Bernoulli Theorem, Minimum Specific Energy, and Water Wave Celerity in Open-Channel Flow

Oscar Castro-Orgaz[1] and Hubert Chanson[2]

Abstract: One basic principle of fluid mechanics used to resolve practical problems in hydraulic engineering is the Bernoulli theorem along a streamline, deduced from the work-energy form of the Euler equation along a streamline. Some confusion exists about the applicability of the Bernoulli theorem and its generalization to open-channel hydraulics. In the present work, a detailed analysis of the Bernoulli theorem and its extension to flow in open channels are developed. The generalized depth-averaged Bernoulli theorem is proposed and it has been proved that the depth-averaged specific energy reaches a minimum in converging accelerating free surface flow over weirs and flumes. Further, in general, a channel control with minimum specific energy in curvilinear flow is not isolated from water waves, as customary state in open-channel hydraulics.

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CE Database subject headings: Open channel flow; Critical flow; Weirs; Flumes; Water waves.

Introduction

One of the most useful principles of fluid mechanics to solve practical problems in hydraulic engineering is the Bernoulli theorem along a streamline, which is deduced from the work-energy form of the Euler equation along a streamline (Rouse 1970). Many applications in open-channel hydraulics are based upon such a theorem that is only valid along a given streamline in first instance. Some confusion exists about the applicability of the Bernoulli theorem, and its generalization to open-channel hydraulics. Very few isolated studies (Liggett 1993; Chanson 2006, 2008) have developed the Bernoulli equation to open-channel flow problems. The extension of this principle to open-channel flows provides a basic equation applicable to the calculation of the minimum specific energy and critical flow conditions, a physical phenomenon that determines the head-discharge relationship in control structures used for water measurement in irrigation and sewage techniques, as flumes and weirs (Fig. 1).

In the present study, a detailed and generalized extension of the Bernoulli theorem to open-channel flow is developed. Using analytical and experimental results, the occurrence of minimum specific energy in open channels is reanalyzed, and general results for the critical flow depth in curvilinear flow are provided. Also, some practical advice for the selection of gauging stations is highlighted in relation to wave motion at the section of minimum specific energy

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Bernoulli Theorem

The integration of the Navier-Stokes equations along a streamline assuming that the flow is steady and the fluid is inviscid and incompressible, yields the Bernoulli equation along a streamline (Rouse 1970; Liggett 1994; Chanson 2004, 2006)

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = \text{const} \quad (1)$$

where p/γ=pressure head; z=vertical elevation of the fluid particle; and V=magnitude of velocity vector. The vorticity causes the constant in Eq. (1) to change from one streamline to another. If both sides of Eq. (1) are multiplied by the elementary discharge across a streamline, $dQ=udA$, with Q=discharge, A=flow cross section area, u=component of velocity vector normal to A, and the resulting expression is integrated across a channel section, one obtains a constant quantity given by

$$\int_0^A \left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) udA = \text{const} \quad (2)$$

Both sides of Eq. (2) may be divided by the total discharge Q, that is assumed a constant for all sections, from which it is obtained that the total head H of a cross section is conserved in the flow direction

$$H = \frac{1}{Q}\int_0^A \left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) udA = \text{const} \quad (3)$$

The total head H gives the total convective flow of energy across A, as discussed in detail by Jaeger (1956). A cross-sectional total piezometric pressure coefficient K_e may be defined as (Jaeger 1956)

$$K_e = \frac{1}{hQ}\int_0^A \left(\frac{p}{\gamma} + y\right) udA \quad (4)$$

where h=flow depth and y=coordinate in the vertical direction above the channel bed, and the extended Coriolis coefficient α for curvilinear flow is (Rouse 1970)

