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Fair-sampling assumption is not necessary for testing local realism

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Almost all Bell inequality experiments to date have used postselection and therefore relied on the fair sampling assumption for their interpretation. The standard form of the fair sampling assumption is that the loss is independent of the measurement settings, so the ensemble of detected systems provides a fair statistical sample of the total ensemble. This is often assumed to be needed to interpret Bell inequality experiments as ruling out hidden-variable theories. Here we show that it is not necessary; the loss can depend on measurement settings, provided the detection efficiency factorizes as a function of the measurement settings and any hidden variable. This condition implies that Tsirelson's bound must be satisfied for entangled states. On the other hand, we show that it is possible for Tsirelson's bound to be violated while the Clauser-Horne-Shimony-Holt (CHSH)-Bell inequality still holds for unentangled states, and present an experimentally feasible example.

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I. INTRODUCTION

When quantum mechanics was first developed it was argued that it might only be an approximation of an underlying classical "hidden-variable" theory [1]. This was put on a testable basis by the development of Bell inequalities [2–4], which should be obeyed by any local hidden variable (LHV) theory. Experimental violation [5–15] of Bell inequalities provides strong evidence against LHV theories, but almost all of these experiments (with the exceptions of those in Refs. [13,15]) have the loophole that the violation of the Bell inequality could, in principle, be caused by loss [16–21]. The interpretation of these experiments as ruling out LHV theories therefore relies on the assumption that the sampling is "fair" [3,22]. For the interpretation of these experiments it is therefore vital to establish what constitutes fair sampling. Here we put fair sampling on a rigorous basis by determining exactly what forms of loss can lead to violation of Bell inequalities.

The standard form of the fair sampling assumption is that the detection efficiency is independent of measurement settings [3,22]. Here we find that assumption is unnecessary. The efficiency can depend on the measurement settings, provided the efficiency factorizes as a function of the measurement settings and any hidden variable. Most experimental tests are of the Clauser-Horne-Shimony-Holt (CHSH) form of the Bell inequality [3], and we therefore concentrate on the CHSH-Bell inequality in this work. Our condition is both necessary and sufficient for the CHSH-Bell inequality to be satisfied for LHV theories. An alternative sufficient condition was previously found by Ref. [23]. We also establish the necessary and sufficient condition for the CHSH-Bell inequality to be satisfied for unentangled states and show that if the sampling is fair it will also prevent violation of Tsirelson's bound [24] with entangled states.

Loss can cause violation of Tsirelson's bound with entangled states [25,26] or with unentangled states [25,27]. It would be useful if the postselection could cause Tsirelson's bound to be violated for entangled states but still ensure that the CHSH-Bell inequality is satisfied for unentangled states [28,29]. Our result means that, in order to achieve this task, it is necessary to examine the specific measurement and not just the form of the loss. We present a scheme that violates Tsirelson's bound with entangled states, but not the CHSH-Bell inequality with unentangled states. The CHSH-Bell inequality is still violated with LHV theories, but the scheme can tolerate greater loss than the bound derived in Ref. [18]. Other methods of constructing Bell inequalities with greater resistance to loss have been proposed in Refs. [20,30–33].

This manuscript is organized as follows. First, the fair sampling assumption is explained in more detail in Sec. II. General Bell inequalities are presented in Sec. III, and then postselection for local hidden variable theories is analyzed in Sec. IV. Postselection for quantum mechanics, including Tsirelson's bound, is analyzed in Sec. V. The use of postselection to enhance violation of Bell inequalities beyond Tsirelson's bound for entangled states is analyzed in Sec. VI. We conclude in Sec. VII.

II. THE FAIR SAMPLING ASSUMPTION

The first work to give a form of the fair sampling assumption was that of Clauser, Horne, Shimony, and Holt [3]. They assumed, in deriving their inequality, that the detection efficiency is independent of the measurement settings. Pearle expressed the assumption alternatively, that "the data recorded [is] representative of the accepted data" [16]. That is, the sample of detected pairs provides a fair statistical sample of all the pairs. These forms of the fair sampling assumption, which superficially appear different, are effectively equivalent.

For the sample to be completely fair, the probability of sampling a pair (i.e., the efficiency) needs to be independent of the pair. For this to be the case, the efficiency needs to be independent of any quantity that varies between the

individual pairs. The only quantities that can vary between the individual pairs are the internal state (the quantum state or any hypothetical hidden variable) and the measurement setting that is used in measuring the pair. Therefore the requirement that the sample is fair is exactly equivalent to requiring the efficiency to be independent of the internal state and measurement setting.

This requirement is slightly stronger than the requirement given by Ref. [3], because it requires that the efficiency is also independent of the state of the pair. However, it is easily seen that the efficiency need not be independent of the state. This is because, if the efficiency is dependent on the state, but independent of the measurement setting, then the postselection simply changes the probabilities for the internal state, yielding a different postselected state. That is, the sampling may not be fair, but the loss is simply yielding a postselected state on which the sampling is fair.

This is taken advantage of in Procrustean entanglement concentration [34], which has been demonstrated to enhance the violation of Bell inequalities [35]. Procrustean entanglement concentration gives loss that depends on the state, but because this loss is independent of the measurement settings of the Bell measurement, it does not invalidate the Bell inequalities for unentangled states. Although the sampling is not entirely fair for the initial state, it is fair for the state produced by the Procrustean entanglement concentration.

It is important to note that the condition that the efficiency is independent of the measurement setting means that it must be *completely* independent of the measurement setting. That is, it can not be a function of the measurement setting, so it is independent of the measurement setting for any internal state. On the other hand, if the efficiency is independent of the measurement setting for just one state, then it is easy to provide examples where the Bell inequality is violated with unentangled states or hidden variables. For a simple example, see Appendix A.

III. GENERAL BELL INEQUALITIES

In a general Bell inequality experiment with multiple parties, these parties each share one component of a state ρ, and each performs one of a number of different measurements. The measurement settings for party k are denoted γ_k, and the measurement results are denoted s_k. We denote the vectors of measurement settings and measurement results $\vec{\gamma}$ and $\vec{s}$. One obtains a set of measurement probabilities $p(\vec{s}|\vec{\gamma}, \rho)$ and can define a Bell quantity as a linear combination of these probabilities. A Bell inequality is then an upper bound on the value of this quantity for LHV theories. With loss, we denote the probability of a successful measurement for settings $\vec{\gamma}$ by $\mathcal{E}(\vec{\gamma}, \rho)$. The postselected probabilities are then given by

$$\tilde{p}(\vec{s}|\vec{\gamma}, \rho) = p(\vec{s}|\vec{\gamma}, \rho)/\mathcal{E}(\vec{\gamma}, \rho). \tag{1}$$

We do not consider complete loss, which would make the postselected probabilities undefined.

In the specific case of CHSH-Bell inequalities, there are two parties, Alice and Bob, and each performs one of two dichotomic measurements. Now using the notation α and β for the measurement settings for Alice and Bob, respectively, we have $\alpha \in \{A, a\}$, $\beta \in \{B, b\}$, and $s_1, s_2 \in \{-, +\}$. The CHSH-Bell quantity without postselection is defined by

$$\mathcal{B}_0 := C(A, B) + C(A, b) + C(a, B) - C(a, b), \tag{2}$$

where

$$C(\alpha, \beta) := p(+, +|\alpha, \beta, \rho) - p(+, -|\alpha, \beta, \rho) - p(-, +|\alpha, \beta, \rho) + p(-, -|\alpha, \beta, \rho). \tag{3}$$

The postselected form of the CHSH-Bell quantity is

$$\mathcal{B} := \frac{C(A, B)}{\mathcal{E}(A, B, \rho)} + \frac{C(a, B)}{\mathcal{E}(a, B, \rho)} + \frac{C(A, b)}{\mathcal{E}(A, b, \rho)} - \frac{C(a, b)}{\mathcal{E}(a, b, \rho)}. \tag{4}$$

IV. LOCAL HIDDEN-VARIABLE THEORIES

First we prove that Bell inequalities must be satisfied for LHV theories provided the detection efficiency factorizes as a function of the measurement setting and a function of the hidden variable. For a stochastic local hidden-variable theory, probabilities are given as

$$p(\vec{s}|\vec{\gamma}, \rho) = \sum_x p(x|\rho) \prod_k p_k(s_k|\gamma_k, x), \tag{5}$$

where x is the hidden variable. The state ρ simply controls the probability distribution for x. With loss, the probabilities $p(\vec{s}|\vec{\gamma}, \rho)$ can sum to less than 1. Summing over the measurement results gives the efficiencies as

$$\mathcal{E}(\vec{\gamma}, \rho) = \sum_x p(x|\rho) \prod_k \mathcal{E}_k(\gamma_k, x), \tag{6}$$

where $\mathcal{E}_k(\gamma_k, x) = \sum_s p(s|\gamma_k, x)$ is the single-party efficiency for party k with measurement setting γ_k and hidden variable x.

