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Equivalent standard DEA models to provide super-efficiency scores

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DEA super-efficiency models were introduced originally with the objective of providing a tie-breaking procedure for ranking units rated as efficient in conventional DEA models. This objective has been expanded to include sensitivity analysis, outlier identification and inter-temporal analysis. However, not all units rated as efficient in conventional DEA models have feasible solutions in DEA super-efficiency models. We propose a new super-efficiency model that (a) generates the same super-efficiency scores as conventional super-efficiency models for all units having a feasible solution under the latter, and (b) generates a feasible solution for all units not having a feasible solution under the latter. Empirical examples are provided to compare the two super-efficiency models.
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Introduction

Charnes *et al*[1] provided the original data envelopment analysis (DEA) constant returns to scale (CRS) model, later extended to variable returns to scale (VRS) by Banker *et al.*[2] These 'standard' models are known by the acronyms CCR and BCC respectively. In standard DEA models, a decision-making unit (DMU) is said to be efficient if its performance relative to other DMUs cannot be improved. In the absence of price data or preferential weightings of inputs and outputs, all efficient DMUs have equal scores of 100%, and rank equally in terms of performance. Inefficient DMUs have scores less than 100% with an input orientation (because they are capable of reducing input use), and greater than 100% with an output orientation (because they are capable of expanding output production).

The area has expanded rapidly[3] with a large number of extensions, modifications and applications of the standard DEA models. An important extension has been the creation during the past decade of 'super-efficiency' models. These deleted domain models exclude the DMU under evaluation from the reference set, which means in the case of an efficient DMU, from the efficient frontier of the production set. The effect of this is to shrink the production set, which allows efficient DMUs to become super-efficient and to have different super-efficiency scores above 100%. Among other things, this permits a ranking of efficient DMUs. Scores for inefficient DMUs remain the same as in the standard models.

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In this paper we introduce an equivalent model in which super-efficiency scores can be obtained using the standard CCR and BCC models. One advantage of our model is that it allows users to employ conventional DEA software. A second advantage is that our model is guaranteed to generate feasible solutions for all DMUs. Dulá and Hickman[4] and Seiford and Zhu[5] have proved theorems providing necessary and sufficient conditions for infeasibility in the conventional super-efficiency model. While our model does not strictly overcome the infeasibility problem in the conventional super-efficiency model, it does identify and provide a feasible solution for all super-efficient DMUs that are infeasible in the conventional super-efficiency model.

We first provide a general description of the super-efficiency model and outline uses suggested for it. Next we describe our approach and provide a mathematical proof of its equivalence with the conventional super-efficiency model. We also specify the determination of our scaling parameter and next provide three empirical examples. We conclude with a discussion of the implications of our model in super-efficiency models.

The super-efficiency model

Figure 1 provides an input-oriented illustration of the super-efficiency model. The efficient frontier consists of the line segments connecting DMUs A, B and C. If DMU B is excluded from the reference set, the effect is to construct a new frontier consisting of the broken line segment connecting DMUs A and C. The super-efficiency of DMU B becomes OB′/OB > 100%. This implies that DMU B could *increase* both inputs and still remain efficient.

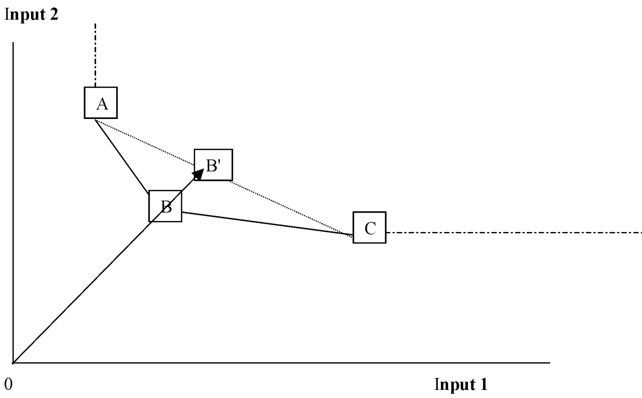

Figure 1 Evaluating the super-efficiency of DMU B.

A number of uses have been proposed for super-efficiency models. These include:

(a) Ranking of efficient DMUs;[6]
(b) Classification of DMUs into extreme-efficient and non-extreme efficient groups;[4,7]
(c) Sensitivity of efficiency classifications;[8–12]
(d) Two-person ratio efficiency games;[13]
(e) Identifying outliers in the data;[14,15]
(f) Overcoming truncation problems in second-stage regressions intended to explain variation in efficiency;[16]
(g) Calculating and decomposing a Malmquist productivity index.[17]

The formulation of the super-efficiency model is reasonably straightforward, whereby the column pertaining to the DMU being scored is excluded from the DEA envelopment linear program (LP) technology matrix. This generates super-efficiency scores for each DMU. However, under certain conditions this procedure can lead to infeasibility. A necessary, but not sufficient, condition for infeasibility is that an excluded DMU be 'extreme-efficient'. Either it has a feasible LP with super-efficiency scores strictly greater than 100%, or it has an infeasible LP.

