail. Certainly, classical active synthesis theory cannot be applied directly to linear quantum stochastic systems because of certain constraints that a noisy quantum integrator must satisfy that are not required of its classical counterpart, and the classical theory is deterministic while in the quantum theory, quantum stochastic noise plays a crucial role, for instance, to preserve the canonical commutation relations in open quantum systems. To highlight another significant difference between the two physical systems, we note that losses in linear electrical systems may be modeled by inserting resistors as dissipative components of the system, while in linear quantum systems, losses are modeled by lossy couplings to quantum noises (heat baths).

6. Systematic synthesis of linear quantum stochastic systems. This section details the construction of arbitrary one degree of freedom open quantum harmonic oscillators and implementation of bilinear direct interactions among the canonical operators of these oscillators, at least approximately, in the context of quantum optics, using various linear and nonlinear quantum optical components.

We begin with a description of some key quantum optical components that will be required for the synthesis. This is followed by a discussion of general synthesis of one degree of freedom open oscillators and finally by a discussion of the implementation of bilinear direct interaction Hamiltonians among different one degree of freedom open oscillators.

6.1. Essential quantum optical components.

6.1.1. Optical cavities. An optical cavity is a system of fully reflecting or partially transmitting mirrors in which a light beam is trapped and is either bounced repeatedly from the mirrors to form a standing wave or circulates inside the cavity (as in a ring cavity); see Figure 6.1. If there are transmitting mirrors present, then light can escape or leak out from the cavity, introducing losses to the cavity.

A cavity is mathematically modeled by a Hamiltonian $H_{cav} = \omega_{cav} a^* a$, where ω_{cav} is the resonance frequency of the cavity and $a = \frac{q+ip}{2}$ is the (non-self-adjoint) cavity annihilation operator or cavity mode satisfying the commutation relation $[a, a^*] = 1$. Here $q = a + a^*$ is the position operator of the cavity mode (also called the *amplitude quadrature* of the mode) and $p = -ia + ia^*$ is the momentum operator of the cavity mode (also called the *phase quadrature* of the mode). If there is a transmission mirror, say, M, then losses through this mirror are modeled as having a vacuum bosonic noise field $A(t)$ incident at this mirror and interacting with the cavity mode via the idealized Hamiltonian H_{Int} given in (2.1) with $L = \sqrt{\kappa}a$, where κ is a positive constant called the mirror *coupling coefficient*. When there are several leaky mirrors, then the losses are modeled by a sum of such interaction Hamiltonians, one for each mirror and with
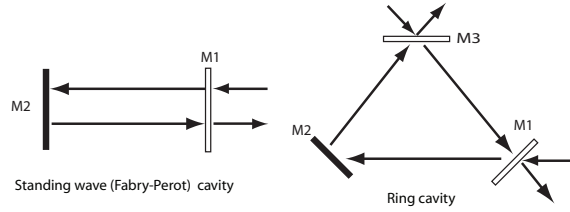
FIG. 6.1. *Two types of optical cavities: a standing wave or Fabry–Perot cavity (left) and a (three mirror) ring cavity (right). Arrows indicate the direction of propagation of light in the cavity. Black rectangles denote fully reflecting mirrors, while white rectangles denote partially transmitting mirrors.*
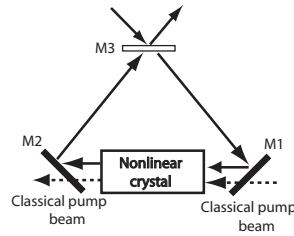

FIG. 6.2. *A DPA consisting of a classically pumped nonlinear crystal in a three mirror ring cavity.*
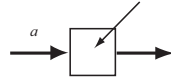

FIG. 6.3. *Schematic representation of a DPA. The white rectangle symbolizes the nonlinear crystal, while the diagonal arrow into the rectangle denotes the pump beam.*

each mirror having its own distinct vacuum bosonic field. The total Hamiltonian of the cavity is then just the sum of H_{cav} and the interaction Hamiltonians. More generally, the field incident at a transmitting mirror need not be a vacuum field, but can be other types of fields, such as a coherent laser field. Nonetheless, the interaction of the cavity mode with such fields via the mirror will still be governed by (2.1) with a coupling operator of the form $L = \sqrt{\kappa}a$.

6.1.2. Degenerate parametric amplifier. In order to amplify a quadrature of the cavity mode, for example, to counter losses in that quadrature caused by light escaping through a transmitting mirror, one can employ a $\chi^{(2)}$ nonlinear optical crystal and a classical pump beam in the configuration of a degenerate parametric amplifier (DPA), following the treatment in [21, section 10.2]. The pump beam acts as a source of additional quanta for amplification and, in the nonlinear crystal, an interaction takes place in which photons of the pump beam are annihilated to create photons of the cavity mode. In an optical cavity, such as a ring cavity shown in Figure 6.2, we place the crystal in one arm of the cavity (for example, in the arm between mirrors M1 and M2) and shine the crystal with a strong coherent pump beam of (angular) frequency ω_p given by $\omega_p = 2\omega_r$, where ω_r is some reference frequency. Here the mirrors at the end the arms should be chosen such that they do not reflect light beams of frequency ω_p. A schematic representation of a DPA (a nonlinear crystal with a classical pump) is shown in Figure 6.3.

Remark 6.1. In the remaining figures, black rectangles will be used to denote mirrors which are fully reflecting at the cavity frequency and fully transmitting at the pump frequency (whenever a pump beam is employed), while white rectangles denote partially transmitting mirrors at the cavity frequency.

Let $a = \frac{q+ip}{2}$ be the cavity mode, and let the cavity frequency ω_{cav} be detuned from ω_r and given by $\omega_{cav} = \omega_r + \Delta$, where Δ is the frequency detuning. The crystal facilitates an energy exchange interaction between the cavity mode and pump beam. By the assumption that the pump beam is intense and not depleted in this interaction, it may be assumed to be classical, in which case the crystal-pump-cavity interaction can be modeled using the (time-varying) Hamiltonian $H(t) = \omega_{cav}a^*a + \frac{i}{2}(\epsilon e^{-i\omega_p t}(a^*)^2 - \epsilon^* e^{i\omega_p t}a^2)$ [21, equation 10.2.1], where ϵ is a complex number representing the *effective* pump intensity. By transforming to a rotating frame with respect to $\omega_r = \frac{\omega_p}{2}$ (i.e., by application of the transformation $a \mapsto ae^{i\frac{\omega_p}{2}t}$; see [21, section 10.2.1] for a derivation of the equations of motion of the DPA in the rotating frame), H can be reexpressed as

$$H = \Delta a^*a + \frac{i}{2}(\epsilon(a^*)^2 - \epsilon^*a^2)$$

and be written compactly as $H = \frac{1}{2}x_0^T R x_0 + c$ (recall $x_0 = (q,p)^T$), where

(6.1)
$$R = \frac{1}{2}\begin{bmatrix} \Delta + \frac{i}{2}(\epsilon - \epsilon^*) & \frac{1}{2}(\epsilon + \epsilon^*) \\ \frac{1}{2}(\epsilon + \epsilon^*) & \Delta - \frac{i}{2}(\epsilon - \epsilon^*) \end{bmatrix}$$

and c is a real number. Since c merely contributes a phase factor that has no effect on the overall dynamics of the system operators, it plays no essential role and can simply be ignored (cf. section 2). Note that transformation to a rotating frame effects the following: If $a(t)$ is the evolution of a under the original time-varying Hamiltonian $H(t) = \omega_{cav}a^*a + \frac{i}{2}(\epsilon e^{-i\omega_p t}(a^*)^2 - \epsilon^* e^{i\omega_p t}a^2)$ and we define $\tilde{a}(t) = a(t)e^{i\omega_r t}$ (i.e., $\tilde{a}(t)$ is $a(t)$ in a frame rotating at frequency ω_r), then $\tilde{a}(t)$ coincides with the time evolution of a under the time-independent Hamiltonian $\tilde{H} = \frac{1}{2}x_0^T R x_0$. In other words, in this rotating frame, the DPA can be viewed as a harmonic oscillator with quadratic Hamiltonian $\frac{1}{2}x_0^T R x_0$.

