PHYSICAL REVIEW A **80**, 032309 (2009)

Continuous-variable entanglement distillation over a general lossy channel

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(Received 21 April 2009; published 9 September 2009)

Any continuous-variable distillation protocol that does not change the effective loss of the transmission channel cannot be used to achieve maximum Einstein-Podolsky-Rosen style correlations after distillation. We analyze the continuous-variable entanglement distillation protocol described by [Browne *et al.*, Phys. Rev. A **67**, 062320 (2003)] and show that it does not change the effective loss of the transmission of entanglement. We extend this scheme to one which does change the effective transmission coefficient of the channel. Our scheme has the added benefit that the states generated exhibit isotropic noise which is best suited to applications of continuous-variable entanglement such as continuous-variable teleportation.

DOI: 10.1103/PhysRevA.80.032309 PACS number(s): 03.67.−a, 03.65.Ud, 42.50.Dv

I. INTRODUCTION

It has been established now for quite some time that the distillation [1] of Gaussian continuous-variable (CV) entanglement from Gaussian states with “Gaussian operations” (i.e., those operations which transform Gaussian states into Gaussian states) is not possible [2]. This of course left open the possibility of distilling Gaussian entanglement using non-Gaussian operations.

Distilling non-Gaussian entanglement from Gaussian entanglement has been shown to be possible [3]. However, the desire here is to have a procedure that outputs Gaussian entanglement so it can be used within the standard CV operations which perform best with Gaussian states. It has also been shown that Gaussian entanglement decohered by a non-Gaussian channel can also be distilled [4]. Unfortunately, though, the most common decohering channels are in fact Gaussian.

An example of a scheme for distilling Gaussian entanglement was presented in [5]. In this scheme, photon counting was used as the non-Gaussian operation used to distill entanglement. This paper begins by describing the important characteristics of the scheme in [5] and finds that the purification power of this scheme is limited. That is, this scheme has very limited ability to mitigate the effects of a Gaussian decohering channel. A scheme built upon this is then described, and its distillation and purification abilities are quantified. The scheme is shown to have superior purification results.

The scheme in [5] and the scheme described here depend on an iterative procedure which must be iterated many times to distill Gaussian entanglement. Some of the convergence properties are explored and it is shown that only a few iterations need be performed to distill a state close to the Gaussian one.

II. GAUSSIAN STATES

Squeezed states and coherent states are examples of states with Gaussian probability distributions in the position and momentum quadratures. Gaussian probability distributions are fully characterized by specifying a vector of means and a covariance matrix. Any state which has a vector of means which is the zero vector and a covariance matrix which has an eigenvalue less than the variance due to vacuum noise is called a squeezed vacuum.

The covariance matrix contains the variances of the position and momentum observable of each mode along the diagonal and the covariances between those observables in the off-diagonal entries. A covariance matrix must be symmetric as the covariance between two observables is independent of the order of the observables.

The canonical pure two-mode squeezed vacuum state has a covariance matrix

$$\begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & \sinh 2r & 0 & \cosh 2r \end{pmatrix}, \tag{1}$$

where r is a parameter representing the strength of the correlation between the two modes and is referred to as the “squeeze parameter.” The same state can be written in a two-mode Fock basis as

$$(\cosh r)^{-1}\sum_{n=0}^{\infty} \tanh^n r|n,n\rangle, \tag{2}$$

which can be verified by computing the variances and covariances of the four quadrature variables of the two modes. This state is entangled for $r > 0$.

III. GAUSSIFICATION AND DISTILLATION

The distillation protocol in [5] consists of two procedures. First, several pairs of weakly entangled Gaussian CV states are collected by the two parties. Then for each pair one of the parties mixes their mode with a single photon at a beamsplitter of reflectivity η^2. Then this party makes photon counting measurements on one of the two output modes for each pair.