Conditions for infeasibility in the CCR super-efficiency model appear in Dulá and Hickman,[4] Seiford and Zhu,[5] Thrall[7] and Zhu.[9] A *necessary and sufficient* condition for infeasibility in an input-oriented model is that the excluded DMU have the only zero value for any input, or the only positive value for any output, among all DMUs in the reference set. Infeasibility cannot arise in an output-oriented CCR super-efficiency model.

Conditions for infeasibility in the BCC super-efficiency model appear in Dulá and Hickman,[4] Seiford and Zhu,[5] Zhu[9] and Xue and Harker.[18] Infeasibility arises in either orientation whenever there is no referent DMU for the excluded DMU. A *necessary* condition for infeasibility is that the excluded DMU be 'extreme-efficient'. A *sufficient* condition for infeasibility is the pattern of zeros mentioned above. When all inputs and all outputs are positive for all DMUs, a *sufficient* condition for infeasibility is that the excluded DMU be 'strongly super-efficient' in the sense that (a) in an input-oriented model it has at least one output strictly larger than the corresponding output for any other DMU in the reference set, or (b) in an output-oriented model it has at least one input strictly smaller than the corresponding input for any other DMU in the reference set. A *necessary and sufficient* condition for infeasibility is that the excluded DMU be 'super-efficient' in the sense that (a) in an input-oriented model it has at least one output strictly larger than a convex combination of that output among all DMUs in the reference set, or (b) in an output-oriented model it has at least one input strictly smaller than a convex combination of that input among all DMUs in the reference set.

An introduction to the two super-efficiency models: conventional and modified

Table 1 and Figure 2 provide the intuition for our modified super-efficiency model, assuming an output orientation. Table 1 lists data for three DMUs, all of which produce efficient combinations of outputs 1 and 2 with the same level of input. The fourth row, DMU B**, represents DMU B with its outputs scaled down by a factor of 10. The effect of this scaling can be seen in Figure 2, where scaled DMU B** lies one-tenth of the distance along a ray extending from the origin to DMU B. The conventional super-efficiency model excludes DMU B from the reference set, changing the efficient frontier to the broken line segment connecting DMUs A and C, and evaluates DMU B against this reduced frontier. In this example DMU B receives a super-efficiency score of 122%.

Our modified super-efficiency model scales DMU B to DMU B** and retains it as part of the reference set. However, as it is now inefficient, it is no longer part of the efficient frontier connecting DMUs A and C. Thus the conventional super-efficiency frontier and our modified super-efficiency frontier coincide. The difference is that scaled DMU B** is inefficient. The radial efficiency of scaled DMU B** is 12.2%, which when rescaled produces the same super-efficiency score of 122%.

An equivalent super-efficiency model

Define notation for outputs $(y_1, y_2, \ldots, y_s)$ and inputs $(x_1, x_2, \ldots, x_m)$ for DMUs $j = 1, 2, \ldots, n$. $\mathbf{Y}$ is an $s \times (n-1)$ matrix of outputs, $\mathbf{X}$ is an $m \times (n-1)$ matrix of inputs, and λ is an $(n-1)$-dimensional vector of intensity

Table 1 Data for Figure 2

DMU	*Output 1*	*Output 2*
A	3.2	8.5
B	7	7
C	8	3.2
B**	0.7	0.7

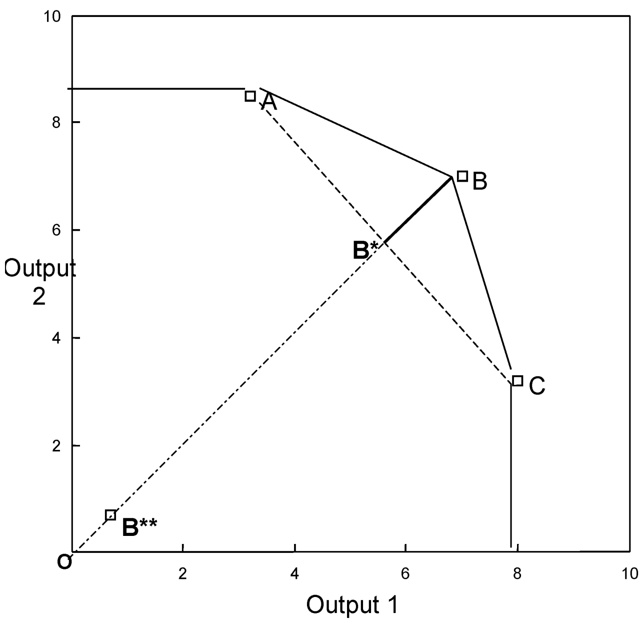

Figure 2 Radial scaling of DMU B.

variables for DMUs j, with $j \neq$ o. $\mathbf{y}_o$ and $\mathbf{x}_o$ are output and input vectors for DMU_o being evaluated; and λ_o is the intensity variable for DMU_o. We assume that inputs and outputs are non-negative with at least one input and one output positive for every DMU. We start with the input-oriented BCC model. Both the envelopment and multiplier forms are shown, but the discussion is confined to the former. (Note that the exposition is identical for the CCR model, after the removal of the convexity constraint and its dual variable).