6.1.3. Two-mode squeezing. If two cavities are positioned in such a way that the beams circulating in them intersect one another, then these beams will merely pass through each other without interacting. One way of making the beams interact is to have their paths intersect inside a $\chi^{(2)}$ nonlinear optical crystal. Typically, to facilitate such an interaction, one or two auxiliary pump beams are also employed as a source of quanta/energy. For instance, in a $\chi^{(2)}$ optical crystal in which the modes of two cavities interact with an undepleted classical pump beam as depicted in Figure 6.4, the interaction can be modeled by the Hamiltonian

$$H(t) = \frac{i}{2}(\epsilon e^{-i\omega_p t}a_1^*a_2^* - \epsilon^* e^{i\omega_p t}a_1 a_2),$$

where ϵ is a complex number representing the effective intensity of the pump beam and ω_p is the pump frequency. Transforming to a rotating frame at half the pump frequency by applying the rotating frame transformation $a_1 \mapsto a_1 e^{i\frac{\omega_p}{2}t}$ and $a_2 \mapsto a_2 e^{i\frac{\omega_p}{2}t}$,
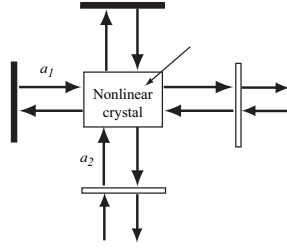
FIG. 6.4. *Two cavity modes interacting in a single classically pumped nonlinear crystal. The diagonal arrow into the crystal denotes the pump beam.*
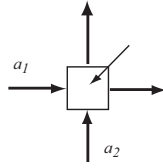

FIG. 6.5. *Schematic representation of a two-mode squeezer.*

$H(t)$ can be expressed in this new frame in the time-invariant form $H = \frac{i}{2}(\epsilon a_1^* a_2^* - \epsilon^* a_1 a_2)$. This type of Hamiltonian is called a *two-mode squeezing Hamiltonian*, as it simultaneously affects squeezing in one quadrature of (possibly rotated versions of) a_1 and a_2 and will play an important role later on in the paper. A two-mode squeezer is schematically represented by the symbol shown in Figure 6.5.

Remark 6.2. It will be implicitly assumed in this paper that the equations for the dynamics of generalized open operators are given with respect to a common rotating frame of frequency ω_r, including the transformation of all bosonic noises $A_i(t)$ according to $A_i(t) \mapsto A_i(t)e^{i\omega_r t}$, and that classical pumps employed are all of frequency $\omega_p = 2\omega_r$. This is a natural setting in quantum optics where a rotating frame is essential for obtaining linear time invariant QSDE models for active devices that require an external source of quanta. In a control setting, this means both the quantum plant and the controller equations have been expressed in the same rotating frame.

6.2. Static linear optical devices and networks. Static linear optical devices implement static linear transformations (meaning that the transformation can be represented by a complex square matrix) of a set of independent incoming single mode fields, such as the field in a cavity, $a = (a_1, a_2, \ldots a_m)^T$ to an equal number $a' = (a'_1, a'_2, \ldots a'_m)^T$ of independent outgoing fields. The incoming fields satisfy the commutation relations $[a_j, a_k] = 0$ and $[a_j, a_k^*] = \delta_{jk}$. The incoming fields may also be vacuum bosonic fields $A(t) = (A_1(t), A_2(t), \ldots, A_m(t))^T$ with outgoing bosonic fields (that need no longer be in the vacuum state) $A'(t) = (A'_1(t), A'_2(t), \ldots, A'_m(t))^T$. In the latter, the commutation relations are $[dA_j(t), dA_k(t)] = 0$ and $[dA_j(t), dA_k(t)^*] = \delta_{jk}dt$. However, to avoid cumbersome and unnecessary repetitions, in the following we shall only discuss the operation of a static linear optical device in the context of single mode fields. The operation is completely analogous for bosonic incoming and outgoing fields and requires only making substitutions such as $a \to A(t)$, $a' \to A'(t)$, $[a_j, a_k] = 0 \to [dA_j(t), dA_k(t)] = 0$, and $[a_j, a_k^*] = \delta_{jk} \to [dA_j(t), dA_k(t)^*] = \delta_{jk}dt$, etc.
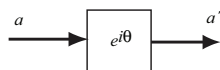
FIG. 6.6. *Phase shifter with a phase shift of θ radians.*

The operation of a static linear optical device can mathematically be expressed as

$$\left[\begin{array}{c} a' \\ a'^{\#} \end{array} \right] = Q \left[\begin{array}{c} a \\ a^{\#} \end{array} \right]; \; Q = \left[\begin{array}{cc} Q_1 & Q_2 \\ Q_2^{\#} & Q_1^{\#} \end{array} \right],$$

where $Q_1, Q_2 \in \mathbb{C}^{m \times m}$ and S is a quasi-unitary matrix [20, section 3.1] satisfying

$$Q \left[\begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right] Q^{\dagger} = \left[\begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right].$$

A consequence of the quasi-unitarity of Q is that it preserves the commutation relations among the fields, that is, to say that the output fields a' satisfy the same commutation relations as a. Another important property of a quasi-unitary matrix is that it has an inverse Q^{-1} given by $Q^{-1} = GQ^{\dagger}G$, where $G = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix} \right]$, and this inverse is again quasi-unitary, i.e., the set of quasi-unitary matrices of the same dimension form a group.

In the case where the submatrix Q_2 of Q is $Q_2 = 0$, the device does not mix creation and annihilation operators of the fields, and it necessarily follows that Q_1 is a complex *unitary* matrix. Such devices are said to be *static passive* linear optical devices because they do not require any external source of quanta for their operation. It is well known that any passive network can be constructed using only beam splitters and mirrors (e.g., see references 2–4 in [28]). In all other cases, the devices are *static active*. Specific passive and static devices that will be utilized in this paper will be discussed in the following.

6.2.1. Phase shifter. A phase shifter is a device that produces an outgoing field that is a phase shifted version of the incoming field. That is, if there is one input field a, then the output field is $a' = e^{i\theta}a$ for some real number θ, called the *phase shift*; a phase shifter is schematically represented by the symbol shown in Figure 6.6. By definition, a phase shifter is a static passive device. The transformation matrix Q_{PS} of a phase shifter with a single input field is given by

$$Q_{PS} = \left[\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right].$$

6.2.2. Beam splitter. A beam splitter is a static and passive device that forms a linear combination of two input fields a_1 and a_2 to produce two output fields a'_1 and a'_2 such that energy is conserved: $a_1^* a_1 + a_2^* a_2 = (a'_1)^* a'_1 + (a'_2)^* a'_2$. The transformation affected by a beam splitter can be written as

$$Q_{BS} = \left[\begin{array}{cc} B & 0 \\ 0 & B^{\#} \end{array} \right],$$

where B is a unitary matrix given by

$$B = e^{i\Xi/2} \left[\begin{array}{cc} e^{i\Psi/2} & 0 \\ 0 & e^{-i\Psi/2} \end{array} \right] \left[\begin{array}{cc} \cos(\Theta)/2 & \sin(\Theta)/2 \\ -\sin(\Theta)/2 & \cos(\Theta)/2 \end{array} \right] \left[\begin{array}{cc} e^{i\Phi/2} & 0 \\ 0 & e^{-i\Phi/2} \end{array} \right].$$
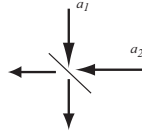
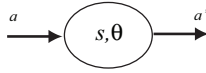Fig. 6.7. *Schematic representation of a beam splitter.*

Fig. 6.8. *Schematic representation of a squeezer.*

Here Ξ, Θ, Φ, Ψ are real numbers. Θ is called the *mixing angle* of the beam splitter, and it is the most important parameter. Φ and Ψ introduce a phase difference in the two incoming and outgoing modes, respectively, while Ξ introduces an overall phase shift in both modes.