We also use the notation $\mathcal{E}_{k\mathrm{S}}(\gamma_k)$ for the efficiency due to measurement setting γ_k for party k, and $\mathcal{E}_{k\mathrm{H}}(x)$ for the efficiency due to the hidden variable for party k. With this notation defined, we can now state our condition rigorously.

Theorem 1. When the efficiency for each party factorizes as

$$\mathcal{E}_k(\gamma_k, x) = \mathcal{E}_{k\mathrm{S}}(\gamma_k)\mathcal{E}_{k\mathrm{H}}(x), \tag{7}$$

the set of postselected probabilities that can be obtained $\{\tilde{p}(\vec{s}|\vec{\gamma}, \rho)\}$ is identical to that which can be obtained without postselection (provided no efficiency is zero).

This result means that this form of loss does not change the type of probability distributions that are possible with LHV models, and in particular any Bell inequality must still be satisfied with postselection.

Proof. The proof follows by showing that there exists a measurement scheme without postselection that yields the same probabilities as the $\tilde{p}(\vec{s}|\vec{\alpha}, \rho)$ for the postselected scheme. The postselected probabilities may alternatively be given by

$$\tilde{p}(\vec{s}|\vec{\gamma}, \rho) = \sum_x \tilde{p}(x|\rho) \prod_k \tilde{p}(s_k|\gamma_k, x), \tag{8}$$

where

$$\tilde{p}(x|\rho) := \frac{p(x|\rho)}{\mathcal{E}_{\mathrm{R}}(\rho)} \prod_k \mathcal{E}_{k\mathrm{H}}(x),$$
$$\mathcal{E}_{\mathrm{R}}(\rho) := \sum_x p(x|\rho) \prod_k \mathcal{E}_{k\mathrm{H}}(x), \tag{9}$$
$$\tilde{p}(s_k|\gamma_k, x) := p(s_k|\gamma_k, x)/\mathcal{E}_k(\gamma_k, x).$$

This means that, by using a different LHV model with different probabilities, we may obtain exactly the same probability distribution for the measurement results as for the postselected case. ■

This result shows that our condition (7) is sufficient for Bell inequalities to hold with postselection. We also have a necessity proof in the case of CHSH-Bell inequalities. Before showing this result, we first show the general form of the CHSH-Bell inequality with postelection.

Lemma 1. Any local hidden variable theory must satisfy

$$|\mathcal{B}| \leqslant 4 - 2\sum_x p(x|\rho) \min_{\alpha^x,\beta^x} \frac{\mathcal{E}_1(\alpha^x, x)\mathcal{E}_2(\beta^x, x)}{\mathcal{E}(\alpha^x, \beta^x, \rho)}. \tag{10}$$

For given $\mathcal{E}$ there exist probabilities $p_k(\pm|\gamma_k, x)$ that saturate this inequality.

Here we have used the same notation as for general Bell inequalities, except we have used $\gamma_1 = \alpha$ and $\gamma_2 = \beta$ for Alice and Bob's measurement settings, respectively. We use the superscript x on α and β to indicate that these are chosen as a function of the hidden variable x.

Proof. For LHV theory, the postselected CHSH-Bell quantity can be rewritten as

$$\mathcal{B} = \sum_x p(x|\rho) \left[\frac{\Delta_1(A, x)\Delta_2(B, x)}{\mathcal{E}(A, B, \rho)} + \frac{\Delta_1(a, x)\Delta_2(B, x)}{\mathcal{E}(a, B, \rho)} + \frac{\Delta_1(A, x)\Delta_2(b, x)}{\mathcal{E}(A, b, \rho)} - \frac{\Delta_1(a, x)\Delta_2(b, x)}{\mathcal{E}(a, b, \rho)} \right], \tag{11}$$

where

$$\Delta_k(\gamma_k, x) = p_k(+|\gamma_k, x) - p_k(-|\gamma_k, x).$$

Because the probabilities are non-negative, we have $|\Delta_k(\gamma_k, x)| \leqslant \mathcal{E}_k(\gamma_k, x)$. Considering arbitrary measurements for a given form of loss, we can choose the $p_k(\pm|\gamma_k, x)$ such that $\Delta_k(\gamma_k, x)$ takes any value in the range $[-\mathcal{E}_k(\gamma_k, x), \mathcal{E}_k(\gamma_k, x)]$.

Because $\mathcal{B}$ is linear in each of the $\Delta_k(\gamma_k, x)$, it is maximized (or minimized) by taking extreme values where $\Delta_k(\gamma_k, x) = \pm\mathcal{E}_k(\gamma_k, x)$. By appropriately choosing the signs, it is possible to make any one of the terms in the square brackets in Eq. (11) negative and the rest positive. Changing the sign preserves the parity of the number of negative terms, so it is not possible to make all terms positive.

In particular, taking $\Delta_k(\gamma_k, x) = \mathcal{E}_k(\gamma_k, x)$ makes the last term negative. Then changing $\Delta_2(b, x)$ to $-\mathcal{E}_2(b, x)$ makes only the third term negative. Alternatively, changing $\Delta_1(a, x)$ to $-\mathcal{E}_1(a, x)$ makes only the second term negative, or changing $\Delta_2(b, x)$ to $-\mathcal{E}_2(b, x)$ and $\Delta_1(A, x)$ to $-\mathcal{E}_1(A, x)$ makes only the first term negative.

To maximize $\mathcal{B}$, we take three terms positive and one negative for each value of x. The value of $\mathcal{B}$ will be maximized with the smallest term taken to be negative. That is,

$$\mathcal{B} = \sum_x p(x|\rho) \left[\frac{\mathcal{E}_1(A, x)\mathcal{E}_2(B, x)}{\mathcal{E}(A, B, \rho)} + \frac{\mathcal{E}_1(a, x)\mathcal{E}_2(B, x)}{\mathcal{E}(a, B, \rho)} + \frac{\mathcal{E}_1(A, x)\mathcal{E}_2(b, x)}{\mathcal{E}(A, b, \rho)} + \frac{\mathcal{E}_1(a, x)\mathcal{E}_2(b, x)}{\mathcal{E}(a, b, \rho)} - 2\min_{\alpha^x,\beta^x} \frac{\mathcal{E}_1(\alpha^x, x)\mathcal{E}_2(\beta^x, x)}{\mathcal{E}(\alpha^x, \beta^x, \rho)} \right]. \tag{12}$$

Summing the first four terms in Eq. (12) gives $\mathcal{B}$ equal to

$$\mathcal{B}_{\max} = 4 - 2\sum_x p(x|\rho) \min_{\alpha^x,\beta^x} \frac{\mathcal{E}_1(\alpha^x, x)\mathcal{E}_2(\beta^x, x)}{\mathcal{E}(\alpha^x, \beta^x, \rho)}. \tag{13}$$

To change the overall sign, so one term is positive and the remaining are negative, we can change the sign of both $\Delta_1(A, x)$ and $\Delta_1(a, x)$. Using this, we can ensure that the largest three terms in Eq. (11) are negative, and the smallest is positive. Doing this for each value of x, we simply obtain the negative of what was obtained before, and so obtain $\mathcal{B} = -\mathcal{B}_{\max}$. We therefore find that the generalization of the CHSH-Bell inequality for the case of postselection is $|\mathcal{B}| \leqslant \mathcal{B}_{\max}$, as given in Eq. (10). As the above argument is constructive, it shows how to choose probabilities $p_k(\pm|\gamma_k, x)$ in order to saturate this inequality. ■

Using this result, we can prove the necessary and sufficient condition for the CHSH-Bell inequality to hold for LHV theories with postselection. We first prove the necessary and sufficient condition in an alternative form.

Lemma 2. The condition that

$$\frac{\mathcal{E}_1(\alpha, x)\mathcal{E}_2(\beta, x)}{\mathcal{E}(\alpha, \beta, \rho)} \tag{14}$$

is independent of α and β is necessary and sufficient for $|\mathcal{B}| \leqslant 2$ to be satisfied for all probabilities $p_k(\pm|\gamma_k, x)$.