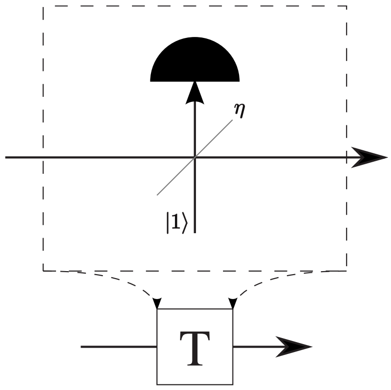

FIG. 1. The distillation step of the protocol described in [5]. The input mode is mixed with a single photon and one output mode is detected. The other output mode is accepted only if a single photon is detected. The beamsplitter ratio η is varied to achieve different ratios of the zero- and one-photon terms in the output. This operation will be labeled T.

The parties decide to accept the distributed entangled pair if the detection registered one and only one photon. The operation that is applied by the acting party to a single pair is shown in Fig. 1.

The effect of this operation can be described by the effect operator

$$\sum_n (-1)^{n-1}\eta^{n-1}[n-(1+n)\eta^2]|n\rangle\langle n| \tag{3}$$

acting on the mode that was transformed by the operation and the other party applies the identity operation.

Starting with a pure two-mode squeezed state transmitted through a channel with no loss, the output state conditional on the measurements giving the desired outcome is

$$|\Psi\rangle \propto |0,0\rangle + \frac{(2\eta^2-1)}{\eta}\tanh r|1,1\rangle + (\text{higher-order terms}), \tag{4}$$

where the last term represents terms of more than two photons. A phase shift has been applied so that the superposition has a plus sign. The ratio of the $|0,0\rangle$ and $|1,1\rangle$ coefficients can be set between 0 and 1 for any given squeezing strength r by choosing the beamsplitter ratio η appropriately.

As pointed out in [5] it is possible to relax the requirement of measuring one and only one photon if the input state is has a very low amplitude. Under these conditions a detector which measures the presence or absence of photons will provide a very good approximation to that of a detector which measures one and only one photon.

In the initial qubit distillation protocols a similar procedure is used to select by measurements those state which have high entanglement [1]. This is known as the Procrustean method of entanglement distillation and hence this first step in the distillation protocol will just be called "distillation."

The second part which we will call "Gaussification" consists of an iterative procedure which converts many copies of the state in Eq. (4) into a Gaussian state with a stronger entanglement than any individual state which the distillation protocol started with. Each party mixes the two modes they have on a 50:50 beamsplitter from two copies of the state in Eq. (4) (Fig. 2). One output mode from either side is measured, and the output is accepted when both parties measure zero photons. The procedure applied by one party to combine two pairs is shown schematically in Fig. 2. After this operation, two entangled pairs are consumed and the output state is closer to a Gaussian state than the input states while keeping the ratio of the $|0,0\rangle$ and $|1,1\rangle$ amplitudes constant.

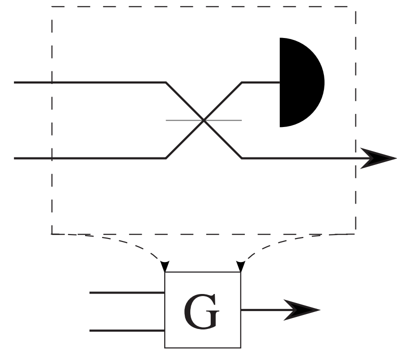

FIG. 2. The basic step of the Gaussification procedure described in 1. Pairs of modes are combined at a 50:50 beamsplitter and one output is detected. The other output is accepted if zero photons are measured. After this, two modes that form the output of two successful iterations are combined and the detection occurs again. This operation is applied by both parties in the same manner. The states which result after many iterations of this procedure are Gaussian states. This operation will be labeled G.