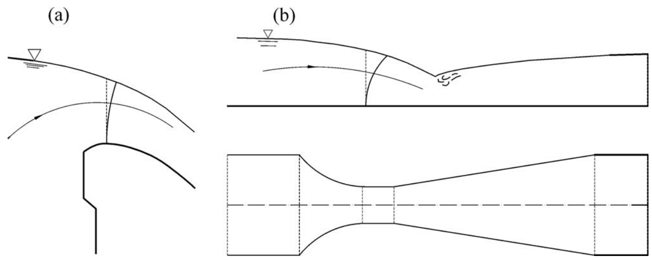

Fig. 1. Critical flow over (a) round-crested weirs; (b) Venturi channels

$$\alpha = \frac{1}{U^2 Q}\int_0^A V^2 u dA \quad (5)$$

where $U=Q/A$=mean flow velocity. It may be remarked that Eq. (5) is a general expression for a Coriolis coefficient, in contrast to the widely used expression

$$\alpha = \frac{1}{U^2 Q}\int_0^A u^3 dA \quad (6)$$

that does not take into account all the velocity components, and, thus, is only accurate for flows with parallel streamlines. The total head H may be rewritten, using Eqs. (4) and (5), as

$$H = z_b + K_e h + \alpha\frac{U^2}{2g} = z_b + E = \text{const} \quad (7)$$

where z_b=channel bed elevation and E=total specific energy. Eq. (7) was developed by Jaeger (1956) and discussed recently by Castro-Orgaz (2008). Using Eq. (7), the equation of motion is simply written as

$$\frac{dH}{dx} = \frac{dz_b}{dx} + \frac{dE}{dh}\frac{dh}{dx} = 0 \quad (8)$$

Eq. (7) was deduced from the application of the Bernoulli theorem along a streamline, and can be viewed as the generalized Bernoulli theorem for open-channel flow in terms of the total head H, and, hence, of the total flow of energy. Critical flow conditions, as given by the minimum specific energy $dE/dh=0$, are deduced from Eq. (8) when $dz_b/dx=0$: i.e., at the crest of a weir (Henderson 1966), with a continuous smooth free surface ($dh/dx<0$) and without any vertical flow profile (Montes 1998) as classically stated by Bélanger (1828).

If we are now interested in making a cross-sectional mean value for the energy head of all the streamlines, H_m, we cannot simply multiply Eq. (1) by dA and then divide the result by A, as the cross-sectional area $A(x)$ varies along the flow according to the local flow depth $h=h(x)$. However, if the Bernoulli theorem for a streamline, given by Eq. (1), is differentiated

$$\frac{d}{dx}\left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) = 0 \quad (9)$$

it is mathematically permissible to write

$$\int_0^A \frac{d}{dx}\left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) dA = 0 \quad (10)$$

Using the Leibnitz rule, Eq. (10) can be rewritten as

$$\frac{d}{dx}\int_0^A \left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) dA - \left(h + z_b + \frac{V_o^2}{2g}\right)\frac{dA}{dx} = 0 \quad (11)$$

where V_o=free surface streamline velocity. The cross-averaged mean head H_m is defined as

$$H_m = \frac{1}{A}\int_0^A \left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) dA \quad (12)$$

It is worth pointing out that the definition of a mean value of the energy head across a section was first defined by Rouse (1932), in relation to curvilinear flows in spillways.

Using Eq. (12), Eq. (11) is rewritten as

$$\frac{dH_m}{dx} = \frac{(H_o - H_m)}{A}\frac{dA}{dx} \quad (13)$$

which is the form of the Bernoulli theorem in terms of the cross-sectional averaged mean head for all the streamlines H_m, with $H_o=z_b+h+V_o^2/2g$=free surface streamline energy head. A first point that deserves attention is that the depth-averaged form of the Bernoulli theorem applied to an open-channel flow does not imply a constant. Indeed, the head $H_m(x)$ changes due to the varying flow area $A(x)$ in the flow direction, as well as due to the local difference between the energy head of the free surface H_o in relation to the mean value H_m across the depth.