Proof. In order for $\mathcal{B}$ to satisfy the usual CHSH-Bell inequality, $|\mathcal{B}| \leqslant 2$, the sum in Eq. (10) must be equal to 1. Given that the condition is satisfied, we may take $\alpha^x = \alpha$ and $\beta^x = \beta$ for some arbitrary α and β in Eq. (10), giving

$$|\mathcal{B}| \leqslant 4 - 2\sum_x p(x|\rho) \frac{\mathcal{E}_1(\alpha, x)\mathcal{E}_2(\beta, x)}{\mathcal{E}(\alpha, \beta, \rho)} = 2. \tag{15}$$

Thus the condition is sufficient.

To prove necessity, we show that if the condition in Lemma 2 is not satisfied, then $|\mathcal{B}| \leqslant 2$ may be violated. If the condition is not satisfied for $x = z$, then select α and β such that

$$\frac{\mathcal{E}_1(\alpha, z)\mathcal{E}_2(\beta, z)}{\mathcal{E}(\alpha, \beta, \rho)} > \frac{\mathcal{E}_1(\alpha^z, z)\mathcal{E}_2(\beta^z, z)}{\mathcal{E}(\alpha^z, \beta^z, \rho)}. \tag{16}$$

We then have

$$\sum_x p(x|\rho) \min_{\alpha^x,\beta^x} \frac{\mathcal{E}_1(\alpha^x, x)\mathcal{E}_2(\beta^x, x)}{\mathcal{E}(\alpha^x, \beta^x, \rho)} \leqslant \sum_x p(x|\rho) \frac{\mathcal{E}_1(\alpha, x)\mathcal{E}_2(\beta, x)}{\mathcal{E}(\alpha, \beta, \rho)}$$

$$- p(z|\rho)\left[\frac{\mathcal{E}_1(\alpha,z)\mathcal{E}_2(\beta,z)}{\mathcal{E}(\alpha,\beta,\rho)} - \min_{\alpha^z,\beta^z}\frac{\mathcal{E}_1(\alpha^z,z)\mathcal{E}_2(\beta^z,z)}{\mathcal{E}(\alpha^z,\beta^z,\rho)}\right] < 1. \quad (17)$$

As there exists a choice of probabilities $p_k(\pm|\gamma_k,x)$ which saturate the inequality (10), we can obtain $|\mathcal{B}|>2$. Hence we may obtain $|\mathcal{B}|>2$ if the condition in Lemma 2 is violated, and it is therefore a necessary condition. ■

We can now use Lemma 2 to show that the factorization condition (7) is sufficient and necessary for the CHSH-Bell inequality. Specifically, we have the following theorem.

Theorem 2. The condition that the efficiency for each party factorizes as

$$\mathcal{E}_k(\gamma_k,x)=\mathcal{E}_{k\mathrm{S}}(\gamma_k)\mathcal{E}_{k\mathrm{H}}(x), \quad (18)$$

is necessary and sufficient for $|\mathcal{B}|\leqslant 2$ to be satisfied for all probabilities $p_k(\pm|\gamma_k,x)$ (provided no efficiency is zero).

Proof. The proof proceeds simply by showing that the condition in Lemma 2 is equivalent to (18). First, it is trivial to show that Eq. (18) implies the condition. Using Eq. (18) gives

$$\frac{\mathcal{E}_1(\alpha,x)\mathcal{E}_2(\beta,x)}{\mathcal{E}(\alpha,\beta,\rho)}=\frac{\mathcal{E}_{1\mathrm{H}}(x)\mathcal{E}_{2\mathrm{H}}(x)}{\mathcal{E}_{\mathrm{R}}(\rho)}, \quad (19)$$

which is independent of α and β.

To show that the condition of Lemma 2 implies (18), we simply need to use it to define the quantities $\mathcal{E}_{k\mathrm{S}}(\gamma_k)$ and $\mathcal{E}_{k\mathrm{H}}(x)$. Let us define, for some x_0,

$$\mathcal{E}_{1\mathrm{S}}(\alpha):=\mathcal{E}_1(\alpha,x_0)/\max_{\alpha'}\mathcal{E}_1(\alpha',x_0), \quad (20)$$

$$\mathcal{E}_{1\mathrm{H}}(x):=\max_{\alpha}\mathcal{E}_1(\alpha,x), \quad (21)$$

$$\mathcal{E}_{2\mathrm{S}}(\beta):=\mathcal{E}_2(\beta,x_0)/\max_{\beta'}\mathcal{E}_1(\beta',x_0), \quad (22)$$

$$\mathcal{E}_{2\mathrm{H}}(x):=\max_{\beta}\mathcal{E}_1(\beta,x). \quad (23)$$

It remains to show that these definitions satisfy (18). We find that

$$\mathcal{E}_{1\mathrm{S}}(\alpha)\mathcal{E}_{2\mathrm{S}}(\beta)=\frac{\mathcal{E}_1(\alpha,x_0)\mathcal{E}_2(\beta,x_0)}{\max_{\alpha,\beta}\mathcal{E}_1(\alpha,x_0)\mathcal{E}_1(\beta,x_0)}=\frac{\mathcal{E}(\alpha,\beta,\rho)}{\max_{\alpha,\beta}\mathcal{E}(\alpha,\beta,\rho)}, \quad (24)$$

where in the second line we use the condition of Lemma 2 and ρ may be arbitrary. Hence we find that $\mathcal{E}_{1\mathrm{S}}(\alpha)\mathcal{E}_{2\mathrm{S}}(\beta)$ is independent of x_0, and we therefore have

$$\mathcal{E}_{1\mathrm{S}}(\alpha)\mathcal{E}_{1\mathrm{H}}(x)\mathcal{E}_{2\mathrm{S}}(\beta)\mathcal{E}_{2\mathrm{H}}(x)=\mathcal{E}_1(\alpha,x)\mathcal{E}_2(\beta,x). \quad (25)$$

Rearranging Eq. (25) to isolate $\mathcal{E}_1(\alpha,x)$ gives

$$\mathcal{E}_1(\alpha,x)=\mathcal{E}_{1\mathrm{S}}(\alpha)\frac{\mathcal{E}_{1\mathrm{H}}(x)\mathcal{E}_{2\mathrm{S}}(\beta)\mathcal{E}_{2\mathrm{H}}(x)}{\mathcal{E}_2(\beta,x)}. \quad (26)$$

Taking the maximum over α then yields

$$\mathcal{E}_{1\mathrm{H}}(x)=\frac{\mathcal{E}_{1\mathrm{H}}(x)\mathcal{E}_{2\mathrm{S}}(\beta)\mathcal{E}_{2\mathrm{H}}(x)}{\mathcal{E}_2(\beta,x)}, \quad (27)$$

where we have used $\max_\alpha \mathcal{E}_{1\mathrm{S}}(\alpha)=1$. Hence

$$\mathcal{E}_2(\beta,x)=\mathcal{E}_{2\mathrm{S}}(\beta)\mathcal{E}_{2\mathrm{H}}(x). \quad (28)$$

Using this expression in Eq. (25) gives

$$\mathcal{E}_1(\alpha,x)=\mathcal{E}_{1\mathrm{S}}(\alpha)\mathcal{E}_{1\mathrm{H}}(x). \quad (29)$$

Thus we have shown that the definitions (20) to (23) satisfy Eq. (18) provided the condition of Lemma 2 holds.

Hence we have shown that the condition of Lemma 2 and the condition (18) are equivalent, and therefore Theorem 2 follows from Lemma 2.

V. QUANTUM MECHANICS

Next we consider the restriction on the loss for quantum mechanics rather than LHV theories. It might be thought that the case of quantum mechanics is equivalent, because a LHV theory can be thought of as an unentangled state, with the x labeling orthogonal basis states. However, the case of quantum mechanics is slightly different, because we also need to consider all linear combinations of orthogonal basis states.

For quantum theory the probabilities of local measurement results are obtained via a positive operator-valued measure with elements $M_{k,\gamma_k}^{s_k}$, which corresponds to successful measurement result s_k for measurement setting γ_k for party k. We also use the notation

$$M_{k,\gamma_k}:=\sum_{s_k}M_{k,\gamma_k}^{s_k}, \quad (30)$$

for the operator corresponding to a successful measurement. Here the sum is over all successful measurement results s_k; the measurement operator for failure is $\mathbb{1}-M_{k,\gamma_k}$.