A particularly useful result on the operation of a beam splitter with $\Xi = \Psi = \Phi = 0$ is that it can be modeled by an effective Hamiltonian H_{BS}^0 given by $H_{BS}^0 = i\Theta(a_1^* a_2 - a_1 a_2^*)$ (see [20, section 4.1] for details). This means that in this case we have the representation

$$Q_{BS} \left[\begin{array}{c} a \\ a^{\#} \end{array} \right] = \exp(iH_{BS}^0) \left[\begin{array}{c} a \\ a^{\#} \end{array} \right] \exp(-iH_{BS}^0),$$

where $a = (a_1, a_2)^T$. More generally, it follows from this, by considering phase shifted inputs $a_1 \rightarrow a_1 e^{i\frac{\theta+\Phi}{2}}$ and $a_2 \rightarrow a_2 e^{i\frac{\theta-\Phi}{2}}$ (θ being an arbitrary real number), that a beam splitter with $\Xi = 0$ and $\Psi = -\Phi$ will have the effective Hamiltonian $H_{BS} = i\Theta(e^{-i\Phi} a_1^* a_2 - e^{i\Phi} a_1 a_2^*) = \alpha a_1^* a_2 + \alpha^* a_1 a_2$, with $\alpha = i\Theta e^{-i\Phi}$. This is the most general type of beam splitter that will be employed in the realization theory of this paper. A beam splitter with a Hamiltonian of the form H_{BS} is represented schematically using the symbol in Figure 6.7.

6.2.3. Squeezer. Let there be a single input mode a. Write a as $a = q' + ip'$, where $q' = (a + a^*)/2$ is the *real* or *amplitude* quadrature of a and $p' = (a - a^*)/2i$ is the *imaginary* or *phase* quadrature of a. *Squeezing* of a field is an operation in which the variance of one quadrature, either q' or p', is squeezed or attenuated (it becomes less noisy) at the expense of increasing the variance of the other quadrature (it becomes noisier). A device that performs squeezing of a field is called a *squeezer*. An ideal squeezer affects the transformation $Q_{squeezer}$ given by

$$Q_{squeezer} = \left[\begin{array}{cc} \cosh(s) & e^{i\theta}\sinh(s) \\ e^{-i\theta}\sinh(s) & \cosh(s) \end{array} \right],$$

where s and θ are real parameters. We shall refer to s as the squeezing parameter and θ as the phase angle. For $s < 0$, the squeezer squeezes the amplitude quadrature of $e^{-i\frac{\theta}{2}}a$ (a phase shifted version of a) while if $s > 0$, it squeezes the phase quadrature and then shifts the phase of the squeezed field by $\frac{\theta}{2}$. A squeezer with parameters s, θ is schematically represented by the symbol shown in Figure 6.8.

A squeezer can be implemented, for instance, by using a combination of a parametric amplifier and a beam splitter for single mode fields [20, section 6.1] or as a
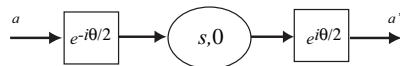
FIG. 6.9. *Implementation of a squeezer with arbitrary phase angle employing a squeezer with a zero phase angle and two phase shifters.*

DPA with a transmitting mirror for bosonic fields [21, section 7.2.9]. It is easy to see that $Q_{squeezer}^{-1}$ is given by

$$Q_{squeezer}^{-1} = \left[\begin{array}{cc} \cosh(-s) & e^{i\theta}\sinh(-s) \\ e^{-i\theta}\sinh(-s) & \cosh(-s) \end{array} \right] = \left[\begin{array}{cc} \cosh(s) & -e^{i\theta}\sinh(s) \\ -e^{-i\theta}\sinh(s) & \cosh(s) \end{array} \right].$$

6.2.4. Static optical linear networks. It is known that an arbitrary static linear optical network can be decomposed as a cascade of simpler networks. In particular, any quasi-unitary matrix Q can be constructively decomposed as [28]:

$$Q = \exp \left[\begin{array}{cc} A_1 & 0 \\ 0 & A_1^\# \end{array} \right] \exp \left[\begin{array}{cc} 0 & D \\ D & 0 \end{array} \right] \exp \left[\begin{array}{cc} A_3 & 0 \\ 0 & A_3^\# \end{array} \right]$$

$$= \left[\begin{array}{cc} \exp A_1 & 0 \\ 0 & \exp A_1^\# \end{array} \right] \left[\begin{array}{cc} \cosh D & \sinh D \\ \sinh D & \cosh D \end{array} \right] \left[\begin{array}{cc} \exp A_3 & 0 \\ 0 & \exp A_3^\# \end{array} \right],$$

where A_1 and A_3 are skew symmetric complex matrices and D is a real diagonal matrix. The first and third matrix exponential represent passive static networks that can be implemented by beam splitters and mirrors, while the second exponential represents an independent collection of squeezers (with trivial phase angles) each acting on a distinct field.

In summary, in any static linear optical network the incident fields can be thought of as going through a sequence of three operations: they are initially mixed by a passive network, then they undergo squeezing, and finally they are subjected to another passive transformation. In the special case where the entire network is passive, the squeezing parameters (i.e., elements of the D matrix) are zero.

For example, a squeezer with arbitrary phase angle θ can be constructed by sandwiching a squeezer with phase angle 0 between a $-\theta/2$ phase shifter at its input and a $\theta/2$ phase shifter at its output, respectively. This is shown in Figure 6.9.

6.3. Synthesis of one degree of freedom open oscillators. One degree of freedom open oscillators are completely described by a real symmetric Hamiltonian matrix $R = R^T \in \mathbb{R}^{2\times2}$ and complex coupling matrix $K \in \mathbb{C}^{m\times2}$. Thus one needs to be able to implement both R and K. Here we shall propose the realization of one degree of freedom open quantum harmonic oscillators based around a ring cavity structure, such as shown in Figure 6.1, using fully reflecting and partially reflecting mirrors and nonlinear optical elements appropriately placed between the mirrors.

The matrix R determines the quadratic Hamiltonian $H = \frac{1}{2}x^T R x$ and in a one-dimensional setup such a quadratic Hamiltonian can be realized with a DPA as discussed in section 6.1.2. From (6.1), it is easily inspected that any real symmetric matrix R can be realized by suitably choosing the complex effective pump intensity parameter ϵ and the cavity detuning parameter Δ of the DPA. In fact, for any particular R, the choice of parameters is *unique*. For example, to realize

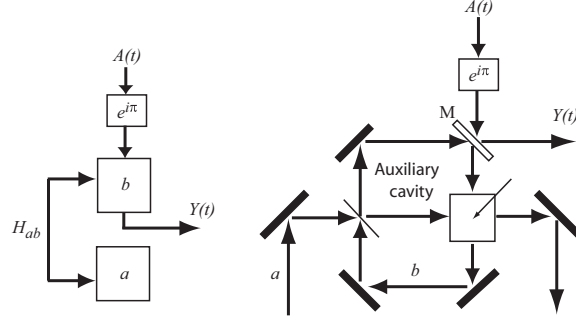$$R = \left[\begin{array}{cc} 1 & -2 \\ -2 & 0.5 \end{array} \right],$$

FIG. 6.10. *Scheme for (approximate) implementation of a coupling $L = \tilde{\alpha}a + \tilde{\beta}a^*$ to cavity mode a using an auxiliary cavity b (whose dynamics is adiabatically eliminated), a two-mode squeezer, and a beam splitter with the appropriate parameters. The left figure is a block diagram showing the fast mode b interacting with the slow mode a via the direct interaction Hamiltonian H_{ab}, implemented by the two-mode squeezer and the beam splitter, and also interacting with a $180°$ phase shifted input field $A(t)$ to produce the output field $Y(t)$. The right figure details the physical implementation of the block diagram.*

one solves the set of equations

$$\Delta - \Im\{\epsilon\} = 2, \quad \Re\{\epsilon\} = -4, \quad \text{and} \quad \Delta + \Im\{\epsilon\} = 1$$

for $\Delta, \Im\{\epsilon\}, \Re\{\epsilon\}$ to yield the unique solution $\Delta = 3/2$ and $\epsilon = -4 - i/2$.