Envelopment model

$$
\begin{aligned}
&\text{Min } \theta_o\\
&\text{s.t. } \mathbf{Y\lambda} + \mathbf{y}_o\lambda_o \geqslant \mathbf{y}_o\\
&\quad \mathbf{X\lambda} + \mathbf{x}_o\lambda_o \leqslant \mathbf{x}_o\theta_o\\
&\quad \Sigma\boldsymbol{\lambda} + \lambda_o = 1\\
&\quad \boldsymbol{\lambda}, \lambda_o \geqslant 0;\ \theta_o \text{ free}
\end{aligned}
$$

Multiplier model

$$
\begin{aligned}
&\text{Max } \boldsymbol{\mu}^T\mathbf{y}_o + w_o\\
&\text{s.t. } \boldsymbol{\mu}^T\mathbf{Y} - \boldsymbol{\nu}^T\mathbf{X} + w_o \leqslant 0\\
&\quad \boldsymbol{\mu}^T\mathbf{y}_o - \boldsymbol{\nu}^T\mathbf{x}_o + w_o \leqslant 0\\
&\quad \boldsymbol{\nu}^T\mathbf{x}_o = 1\\
&\quad \boldsymbol{\mu}, \boldsymbol{\nu} \geqslant 0, w_o \text{ free} \qquad (P_o)
\end{aligned}
$$

An optimal feasible solution for P_o is $0 \leqslant \theta_o^* \leqslant 1$. For $\theta_o^* < 1$, DMU_o is designated inefficient and, following Charnes *et al*,[19] assigned to category N; for $\theta_o^* = 1$, DMU_o is efficient, with subcategories E, E′ and F denoting extreme-efficient, efficient but not extreme-efficient, and weakly efficient, respectively. For DMUs belonging to E, $\lambda_o^* = 1$ and they are their own referents. For DMUs belonging to E′ there exists an optimal basic feasible solution such that $\lambda_o^* = 0$, implying the existence of multiple optima.[4] In other words, there is one or more $\boldsymbol{\lambda}^*$ of DMUs belonging to E that are positive (and sum to one) which are the referents for DMU_o. DMUs belonging to F have positive slack in at least one dimension in some optimal solution.

The super-efficiency modification to the standard BCC model involves excluding the column of the DMU being scored from the coefficient matrix (LHS), whilst retaining its inputs and outputs in the parameter vector (RHS). The conventional super-efficiency model P_1 excludes $\mathbf{y}_o\lambda_o$, $\mathbf{x}_o\lambda_o$ and λ_o from the coefficient matrix (LHS) and the corresponding constraint from the multiplier model.

Envelopment model

$$
\begin{aligned}
&\text{Min } \theta_1\\
&\text{s.t. } \mathbf{Y\lambda} \geqslant \mathbf{y}_o\\
&\quad \mathbf{X\lambda} \leqslant \mathbf{x}_o\theta_1\\
&\quad \Sigma\boldsymbol{\lambda} = 1\\
&\quad \boldsymbol{\lambda} \geqslant 0, \theta_1 \text{ free}
\end{aligned}
$$

Multiplier model

$$
\begin{aligned}
&\text{Max } \boldsymbol{\mu}^T\mathbf{y}_o + w_o\\
&\text{s.t. } \boldsymbol{\mu}^T\mathbf{Y} - \boldsymbol{\nu}^T\mathbf{X} + w_o \leqslant 0\\
&\quad \boldsymbol{\nu}^T\mathbf{x}_o = 1\\
&\quad \boldsymbol{\mu}, \boldsymbol{\nu} \geqslant 0, w_o \text{ free} \qquad (P_1)
\end{aligned}
$$

The question being addressed in the input-oriented model is whether the remaining DMUs can produce the outputs of DMU_o and what input values will be needed to accomplish this. As reflected in the proportional increases in inputs required to produce the outputs of DMU_o, the solution will always have $\min \theta_1 = \theta_1^* \geqslant 1$ with $\theta_1^* > 1$ indicating the input augmentations that are needed. This result can be used for ranking, with higher values of θ_1^* associated with DMUs that were most super-efficient. The case of infeasibility corresponds to a situation in which it is not possible for the remaining DMUs to attain the wanted output levels at all.