The restriction on the loss in terms of hidden variables (7) implies that

$$\mathcal{E}(\vec{\gamma},\rho)=\mathcal{E}_{\mathrm{R}}(\rho)\prod_k\mathcal{E}_{k\mathrm{S}}(\gamma_k). \quad (31)$$

This expression may be taken to be the definition of the restriction on the loss for quantum mechanics. This restriction may alternatively be expressed as

$$\mathrm{Tr}\left[\left(\bigotimes_k M_{k,\gamma_k}\right)\rho\right]=\mathcal{E}_{\mathrm{R}}(\rho)\prod_k\mathcal{E}_{k\mathrm{S}}(\gamma_k). \quad (32)$$

Let

$$M:=\frac{\bigotimes_k M_{k,\gamma_k}}{\prod_k\mathcal{E}_{k\mathrm{S}}(\gamma_k)}. \quad (33)$$

The restriction (32) implies that $\mathrm{Tr}(M\rho)=\mathcal{E}_{\mathrm{R}}(\rho)$ independent of $\vec{\gamma}$, and therefore M is independent of $\vec{\gamma}$. This is why no subscript $\vec{\gamma}$ is given for M. Thus the restriction in Eq. (31) implies that the $M_{k,\gamma_k}/\mathcal{E}_{k\mathrm{S}}(\gamma_k)$ are independent of γ_k.

Using a similar method as for hidden variables, we can show that postselection with this form of loss cannot change the form of probability distributions obtained, either for the case of entangled or unentangled states. In particular, the result is as in the following theorem.

Theorem 3. For a Bell experiment on a quantum mechanical system, provided the loss is restricted by

$$\mathcal{E}(\vec{\gamma},\rho)=\mathcal{E}_{\mathrm{R}}(\rho)\prod_k\mathcal{E}_{k\mathrm{S}}(\gamma_k), \quad (34)$$

the set of postselected probabilities it is possible to obtain without entanglement $\{\tilde{p}(\vec{s}|\vec{\gamma},\rho)\}$ is identical to that which can be obtained without postselection or entanglement, and

the set of postselected probabilities it is possible to obtain with entanglement is identical to that which can be obtained without postelection but with entanglement (provided no efficiency is zero).

Proof. We define new measurement operators as

$$\tilde{M}^{s_k}_{k,\gamma_k} := (M_{k,\gamma_k})^{-1/2} M^{s_k}_{k,\gamma_k} (M_{k,\gamma_k})^{-1/2}. \tag{35}$$

These measurement operators now give unit efficiency (so there is no postselection). We also define

$$\tilde{\rho} := \frac{M^{-1/2}\rho M^{-1/2}}{\mathrm{Tr}(M^{-1/2}\rho M^{-1/2})}, \tag{36}$$

where M is defined as in Eq. (33). Because we restrict to the case of incomplete loss, the measurement operators are positive and can be inverted. The new measurement operators and states give exactly the same probabilities *without* postselection as the original measurement operators and states did with postselection.

Furthermore, because M is a local operator, it cannot produce entangled states from unentangled states. Therefore the postselected Bell experiment with an unentangled state gives postselected probabilities identical to the non-postselected probabilities for a different Bell experiment with different measurement operators and a different unentangled state $\tilde{\rho}$. ■

This result means that, for an unentangled state, all Bell inequalities must still be satisfied. This is exactly as we expect, because unentangled states may be regarded as equivalent to hidden variables. Similarly, inequalities that hold for entangled states, such as Tsirelson's bound, will be unaffected by the postselection. It is easily checked that schemes for violating Tsirelson's bound [25–27] also violate Eq. (31).

For CHSH-Bell inequalities, it would be expected that Eq. (31) is also a necessary condition due to the result for hidden-variable theories. However, it does not directly follow, and necessity needs to be proven separately. This is because one cannot arbitrarily choose the probabilities $p(\vec{s}|\vec{\gamma}, x)$ for given efficiencies. Therefore it is not necessarily possible to achieve the maximum value of Eq. (10), as in the case of a general hidden-variable theory. Nevertheless, it is possible to show the result in the following theorem.

Theorem 4. For a quantum mechanical system, the condition that the efficiency for each party factorizes as

$$\mathcal{E}(\vec{\gamma}, \rho) = \mathcal{E}_{\mathrm{R}}(\rho) \prod_k \mathcal{E}_{k\mathrm{S}}(\gamma_k), \tag{37}$$

is necessary and sufficient for $|\mathcal{B}| \leqslant 2$ to be satisfied for all probabilities $p_k(\pm|\gamma_k, x)$ (provided no efficiency is zero).

Proof. The sufficiency follows immediately from Theorem 3, so it only remains to show necessity. The condition in the theorem is equivalent to Eq. (32), so we show necessity for Eq. (32). First consider the case where Eq. (32) is violated. In that case, either $M_\alpha/\mathcal{E}_{1\mathrm{S}}(\alpha)$ is dependent on α, or $M_\beta/\mathcal{E}_{2\mathrm{S}}(\beta)$ is dependent on β (or both). We omit the subscripts 1 and 2 in the notation M_α and M_β for simplicity. The party is simply indicated by the symbol used for the measurement setting (α for party 1 and β for party 2).

If $M_\alpha/\mathcal{E}_{1\mathrm{S}}(\alpha)$ (for example) is not independent of α, then there exist orthogonal states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ such that

$$\langle\varphi_0|M_a|\varphi_0\rangle\langle\varphi_1|M_A|\varphi_1\rangle \neq \langle\varphi_0|M_A|\varphi_0\rangle\langle\varphi_1|M_a|\varphi_1\rangle. \tag{38}$$

This result may be proven in the following way. Let us assume that all orthogonal states give equality in (38). Then, for any orthogonal basis $\{|\varphi_j\rangle\}$, we have

$$\langle\varphi_j|M_a|\varphi_j\rangle = \mu\langle\varphi_j|M_A|\varphi_j\rangle, \tag{39}$$

for some μ. Therefore $\mathrm{Tr}(M_a) = \mu\mathrm{Tr}(M_A)$. Now we can take $\{|\varphi'_j\rangle\}$ to be the basis which diagonalizes $M_a - \mu M_A$. Then we have

$$\langle\varphi'_j|M_a|\varphi'_j\rangle = \mu'\langle\varphi'_j|M_A|\varphi'_j\rangle, \tag{40}$$

so $\mathrm{Tr}(M_a) = \mu'\mathrm{Tr}(M_A)$. Because $\mathrm{Tr}(M_\alpha)$ is nonzero (these are positive operators), we have $\mu = \mu'$. Therefore the diagonal elements of $M_a - \mu M_A$ must be zero in the basis $\{|\varphi'_j\rangle\}$. As this is the basis which diagonalizes $M_a - \mu M_A$, we must have $M_a = \mu M_A$. Hence we find that equality in (38) implies $M_a = \mu M_A$, so if M_a is not proportional to M_A, then there must exist orthogonal states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ such that (38) is satisfied.

We obtain the exact equivalent result if $M_\beta/\mathcal{E}_{2\mathrm{S}}(\beta)$ is not independent of β. Therefore, if Eq. (32) is violated, we can select $|\varphi_j\rangle$ and $|\chi_j\rangle$ such that

$$\begin{aligned}\langle\varphi_0|M_a|\varphi_0\rangle\langle\varphi_1|M_A|\varphi_1\rangle &\leqslant \langle\varphi_0|M_A|\varphi_0\rangle\langle\varphi_1|M_a|\varphi_1\rangle,\\ \langle\chi_0|M_b|\chi_0\rangle\langle\chi_1|M_B|\chi_1\rangle &\leqslant \langle\chi_0|M_B|\chi_0\rangle\langle\chi_1|M_b|\chi_1\rangle,\end{aligned} \tag{41}$$

with strict inequality in at least one of these cases. We therefore have

$$\begin{aligned}&\langle\varphi_0|M_a|\varphi_0\rangle\langle\varphi_1|M_A|\varphi_1\rangle\langle\chi_0|M_b|\chi_0\rangle\langle\chi_1|M_B|\chi_1\rangle\\ &\quad< \langle\varphi_0|M_A|\varphi_0\rangle\langle\varphi_1|M_a|\varphi_1\rangle\langle\chi_0|M_B|\chi_0\rangle\langle\chi_1|M_b|\chi_1\rangle.\end{aligned} \tag{42}$$

Now consider the density operator

$$\rho = \left(|\varphi_0\rangle\langle\varphi_0| \otimes |\chi_0\rangle\langle\chi_0| + |\varphi_1\rangle\langle\varphi_1| \otimes |\chi_1\rangle\langle\chi_1|\right)/2. \tag{43}$$