Now, we turn to consider realization of the coupling operator $L = Kx_0$. Let us write $K = [\ K_1^T \ \ \dots \ \ K_m^T \ ]^T$, where $K_l \in \mathbb{C}^{1 \times 2}$ for each $l = 1, \dots, m$. Each K_l represents the coupling of the oscillator to the bosonic field A_l, and so it suffices to study how to implement a single linear coupling to just one field. To this end, suppose now that there is only one field $A(t)$ coupled to the oscillator via a linear coupling operator $L = Kx_0$ for some $K \in \mathbb{C}^{1 \times 2}$. It will be more convenient to express $L = \alpha q + \beta p$ in terms of the oscillator annihilation operator a and creation operator a^* defined by $a = (q + ip)/2$ and $a^* = (q - ip)/2$. Therefore, we write $L = \tilde{\alpha}a + \tilde{\beta}a^*$, with $\tilde{\alpha} = \frac{\alpha - i\beta}{2}$ and $\tilde{\beta} = \frac{\alpha + i\beta}{2}$. Consider the physical scheme shown in Figure 6.10, partly inspired by a scheme proposed by Wiseman and Milburn for quantum nondemolition measurement of the position operator, treated at the level of master equations [32] (whereas here we consider unitary models and QSDEs). In this scheme, additional mirrors are used to implement an auxiliary cavity mode b of the same frequency as the reference frequency ω_r (cf. Remark 6.2). The auxiliary cavity b interacts with a via a cascade of a two-mode squeezer and a beam splitter. The combination of the nonlinear crystal and beam splitter implements an overall interaction Hamiltonian H_{ab}, in a rotating frame at frequency ω_r (equal to half the pump frequency of the two-mode squeezer) of the form

$$(6.2) \qquad H_{ab} = \frac{i}{2}(\epsilon_1 a^* b^* - \epsilon_1^* ab) + \frac{i}{2}(\epsilon_2 a^* b - \epsilon_2^* ab^*),$$

where ϵ_1 is the effective pump intensity of the two-mode squeezer and ϵ_2 is given by $\epsilon_2 = 2\Theta e^{-i\Phi}$, where Θ is the mixing angle of the beam splitter and Φ is the relative phase introduced between the input fields by the beam splitter. Assuming that the coupling coefficient γ_2 of the partially transmitting mirror M on b is such that b is heavily damped compared to a, b will have much faster dynamics than a and thus allows one to adiabatically eliminate b to obtain a reduced dynamics for a only. A

rigorous foundation for such adiabatic elimination or singular perturbation procedure has recently been developed in [33]. Based on this theory, the adiabatic elimination results developed in Appendix A show that, after the elimination of b, the resulting coupling operator to a will be given by

$$L = \frac{1}{\sqrt{\gamma_2}}(-\epsilon_2^* a + \epsilon_1 a^*).$$

Therefore, it becomes clear that by choosing the parameters $\epsilon_1, \epsilon_2, \gamma_2$ with γ_2 large and such that

(6.3) $$\tilde{\alpha} = -\frac{\epsilon_2^*}{\sqrt{\gamma_2}} \text{ and } \tilde{\beta} = \frac{\epsilon_1}{\sqrt{\gamma_2}},$$

it is possible to approximately implement any coefficients $\tilde{\alpha}$ and $\tilde{\beta}$ in a linear coupling operator $L = \tilde{\alpha}a + \tilde{\beta}a^*$. Note that a π radian phase shifter in front of A in Figure 6.10 is required to compensate for the scattering term in the unitary model that is obtained after adiabatic elimination (cf. Appendix A).

Moreover, for the special case where $\tilde{\alpha}, \tilde{\beta}$ satisfy $\tilde{\alpha}$ is real and $\tilde{\alpha} > |\tilde{\beta}| \geq 0$ we also propose an alternative implementation of the linear coupling based on preprocessing and postprocessing with squeezed bosonic fields (see Appendix B for details). To this end, let $\gamma = \tilde{\alpha}^2 - |\tilde{\beta}|^2 > 0$, and consider the interaction Hamiltonian

$$H_{Int}(t) = i(L\eta(t)^* - L^*\eta(t))$$
$$= i((\tilde{\alpha}a + \tilde{\beta}a^*)\eta(t)^* - (\tilde{\alpha}a^* + \tilde{\beta}^*a)\eta(t)).$$

Let us rewrite this Hamiltonian as follows:

$$H_{Int}(t) = i(a(\tilde{\alpha}\eta(t)^* - \tilde{\beta}^*\eta(t)) - a^*(\tilde{\alpha}\eta(t) - \tilde{\beta}\eta(t)^*))$$
$$= i\sqrt{\gamma}(a\eta'(t)^* - a^*\eta'(t)),$$

where $\eta'(t) = \frac{1}{\sqrt{\gamma}}(\tilde{\alpha}\eta(t) - \tilde{\beta}\eta(t)^*)$. Letting $Z(t) = \int_0^t \eta'(s)ds$, we have that $Z(t) = \frac{1}{\sqrt{\gamma}}(\tilde{\alpha}A(t) - \tilde{\beta}A(t)^*)$, and

$$\begin{bmatrix} Z(t) \\ Z(t)^* \end{bmatrix} = Q \begin{bmatrix} A(t) \\ A(t)^* \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{\tilde{\alpha}}{\sqrt{\gamma}} & -\frac{\tilde{\beta}}{\sqrt{\gamma}} \\ -\frac{\tilde{\beta}^*}{\sqrt{\gamma}} & \frac{\tilde{\alpha}}{\sqrt{\gamma}} \end{bmatrix}.$$

The main idea is that instead of considering an oscillator interacting with $A(t)$, we consider the same oscillator interacting with the new field $Z(t)$ via the interaction Hamiltonian $H_{Int}(t) = i\sqrt{\gamma}(a\eta'(t)^* - a^*\eta'(t))$. Since $\tilde{\alpha}^2 - |\tilde{\beta}|^2 = \gamma > 0$, we see that $(\alpha/\sqrt{\gamma})^2 - |\beta/\sqrt{\gamma}|^2 = 1$, from which it follows that Q is a quasi-unitary linear transformation (cf. section 6.2) that preserves the field commutation relations. In fact, $Z(t)$ by definition is a squeezed version of $A(t)$ that can be obtained from $A(t)$ by passing the latter through a squeezer with the appropriate parameters (cf. section 6.2.3); in this case the squeezer would have the parameters $s = -\text{arccosh}(\tilde{\alpha}/\sqrt{\gamma})$ and $\theta = \arg\tilde{\beta}$. $Z(t)$ satisfies $[dZ(t), dZ(t)^*] = dt$ and the Ito rules for a squeezed field that can be generated from the vacuum (the theoretical basis for these manipulations are discussed in Appendix B) are

$$\begin{bmatrix} dZ(t) \\ dZ(t)^* \end{bmatrix} \begin{bmatrix} dZ(t) & dZ(t)^* \end{bmatrix} = Q \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Q^T dt.$$
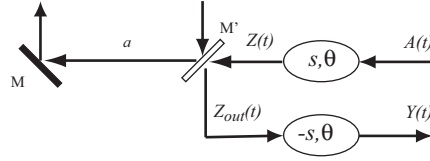
FIG. 6.11. *Scheme for implementation of a coupling* $L = \tilde{\alpha}a + \tilde{\beta}a^*$ *with* $\tilde{\alpha} > 0$ *and* $\tilde{\alpha} > |\tilde{\beta}|$. *Here* $s = -\text{arccosh}(\tilde{\alpha}/\sqrt{\gamma})$, $\theta = \arg(\tilde{\beta})$ *and the mirror M' has coupling coefficient* $\gamma = \sqrt{\tilde{\alpha}^2 - |\tilde{\beta}|^2}$.