For output-oriented models, the question being addressed is the following: What is the proportion of the outputs of DMU_o that the remaining DMUs can produce without exceeding the inputs used by DMU_o? For the reduced production possibility set the solution will be max $\phi_1 = \phi_1^* \leqslant 1$. As reflected in the smaller proportions of outputs with which they are associated, smaller values of ϕ_1^* are associated with deleted DMU_os that were most super-efficient. In this, the output-oriented case, a value of $\phi_1^* = 0$ with all slacks zero yields a solution that shows that a zero amount of all of the outputs produced by DMU_o is the best that can be done while remaining within these input limitations. It is possible, of course, that the solution may have some non-zero slacks, in which case it is shown to be possible to produce some non-zero outputs but not in the proportion—ie, the output mix—of DMU_o. For a discussion of mix (as distinguishable from technical) inefficiencies and how they may be reflected in DEA measures, see Cooper *et al.*[20]

Optimal solutions for inefficient DMUs in P_o remain the same in P_1. For DMUs identified as efficient in P_o, either $\theta_1^* \geqslant 1$ in P_1, so that a DMU is 'super-efficient' in the sense that its efficiency score is bounded *below* by 100%, or no feasible solution exists in P_1. A key observation is that, for a feasible solution to P_1, at least one element of $\boldsymbol{\lambda}$ must be strictly positive. However a feasible solution is not always obtainable, for the reasons outlined above, which in essence result from the exclusion of the column belonging to the DMU being evaluated.

Our proposed modification appears as model P_2 below. Inputs for each DMU that was efficient in P_o are multiplied by a scalar $\alpha > 1$ sufficiently large to make it inefficient in P_2, with $\theta_2^* < 1$. (Inputs for inefficient DMUs can be scaled in a similar fashion. The results after adjustment are the same as those obtained from P_o.)

Envelopment model

$$\begin{aligned} &\text{Min } \theta_2 \\ &\text{s.t. } \mathbf{Y}\boldsymbol{\rho} + \mathbf{y}_o\rho_o \geqslant \mathbf{y}_o \\ &\quad \mathbf{X}\boldsymbol{\rho} + \alpha\mathbf{x}_o\rho_o \leqslant \alpha\mathbf{x}_o\theta_2 \\ &\quad \Sigma\boldsymbol{\rho} + \rho_o = 1 \\ &\quad \boldsymbol{\rho}, \rho_o \geqslant 0, \theta_2 \text{ free} \end{aligned}$$

Multiplier model

$$\begin{aligned} &\text{Max } \boldsymbol{\mu}^T\mathbf{y}_o + w_o \\ &\text{s.t. } \boldsymbol{\mu}^T\mathbf{Y} - \boldsymbol{\nu}^T\mathbf{X} + w_o \leqslant 0 \\ &\quad \boldsymbol{\mu}^T\mathbf{y}_o - \boldsymbol{\nu}^T\alpha\mathbf{x}_o + w_o \leqslant 0 \\ &\quad \mathbf{v}^T\alpha\mathbf{x}_o = 1 \\ &\quad \boldsymbol{\mu}, \boldsymbol{\nu} \geqslant 0, w_o \text{ free} \end{aligned} \qquad (P_2)$$

Note that α must be sufficiently large to ensure that $\theta_2^* < 1$ and $\rho_o^* = 0$. This is not guaranteed if DMU_o is extreme-efficient, as Figure 3 (adapted from Seiford and Zhu[5]) illustrates. DMUs A, B and C are all extreme-efficient. Under an input orientation DMUs A and B can expand input (shown by the horizontal broken arrows) only up to the input levels of B and C respectively and remain radially efficient. Thus infeasibility is not a problem for DMUs A and B. DMU C, however, can expand input *ad infinitum* and remain radially efficient. Thus under an input orientation, infeasibility occurs only for DMU C.

In this case, regardless of how large we make α, DMU C remains radially efficient and there exists no feasible solution under P_1. Note however that P_2 remains feasible for DMU C, with solution ($\rho_0^* = 1, \theta_2^* = 1$). Accordingly, if

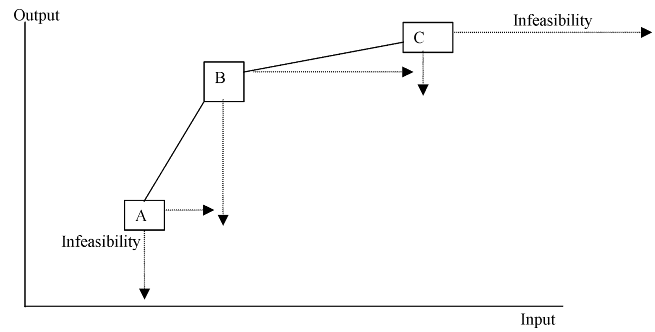

Figure 3 VRS frontier for super-efficiency models

$\alpha\theta_2^* = \alpha \Rightarrow \theta_2^* = 1$, then either α is not large enough, or DMU_o is infeasible (and super-efficient). We consider the specification of α below.

We now prove that the solution to P_2 is equivalent to the solution to P_1 for all DMUs having a feasible solution to P_1. We require only that we can choose a value for α sufficiently large to make DMU_o inefficient in P_2. We also prove that, for super-efficient DMUs not having a feasible solution to P_1, P_2 nonetheless returns a feasible solution.

Theorem 1 If a feasible solution to P_1 exists for DMU_o, then P_1 and P_2 have the same optimal solutions for DMU_o, with $\alpha\theta_2^* = \theta_1^*$.