For this density operator

$$\begin{aligned}\mathcal{B} = \frac{1}{2}\sum_{j=0}^{1}\Bigg\{&\frac{\langle\Delta M_A\rangle_j\langle\Delta M_B\rangle_j}{\mathrm{Tr}[(M_A\otimes M_B)\rho]} + \frac{\langle\Delta M_A\rangle_j\langle\Delta M_b\rangle_j}{\mathrm{Tr}[(M_A\otimes M_b)\rho]}\\ &+ \frac{\langle\Delta M_a\rangle_j\langle\Delta M_B\rangle_j}{\mathrm{Tr}[(M_a\otimes M_B)\rho]} - \frac{\langle\Delta M_a\rangle_j\langle\Delta M_b\rangle_j}{\mathrm{Tr}[(M_a\otimes M_b)\rho]}\Bigg\},\end{aligned} \tag{44}$$

where $\Delta M_{k,\gamma_k} = M^+_{k,\gamma_k} - M^-_{k,\gamma_k}$, and $\langle\cdots\rangle_j$ indicates the expectation value using the state $|\varphi_j\rangle$ (for Alice) or $|\chi_j\rangle$ (for Bob). Given the M_{k,γ_k}, consider the measurement operators for the individual results given by

$$\begin{aligned}M^+_A &= M_A, \quad M^+_a = |\varphi_0\rangle\langle\varphi_0|\langle M_a\rangle_0,\\ M^-_A &= 0, \quad M^-_a = M_a - |\varphi_0\rangle\langle\varphi_0|\langle M_a\rangle_0,\\ M^+_B &= |\varphi_0\rangle\langle\varphi_0|\langle M_B\rangle_0, \quad M^+_b = M_B,\\ M^-_B &= M_B - |\varphi_0\rangle\langle\varphi_0|\langle M_B\rangle_0, \quad M^-_b = 0.\end{aligned} \tag{45}$$

Using these measurement operators gives

$$\begin{aligned}\langle\Delta M_a\rangle_j &= (-1)^j\langle M_a\rangle_j, \quad \langle\Delta M_A\rangle_j = \langle M_A\rangle_j,\\ \langle\Delta M_b\rangle_j &= \langle M_b\rangle_j, \quad \langle\Delta M_B\rangle_j = (-1)^j\langle M_B\rangle_j.\end{aligned} \tag{46}$$

This gives the postselected Bell quantity as

$$\mathcal{B} = 4 - \frac{2\langle M_a\rangle_0\langle M_b\rangle_0}{\langle M_a\rangle_0\langle M_b\rangle_0 + \langle M_a\rangle_1\langle M_b\rangle_1} - \frac{2\langle M_A\rangle_1\langle M_B\rangle_1}{\langle M_A\rangle_0\langle M_B\rangle_0 + \langle M_A\rangle_1\langle M_B\rangle_1}. \quad (47)$$

Equation (42) may be rearranged to give

$$\frac{\langle M_a\rangle_0\langle M_b\rangle_0}{\langle M_a\rangle_0\langle M_b\rangle_0 + \langle M_a\rangle_1\langle M_b\rangle_1} + \frac{\langle M_A\rangle_1\langle M_B\rangle_1}{\langle M_A\rangle_0\langle M_B\rangle_0 + \langle M_A\rangle_1\langle M_B\rangle_1} < 1. \quad (48)$$

We therefore obtain $\mathcal{B} > 2$.

Thus we find that, if Eq. (31) is not satisfied, for given M_{k,γ_k} there exists a separable state and a set of measurement operators $M^{\pm}_{k,\gamma_k}$ such that $\mathcal{B} > 2$. Equation (31) is therefore a necessary condition for the CHSH-Bell inequality to be satisfied for separable states. As Eq. (31) is equivalent to the condition in the theorem, we have proven both necessity and sufficiency as required. ■

For entangled states, $\mathcal{B}$ is limited by Tsirelson's bound of $2\sqrt{2}$ [24]. As explained above, the restriction given by Eq. (31) is sufficient for Tsirelson's bound to be satisfied due to the result in Theorem 3. However, it turns out that it is not necessary. In particular, we find that Tsirelson's bound is not violated if there is no loss on one side and moderate loss on the other side. Let $M_a = M_A = M_b = \mathbb{1}$, and let

$$M_B = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}. \quad (49)$$

We have performed numerical maximizations over the measurements and states for a range of values of p, and the results are shown in Fig. 1. For small values of p, below about 0.17, there is violation of Tsirelson's bound, but for larger values no violation of Tsirelson's bound is achieved. This indicates that there is not a simple necessary and sufficient condition in the case of Tsirelson's bound. Whether Tsirelson's bound can be violated depends on the particular value of the loss.

An interesting fact is that if the condition (31) is violated for both parties, in the sense that $M_{k,\gamma_k}/\mathcal{E}_{kS}(\gamma_k)$ is dependent on γ_k for both Alice and Bob, then it is always possible to find measurements such that Tsirelson's bound is violated. A scheme for doing this is presented in Appendix B.

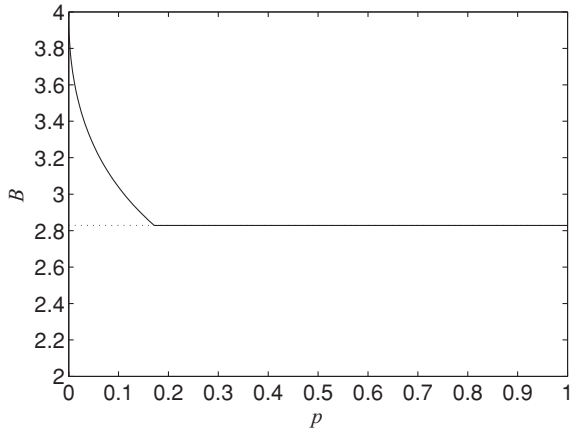

FIG. 1. The numerically found maximal values of $\mathcal{B}$ with $M_a = M_A = M_b = \mathbb{1}$ and $M_B = \mathrm{diag}(1, p)$. Tsirelson's bound is shown as the dotted line for comparison.

VI. POSTSELECTION THAT VIOLATES TSIRELSON'S BOUND

One motivation for this work was to determine forms of loss that can violate Tsirelson's bound while ensuring that the CHSH-Bell inequality is still valid, in order to generalize Ref. [28]. The result of the previous section is that the form of postselection that is necessary for the CHSH-Bell inequality to hold for unentangled states (as shown in Theorem 4) also implies that Tsirelson's bound holds for entangled states (as follows from Theorem 3).

This raises the question of how the postselection in Ref. [28] differs from the postselection used here. The difference is that the postselection in Ref. [28] relies on knowledge of the hidden variables. That is, the postselection in Ref. [28] would not actually be performed in an experiment, it is a method of deriving three-party inequalities that can be experimentally tested, as in Ref. [29]. If one attempted to perform the postselection without knowledge of the hidden variables or state, then one could obtain a violation of the Bell inequality with an unentangled state (see Appendix C). We do not use that approach here and simply allow postselection that may depend on the state and measurement setting.

A crucial subtlety in our results is that the condition (31) is necessary for the CHSH-Bell inequality to be satisfied for unentangled states, provided we consider arbitrary measurement schemes for the given form of loss. This does not eliminate the possibility that there are particular measurement schemes with postselection such that the Bell inequality is satisfied for unentangled states but Tsirelson's bound may be violated for entangled states. We consider such a scheme in this section.

We emphasize that this scheme does not violate the Bell inequality with unentangled states *provided* the measurements are acting as expected. But, if there is an underlying hidden variable theory, or equivalently if the measurements are not acting as expected on the underlying quantum state, then the CHSH-Bell inequality can be violated without entanglement.

The advantage of this postselection is that it also increases the Bell quantity that may be obtained with entanglement. The standard result that the efficiency must be at least $2(\sqrt{2}-1) \approx 82.8\%$ with maximally entangled states [18] is based on the assumption that Tsirelson's bound still holds for entangled states. If the Bell quantity is also enhanced for entangled states, then the efficiency may be lower before the value that is possible with a LHV theory reaches that possible with entangled states.

The example we consider is where Alice and Bob share the two-qubit entangled state

$$|\psi_-\rangle_{12} = \mathcal{N}(|u\rangle_1|u\rangle_2 - |v\rangle_1|v\rangle_2), \quad (50)$$

where $\mathcal{N}$ is the normalization factor. The states $|u\rangle$ and $|v\rangle$ are assumed to be nonorthogonal, with the real inner product κ. We define a local transformation $R(\theta)$ which acts as $R(\theta)|u\rangle = |u\rangle$ and $R(\theta)|v\rangle = e^{i\theta}|v\rangle$.