When this operation is applied successfully by both parties the effect on an input composed of two copies of an arbitrary input density matrix was found in [5]. It is

$$\bar{\rho}_{a,b;c,d} = \sum_{s=0}^{a}\sum_{t=0}^{b}\sum_{n=0}^{c}\sum_{m=0}^{d} M^{s,t;n,m}_{a,b;c,d}\rho_{s,t;n,m}\rho_{a-s,b-t;c-n,d-m}, \tag{5}$$

where the coefficients $M^{s,t;n,m}_{a,b;c,d}$ are given by

$$M^{s,t;n,m}_{a,b;c,d} = 2^{-(a+b+c+d)/2}(-1)^{(a+b+c+d)-(s+t+n+m)} \times \left[\binom{a}{s}\binom{b}{t}\binom{c}{n}\binom{d}{m}\right]^{1/2}, \tag{6}$$

where the density operator element indices are defined as

$$(\langle a|_1\langle b|_2)\hat{\rho}(|c\rangle_1|d\rangle_2) = \rho_{a,b;c,d} \tag{7}$$

and the subscripts of the kets define which mode the state describes. As shown in [5], if this mapping is iterated then the states which this map converge to are Gaussian states. This is what gives rise to the name Gaussification.

So combining these procedures results in a process which has many copies of a weakly entangled Gaussian state as input and a single (potentially stronger) Gaussian entangled state as the output. The combination of these procedures is shown in Fig. 3.

The convergence of the Gaussification procedure is conditional on the covariance matrix that the output state is expected to converge to being that of a physical state. The

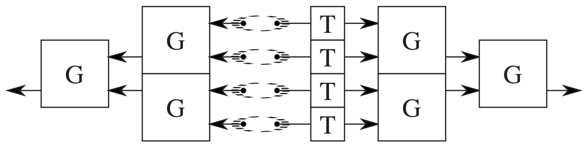

FIG. 3. A schematic summary of the distillation protocol in [5]. A source of weakly entangled squeezed state is depicted in the middle. The distillation operation T from Fig. 1 is performed by one party on all distributed pairs. Then the Gaussification steps G as shown in Fig. 1 are repeated until a single mode is left as the output. If the expected covariance matrix is physical then the output will converge toward a Gaussian state in the limit that infinitely many pairs are used. Note that the distillation step is only performed on one side, and any channel loss considered is contained to the side not performing the distillation.

particular Gaussian state that the procedure is expected to converge to is completely determined by the following elements of the density operator *after the first iteration*:

$$\rho_{00;00}, \rho_{10;10}, \rho_{01;01}, \rho_{10;01}, \rho_{20;00}, \rho_{02;00}, \rho_{11;00}.$$

This is because these elements relative to each other remain unchanged after the first iteration of the map. By choosing these elements for the initial input state using non-Gaussian operations, distillation of Gaussian entanglement is achieved.

The covariance matrix Γ for the above elements of the input state was computed in [5]. Writing $\Gamma = B^{-1} - 1$ then the matrix elements of B are

$$\rho_{10;10}/\rho_{00;00} = 1 - B_{1,1} - B_{2,2}, \tag{8}$$

$$\rho_{01;01}/\rho_{00;00} = 1 - B_{3,3} - B_{4,4}, \tag{9}$$

$$\rho_{10;01}/\rho_{00;00} = -B_{1,3} - B_{2,4} + i(B_{1,4} - B_{2,3}), \tag{10}$$

$$\rho_{20;00}/\rho_{00;00} = \sqrt{\frac{1}{2}}(-B_{1,1} + B_{2,2} - 2iB_{1,2}), \tag{11}$$

$$\rho_{02;00}/\rho_{00;00} = \sqrt{\frac{1}{2}}(-B_{3,3} + B_{4,4} - 2iB_{3,4}), \tag{12}$$

$$\rho_{11;00}/\rho_{00;00} = -B_{1,3} + B_{2,4} - i(B_{1,4} + B_{2,3}). \tag{13}$$

By inverting these relations one can calculate the covariance matrix. As it is only the ratio with $\rho_{00;00}$ that defines Γ it is convenient to work with ρ un-normalized such that $\rho_{00;00} = 1$.

The covariance matrix Γ represents a true quantum state if and only if $\Gamma + i\Sigma$ is non-negative definite where Σ is the two-mode symplectic matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{14}$$

IV. TWO-MODE SQUEEZED STATES WITH LOSS

It is useful at this point to look at the covariance matrix for a pure two-mode squeezed state which has traveled through loss. We are going to consider the general case where the loss the two-mode traverse may be unequal. We shall call this distribution of the total channel loss from the source to the two outputs the *loss profile*. We are also going to work with variables which represent the channel transmissions instead of the loss but as the sum of the loss and transmission is unity, one can move between the two representations. Channel transmissions will be represented by τ.