Proof

(1) Suppose $(\theta_2^*, \boldsymbol{\rho}^*, 0)$ solves P_2. Then the constraints to P_2 become

$$\begin{aligned} &\mathbf{Y}\boldsymbol{\rho}^* \geqslant \mathbf{y}_o \\ &\mathbf{X}\boldsymbol{\rho}^* \leqslant \alpha\mathbf{x}_o\theta_2^* \\ &\Sigma\boldsymbol{\rho}^* = 1 \\ \Rightarrow &(\boldsymbol{\rho}^*, \alpha\theta_2^*) \text{ is feasible for } P_1 \Rightarrow \alpha\theta_2^* \geqslant \theta_1^*. \end{aligned}$$

(2) Suppose $(\theta_1^*, \boldsymbol{\lambda}^*)$ solves P_1. Then for $\alpha > 0$ the constraints to P_1 can be written as

$$\begin{aligned} &\mathbf{Y}\boldsymbol{\lambda}^* + \mathbf{y}_o 0 \geqslant \mathbf{y}_o \\ &\mathbf{X}\boldsymbol{\lambda}^* + \alpha\mathbf{x}_o 0 \leqslant \alpha\mathbf{x}_o\theta_1^*/\alpha \\ &\Sigma\boldsymbol{\lambda}^* + 0 = 1 \end{aligned}$$

$\Rightarrow (\boldsymbol{\lambda}^*, 0, \theta_1^*/\alpha)$ is feasible for $P_2 \Rightarrow \theta_1^*/\alpha \geqslant \theta_2^* \Rightarrow \alpha\theta_2^* \leqslant \theta_1^*$.

Combining (1) and (2) yields $\alpha\theta_2^* = \theta_1^*$.

□

Theorem 2 A feasible solution exists for all DMUs to problem P_2.

Proof We consider categories (1) N (inefficient), (2) E (extreme efficient), (3) $E' \cup F$ (efficient but not extreme efficient or weak efficient).

(1) If $DMU_o \in N$ in P_2, then $\rho_o = 0$ and $(\theta_2^*, \boldsymbol{\rho}^*, 0)$ is feasible for P_2.

(2) If $DMU_o \in E$ in P_2, then $\rho_o = 1$ and $(\theta_2^*, \mathbf{0}, 1)$ is feasible for P_2.

$$\mathbf{Y_o 0} + \mathbf{y}_o\rho_o \geqslant \mathbf{y}_o$$
$$\mathbf{X_o 0} + \alpha\mathbf{x}_o\rho_o \leqslant \alpha\mathbf{x}_o\theta_2$$
$$\Sigma\mathbf{0} + \rho_o = 1$$
$$\boldsymbol{\rho}, \rho_o \geqslant 0, \theta_2 \text{ free}$$

All constraints are satisfied and a feasible solution is obtained.

(3) If $DMU_o \in E' \cup F$ in P_2, then there exists an optimal basic feasible solution with $\Sigma\boldsymbol{\rho}^* = 1$ and $\rho_o = 0$.

$$\mathbf{Y}\boldsymbol{\rho} + \mathbf{y}_o 0 \geqslant \mathbf{y}_o$$
$$\mathbf{X}\boldsymbol{\rho} + \alpha\mathbf{x}_o 0 \leqslant \alpha\mathbf{x}_o\theta_2$$
$$\Sigma\boldsymbol{\rho} + 0 = 1$$
$$\boldsymbol{\rho}, \rho_o \geqslant 0, \theta_2 \text{ free}$$

All constraints are satisfied and a feasible solution is obtained. □

Output orientation

Infeasibility under an output orientation is also illustrated in Figure 3. In this orientation, the infeasibility problem does not occur for DMUs B and C. However super-efficient DMU A can reduce output toward zero and still remain efficient, and so no feasible solution exists to problem P_1 for DMU A. This emphasises a key point: super-efficiency, which is necessary and sufficient for infeasibility, is conditional on model orientation.

The output-oriented models corresponding to P_1 and P_2 are as follows. In the model corresponding to P_2, the outputs of each DMU that was efficient in P_o are multiplied by a scalar $0 < \beta < 1$ sufficiently small to make it inefficient in P_2, with $\varphi_2^{*-1} < 1$. (Note that the exposition is identical for the CCR model taking into consideration the removal of the convexity constraint and its dual variable.)