A cross-sectional averaged piezometric pressure coefficient K_m can be defined as (Rouse 1932; Chanson 2006)

$$K_m = \frac{1}{hA}\int_0^A \left(\frac{p}{\gamma} + y\right) dA \quad (14)$$

and the "apparent" Boussinesq coefficient for curvilinear flow is (Chanson 2006)

$$\beta = \frac{1}{U^2 A}\int_0^A V^2 dA \quad (15)$$

Eq. (15) is refereed to as an "apparent" Boussinesq coefficient, as it contains the magnitude of the velocity vector V, which is a scalar magnitude arising from the energetic nature of the Bernoulli theorem along a streamline (Rouse 1970). The Boussinesq coefficient is a tensorial magnitude, defined as a vector in a given direction, along which the conservation of momentum is applied (Yen 1973). Therefore, the Boussinesq coefficient is defined in the x direction as (Yen 1973; Liggett 1993)

$$\beta_{xx} = \frac{1}{U^2 A}\int_0^A u^2 dA \quad (16)$$

Using Eqs. (14) and (15), the total mean head H_m is rewritten as

$$H_m = z_b + K_m h + \beta\frac{U^2}{2g} = z_b + E_m \quad (17)$$

where E_m=depth-averaged specific energy, as given by Chanson (2006). Expanding Eq. (13), yields

$$\frac{dz_b}{dx} + \frac{dE_m}{dx} = \frac{(H_o - H_m)}{A}\frac{dA}{dx} \quad (18)$$

For plane channel flow, Eq. (18) can be written as

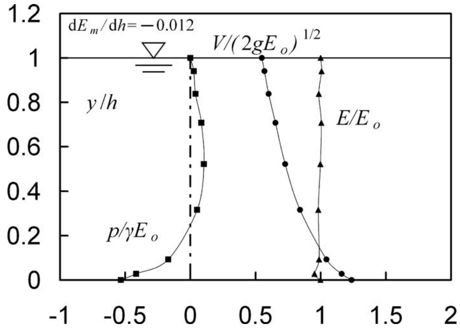

Fig. 2. Experimental data (Fawer 1937) over cylindrical weir: (■) dimensionless pressure head distribution $p/\gamma E_o(y/h)$, (●) dimensionless total velocity distribution $V/(2gE_o)^{1/2}(y/h)$, (▲) dimensionless total head distribution $E/E_o(y/h)$

$$\frac{dz_b}{dx} + \frac{dE_m}{dh}\frac{dh}{dx} = \frac{H_o - H_m}{h}\frac{dh}{dx} \tag{19}$$

which is the generalized depth-averaged Bernoulli theorem for open-channel flows. At the crest of a weir, $dz_b/dx=0$ and Eq. (19) yields

$$\frac{dE_m}{dh} = \frac{H_o - H_m}{h} \tag{20}$$

Eq. (20) implies that, strictly speaking, the depth-averaged mean specific energy is only minimal ($dE_m/dh=0$) at the crest of a weir when the flow is irrotational and, therefore, $H_o=H_m$. In a real flow the vorticity causes a difference between H_o and H_m. However, in short transitions with accelerating converging streamlines, as in the case of flumes and weirs, the flow is nearly irrotational and thus one obtains

$$\frac{dE_m}{dh} = \frac{H_o - H_m}{h} \approx 0 \tag{21}$$

from which

$$H_m = z_b + K_m h + \beta\frac{U^2}{2g} = z_b + E_m \approx \text{const} \tag{22}$$

Eq. (22) was successfully verified by Chanson (2006) with test data on round-crested weirs, computing the coefficients β and K_m using flow net diagrams. Montes (1998) estimated the coefficients β and K_m of Eq. (22) with a Boussinesq-type approach, and compared successfully the results with test data on parabolic weirs. In flows with nearly parallel streamlines, the flow is "gradually varied," and the pressure is hydrostatic ($K_m=1$), the vertical velocity is negligible ($u\approx V$) and the free surface slope is very small ($dh/dx\approx 0$). Under these conditions, the generalized depth-averaged Bernoulli theorem [Eq. (19)] gives

$$z_b + h + \beta_{xx}\frac{U^2}{2g} = \text{const} \tag{23}$$

which was derived by Liggett (1993). It is of interest to remark that the result of Liggett (1993) was obtained from the depth-averaged form of the Euler equations for flows with parallel streamlines and a velocity shape almost invariant with distance ($\beta_{xx}\approx$const). The depth-averaging process of the momentum equation yields an equation with a Boussinesq coefficient [Eq. (23)] rather than with a Coriolis coefficient [Eq. (7)], in agreement with the full integration of the energy and momentum