H_{Int} can be implemented in one arm of a ring cavity with a fully reflecting mirror M and a partially transmitting mirror M' with coupling coefficient γ, with $Z(t)$ incident on M'. After the interaction, an output field $Z_{out}(t)$ is reflected by M' given by

$$Z_{out}(t) = U(t)^* Z(t) U(t)$$

$$= \frac{\tilde{\alpha}}{\sqrt{\gamma}} U(t)^* A(t) U(t) - \frac{\tilde{\beta}}{\sqrt{\gamma}} U(t)^* A(t)^* U(t).$$

However, the actual output that is of interest is the output $Y(t) = U(t)^* A(t) U(t)$ when the oscillator interacts directly with the field $A(t)$. To recover $Y(t)$ from $Z_{out}(t)$, notice that since Q is a quasi-unitary transformation, it has an inverse Q^{-1} which is again quasiunitary. Hence $Y(t)$ can be recovered from $Z_{out}(t)$ by exploiting the following relation that follows directly from the fact that $(Z_1(t), Z_1(t)^*)^T = Q(A_1(t), A_1(t)^*)^T$:

$$\begin{bmatrix} Y(t) \\ Y(t)^* \end{bmatrix} = Q^{-1} \begin{bmatrix} Z_{out}(t) \\ Z_{out}(t)^* \end{bmatrix}.$$

That is, $Y(t)$ is the output of a squeezer that implements the quasi-unitary transformation Q^{-1} with $Z_{out}(t)$ as its input field. The complete implementation of this linear coupling is shown in Figure 6.11.

6.4. Engineering the interactions between one-dimensional open quantum harmonic oscillators. The second necessary ingredient to synthesizing a general generalized open oscillator according to Theorem 5.1 is to be able to implement a direct interaction Hamiltonian H^d given by (5.4) between one-dimensional harmonic oscillators. The only exception to this, where field-mediated interactions suffice, is in the fortuitous instance where R_{jk} and L_j and S_j, $j, k = 1, \ldots, n$, are such that $H^d = 0$. The Hamiltonian H^d is essentially the sum of direct interaction Hamiltonians between pairs of one-dimensional harmonic oscillators of the form $H_{kl} = x_k^T C_{kl} x_l$ ($k \neq l$) with C_{kl} a real matrix. Under the assumption that the time it takes for the light in a ring cavity to make a round trip is much faster than the time scales of all processes taking place in the ring cavity (i.e., the cavity length should not be too long), it will be sufficient for us to only consider how to implement H_{kl} for any two pairs of one-dimensional harmonic oscillators and then implementing all of them simultaneously in a network. To this end, let $a_j = (p_j + iq_j)/2$ and $a_j^* = (p_j - iq_j)/2$ for $j = k, l$, and rewrite H_{kl} as

$$H_{kl} = \epsilon_1 a_k^* a_l + \epsilon_1^* a_k a_l^* + \epsilon_2 a_k^* a_l^* + \epsilon_2^* a_k a_l$$

for some complex numbers ϵ_1 and ϵ_2. The first part $H_{kl}^1 = \epsilon_1 a_k^* a_l + \epsilon_1^* a_k a_l^*$ can be simply implemented by a beam splitter with a mixing angle $\Theta = |\epsilon_1|$, $\Phi = -\arg(\epsilon_1) +$
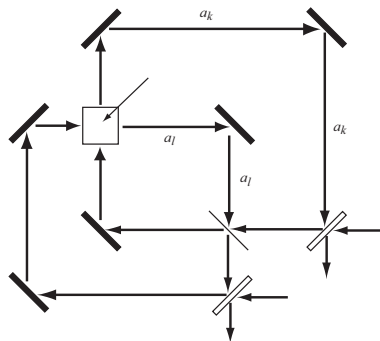
FIG. 6.12. *Example implementation of the total direct interaction $H_{kl} = H_{kl}^1 + H_{kl}^2$ between the modes a_k and a_l of two ring cavities.*

$\frac{\pi}{2}$, $\Psi = -\Phi$, and $\Xi = 0$ (see section 6.2.2). On the other hand, the second part $H_{kl}^2 = \epsilon_2 a_k^* a_l^* + \epsilon_2^* a_k a_l$ can be implemented by having the two modes a_k and a_l interact in a suitable $\chi^{(2)}$ nonlinear crystal using a classical pump beam of frequency $2\omega_r$ and effective pump intensity $-2i\epsilon_2$ in a two-mode squeezing process as described in section 6.1.3. The overall Hamiltonian H_{kl} can be achieved by positioning the arms of the two ring cavities (with canonical operators x_k and x_l) to allow their circulating light beams to "overlap" at two points where a beam splitter and a nonlinear crystal are placed to implement H_{kl}^1 and H_{kl}^2, respectively. An example of this is scheme is depicted in Figure 6.12.

7. Illustrative synthesis example. Consider a two degrees of freedom open oscillator G coupled to a single external bosonic noise field $A(t)$ given by $G = (I_{4 \times 4}, Kx, x^T \mathrm{diag}(R_1, R_2)x)$, with $x = (q_1, p_1, q_2, p_2)^T$, $K = [\begin{array}{cccc} 3/2 & 1/2i & 1 & i \end{array}]$, $R_1 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 3 \end{bmatrix}$, and $R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let G_1 and G_2 be two independent one degree of freedom open oscillators given by $G_1 = (I_{2 \times 2}, K_1 x_1, \frac{1}{2} x_1^T R_1 x_1)$ and $G_2 = (I_{2 \times 2}, K_2 x_2, \frac{1}{2} x_2^T R_2 x_2)$, with $x_1 = (q_1, p_1)^T$, $x_2 = (q_2, p_2)^T$, $K_1 = [\begin{array}{cc} 3/2 & i/2 \end{array}]$, and $K_2 = [\begin{array}{cc} 1 & i \end{array}]$. Since the scattering matrix for G is an identity matrix, it follows from Theorem 5.1 that G may be constructed as a reducible network given by $G = \{\{G_1, G_2\}, H_{12}^d, G_2 \triangleleft G_1\}$ with the direct interaction Hamiltonian H_{12}^d between G_1 and G_2 given by (cf. (5.4))

$$H_{12}^d = -\frac{1}{2i} x_2^T (K_2^\dagger K_1 - K_2^T K_1^\#) x_1$$
$$= \frac{1}{2} x_2^T \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} x_1.$$

This network is depicted in Figure 5.1.

In the following we shall illustrate how to build G_1 and G_2 and how H_{12}^d can be implemented to synthesize the overall system G.

7.1. Synthesis of G_1 and G_2. Let us now consider the synthesis of $G_1 = (I_{2 \times 2}, K_1 x_1, \frac{1}{2} x_1^T R_1 x_1)$. From the discussion in section 6.3, $R_1 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 3 \end{bmatrix}$ can be realized as a DPA with parameters $\Delta = 5$ and $\epsilon = 1 + i$, while the coupling operator $L_1 = K_1 x_1$ can be realized by the first scheme proposed in section 6.3 and shown in Figure 6.10 by the combination of a two-mode squeezer, a beam splitter, and an auxiliary cavity mode. Suppose that the coupling coefficient of the mirror M is
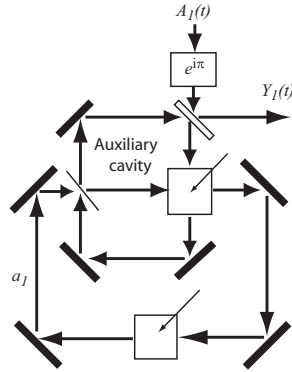
FIG. 7.1. *Realization of G_1.*
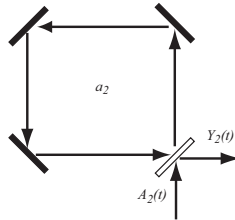
FIG. 7.2. *Realization of G_2.*

$\gamma_2 = 100$; then the effective pump intensity of the two-mode squeezer is set to be 10 and the beam splitter should have a mixing angle of -10 with all other parameters equal to 0. Overall, the open oscillator G_1 with Hamiltonian $H_1 = \frac{1}{2}x_1^T R_1 x_1$ and coupling operator L_1 can be implemented around a ring cavity structure, as shown in Figure 7.1. The open oscillator G_2 can be implemented in a similar way to G_1. The Hamiltonian $H_2 = \frac{1}{2}x_2^T R_2 x_2$ can be implemented in the same way as H_1 with the choice $\Delta = 2$ and $\epsilon = 0$. Since $\epsilon = 0$, this means no optical crystal and pump beam are required to implement R_2, but it suffices to have a cavity that is detuned from ω_r, the reference frequency in Remark 6.2, by an amount $\Delta = 2$. The coupling operator $L_2 = q_2 + ip_2 = 2a_2$, where a_2 is the annihilation operator/cavity mode of cavity is standard and can be implemented simply with a partially transmitting mirror with coupling coefficient $\kappa = 4$, on which an external vacuum noise field $A_2(t)$ interacts with the cavity mode a_2 to produce an outgoing field $Y_2(t)$. The implementation of G_2 is shown in Figure 7.2.