Envelopment model	**Multiplier model**
Max φ_1	Min $\boldsymbol{\nu}^T\mathbf{x}_o - v_o$
s.t. $\mathbf{Y}\boldsymbol{\lambda} \geqslant \mathbf{y}_o\varphi_1$	s.t. $\boldsymbol{\nu}^T\mathbf{X} - \boldsymbol{\mu}^T\mathbf{Y} - v_o \geqslant 0$
$\mathbf{X}\boldsymbol{\lambda} \leqslant \mathbf{x}_o$	$\boldsymbol{\mu}^T\mathbf{y}_o = 1$
$\Sigma\boldsymbol{\lambda} = 1$	$\boldsymbol{\mu}, \boldsymbol{\nu} \geqslant 0, v_o$ free
$\boldsymbol{\lambda} \geqslant 0, \varphi_1$ free	

(OP_1)

Envelopment model	**Multiplier model**
Max φ_2	Min $\boldsymbol{\nu}^T\mathbf{x}_o - v_o$
s.t. $\mathbf{Y}\boldsymbol{\rho} + \beta\mathbf{y}_o\rho_o \geqslant \beta\mathbf{y}_o\varphi_2$	s.t. $\boldsymbol{\nu}^T\mathbf{X} - \boldsymbol{\mu}^T\mathbf{Y} - v_o \geqslant 0$
$\mathbf{X}\boldsymbol{\rho} + \mathbf{x}_o\rho_o \leqslant \mathbf{x}_o$	$\boldsymbol{\nu}^T\mathbf{x}_o - \boldsymbol{\mu}^T\beta\mathbf{y}_o - v_o \geqslant 0$
$\Sigma\boldsymbol{\rho} + \rho_o = 1$	$\boldsymbol{\mu}^T\beta\mathbf{y}_o = 1$
$\boldsymbol{\rho}, \rho_o \geqslant 0, \varphi_2$ free	$\boldsymbol{\mu}, \boldsymbol{\nu} \geqslant 0, v_o$ free

(OP_2)

An analogous pair of theorems holds for the output-oriented BCC models. A DMU identified as super-efficient in OP_1 has $\varphi_1^{-1} > 1$ or no feasible solution exists in OP_1. In OP_2 its outputs are multiplied by a scalar $\beta < 1$ sufficiently small to make it inefficient with $\varphi_2^{-1} < 1$. The super-efficient score is obtained from $\varphi_2^*\beta = \varphi_1$. The proof is identical to the input-orientated models above with minor adjustments (a copy is available from the authors).

Similar theorems also apply to the input-oriented CCR model (infeasibility does not occur in the output-oriented CCR model). In the case where Zhu's[9] pattern of zeros is present in the data, a feasible solution is always obtainable by setting $\boldsymbol{\rho} = 0$ and $\rho_o = 1$.

Slacks may occur at optimal solutions to models P_2 and OP_2. The part of the frontier to which a super-efficient DMU is projected may be weak-efficient (eg, DMU C in Figure 3 with an output orientation) under both the original Andersen and Petersen[6] model and the equivalent models described in this paper. This may not be an issue for many of the uses listed at the beginning of this paper that use only the radial efficiency measure. Nonetheless, both methods provide the same information on slacks (the proof follows from Theorem 1): $(\mathbf{Y}\boldsymbol{\rho}^* - \mathbf{s}, \mathbf{X}\boldsymbol{\rho}^* + \mathbf{s})$ where $\boldsymbol{\rho}$ does not include DMU_o. If strong efficiency is required, then there are a number ways in which slacks can be incorporated to adjust the radial efficiency score.[21]

Specification of the scaling parameter

In this section we specify a scaling parameter α (or β) that is sufficient for model P_2 (or OP_2). The intuition for our approach, using an input orientation, for example, is that we scale the inputs of the DMU under evaluation so that its inputs exceed those for all producers in the reference set. It should be apparent that setting the scaling parameter to slightly exceed the maximum ratio of the highest to lowest inputs will achieve this.

Input orientation

For each input $i = 1, \ldots, m$, and for $j = 1, \ldots, n$ DMUs, select min $x_{ij} > 0$ to discard any zero values, and calculate $\alpha_i = \max x_{ij}/\min x_{ij}$ and set $\alpha = \max(\alpha_1, \ldots, \alpha_m) + 1$. If a super-efficient DMU remains efficient after scaling its inputs by α, then it belongs to the super-efficient category of DMUs (SE) identified as infeasible by Xue and Harker;[18] ie, at least one output belonging to DMU_o is strictly larger than a convex combination of that output among all other DMUs in the reference set.

Theorem 3 For an input orientation, α is a sufficient scalar for x_o s.t. $DMU_o \in N \cup SE$.

Proof

(1) If $DMU_o \in N$ then $\lambda_o = 0$ and

$$\mathbf{Y}\boldsymbol{\lambda} + \mathbf{y}_o(0) \geqslant \mathbf{y}_o$$
$$\mathbf{X}\boldsymbol{\lambda} + \alpha\mathbf{x}_o(0) \leqslant \alpha\mathbf{x}_o\theta$$
$$\Sigma\boldsymbol{\lambda} + (0) = 1$$

Since $\mathbf{X\lambda} < \alpha \mathbf{x}_o$ by construction $[\alpha = \max\{\max x_{ij}/\min x_{ij}\} + 1]$ then $\theta > 1$.