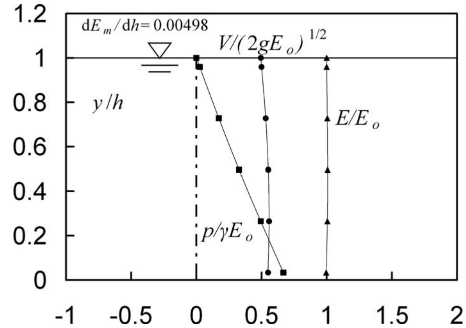

Fig. 3. Experimental data (Khafagi 1942) in a Venturi channel for $Q=22$ l/s: (■) dimensionless pressure head distribution $p/\gamma E_o(y/h)$, (●) dimensionless total velocity distribution $V/(2gE_o)^{1/2}(y/h)$, (▲) dimensionless total head distribution $E/E_o(y/h)$

fluid flow equations in open-channel flow (Yen 1973). Eq. (23) proved to be a particular case of the more general relation [Eq. (19)] developed herein.

The test data of Fawer (1937) with flow over round-crested weirs and of Khafagi (1942) in Venturi flumes were used to verify Eq. (21). Fig. 2 presents the data of Fawer (1937) for a circular weir of radius 3.25 and 30.3 cm width under a discharge of 0.01525 m^3/s, a flow depth $h=5.37$ cm and a specific energy head over the weir of $E=7.68$ cm. The test data show that the flow is nearly irrotational, with a computed value $dE_m/dh=-0.012$ using Eq. (20). Figs. 3–5 show the data of Khafagi for a Venturi channel of throat width 12 cm, inlet width 30 cm, radius of channel sidewalls 54.5 cm and discharges of 22, 17.5, and 14 L/s, respectively. The test data yield $dE_m/dh=0.00498$, $dE_m/dh=0.00547$, and $dE_m/dh=0.00796$, for discharges of 22, 17.5, and 14 L/s, respectively. Thus, theory and experiments support the occurrence of minimum depth-averaged specific energy at channel controls.

Minimum Specific Energy

The concept of critical flow was historically developed as the singularity in the backwater equation for open-channel flows (Bélanger 1828; Chanson 2004). Later, it was concluded by Bakhmeteff (1932) what the conditions are at which the specific

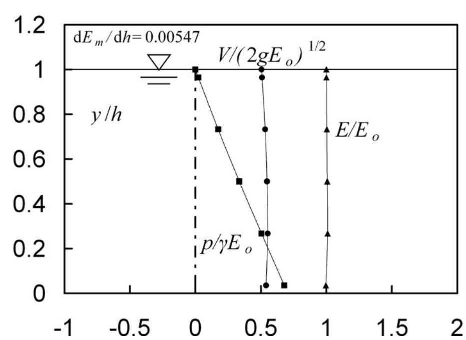

Fig. 4. Experimental data (Khafagi 1942) in a Venturi channel for $Q=17.5$ l/s: (■) dimensionless pressure head distribution $p/\gamma E_o(y/h)$, (●) dimensionless total velocity distribution $V/(2gE_o)^{1/2}(y/h)$, (▲) dimensionless total head distribution $E/E_o(y/h)$