7.2. Synthesis of H_{12}^d. We now consider the implementation of the direct interaction Hamiltonian H_{12}^d given by $H_{12}^d = \frac{1}{2}x_2^T \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} x_1$. To proceed, we first note that H_{12}^d may be reexpressed in terms of the cavity modes a_1 and a_2 as $H_{12}^d = -i(a_1^* a_2 - a_1 a_2^*) + 2i(a_1^* a_2^* - a_1 a_2)$. Define $H_{12,1}^d = -i(a_1^* a_2 - a_1 a_2^*)$ and $H_{12,2}^d = 2i(a_1^* a_2^* - a_1 a_2)$ so that $H_{12}^d = H_{12,1}^d + H_{12,2}^d$. The first part $H_{12,1}^d = -i(a_1^* a_2 - a_1 a_2^*)$ can be simply implemented as a beam splitter with a rotation/mixing angle $\Theta = -1$ and all other parameters set to 0 (cf. section 6.2.2). On the other hand, the second part $H_{12,2}^d = 2i(a_1^* a_2^* - a_1 a_2)$ can be implemented by having the two modes a_k and a_l interact in a suitable $\chi^{(2)}$ nonlinear crystal using a classical pump beam of frequency $\omega_p = 2\omega_r$ and effective intensity $\epsilon = 4$.
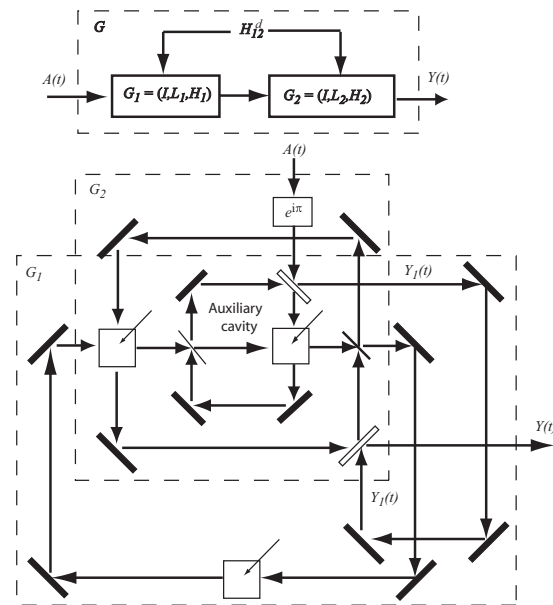
FIG. 7.3. *Realization of G. The block diagram at the top shows how G is realized by a series connection of G_1 into G_2 and a bilinear direct interaction H_{12}^d between the canonical operators of G_1 and G_2. The bottom figure shows the physical implementation of G based on the block diagram.*

7.3. Complete realization of $G = \{\{G_1, G_2\}, H_{12}^d, G_2 \lhd G_1\}$. The overall two degrees of freedom open oscillator G can now be realized by (i) positioning the arms of the two (ring) cavities of G_1 and G_2 to allow their internal light beams to "overlap" at two points where a beam splitter and a nonlinear crystal are placed to implement $H_{12,1}^d$ and $H_{12,2}^d$, respectively, and (ii) passing the output $Y_1(t)$ of G_1 as input to G_2. This implementation is shown in Figure 7.3.

8. Conclusions. In this paper we have developed a network theory for synthesizing arbitrarily complex linear dynamical quantum stochastic systems from one degree of freedom open quantum harmonic oscillators in a systematic way. We also propose schemes for building the one degree of freedom oscillators and the required interconnections and interactions among them, in the setting of quantum optics.

Together with advances in experimental physics and the availability of high-quality basic quantum devices, it is hoped the results of this work will assist in the construction of high-performance coherent linear quantum stochastic controllers and linear photonic circuits in the laboratory for applications in quantum control and quantum information science.

Appendix A. In this section, we shall derive formulas for two coupled cavity modes in which one of the cavities has very fast dynamics compared to the other and can be adiabatically eliminated, leaving only the dynamics of the slow cavity mode. The cavities are each coupled to separate bosonic fields and are interacting with one another in a classically pumped nonlinear crystal. A mathematically rigorous theory for the type of adiabatic elimination/singular perturbation that we are interested in here has recently been developed in [33].

The two cavity modes will be denoted by a and b, each defined on two distinct copies of the Hilbert space $l^2(\mathbb{Z}_+)$ of square-integrable sequences ($\mathbb{Z}_+$ denotes the set

of all nonnegative integers). Thus the composite Hilbert space for the two cavity modes is $\mathcal{H} = l^2(\mathbb{Z}_+) \otimes l^2(\mathbb{Z}_+)$. The interaction in a nonlinear crystal is given, in some rotating frame, by an interaction Hamiltonian H_{ab} of the form $H_{ab} = \alpha a^* b + \beta a^* b^* + \alpha^* a b^* + \beta^* a b$ for some complex constants α and β. The mode a is coupled to a bosonic field A_1, while b is coupled to the bosonic field A_2, both fields in the vacuum state. The fields A_1 and A_2 live, respectively, on boson Fock spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, and we denote $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$. We take a to be the slow mode to be retained and b to be the fast mode to be eliminated.

We consider a sequence of generalized open oscillators $G_k = (I, \tilde{L}^{(k)}, H_{ab}^{(k)})$, with $\tilde{L}^{(k)} = (\sqrt{\gamma_1}a, k\sqrt{\gamma_2}b)^T$ and $H_{ab}^{(k)} = \Delta_1 a^* a + k^2 \Delta_2 b^* b + k(\alpha a^* b + \beta a^* b^* + \alpha^* a b^* + \beta^* a b)$ each evolving according to the unitary U_k satisfying the left H-P QSDE (as opposed to the right H-P QSDE in (2.2)):

$$dU_k(t) = U_k(t)\left(\tilde{L}^{(k)\dagger}(dA_1(t), dA_2(t))^T - \tilde{L}^{(k)T}(dA_1(t), dA_2(t))^\dagger + iH_{ab}^{(k)} \right.$$

$$\left. - \frac{1}{2}\tilde{L}^{(k)\dagger}\tilde{L}^{(k)}dt \right).$$

Here we are using the left QSDE following the convention used in [33] (see Remark 2 therein) so that the interaction picture dynamics of an operator x is given by $x(t) = U_k(t)xU_k(t)^*$. We shall use the results of [33] to show, in a similar treatment to section 3.2 therein, that in the limit as $k \to \infty$:

(A.1) $$\lim_{k\to\infty} \sup_{0 \le t \le T} \|U_k(t)^*\phi - U(t)^*\phi\| = 0 \quad \forall \phi \in \mathcal{H}_0 \otimes \mathcal{F}$$

for any fixed time $T > 0$, where $\mathcal{H}_0$ is an appropriate Hilbert subspace of $\mathcal{H}$ (to be precisely specified in the next paragraph) for a limiting unitary $U(t)$ (again as a left H-P QSDE) satisfying

(A.2) $$dU(t) = U(t)\left(\left(\frac{i2\Delta_2 + \gamma_2}{i2\Delta_2 - \gamma_2} - 1 \right) d\Lambda_{22} + \sqrt{\gamma_1}a^* dA_1(t) - \sqrt{\gamma_1}a dA_1(t)^* \right.$$

$$- i\sqrt{\gamma_2}\left(i\Delta_2 - \frac{\gamma_2}{2} \right)^{-1} (\alpha a^* + \beta^* a)dA_2(t)$$

$$+ i\frac{2\sqrt{\gamma_2}}{i2\Delta_2 - \gamma_2}(\alpha^* a + \beta a^*)dA_2(t)^* + \left(i\Delta_1 - \frac{\gamma_1}{2} \right) a^* a dt$$

$$\left. + \left(i\Delta_2 - \frac{\gamma_2}{2} \right)^{-1} (\alpha a^* + \beta^* a)(\alpha^* a + \beta a^*)dt \right)$$

on $\mathcal{H}_0 \otimes \mathcal{F}$. Note that (A.2) is a left H-P QSDE corresponding to the right form in section 2 by noting that we may write

$$\left(i\Delta_1 - \frac{\gamma_1}{2} \right) a^* a + \left(i\Delta_2 - \frac{\gamma_2}{2} \right)^{-1} (\alpha a^* + \beta^* a)(\alpha^* a + \beta a^*)$$

$$= i\left(\Delta_1 a^* a - \frac{\Delta_2}{\Delta_2^2 + (\frac{\gamma_2}{2})^2}(\alpha a^* + \beta^* a)(\alpha^* a + \beta a^*) \right) - \frac{1}{2}(\tilde{L}_1^\dagger \tilde{L}_1 + \tilde{L}_2^\dagger \tilde{L}_2),$$

with $\tilde{L}_1 = \sqrt{\gamma_1}a$ and $\tilde{L}_2 = i\sqrt{\gamma_2}(-i\Delta_2 - \frac{\gamma_2}{2})^{-1}(\alpha^* a + \beta a^*)$. As such, it satisfies the H-P Condition 1 of [33].