The transformation $R(\theta)$ is nonunitary and cannot be realized deterministically with a nonzero κ. Alice performs

the local operation $R(\theta)$, and Bob applies $R(\phi)$. The different measurement settings for Alice and Bob are achieved by using different rotation angles θ and ϕ. This nondeterministically transforms the state $|\psi_-\rangle_{12}$ to

$$|\psi'_-\rangle_{12} = \mathcal{N}'(|u\rangle_1|u\rangle_2 - e^{i(\theta+\phi)}|v\rangle_1|v\rangle_2). \tag{51}$$

Alice and Bob then perform orthogonal measurements on their respective qubits using the basis $|\pm\rangle = \mathcal{N}_\pm(|u\rangle \pm |v\rangle)$, and there are four possible cases of the combined measurement results. The probabilities for these four cases are

$$p(+,+|\theta,\phi) = \frac{(1+\kappa)^2[1-\cos(\theta+\phi)]}{4[1-\kappa^2\cos(\theta+\phi)]},$$
$$p(+,-|\theta,\phi) = p(-,+|\theta,\phi) = \frac{(1-\kappa^2)[1+\cos(\theta+\phi)]}{4[1-\kappa^2\cos(\theta+\phi)]},$$
$$p(-,-|\theta,\phi) = \frac{(1-\kappa)^2[1-\cos(\theta+\phi)]}{4[1-\kappa^2\cos(\theta+\phi)]}, \tag{52}$$

where $p(s_1,s_2|\theta,\phi) = |\langle\psi'_-|s_1\rangle_1|s_2\rangle_2|^2$.

The Bell function $\mathcal{B}$ is then obtained as

$$\mathcal{B} = C(\theta_A,\phi_B) + C(\theta_A,\phi_b) + C(\theta_a,\phi_B) - C(\theta_a,\phi_b), \tag{53}$$

where

$$\begin{aligned} C(\theta,\phi) &= p(+,+|\theta,\phi) - p(+,-|\theta,\phi) \\ &\quad - p(-,+|\theta,\phi) + p(-,-|\theta,\phi) \\ &= \frac{\kappa^2 - \cos(\theta+\phi)}{1-\kappa^2\cos(\theta+\phi)}. \end{aligned} \tag{54}$$

The maximum is obtained for

$$(\theta_A + \theta_a + \phi_B + \phi_b) = \theta_a - \theta_A = \phi_b - \phi_B = \Theta, \tag{55}$$

for some parameter Θ. The Bell function then depends only on Θ:

$$\mathcal{B} = 3C(\Theta/2,0) - C(3\Theta/2,0). \tag{56}$$

There does not appear to be an analytic solution for Θ, but a good approximation is given by $\Theta = \pi[17+\cos(\pi\kappa)]/12$.

To determine the loss that is required for the operation $R(\theta)$ on $|u\rangle$ and $|v\rangle$, consider the singular value decomposition [36] of the matrix representing $R(\theta)$, i.e., $U_+ W U_-^\dagger$ (omitting a global phase) where

$$U_\pm = \frac{1}{\sqrt{b^2+(r\pm d)^2}}\begin{bmatrix} -ib & r\pm d \\ -(r\pm d) & ib \end{bmatrix}, \tag{57}$$

$$W = \begin{bmatrix} \sqrt{\frac{a-r}{2}} & 0, \\ 0 & \sqrt{\frac{a+r}{2}} \end{bmatrix}, \tag{58}$$

with $b = \sin\theta[\sqrt{(1+\kappa)/(1-\kappa)} - \sqrt{(1-\kappa)/(1+\kappa)}]$, $d = 4\kappa\sin^2(\theta/2)/(1-\kappa^2)$, $a = 2+\kappa d$, and $r = \sqrt{a^2-4}$. This therefore gives the minimum detector efficiency as

$$\eta = \sqrt{\frac{a-r}{a+r}}. \tag{59}$$

The maximum value of $\mathcal{B}$ possible with LHV theories is then $4/\eta - 2$ [18].

The values of the quantity $\mathcal{B}$ for the initial state $|\psi_-\rangle_{12}$, as well as the maximum values for separable states, and the maximum values for LHV theories, are plotted in Fig. 2. The maximum value for separable states was obtained by numerical optimization, and the maximum value for LHV theories was obtained using the formula $4/\eta - 2$. For $|\psi_-\rangle_{12}$, the value for entangled states varies from Tsirelson's bound for $\kappa = 0$ to a maximum of 4 in the limit $\kappa \to 1$. On the other hand, $\mathcal{B}$ initially has a maximum of $\sqrt{2}$ for separable states. It increases with κ, but still does not exceed 2 for $\kappa \lesssim 0.357$. For this value of κ, $\mathcal{B}$ is almost 3 (2.966) for $|\psi_-\rangle_{12}$. On the other hand, the maximum value of $\mathcal{B}$ for LHV theories increases from 2 for $\kappa = 0$ (no loss) and crosses the line for entangled states at $\kappa \approx 0.124$. At this value of κ, $\eta \approx 82.6\%$, slightly below the limit of 82.8% derived in Ref. [18].

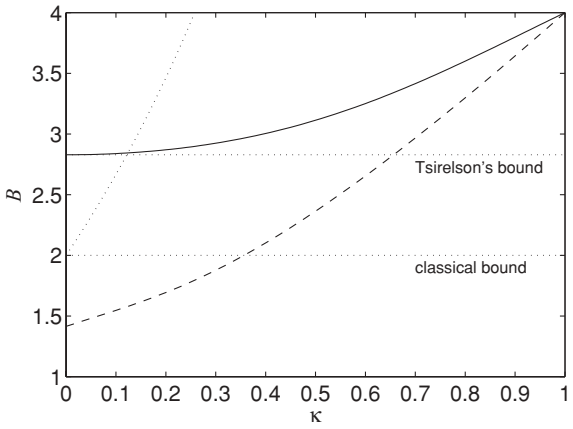

FIG. 2. The CHSH-Bell quantity $\mathcal{B}$ using postselection. The value for $|\psi_-\rangle_{12}$ is shown as the solid line, the maximum value for separable states is shown as the dashed line, and the maximum value for LHV theories is shown as the dotted curve.

We have also considered more general measurements that do not have the simple interpretation given above. These measurements are numerically optimized to maximize the value of $\mathcal{B}$ for entangled states with a given level of loss. The results for this scheme are shown in Fig. 3. For this numerically optimized scheme the value of $\mathcal{B}$ can be greater than 3 (3.0046) for entangled states before it reaches 2 for separable states. The

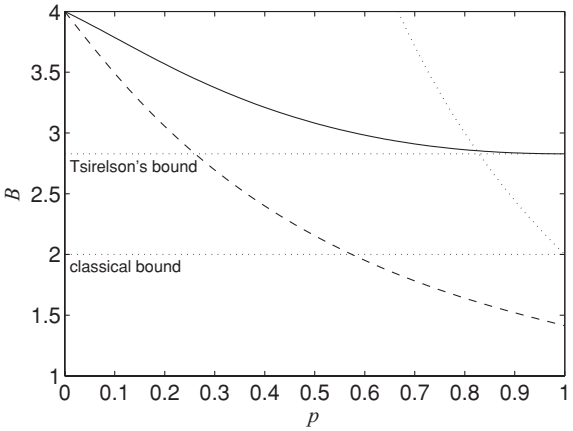

FIG. 3. The CHSH-Bell quantity $\mathcal{B}$ using postselection with a numerically optimized scheme. The value for the maximally entangled state is shown as the solid line, the maximum value for separable states is shown as the dashed line, and the maximum value for LHV theories is shown as the dotted curve.

value of η when the value of $\mathcal{B}$ for LHV theories reaches that for entangled states is slightly lower than before, at 82.4%.