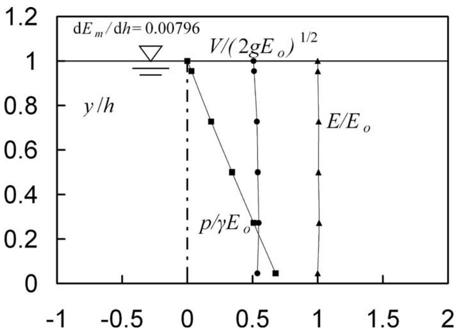

Fig. 5. Experimental data (Khafagi 1942) in a Venturi channel for $Q=14$ l/s: (■) dimensionless pressure head distribution $p/\gamma E_o(y/h)$, (●) dimensionless total velocity distribution $V/(2gE_o)^{1/2}(y/h)$, (▲) dimensionless total head distribution $E/E_o(y/h)$

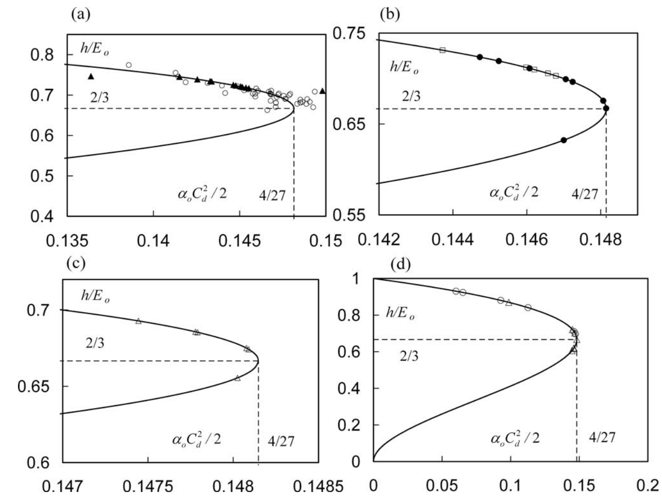

Fig. 6. Discharge curve of flow in open channels (a) (—) Eq. (26), (○) experimental data Chanson and Montes (1997, 1998) flow over cylindrical weirs, (▲) experimental data Khafagi (1942) flow in Venturi channels; (b) (—) Eq. (26), (●) experimental data Fawer (1937) flow over round-crested weirs, (□) experimental data Blau (Montes 1998) flow over parabolic weirs; (c) (—) Eq. (26), (△) experimental data Kindsvater (1964) flow over trapezoidal-profile weirs; and (d) (—) Eq. (26), (△) experimental data Gonzalez and Chanson (2007) flow over broad-crested weirs, (○) experimental data Chanson (2005) in near critical flows

energy reaches a minimum value. The developments herein prove that, in curvilinear flow, it is possible to define the concept of critical flow either using a convective energy flux total specific head E or a depth-averaged specific head for all streamlines across the depth, E_m. If the flow is irrotational, it is also permissible, and even simpler, to write

$$E_m \approx E_o = h + \frac{V_o^2}{2g} \tag{24}$$

where E_o=free surface specific energy

Thus, the depth-averaged specific energy is accurately represented by the specific energy of the free surface streamline. This approach avoids the use of depth-averaging coefficients, and permits one to represent the head-discharge relationship with only one parameter, a "fictitious" Coriolis coefficient α_o defined as

$$\alpha_o = \left(\frac{V_o}{U}\right)^2 \tag{25}$$

that represents the quotient between the free surface and the mean velocities. From Eq. (24), the generalized channel flow relation is, using Eq. (25) (Castro-Orgaz 2008)

$$\left(\frac{h}{E_o}\right)^2\left(\frac{h}{E_o} - 1\right) + \frac{\alpha_o C_d^2}{2} = 0 \tag{26}$$

where C_d=discharge coefficient=$Q/[b(gH_o^3)^{1/2}]$ and b=channel width.