Let $\phi_0, \phi_1, \ldots$ be the standard orthogonal bases of $l^2(\mathbb{Z}_+)$, i.e., ϕ_l is an infinite sequence (indexed starting from 0) of complex numbers with all zeros except a 1 in

the lth place. First, let us specify that $\mathcal{H}_0 = l^2(\mathbb{Z}_+) \otimes \mathbb{C}\phi_0$; this is the subspace of $\mathcal{H}$ where the slow dynamics of the system will evolve. Next, we define a dense domain $\mathcal{D} = \mathrm{span}\{\phi_j \otimes \phi_l; \; j, l = 0, 1, 2, \ldots\}$ of $\mathcal{H}$. The strategy is to show that [33, Assumptions 2–3] are satisfied, from which the desired result will follow from [33, Theorem 11].

From the definition of $H_{ab}^{(k)}$, $\tilde{L}^{(k)}$ and U_k given above, and we can define the operators Y, A, B, G_1, G_2, and W_{jl} $(j, l = 1, 2)$ in [33, Assumption 1] as follows: $Y = (i\Delta_2 - \frac{\gamma_2}{2})b^*b$, $A = i(\alpha a^*b + \alpha^*ab^* + \beta a^*b^* + \beta^*ab)$, $B = (i\Delta_1 - \frac{\gamma_1}{2})a^*a$, $G_1 = \sqrt{\gamma_1}a^*$, $G_2 = 0$, $F_1 = 0$, $F_2 = \sqrt{\gamma_2}b^*$, $W_{jl} = \delta_{jl}$. Then we can define the operators $K^{(k)}$, $L_j^{(k)}$ in this assumption as

$$K^{(k)} = k^2 Y + kA + B, \quad L_j^{(k)} = kF_j + G_j \; (j = 1, 2).$$

Let P_0 be the projection operator to $\mathcal{H}_0$. Let us now address Assumption 2 of [33]. From our definition of $\mathcal{H}_0$, it is clear that we have that (a) $P_0\mathcal{D} \subset \mathcal{D}$. Any element of $P_0\mathcal{D}$ is of the form $f \otimes \phi_0$ for some $f \in \mathrm{span}\{\phi_l; \; l = 0, 1, 2, \ldots\}$; therefore, since $Y = (i\Delta_2 - \frac{\gamma_2}{2})b^*b$ and $b\phi_0 = 0$, we find that (b) $YP_0d = 0 \; \forall \; d \in \mathcal{D}$. Define the operator $\tilde{Y}$ on $\mathcal{D}$ defined by $\tilde{Y}f \otimes \phi_0 = 0$ and $\tilde{Y}f \otimes \phi_l = l^{-1}(i\Delta_2 - \frac{\gamma_2}{2})^{-1}f \otimes \phi_l$ for $l = 1, 2, \ldots$ ($\tilde{Y}$ can then be defined to all of $\mathcal{D}$ by linear extension). From the definition of Y and $\tilde{Y}$, it is easily inspected that (c1) $Y\tilde{Y}f = \tilde{Y}Yf = P_1f \; \forall \; f \in \mathcal{D}$, where $P_1 = I - P_0$ (i.e., the projection onto the subspace of $\mathcal{H}$ complementary to $\mathcal{H}_0$). Moreover, because of the simple form of $\tilde{Y}$, it is also readily inspected that (c2) $\tilde{Y}$ has an adjoint $\tilde{Y}^*$ with a dense domain that contains $\mathcal{D}$. Since $F_1 = 0$, we have that (d1) $F_1^*P_0 = 0$ on $\mathcal{D}$, while since $F_2^*f \otimes \phi_0 = \sqrt{\gamma_2}bf \otimes \phi_0 = 0 \; \forall f \in l^2(\mathbb{Z}_+)$, we also have (d2) $F_2^*P_0 = 0$ on $\mathcal{D}$. Finally, from the expression for A and the orthogonality of the bases $\phi_0, \phi_1, \ldots$, a little algebra reveals that (e) $P_0AP_0d = 0 \; \forall \; d \in \mathcal{D}$. From (a), (b), (c1–c2), (d1–d2), and (e), we have now verified that Assumption 2 of [33] is satisfied.

Finally, let us check that the limiting operator coefficients K, L_1, L_2, M_1, M_2, and N_{jk} $(i, j = 1, 2)$ (as operators on $\mathcal{H}_0$) of Assumption 3 of [33] coincide with the corresponding coefficients of (A.2). These operator coefficients are defined as $K = P_0(B - A\tilde{Y}A)P_0$, $L_j = P_0(G_j - A\tilde{Y}F_j)P_0$, $M_j = -\sum_{r=1}^{2} P_0W_{jr}(G_r^* - F_r^*\tilde{Y}A)P_0$, and $N_{jl} = \sum_{r=1}^{2} P_0W_{jr}(F_r^*\tilde{Y}F_l + \delta_{rl})P_0$. From these definitions and some straightforward algebra, we find that for all $f \in \mathrm{span}\{\phi_l; \; l = 0, 1, 2, \ldots\}$

$$Kf \otimes \phi_0 = \left(\left(i\Delta_1 - \frac{\gamma_1}{2}\right)a^*a + \left(i\Delta_2 - \frac{\gamma_2}{2}\right)^{-1}(\alpha a^* + \beta^*a)(\alpha^*a + \beta a^*) \right) f \otimes \phi_0,$$

$$L_1f \otimes \phi_0 = \sqrt{\gamma_1}a^*f \otimes \phi_0,$$

$$L_2f \otimes \phi_0 = -i\sqrt{\gamma_2}\left(i\Delta_2 - \frac{\gamma_2}{2}\right)^{-1}(\alpha a^* + \beta^*a)f \otimes \phi_0,$$

$$M_1f \otimes \phi_0 = -\sqrt{\gamma_1}af \otimes \phi_0,$$

$$M_2f \otimes \phi_0 = \sqrt{\gamma_2}\left(i\Delta_2 - \frac{\gamma_2}{2}\right)^{-1}(\alpha^*a + \beta a^*)f \otimes \phi_0,$$

and

$$N_{11}f \otimes \phi_0 = f \otimes \phi_0, \quad N_{12}f \otimes \phi_0 = 0, \quad N_{21}f \otimes \phi_0 = 0,$$

$$N_{22}f \otimes \phi_0 = \frac{\gamma_2 + i2\Delta_2}{-\gamma_2 + i2\Delta_2}f \otimes \phi_0.$$

Therefore, we see that $U(t)$ may be written as

$$dU(t) = U(t) \left(\sum_{j,l=1}^{2} (N_{jl} - \delta_{jl}) d\Lambda_{jl} + \sum_{j=1}^{2} M_j dA_j^* + \sum_{j=1}^{2} L_j dA_j + K dt \right).$$

Since we have already verified that (A.2) is a bona fide right-QSDE equation, it now follows that Assumption 3 of [33] is satisfied. Now (A.1) follows from [33, Theorem 11], and the proof is complete.