It is easiest to calculate the covariance matrix by considering the evolution of the state in the Heisenberg picture. If the two modes from the squeezed state are labeled a and b then the transmitted modes at the output of the channels are

$$X'_a = \sqrt{\tau_a} X_a + \sqrt{(1-\tau_a)} v_a, \tag{15}$$

$$X'_b = \sqrt{\tau_b} X_b + \sqrt{(1-\tau_b)} v_b, \tag{16}$$

with τ_a and τ_b being the transmission of channels a and b, respectively. Using these equations one finds the variance of the mode at output a (in units where the variance of vacuum noise is unity) is

$$\langle (X'_a)^2 \rangle = \tau_a \langle X_a^2 \rangle + (1-\tau_a) \tag{17}$$

and similarly for mode b. The cross correlations can also be computed

$$\langle X'_a X'_b \rangle = \sqrt{\tau_a \tau_b} \langle X_a X_b \rangle. \tag{18}$$

Incorporating the two-mode squeezed state from Eq. (1) and the above two-mode loss transformations generates the covariance matrix for a lossy two-mode squeezed state

$$\begin{pmatrix} C_1 & 0 & -S & 0 \\ 0 & C_1 & 0 & S \\ -S & 0 & C_2 & 0 \\ 0 & S & 0 & C_2 \end{pmatrix}, \tag{19}$$

where

$$C_1 = 1 + \tau_a(\cosh 2r - 1), \tag{20}$$

$$C_2 = 1 + \tau_b(\cosh 2r - 1), \tag{21}$$

$$S = \sqrt{\tau_a \tau_b} \sinh 2r. \tag{22}$$

Alternatively if one is given a covariance matrix of the general form shown in Eq. (19) defined by only three parameters C_1, C_2, and S, one may try and write it down in terms of the parameters which define a lossy two-mode squeezed state r', τ'_a, and τ'_b. That is, one may want to invert the relationship given in Eqs. (20)–(22). This is achieved by the following relations:

$$\cosh(2r') = \frac{S^2 + (C_1 - 1)(C_2 - 1)}{S^2 - (C_1 - 1)(C_2 - 1)}, \tag{23}$$

$$\tau'_a = \frac{C_1 - 1}{\cosh(2r') - 1}, \tag{24}$$

$$\tau_b' = \frac{C_2 - 1}{\cosh(2r') - 1}. \tag{25}$$

It is important to note that the Gaussian state under consideration in these relations may not have been generated from a lossy squeezed source, but this description will give the parameters of an equivalent lossy squeezed source which will behave identically. Note that r' must be real and $0 \leq \tau_a'$, $\tau_b' \leq 1$. This puts a restriction on the type of covariance matrices of the form in Eq. (19) which can be considered equivalent to those generated from a pure squeezed state and loss.

It is quite common in experiments to build a symmetric configuration such that the output state has isotropic noise. We will represent this case by $C_1 = C_2 = C$. For this isotropic case, the physicality condition for the covariance matrix simplifies to

$$C^2 - S^2 \geq 1. \tag{26}$$

If one wants to write an isotropic covariance matrix as from a lossy squeezed source then to satisfy the criterion that r' be real, C and S must be constrained to $C > 1$ and either $C - S < 1$ or $C + S < 1$ must be true to keep the denominator of Eq. (23) positive. Also the isotropic nature requires $\tau_a' = \tau_b' = \tau'$ and the previous condition on C ensure that $\tau' \geq 0$. $\tau' \leq 1$ is the same condition as that in Eq. (26) provided that the restrictions $C - S < 1$ and $C > 1$ are already satisfied.

In the Gaussification procedure, the low photon number components of the density operator describing the state received by the two parties are the most important. These will now be calculated for the general two-mode lossy squeezed state.