Eq. (26) and the experimental data of Chanson and Montes (1997, 1998), series QIIA, for flow over circular-crested weirs, the data of Fawer (1937) of flow over round-crested weirs, the data of Blau of parabolic weirs (Montes 1998), the data of Kindsvater (1964) (model 2, free flow conditions) on flow over trapezoidal-profile weirs, and the data of Khafagi (1942) for circular-shaped inlet flume of rectangular cross section are plotted in Figs. 6(a–c). As shown in Figs. 6(a–c), all the data of critical flow in flumes and weirs collapse in the upper branch of the curve. Then, of the two branches of the discharge curve, only the upper branch has a physical meaning of critical flow at a weir crest and at a flume throat, corresponding to relations $h/E_o > 2/3$ for curvilinear flows. In parallel flows, $h/E_o=2/3$ and, consequently, the particular point $\alpha_o C_d^2/2=4/27$ in Fig. 6 is obtained. However, the lower part of the curve was not close to any experimental data. This is because Eq. (26) is a mathematical relation between the discharge, the specific energy, and the flow depth at the crest, with up to three real values of the flow depth-energy ratio for any product of the square of discharge coefficient and fictitious Coriolis coefficient. For the establishment of critical flow conditions, two simultaneous conditions are required. First, an extreme in the channel geometry (maximum elevation in a weir, minimum width in a flume, Henderson 1966) is needed to create a potential section for the appearance of critical flow. Second, in the extreme section, a sufficient condition given by the derivative $dE_o/dh=0$ is required to provide a unique relation between E_o and h for a given Q, avoiding two of the solutions of Eq. (26) that do not imply critical flow conditions. A comparison of test data with the whole curve [see Figs. 6(a–c)] simply shows that the upper branch is the critical flow solution of the three possible roots. However, although the critical points can only lie in the upper branch, other possible types of flows can also lie there. Subcritical flow over the whole weir profile implies higher tailwater levels than the modular limit of the weir (Dominguez 1959; Montes 1998). Under these conditions, the relation h/E_o at the weir crest increases above the values for free flow. As the tailwater level increases for a given upstream head E_o, the flow that passes over the weir is reduced. Extreme submergence conditions imply $h/E_o=1$, and, consequently, $\alpha_o C_d^2/2=0$, which is the initial point of the upper branch.

The geometry of a broad-crested weir does not allows for an extreme in the channel geometry, given by a channel bottom elevation or a width contraction, and thus, critical depth and its position on the weir are governed mainly by frictional effects and streamline curvature (Rouse 1932). It is then futile to attempt to define the discharge characteristics of the broad-crested weir trying to locate the real critical depth section, which, in the more general case, is not necessarily equal to the hydrostatic pressure critical depth, a case in which computations become complex (Castro-Orgaz 2008). Then, although in the strictest sense one cannot find the critical depth section on a broad-crested weir with

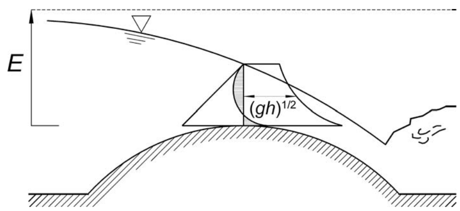

Fig. 7. Minimum specific energy and celerity of shallow-water wave over a round-crested weir

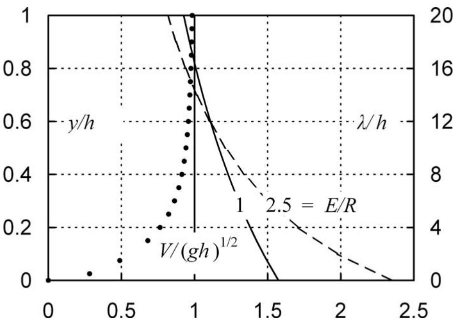

Fig. 8. Water wave celerity and velocity profiles, (——) normalized velocity profile $V/(gh)^{1/2}(y/h)$ for $E/R=1$, (– –) normalized velocity profile $V/(gh)^{1/2}(y/h)$ for $E/R=2.5$, (●) dimensionless celerity $c/(gh)^{1/2}(\lambda/h)$ of linear Airy waves [Eq. (28)]