Moreover, we can observe from the derivation above that the coupling of a to $A_2(t)$ after adiabatic elimination will not change if a is also coupled to other cavities modes $b_3, \ldots, b_m$ via an interaction Hamiltonian of the form $\sum_{i=j}^{m} (\alpha_{j1} a b_j^* + \alpha_{j1}^* a^* b_j + \alpha_{j2} a^* b_j^* + \alpha_{j2}^* a b_j)$, and each additional mode may also linearly coupled to distinct bosonic fields $A_3, \ldots, A_m$, respectively, as long as these other modes are *not* interacting with b and with one another (this amounts to just introducing additional operators $F_j, G_j, j \geq 3$, etc.). Moreover, under these conditions one can also adiabatically eliminate any of the additional modes, and the only effect will be the presence of additional sum terms in $U(t)$ that do not involve b, $A_1(t)$, and $A_2(t)$.

Appendix B. The purpose of this appendix is to briefly recall results from the theory squeezed white noise calculus [34] that are relevant as a basis for some formal calculations presented in section 6.3. As the theory is quite involved, it is not our intention here to discuss any aspects of it in detail, but instead to point the reader to specific results of [34].

Let $\mathcal{F}(L^2(\mathbb{R}))$ denote the usual (symmetric) boson Fock space over the Hilbert space $L^2(\mathbb{R})$ of complex-valued square integrable functions on $\mathbb{R}$. Let $\Omega_{\mathcal{F}}$ be the Fock vacuum vector, and let $a_0(f)$ and $a_0^*(g)$ for $f, g \in L^2(\mathbb{R})$ be the vacuum creation and annihilation operators on $\mathcal{F}(L^2(\mathbb{R}))$, respectively. Let $n \in \mathbb{R}$ and $c \in \mathbb{C}$ satisfy $n \geq 0$, $n \geq 0$, and $n(n+1) \geq c$. The parameters n, c characterize the so-called squeezed white noise states $\omega_{n,c}$ [34, section 2.1] that are postulated to satisfy the properties (2.1)–(2.5) therein (see also [21, Chapter 10]). However, here we will only be interested in the special case of squeezed states with n, c satisfying the constraint $n(n+1) = |c|^2$, as this is the special case of squeezed states that can be generated from the vacuum state ω_0 by an appropriate squeezing Bogoliubov transformation [34, equation (2.16)]; see [34, Theorem 2.3]. It has been shown that the annihilation and creation operators $a_{n,c}(f)$ and $a_{n,c}^*(g)$ $(f, g \in L^2(\mathbb{R}))$ corresponding to such a squeezed states can be concretely realized as operators on $\mathcal{F}(L^2(\mathbb{R}))$ [34, Theorem 2.11 part (b)] and are given in terms of the vacuum creation and annihilation operators $a_0(f)$ and $a_0(g)$, as (this follows from [34, Theorem 2.3 and equation (3.12)])

$$a_{n,c}(f) = \cosh(s) a_0(f) + e^{i\theta} \sinh(s) a_0^*(Jf),$$
$$a_{n,c}^*(f) = \cosh(s) a_0^*(f) + e^{-i\theta} \sinh(s) a_0(Jf),$$

where $J : f \mapsto f^*$, $s = \operatorname{arctanh}(\frac{2|c|}{2n+1})$, and $\theta = \arg(c)$. Conversely, we have $n = \frac{1}{2} \cosh(2s) - \frac{1}{2}$ and $c = \frac{1}{2} e^{i\theta} \sinh(2s)$. The squeezed white noise state $\omega_{n,c}$ acts on an operator A affiliated to the von Neumann algebra $\Pi_{n,c}(\mathcal{W}(L^2(\mathbb{R})))''$ of operators on $\mathcal{F}(L^2(\mathbb{R}))$ (here $\Pi_{n,c}(\mathcal{W}(L^2(\mathbb{R})))$ denotes the Gelfand–Naimark–Segal representation of the Weyl C*-algebra $\mathcal{W}(L^2(\mathbb{R}))$ on $\mathcal{F}(L^2(\mathbb{R}))$ corresponding to the state $\omega_{n,c}$, and $''$ denotes the double commutant) as

$$\omega_{n,c}(A) = \langle \Omega_{\mathcal{F}}, A\Omega_{\mathcal{F}} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the complex inner-product on $\mathcal{F}(L^2(\mathbb{R}))$ (antilinear in the first slot and linear in the second).

Let $A(t) = a(\chi_{[0,t]})$ be a vacuum bosonic field, where $\chi_{[0,t]}$ denotes the indicator function for the interval $[0, t]$, and define the squeezed bosonic field $A_{n,c}(t) = a_{n,c}(\chi_{[0,t]})$ with $a_{n,c}$ as defined above. Then $A_{n,c}$ and its adjoint $A_{n,c}^*$ are related to A and A^* by

(B.1)
$$A_{n,c}(t) = \cosh(s)A(t) + e^{i\theta}\sinh(s)A^*(t),$$
$$A_{n,c}^*(t) = \cosh(s)A^*(t) + e^{-i\theta}\sinh(s)A(t).$$

Now, consider an open oscillator whose dynamics are given by the H-P QSDE:

(B.2)
$$dU(t) = \left(-iH + dA(t)^\dagger L - L^\dagger dA(t) - \frac{1}{2}L^\dagger L dt\right)U(t),$$

where H is the quadratic Hamiltonian of the oscillator and L is the linear coupling operator to $A(t)$. By using (B.1) and substituting this into the above QSDE, we may rewrite it in terms of the $A_{n,c}$ and $A_{n,c}^*$ as follows:

(B.3)
$$dU(t) = \Bigg(-iH + dA_{n,c}(t)^\dagger M - M^\dagger dA_{n,c}(t)$$
$$- \frac{1}{2}(nMM^* + (n+1)M^*M - c^*M^2 - cM^*M)dt\Bigg)U(t),$$

where M is a new linear coupling operator given by

$$M = \cosh(s)L + e^{i\theta}\sinh(s)L^*.$$

As shown in [34], (B.3) can be interpreted on its own as the unitary evolution of a harmonic oscillator and a squeezed bosonic field linearly coupled via the coupling operator M, and this defines a quantum Markov process on the oscillator algebra (by projecting to the oscillator algebra; see [34, section 3]). In this interpretation of (2), the squeezed bosonic fields $A_{n,c}$ and $A_{n,c}^*$ satisfy the squeezed Ito multiplication rules given by

$$dA_{n,c}^2 = c\,dt, \quad dA_{n,c}dA_{n,c}^* = (n+1)dt, \quad dA_{n,c}^*dA_{n,c} = n, \quad (dA_{n,c}^*)^2 = c^*dt,$$
$$dA_{n,c}dt = 0, \quad dA_{n,c}^*dt = 0$$

that forms a basis for a quantum stochastic calculus for squeezed bosonic fields. A formal interpretation of this is that (B.3) defines the evolution of a system coupled to $A_{n,c}$ via the formal interaction Hamiltonian (see [34, section 3.6]):

(B.4)
$$H_{Int}(t) = i(M\eta_{n,c}^*(t) - M^*\eta_{n,c}(t)),$$

where $\eta_{n,c}$ is a squeezed quantum white noise that can be formally written as $\eta_{n,c} = a_{n,c}(\delta(t))$.[1] The connection with the discussion in section 6.3 is made by identifying the field $Z(t)$ introduced therein with $A_{n,c}(t)$, and $\eta'(t)$ with $\eta_{n,c}(t)$.

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[1] As is often the case, there is technical caveat in that for mathematical convenience the results of [34] are derived on the assumption that H and M are bounded operators on the oscillator Hilbert space. Here we do not concern ourselves too much with such detail and assume the optimistic view that these results can be extended to unbounded coupling operators M, which are linear combinations of the canonical operators of the harmonic oscillator, in view of the fact that the left form (cf. Appendix A) of (B.2), from which the left form of (B.3) can be recovered, still makes sense for a quadratic H and the unbounded operator L associated with M (i.e., $L = \cosh(s)M - e^{-i\theta}\sinh(s)M^*$) [27]. Moreover, singular interaction Hamiltonians of the form (B.4) between the unbounded canonical operators of a harmonic oscillator and a vacuum or squeezed quantum white noise are physically well motivated and widely used in the physics community. See, e.g., [21, Chapters 5 and 10] and related references from [34, section 3.6].

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