Methods for general nonunitary transformations such as $R(\theta)$ have been presented in Ref. [37], and methods for performing single-qubit nonunitary transformations in optical systems in Ref. [38]. An experiment for our proposal can be performed using current technology with the photon-polarization qubit basis, namely $|H\rangle$ and $|V\rangle$, where $|H\rangle$ is the horizontal-polarization state of a photon and $|V\rangle$ vertical. The two nonorthogonal states $|u\rangle$ and $|v\rangle$ can be defined as $|u\rangle = \lambda|H\rangle + \sqrt{1-\lambda^2}|V\rangle$ and $|v\rangle = \lambda|H\rangle - \sqrt{1-\lambda^2}|V\rangle$, where λ is assumed to be a real value larger than $1/\sqrt{2}$. In this case, the orthogonal basis states, $|+\rangle = |H\rangle$ and $|-\rangle = |V\rangle$, are obtained and $\kappa = \langle u|v\rangle = 2\lambda^2 - 1$. The entangled state $|\psi_-\rangle_{12}$ turns out to be equal to $\mathcal{N}(|u\rangle_1|u\rangle_2 - |v\rangle_1|v\rangle_2) = (|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2)/\sqrt{2}$. It is well known that such a two-qubit entangled state can be generated using spontaneous parametric down-conversion [39].

In the singular value decomposition $R(\theta) = U_+ W U_-^\dagger$, the $U_\pm$ are unitary and W is diagonal. The unitary operations correspond to single qubit rotations and can be implemented using linear optical elements [40]. After $U_-^\dagger$ is performed on the corresponding qubit, the nonunitary component W can be achieved using selective absorption at one polarization. Provided the photon is detected in the final measurement, W will have been successfully applied to the qubit. Then U_+ is applied as discussed above. The final measurement is then just a measurement in the polarization basis, which can be achieved using a polarizing beam splitter and photodetectors.

VII. CONCLUSIONS

Our results provide clear guidelines for determining if the sampling is fair in any particular Bell experiment. If the goal is simply to demonstrate entanglement, rather than disprove LHV theories, then the condition required is given by Eq. (31), which can be experimentally tested. If it can be demonstrated to hold, then any violation of Bell inequalities is due to entanglement. This is relevant to quantum key distribution (QKD), where security may be proven by violation of Bell inequalities [41–45]. In cases where there is loss, but the sampling is tested and found to be fair, then this indicates that the QKD should still be secure. On the other hand, unfair sampling can break the security of QKD [46].

In the case where hidden variables are allowed, then the relevant condition is that the efficiency factorizes as in Eq. (7). Any condition that depends on the hidden variables can not be proven to hold, because it is possible that it might be violated for values of the hidden variable that it is not possible to prepare. However, it is possible to falsify it. The great advantage of providing a *necessary* condition, as we have done, is that if it can be shown not to hold, then the sampling is shown to be of a form that invalidates the CHSH-Bell inequality. In contrast, if the condition that is tested is *not* necessary, then testing it is not useful. Showing that it does not hold does not show that the sampling is of a form that invalidates the CHSH-Bell inequality, and it cannot be conclusively shown to hold. Thus our results put testing of the sampling in Bell experiments [47] on a rigorous basis.

These results also cast light on the effort to develop ways of enhancing Bell violations using postselection, as discussed in Refs. [28,48]. Here we have found that the condition on the loss that is necessary to prevent violation of the CHSH-Bell inequality with postselection is also sufficient to ensure that Tsirelson's bound is not violated. Our results show that this form of sampling cannot be used to enhance violation of Bell inequalities beyond what is possible without postselection.

This suggests that is should not be possible to violate Tsirelson's bound with postselection while retaining a valid Bell inequality. On the other hand, it is possible for postselection to enhance the violation of Bell inequalities up to Tsirelson's bound without invalidating them. For example, Procrustean entanglement concentration [34] has been demonstrated to enhance the violation of Bell inequalities [35]. Because Procrustean entanglement concentration corresponds to fair sampling, this postselection does not invalidate the Bell inequalities for unentangled states and does not lead to violation of Tsirelson's bound for entangled states.

Our results do not entirely rule out the possibility of violating Tsirelson's bound while the CHSH-Bell inequality still holds for unentangled states. However, this requires examining the measurement used rather than simply relying on the form of loss. We have found schemes that allow the Bell quantity to be as large as 3 before the CHSH-Bell inequality is violated with separable states. These schemes also allow the efficiency to be below the limit of 82.8% derived by Ref. [18] before the value of $\mathcal{B}$ possible with LHV theories reaches that obtained for entangled states. This is because it is assumed in Ref. [18] that the loss cannot enhance violation of the Bell inequality beyond Tsirelson's bound.

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APPENDIX A: EFFICIENCY INDEPENDENT OF THE MEASUREMENT SETTING FOR ONE STATE

As an example of a case where $|\mathcal{B}| > 2$ when the efficiency is independent of the measurement settings for the state under consideration, but Eq. (31) is violated, consider the state

$$\rho = \tfrac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|). \tag{A1}$$

For Bob we have the measurement operators

$$M_B^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_B^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{A2}$$

$$M_b^- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_b^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{A3}$$

That is, Bob either always returns the result $+$, or performs a measurement in the computational basis, and returns $+$ or $-$

for 0 or 1, respectively. Alice's measurement operators are

$$M_A^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_A^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{A4}$$

$$M_a^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_a^+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{A5}$$

That is, for measurement A Alice returns the result $+$ for $|0\rangle$ or there is failure for $|1\rangle$. For measurement a Alice returns $+$ for $|1\rangle$ or failure for $|0\rangle$. Experimentally this could correspond to a beam splitter sending two different polarizations to different photodetectors, one of which always fails.

For this example, the probability of success is always $1/2$, and we obtain $\mathcal{B} = 4$. Thus we obtain the maximum possible violation of the Bell-CHSH inequality and Tsirelson's bound with a separable state and with equal probabilities of success for each measurement setting. This demonstrates that equal probabilities of success for each measurement setting for one state does not imply that the loss preserves the CHSH-Bell inequality. It is also important to note that unequal efficiencies for detectors is sufficient to invalidate the CHSH-Bell inequality and Tsirelson's bound. This is an important case, because one would not expect detectors to be identical experimentally.

We may also use this case to provide a counterexample which demonstrates that the condition (31) does not imply (7). Let the hidden variable take the values 0 and 1, and give the probabilities of the measurement results and successes as above. However, take the quantum system to be one-dimensional, so there are no measurable features. The probability of failure for all combinations of measurement settings is equal to $1/2$, regardless of the state, so it is clear that (31) is satisfied. However, the hidden variables do not satisfy (7), and the Bell inequality is violated. In fact, for this hidden variable theory, Tsirelson's bound is violated as well, despite (31) being satisfied. This is because the proof in Sec. V assumes that probabilities are obtained using quantum measurement theory, not a hidden variable theory.

APPENDIX B: VIOLATION OF TSIRELSON'S BOUND

Here we show that Tsirelson's bound may be violated when $M_{k,\gamma_k}/\mathcal{E}_{kS}(\gamma_k)$ is dependent on γ_k for both Alice and Bob. Note that this is a slightly stronger condition than violation of (31), because (31) can be violated with dependence on just one side.

Define the unitary operators U_A and U_B to be those that diagonalize $(M_a)^{-1/2}M_A(M_a)^{-1/2}$ and $(M_b)^{-1/2}M_B(M_b)^{-1/2}$, respectively. We define the new operators

$$\begin{aligned}
\tilde{M}_A &\equiv \frac{U_A(M_a)^{-1/2}M_A(M_a)^{-1/2}U_A^\dagger}{\max[\mathrm{eig}(M_a)^{-1/2}M_A(M_a)^{-1/2}]}, \\
\tilde{M}_A^\pm &\equiv \frac{U_A(M_a)^{-1/2}M_A^\pm(M_a)^{-1/2}U_A^\dagger}{\max[\mathrm{eig}(M_a)^{-1/2}M_A(M_a)^{-1/2}]}, \\
\tilde{M}_B &\equiv \frac{U_B(M_b)^{-1/2}M_B(M_b)^{-1/2}U_B^\dagger}{\max[\mathrm{eig}(M_b)^{-1/2}M_B(M_b)^{-1/2}]}, \\
\tilde{M}_B^\pm &\equiv \frac{U_B(M_b)^{-1/2}M_B^\pm(M_b)^{-1/2}U_B^\dagger}{\max[\mathrm{eig}(M_b)^{-1/2}M_B(M_b)^{-1/2}]},
\end{aligned}$$

$$\begin{aligned}
\tilde{M}_a &\equiv U_A(M_a)^{-1/2}M_a(M_a)^{-1/2}U_A^\dagger = \mathbb{1}, \\
\tilde{M}_a^\pm &\equiv U_A(M_a)^{-1/2}M_a^\pm(M_a)^{-1/2}U_A^\dagger, \\
\tilde{M}_b &\equiv U_B(M_b)^{-1/2}M_b(M_b)^{-1/2}U_B^\dagger = \mathbb{1}, \\
\tilde{M}_b^\pm &\equiv U_B(M_b)^{-1/2}M_b^\pm(M_b)^{-1/2}U_B^\dagger.
\end{aligned} \tag{B1}$$

In addition, the system state ρ is replaced with

$$\tilde{\rho} \equiv \frac{(U_A \otimes U_B)(M_a \otimes M_b)^{1/2}\rho(M_a \otimes M_b)^{1/2}(U_A^\dagger \otimes U_B^\dagger)}{\mathrm{Tr}[(M_a \otimes M_b)\rho]}. \tag{B2}$$

The measurements in terms of these new operators on the new state $\tilde{\rho}$ must give the same normalized $C(\alpha, \beta)$ as the original operators, and therefore the same value for the CHSH-Bell quantity. In addition, the operators $\tilde{M}_A$, $\tilde{M}_a$, $\tilde{M}_B$, and $\tilde{M}_b$ are all diagonal (and $\tilde{M}_a = \mathbb{1}$ and $\tilde{M}_b = \mathbb{1}$).