any accuracy, it is relevant to analyze data on flow over broad-crested weirs. The test data of Gonzalez and Chanson (2007) for a large broad-crested weir are also plotted in Fig. 6(d), corresponding to flow depths in gauging stations on the first half of broad-crested weir models. As seen in Fig. 6(d), the flow over a broad-crested weir may be properly called transcritical, rather than critical, as the flow changes between the two real branches of Eq. (26) without a definite flow pattern in terms of a critical depth relationship h/E_o. In this regard, it is also interesting to plot data with near critical flows (Chanson 2005) in Fig. 6(d). In those cases, all the experiment points lie in the upper branch, but, in most cases, the relationship h/E_o is exceedingly high (≈ 0.85) for representing critical flows, even when including streamline curvature effects. Note that flow over round-crested weirs (Fawer 1937) imply h/E_o around 0.7. It proves again that although curved streamline critical flows lie only in the upper branch, with, typically, $h/E_o \approx 0.7$ as a mean, other types of curvilinear flows can also be found there, as near critical flows.

Water Wave Celerity

In the previous section, a set of expressions for the Bernoulli theorem in an open-channel flow were developed, each of which could be used to define critical flow based on the concept of minimum specific energy. Recently, it was shown that in general critical flow is not single magnitude when it is defined with the energy and momentum principles (Castro-Orgaz 2008). A simple and relevant case can be explained in relation to the weir flow case, and the Bernoulli theorem. Consider the weir flow drawing of Fig. 7. At the weir crest, the streamlines are curved and sloped, and the velocity distribution increases from the free surface to the channel bottom. According to the Bernoulli theorem for a streamline, an increase in the velocity head causes a drop in the pressure, which is no longer hydrostatic across the depth. The increase in the velocity causes an increase in the discharge for a given head, and the test data of Fawer (1937) and the computations of Chanson (2006) proved this flow feature. Critical flow can also be defined using momentum considerations, in relation to the shallow-water wave celerity of a one-dimensional (1D) flow (Montes 1998; Chanson 2004)

$$c=(gh)^{1/2} \tag{27}$$

with c=shallow-water wave velocity of a 1D flow. Eq. (27) was first proposed by Joseph-Louis Lagrange (1781). Fig. 7 shows that, under critical flow conditions, only a particular streamline has a velocity equal to the celerity c. As a result, the flow region above the section of minimum specific energy (weir crest) is not isolated from shallow-water waves. This shows that minimum specific energy considerations are not necessarily in agreement with momentum concepts when defining critical flow conditions.

Similar conclusions were outlined by Castro-Orgaz (2008), who improved the celerity c by incorporating the nonuniform velocity and nonhydrostatic pressure effects. Therefore, only in the descending branch of the weir, where the flow is parallel, one can find sections isolated from shallow-water waves. The flow feature discussed is relevant in such irrigation works as the flow dividers analyzed by Dominguez (1959).

The conclusions of this discussion demand caution in engineering practice when trying to define whether the flow is subcritical or supercritical by means of causing surface waves in a channel and observing the direction in which these waves travel. It was proved that the section of minimum specific energy (i.e., critical flow section) is in general not a section which equals the shallow-water wave celerity. For the test data of Fawer (Fig. 2) the dimensionless critical depth at the weir crest is $h/H_o=0.699$, and the dimensionless celerity is $c/(2gE_o)^{1/2}=(0.699/2)^{1/2}=0.5912$. This value is very near to the dimensionless velocity at the free surface $[V_o/(2gE_o)^{1/2}=0.548$, see Fig. 2], so, in this case, nearly the whole flow section is isolated from "shallow-water waves." More questionable is the fact that Eq. (27) is the "computational velocity of floods" when the method of characteristics is applied to the Saint-Venant gradually varied unsteady shallow-flow equations, but a two-dimensional (2D) water wave celerity may differ from Eq. (27) (Montes 1998) when streamline curvature is important. Moreover, when nearly the whole section is isolated from shallow-water waves, as in the case of the experiments of Fawer (1937), it does not give any guarantee of a similar performance under a real 2D water wave motion. This flow feature deserves some discussion. Hager (1991) showed that the velocity distribution over round-crested weirs can be described using the free vortex approach $V(R+y)$=const, with R=curvature radius at channel bed. Normalized velocity profiles $V/(gh)^{1/2}(y/h)$ for two typical values of the dimensionless head, $E/R=1$ and 2.5, are plotted in Fig. 8. At the elevation y/h where $V/(gh)^{1/2}=1$ the velocity profile equals the shallow-water wave celerity. Fig. 8 shows that the incipient shallow-flow celerity appears at $y/h=0.71$ and 0.8 for $E/R=2.5$ and 1, respectively, indicating that in general, as a mean, the upper 25% of the flow depth is not isolated from shallow-water waves. Some additional data can be added to Fig. 8, considering the general dimensionless celerity $c/(gh)^{1/2}(\lambda/h)$ of linear Airy waves (Montes 1998)