The calculation proceeds by considering the same arrangement for loss as was considered when calculating the covariance matrix. The pure two-mode squeezed state can be written

$$\sum_{n=0}^{\infty} \frac{\tanh^n r}{n!} a^{\dagger n} b^{\dagger n} |0,0\rangle, \tag{27}$$

where a and b are the annihilation operators for the two squeezed modes and the state written such that the $|0,0\rangle$ term has coefficient 1. By introducing annihilation operators for loss modes l_a and l_b, after the two squeezed modes have passed through the loss then the output state can be written

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n r}{n!} (\sqrt{\tau_a} a^\dagger + \sqrt{1-\tau_a} l_a^\dagger)^n \times (\sqrt{\tau_b} b^\dagger + \sqrt{1-\tau_b} l_b^\dagger)^n |0,0;0,0\rangle, \tag{28}$$

where the numbers after the semicolon in the Fock state represent the two loss modes. Expanding this expression

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n r}{n!} \sum_{k=0}^{n} \binom{n}{k} (\sqrt{\tau_a} a^\dagger)^{n-k} (\sqrt{1-\tau_a} l_a^\dagger)^k \sum_{p=0}^{n} \binom{n}{k} \times (\sqrt{\tau_b} b^\dagger)^{n-p} (\sqrt{1-\tau_b} l_b^\dagger)^p |0,0;0,0\rangle, \tag{29}$$

$$|\Psi\rangle = \sum_{n=0}^{\infty} \tanh^n r \sum_{k,p=0}^{n} \sqrt{\binom{n}{k}\binom{n}{p}} (1-\tau_a)^{k/2} \tau_a^{(n-k)/2} \times (1-\tau_b)^{p/2} \tau_b^{(n-p)/2} |n-k, n-p; k, p\rangle. \tag{30}$$

This expression can be simplified by calculation of the coefficients of the wave function $\Psi_{\alpha,\beta}^{c,d} = \langle \alpha, \beta; c, d | \Psi \rangle$, where c and d are the number of photons in the loss modes from modes a and b, respectively, and α and β label the photon number in the final output modes,

$$\Psi_{\alpha,\beta}^{c,d} = \sum_{n=0}^{\infty} \sum_{k,p=0}^{n} \tanh^n(r) \times \sqrt{\binom{n}{k}\binom{n}{p} (1-\tau_a)^k \tau_a^{n-k} (1-\tau_b)^p \tau_b^{n-p}} \times \delta_{\alpha,n-k} \delta_{\beta,n-p} \delta_{c,k} \delta_{d,p} \tag{31}$$

By defining $\binom{n}{k} = 0$ for $k > n$ and eliminating the k,p sum

$$\Psi_{\alpha,\beta}^{c,d} = \sum_{n=0}^{\infty} \tanh^n(r) \times \sqrt{\binom{n}{c}\binom{n}{d} (1-\tau_a)^c \tau_a^{n-c} (1-\tau_b)^d \tau_b^{n-d}} \, \delta_{\alpha,n-c} \delta_{\beta,n-d} \tag{32}$$

and finally eliminating the n sum

$$\Psi_{\alpha,\beta}^{c,d} = \delta_{\alpha+c,\beta+d} \tanh^{\alpha+c}(r) \times \sqrt{\binom{\alpha+c}{c}\binom{\beta+d}{d} (1-\tau_a)^c \tau_a^{\alpha} (1-\tau_b)^d \tau_b^{\beta}}. \tag{33}$$

To calculate density operator elements, one must trace over the loss modes. Remembering that the above expressions describes the coefficients of the wave function, the density operator element for the vacuum state can then be written

$$\rho_{00,00} = \sum_{c,d=0}^{\infty} \Psi_{0,0}^{c,d} \Psi_{0,0}^{*c,d}. \tag{34}$$