The normalizations given for M_A, $M_A^\pm$, M_B, and $M_B^\pm$ ensure that these operators do not have eigenvalues larger than 1. The unitaries U_A and U_B can also be taken to be those that sort the eigenvalues in decreasing order. Now we restrict to a two-dimensional subspace where M_A and M_B are of the form

$$M_A = \begin{pmatrix} 1 & 0 \\ 0 & p_A \end{pmatrix}, \tag{B3}$$

$$M_B = \begin{pmatrix} 1 & 0 \\ 0 & p_B \end{pmatrix}. \tag{B4}$$

Consider taking the state $\tilde{\rho}$ to be a maximally entangled state

$$\begin{aligned}
\tilde{\rho} &= |\psi\rangle\langle\psi|, \\
|\psi\rangle &= -i(|00\rangle + |11\rangle)/2 + (|01\rangle + |10\rangle)/2.
\end{aligned} \tag{B5}$$

We can also take the operators $M_\alpha^\pm$ such that

$$\begin{aligned}
\Delta\tilde{M}_A &= -M_A^{1/2}ZM_A^{1/2}, \\
\Delta\tilde{M}_a &= Y, \\
\Delta\tilde{M}_B &= \frac{1}{\sqrt{2}}M_B^{1/2}(Y - Z)M_B^{1/2}, \\
\Delta\tilde{M}_b &= \frac{1}{\sqrt{2}}(Y + Z).
\end{aligned} \tag{B6}$$

Here Y and Z are the usual Pauli operators σ_y and σ_z. Using these operators we find that

$$\begin{aligned}
&\frac{\mathrm{Tr}(\Delta\tilde{M}_A\Delta\tilde{M}_B\tilde{\rho})}{\mathrm{Tr}(\tilde{M}_A\tilde{M}_B\tilde{\rho})} \\
&\quad = \frac{1 - p_A + 2(1 + p_A)\sqrt{p_B} + (p_A - 1)p_B}{\sqrt{2}(1 + p_A)(1 + p_B)} \\
&\frac{\mathrm{Tr}(\Delta\tilde{M}_A\Delta\tilde{M}_b\tilde{\rho})}{\mathrm{Tr}(\tilde{M}_A\tilde{M}_b\tilde{\rho})} = \frac{1}{\sqrt{2}} \\
&\frac{\mathrm{Tr}(\Delta\tilde{M}_a\Delta\tilde{M}_B\tilde{\rho})}{\mathrm{Tr}(\tilde{M}_a\tilde{M}_B\tilde{\rho})} = \frac{1}{\sqrt{2}} \\
&\frac{\mathrm{Tr}(\Delta\tilde{M}_a\Delta\tilde{M}_b\tilde{\rho})}{\mathrm{Tr}(\tilde{M}_a\tilde{M}_b\tilde{\rho})} = -\frac{1}{\sqrt{2}}.
\end{aligned} \tag{B7}$$

The Bell quantity is then

$$\mathcal{B} = 2\sqrt{2} + \frac{\sqrt{2}(1 - \sqrt{p_B})(\sqrt{p_B} - p_A)}{(1 + p_A)(1 + p_B)}. \tag{B8}$$

This is greater than Tsirelson's bound provided that $p_A < \sqrt{p_B} < 1$.

In the case that $p_B < \sqrt{p_A} < 1$, it is possible to obtain violation of Tsirelson's bound by exchanging the roles of Alice and Bob. That is,

$$\Delta\tilde{M}_A = \frac{1}{\sqrt{2}} M_A^{1/2}(Y - Z)M_A^{1/2},$$
$$\Delta\tilde{M}_a = \frac{1}{\sqrt{2}}(Y + Z),$$
$$\Delta\tilde{M}_B = -M_B^{1/2} Z M_B^{1/2},$$
$$\Delta\tilde{M}_b = Y, \tag{B9}$$

so

$$\mathcal{B} = 2\sqrt{2} + \frac{\sqrt{2}(1 - \sqrt{p_A})(\sqrt{p_A} - p_B)}{(1 + p_A)(1 + p_B)}. \tag{B10}$$

Hence, provided $p_A \neq 1$ and $p_B \neq 1$ (so $M_{k,\gamma_k}/\mathcal{E}_{k\mathrm{S}}(\gamma_k)$ is dependent on γ_k for both Alice and Bob), it is possible to violate Tsirelson's bound. If one of p_A or p_B is equal to 1, then this approach does not yield violation of Tsirelson's bound.

APPENDIX C: POSTSELECTION OF REF. [28]

Here we show that, if the postselection in Ref. [28] is used without knowledge of the state or hidden variables, then it allows violation of the CHSH-Bell inequality with unentangled states. Reference [28] considers the Greenberger-Horne-Zeilinger (GHZ) state $|\Psi\rangle = (|+++\rangle_y + |---\rangle_y)/\sqrt{2}$, where $|\pm\rangle_y$ are Y eigenstates. The qubits are labeled i, j, and k according to the following prescription. When Z is measured on each qubit, either two results are -1 and one is $+1$ or all three are $+1$. If two results are -1, the corresponding qubits are taken to be i and j. If all three results are $+1$, the labels i, j, and k are assigned randomly.

In practice, Z is not always measured on all three qubits, so it is not possible to apply this method directly. Instead, one could force qubit 3 to be k by measuring Z on it and postselecting on a measurement result of 1 (as proposed by Ref. [28]). Alternatively, one could randomly select a qubit and force it to be k by this method.

The form of the CHSH-Bell inequality in Ref. [28] is $|\mathcal{B}| \leqslant 2$, with

$$\mathcal{B}_x = C(Z_i, Z_j) - x_k C(Z_i, X_j) - x_k C(X_i, Z_j) - C(X_i, X_j). \tag{C1}$$

Here subscripts are used to indicate the operator acting on the corresponding qubit, and x_k is the result which would be obtained by measuring X_k. The difficulty is that X_k is not the quantity that is measured on qubit k. If the results were given by a hidden variable theory, then x_k may be something which can be determined from the hidden variables.

In practice, hidden variables are not known, so the inequality (C1) is not something which can be tested directly. One approach suggested in Ref. [28] is to use knowledge of the state, and the measurement results on qubits i and j, to infer x_k. One takes x_k to be the negative of the product of the measurement results for qubits i and j (if X is measured on one and Z on the other). Because of this method for selecting x_k,

$$-x_k C(Z_i, X_j) = -x_k C(X_i, Z_j) = 1. \tag{C2}$$

For the GHZ state, one also obtains $C(Z_i, Z_j) = 1$ and $C(X_i, X_j) = -1$, so $\mathcal{B}_x = 4$.

On the other hand, if the state is *not* known, but is still assumed to be the GHZ state, then $\mathcal{B}_x > 2$ can be obtained with an unentangled state. Consider, for example, the state

$$\rho = \tfrac{1}{4}(|--+\rangle_z\langle--+| + |-+-\rangle_z\langle-+-| + |+--\rangle_z\langle+--| + |+++\rangle_z\langle+++|). \tag{C3}$$

It is found that, regardless of which qubit is taken to be k, the Z_i and Z_j measurements are perfectly correlated, so $C(Z_i, Z_j) = 1$. There are no correlations in the X measurements, so $C(X_i, X_j) = 0$, and Eq. (C2) still holds (because that is due to the method of determining x_k). We therefore find $\mathcal{B}_x = 3$, which violates both the Bell inequality and Tsirelson's bound.

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