(2) If $DMU_o \in SE$ then $\mathbf{Y\lambda} < \mathbf{y}_o$ for at least one output and $\lambda_o = 1$ is the only feasible solution. We prove this by contradiction. Suppose there is a scalar $\gamma_x > \alpha$ that will make DMU_o inefficient. Then for $DMU_o \in N$, λ_o must equal zero and

$$\mathbf{Y\lambda} + \mathbf{y}_o(0) \geqslant \mathbf{y}_o$$
$$\mathbf{X\lambda} + \gamma_x \mathbf{x}_o(0) \leqslant \gamma_x \mathbf{x}_o \theta$$
$$\Sigma\boldsymbol{\lambda} + (0) = 1$$

But $\mathbf{Y\lambda} < \mathbf{y}_o$ for at least one output and no feasible solution is obtainable. Hence $\lambda_o = 1$ for $\gamma_x \to \infty$.

Combining (1) and (2) α is a sufficient scalar for x_o s.t. $DMU_o \in N \cup SE$ □

Output orientation

For each output $r = 1, \ldots, s$ and for $j = 1, \ldots, n$ DMUs, select min $y_{rj} > 0$ to remove any zero values and calculate $\beta_r = (\max y_{rj}/\min y_{rj}) + 1$ and set $\beta = \{\max(\beta_1, \ldots, \beta_s)\}^{-1}$.

Theorem 4 For an output orientation, β is a sufficient scalar for $\mathbf{y}_o$ s.t. $DMU_o \in N \cup SE$

Proof

(1) If $DMU_o \in N$ then $\lambda_o = 0$ and

$$\mathbf{Y\lambda} + \beta \mathbf{y}_o(0) \geqslant \beta \mathbf{y}_o \phi$$
$$\mathbf{X\lambda} + \mathbf{x}_o(0) \leqslant \mathbf{x}_o$$
$$\Sigma\boldsymbol{\lambda} + (0) = 1$$

Since $\mathbf{Y\lambda} > \beta y_o$ by construction $[\beta = \{\max(\max y_{ij}/\min y_{ij}) + 1\}^{-1}]$ then $\phi > 1$.

(2) If $DMU_o \in SE$ then $X\lambda > x_o$ for at least one input and $\lambda_o = 1$ is the only feasible solution. We prove this by contradiction. Suppose there is a scalar $\gamma_y < \beta$ that will make DMU_o inefficient. Then for $DMU_o \in N$, λ_o must equal zero and

$$\mathbf{Y\lambda} + \gamma_y \mathbf{y}_o(0) \geqslant \gamma_y \mathbf{y}_o \phi$$
$$\mathbf{X\lambda} + \mathbf{x}_o(0) \leqslant \mathbf{x}_o$$
$$\Sigma\boldsymbol{\lambda} + (0) = 1$$

But $\mathbf{X\lambda} > \mathbf{x}_o$ for at least one input and no feasible solution is obtainable. Hence $\lambda_o = 1$ for $\lambda_y \to \epsilon > 0$.

Combining (1) and (2), β is a sufficient scalar for $\mathbf{y}_o$ s.t. $DMU_o \in N \cup SE$ □

Examples in which infeasibility arises

We now show how our modified super-efficiency model P_2 copes with some of the infeasibility examples provided by Dulá and Hickman[4] and Seiford and Zhu;[11] data for these examples are provided in the Appendix. We have used the Warwick DEA software package,[22] which contains the super-efficiency model P_1, to act as the comparison for our own results using our modified super-efficiency model P_2. We have confirmed the results using ordinary LP software.

Our first example uses data from Table 1 of Dulá and Hickman,[4] which the original authors used to illustrate the infeasibility of DMU 8 in P_1. DMUs 6 and 7 are also infeasible in P_1. In accordance with Theorem 1, the modified super-efficiency model P_2 produces the same results as the super-efficiency model P_1 for DMUs 1 through 5 (Table 2). For DMUs 6, 7 and 8 Warwick reports infeasible solutions with an input orientation, and flags them with '999' with an output orientation (Table 3). The modified super-efficiency model P_2 provides all three DMUs with an efficiency score for an input orientation (1000%) $[\alpha = 10]$ and 200% for an output orientation $[\beta = 2]$.

Our second example is also taken from Dulá and Hickman,[4] using data from their Table 2. Whereas results under a VRS input orientation are infeasible for DMUs 2, 3 and 4 when the super-efficiency model P_1 is used, our modified super-efficiency model P_2 still produces feasible solutions (200% with $\alpha = 2$) (Tables 4 and 5). For an output orientation ($\beta = 7$), infeasibility does not occur for any DMU, and in accordance with Theorem 1 the two methods produce identical results.