The contributions from Ψ are only nonzero if $c = d$ and Ψ is real. We can write

$$\rho_{00,00} = \sum_{n=0}^{\infty} (\Psi_{0,0}^{n,n})^2. \tag{35}$$

Hence,

$$\rho_{00,00} = \sum_{n=0}^{\infty} \tanh^{2n}(r)(1-\tau_a)^n(1-\tau_b)^n = \frac{1}{1 - \tanh^2(r)(1-\tau_a)(1-\tau_b)}. \tag{36}$$

As the only nonzero terms in the sum which forms the density operator are those where $\alpha + c = \beta + d$ for both terms, we find the following density operator elements are zero:

$$\rho_{20,00} = \sum_{c,d=0}^{\infty} \Psi_{2,0}^{c,d}\Psi_{0,0}^{c,d} = 0, \tag{37}$$

$$\rho_{02,00} = \sum_{c,d=0}^{\infty} \Psi_{0,2}^{c,d}\Psi_{0,0}^{c,d} = 0, \tag{38}$$

$$\rho_{10,01} = \sum_{c,d=0}^{\infty} \Psi_{1,0}^{c,d}\Psi_{0,1}^{c,d} = 0. \tag{39}$$

The nonzero terms are

$$\rho_{11,00} = \frac{\tanh r\sqrt{\tau_a\tau_b}}{1-\tanh^2(r)(1-\tau_a)(1-\tau_b)}\rho_{00,00}, \tag{40}$$

$$\rho_{01,01} = \frac{\tanh^2(r)(1-\tau_a)\tau_b}{1-\tanh^2(r)(1-\tau_a)(1-\tau_b)}\rho_{00,00}, \tag{41}$$

$$\rho_{10,10} = \frac{\tanh^2(r)\tau_a(1-\tau_b)}{1-\tanh^2(r)(1-\tau_a)(1-\tau_b)}\rho_{00,00}. \tag{42}$$

V. GAUSSIFICATION AND LOSS

Applying the distillation step shown in Eq. (3) to the lossy squeezed state, the nonzero density operator elements become

$$\rho_{00,00} = \frac{\eta^2}{1-\tanh^2(r)(1-\tau_a)(1-\tau_b)}, \tag{43}$$

$$\rho_{11,00} = \frac{(2\eta^2-1)\tanh r\sqrt{\tau_a\tau_b}}{\eta[1-\tanh^2(r)(1-\tau_a)(1-\tau_b)]}\rho_{00,00}, \tag{44}$$

$$\rho_{01,01} = \frac{(2\eta^2-1)^2\tanh^2(r)(1-\tau_a)\tau_b}{\eta^2[1-\tanh^2(r)(1-\tau_a)(1-\tau_b)]}\rho_{00,00}, \tag{45}$$

$$\rho_{10,10} = \frac{\tanh^2(r)\tau_a(1-\tau_b)}{1-\tanh^2(r)(1-\tau_a)(1-\tau_b)}\rho_{00,00}. \tag{46}$$

The work in [5] considered a channel with an asymmetric loss profile. The side doing the single-photon distillation had no loss (i.e., $\tau_b=1$), and the other side had loss τ. This then gave the simpler density operator elements

$$\rho_{00,00} = \eta^2, \tag{47}$$

$$\rho_{11,00} = \frac{(2\eta^2-1)\tanh r\sqrt{\tau}}{\eta}\rho_{00,00}, \tag{48}$$

$$\rho_{01,01} = \frac{(2\eta^2-1)^2\tanh^2(r)(1-\tau)}{\eta^2}\rho_{00,00}, \tag{49}$$

with all other elements zero. They then wrote this as

$$\rho_{11,00}/\rho_{00,00} = \lambda, \tag{50}$$

$$\rho_{01,01}/\rho_{00,00} = \frac{1-\tau}{\tau}\lambda^2, \tag{51}$$

where $\lambda=(2\eta^2-1)\tanh r\sqrt{\tau}/\eta$ which could be set to any value between zero and $\sqrt{\tau}$ by choosing η. This is similar to the zero loss case, with the exception that the maximum λ is now $\sqrt{\tau}$.