$$\frac{c}{(gh)^{1/2}}=\left[\frac{\tanh(2\pi h/\lambda)}{2\pi h/\lambda}\right]^{1/2} \tag{28}$$

where λ=wavelength. For shallow flows, $\lambda/h\rightarrow\infty$ and Eq. (28) equals Eq. (27). As seen in Fig. 8 the celerity of linear waves support substantively the previous conclusions, proving that, in general, the section of minimum specific energy is not isolated from water wave motion.

Conclusions

A detailed analysis of the Bernoulli theorem and its extension to flow in open channels has been developed. From the analytical results of the extension of the Bernoulli principle to open-channel flow, the generalized depth-averaged Bernoulli theorem is proposed, extending the earlier works of Liggett (1993) and Chanson (2006,2008). From the new depth-averaged Bernoulli equation, it was shown that the depth-averaged specific energy reaches a minimum in converging accelerating free surface flow over weirs and flumes. A generalized open-channel flow diagram based on the Bernoulli theorem was used to show analytically the critical depth relationships in curvilinear flows. A comparison with experiment data of round-crested weirs and Venturi channels demonstrated the validity of the analytical findings. In general, a channel control with minimum specific energy in curvilinear flow is not isolated from water waves. Hence, any method for producing waves in water is usually not appropriate for deciding whether the flow is subcritical or supercritical.

Notation

The following symbols are used in this paper:

A = cross-sectional area (m^2);
b = channel width (m);
C_d = discharge coefficient (-);
c = velocity of a shallow-water 1D wave ($m\ s^{-1}$);
E = total specific energy of flow, also specific energy of a streamline (m);
E_m = mean specific energy of flow (m);
E_o = specific energy of free surface streamline (m);
g = acceleration of gravity ($m\ s^{-2}$);
H = total energy head of flow (m);
H_m = mean energy head of flow (m);
H_o = free surface streamline energy head (m);
h = flow depth (m);
K_m = cross-sectional averaged piezometric pressure coefficient (-);
K_e = total head piezometric pressure coefficient (-);
p/γ = pressure head (m);
Q = discharge ($m^3\ s^{-1}$);
R = curvature radius at channel bed (m);
U = mean flow velocity=Q/A ($m\ s^{-1}$);
u = component of local velocity normal to A ($m\ s^{-1}$);
V = magnitude of velocity vector ($m\ s^{-1}$);
V_o = free surface streamline velocity ($m\ s^{-1}$);
x = streamwise distance (m);
y = coordinate in the vertical direction above the channel bed (m);
z_b = elevation of the channel bed (m);
z = vertical elevation of a fluid particle (m);
α = Coriolis coefficient (-);
α_o = fictitious Coriolis coefficient (-);
β = apparent Boussinesq coefficient (-);
β_{xx} = Boussinesq coefficient (-);
γ = specific fluid weight ($N\ m^{-3}$); and
λ = wavelength (m).

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