Our third example uses data set 3 from Seiford and Zhu.[11] Again for DMUs with feasible solutions under P_1, results are identical under both super-efficiency models P_1 and P_2,

Table 2 Modified super-efficiency model (P_2)

DMU	*CRS input* (%)	*CRS output* (%)
D01	40.37	40.37
D02	150.00	150.00
D03	150.00	150.00
D04	100.00	100.00
D05	33.33	33.33
D06	1000.00	200.00
D07	1000.00	200.00
D08	1000.00	200.00

Table 3 Warwick DEA software (P_1)

DMU	*CRS input* (%)	*CRS output* (%)
D01	40.37	40.37
D02	150.00	150.00
D03	150.00	150.00
D04	100.00	100.00
D05	33.33	33.33
D06		999
D07		999
D08		999

Table 4 Modified super-efficiency model (P_2)

DMU	*Model P_2: VRS input orientation* (%)	*Model P_2: VRS output orientation* (%)
D01	100.00	58.82
D02	200.00	120.00
D03	200.00	133.33
D04	200.00	130.43

Table 5 Warwick DEA software (P_1)

DMU	*Model P_1:VRS input orientation* (%)	*Model P_1: VRS output orientation* (%)
D01	100.00	58.82
D02		120.00
D03		133.33
D04		130.43

and correspond to those reported in Seiford and Zhu.[11] For infeasible DMUs (DMUs 2 and 6 with an output orientation, and DMU 8 with an input orientation), no results are provided by Warwick (Tables 6 and 7). Our modified method provides a 600% score for input orientation and 1100% score for an output orientation corresponding to the scalars.

Table 6 Modified super-efficiency model (P_2)

DMU	*CRS* (%)	*VRS input* (%)	*VRS output* (%)
D01	97.77	106.26	105.51
D02	126.25	152.77	1100.00
D03	90.97	97.65	97.96
D04	72.81	73.54	76.17
D05	95.43	97.52	97.77
D06	75.42	107.25	1100.00
D07	76.90	78.52	82.16
D08	136.81	600.00	162.23
D09	91.88	92.46	92.24
D10	91.57	106.02	108.11

Table 7 Warwick DEA software (P_1)

DMU	*CRS* (%)	*VRS input* (%)	*VRS output* (%)
D01	97.77	106.26	105.51
D02	126.25	152.77	
D03	90.97	97.65	97.96
D04	72.81	73.54	76.17
D05	95.43	97.52	97.77
D06	75.42	107.25	
D07	76.90	78.52	82.16
D08	136.81		162.23
D09	91.88	92.46	92.24
D10	91.57	106.02	108.11

Summary and discussion

The new super-efficiency model described in this paper has a number of useful and interesting features. First, it enables super-efficiency scores to be obtained using standard DEA models and software. Second, we have shown how to calculate a scalar that is sufficient either to obtain scores for those super-efficient DMUs that have feasible solutions under Xue and Harker[18] or to assign efficiency scores equal to the scalar for strong super-efficient (SSE) DMUs that have infeasible solutions under Xue and Harker.[18] Theorems 2 and 7 of Seiford and Zhu[5] prove that conceptually there is no scalar that will render an SSE inefficient in one orientation. Our scalar is derived empirically from the sample, and provides a bound to an otherwise unbounded scalar. Since it is set by the maximum increase in inputs or decrease in outputs, the ranking of super-efficient DMUs proposed by Xue and Harker[18] is preserved: "Generally, the relative efficiency of units in the four classes can be ranked from higher to lower as: Super Efficient (including Strongly Super Efficient)→Strongly Efficient→Efficient→Weakly Efficient. That is, SE (including SSE)→ E → E′ → F while $E \subseteq SE \subseteq SSE$." Finally, because our scalar is defined in terms of the maximum of variable ratios observed in the sample, it is consistent with the Xue and Harker[18] notion of strong super-efficiency.

Further research is required to provide an empirical interpretation to the scalar, as a first step in better understanding SSE DMUs. Furthermore, our scalar is merely sufficient; a smaller (input orientation) or larger (output orientation) scalar might be obtained if the calculations were restricted to the inputs/outputs for extreme efficient DMUs. Naturally this would involve additional processing. We conclude by observing the importance of these models in light of the uses detailed at the beginning of this paper and emphasise the insights that are frequently obtained when outliers are examined more closely.

Appendix

Dulá and Hickman[4]

Table A1 Example from Table 1

DMU	*Input 1*	*Input 2*	*Input 3*	*Input 4*	*Output 1*	*Output 2*
D01	7	1	5	6	1	1
D02	3	0	4	1	1	1
D03	4	5	3	0	1	1
D04	2	9	7	3	1	1
D05	5	0	6	0	0	1
D06	1	0	0	5	1	1
D07	0	0	2	0	0	1
D08	6	0	3	0	1	1

Table A2 Example from Table 2

DMU	*Input*	*Output 1*	*Output 2*
D01	1	2	2
D02	1	1	6
D03	1	4	1
D04	1	3	5

Seiford and Zhu (1998)[11]

Table A3 Example from Table 7, data set 3

DMU	*Input 1*	*Input 2*	*Input 3*	*Output 1*	*Output 2*
D01	182	237	468	5008	5303
D02	74	82	148	1857	2336
D03	160	195	400	4041	5001
D04	183	150	339	2779	2418
D05	133	155	329	3506	3602
D06	106	120	138	1306	956
D07	109	110	188	1515	2282
D08	240	243	806	7763	9601
D09	276	188	574	4577	6493
D10	191	117	466	3322	4233

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