The covariance matrix which results after the entanglement distillation and Gaussification using this single-sided loss is

$$\Gamma = \frac{1}{\tau-\lambda^2}\begin{pmatrix} \tau+\lambda^2(2\tau-1) & 0 & 2\tau\lambda & 0 \\ 0 & \tau+\lambda^2(2\tau-1) & -2\tau\lambda & 0 \\ 2\tau\lambda & 0 & \lambda^2+\tau & 0 \\ 0 & -2\tau\lambda & 0 & \lambda^2+\tau \end{pmatrix}. \tag{52}$$

This result was computed in [5]. Using Eqs. (23) through (25) it can be shown that this covariance matrix is equivalent to a lossy-squeezed state with parameters

$$r = \tanh^{-1}(\lambda/\sqrt{\tau}), \tag{53}$$

$$\tau_a = \tau, \tag{54}$$

$$\tau_b = 1. \tag{55}$$

So the effect of the distillation protocol is to have an effective source whose squeezing parameter is increased by a factor involving the chosen beamsplitter reflectivity. However, the loss profile of the transmission channel remains unchanged.

VI. CONTINUOUS-VARIABLE TELEPORTATION, ENTANGLEMENT SWAPPING, AND REPEATERS

CV teleportation is an immediate application of Gaussian CV entanglement. The aim of teleportation is to recreate an arbitrary input state by sending classical signals and utilizing shared entanglement without actually sending the state itself.

The unity gain fidelity of CV teleportation of a coherent state performed through a channel with loss of τ equal on both sides is given by

$$\mathcal{F}_{equal,loss} = \frac{1}{2-\tau}. \tag{56}$$

This fidelity is achieved if the entanglement source had maximum squeezing. For a channel with one sided loss the maximum fidelity is

$$\mathcal{F}_{unequal,loss} = \frac{1}{2-\tau^2}. \tag{57}$$

The fidelity on the unequally distributed loss is always lower than that of the equal loss.

The fidelity for an input state of non-minimum uncertainty is more complex than these equations just given. As a

CV repeater arrangement would use mixed states in general, these expressions for the fidelity cannot be used to calculate the overall fidelity of a CV repeater. More generally under repeated application of any channel the final fidelity is not the product of fidelities of each constituent channel.

Another measure of performance of CV teleportation is the amount of noise added to the output state [6]. In order to achieve a high fidelity for any input state, the teleportation protocol must be operated at unity gain. The noise added under these conditions is

$$\Delta^2(X_2 \pm X_1), \tag{58}$$

where the subscripts 1 and 2 represent the modes of the entanglement and X represents the particular quadrature under consideration. The plus or minus sign represents the sign of the signal sent on the classical channel. The sign can be chosen for each quadrature and should be chosen to give the minimum value for the added noise. This expression for the added noise holds for both pure and mixed input states. Also, the effect of multiple teleportations chained together can be found by adding the noise added by each teleportation.

If the entanglement in the CV teleporter is generated from a two-mode squeezed state under general lossy conditions [Eqs. (20)–(22)] the added noise is

$$2 + (\tau_a + \tau_b)[\cosh(2r) - 1] - 2\sqrt{\tau_a\tau_b}\sinh(2r). \tag{59}$$

If the total channel throughput is constrained (i.e., $\tau_a+\tau_b-\tau_a\tau_b$ is constant) but τ_a and τ_b are varied then this expression is minimized when $\tau_a=\tau_b$. That is, the least noise is added when the squeezing source is placed in the middle of the channel. When this is the case, the added noise is

$$2(1-\tau) + \tau e^{-2r}, \tag{60}$$

and the least added noise is $2(1-\tau)$ occurring when $r\to\infty$.

In qubit based protocols, a threshold level of purity exists below which no entanglement distillation can occur [7]. The idea of the quantum repeater protocol is to perform entanglement distillation before this level is reached and use entanglement swapping to achieve entanglement at the end points.

In the case of maximum squeezing, the added noise is a linear function of the loss between end points. Attempting to use CV entanglement distillation and entanglement swapping would not decrease the noise below that of merely directly transmitting the state unless the distillation protocol decreases the effective loss of the channel. As shown in the previous sections the protocol described in [5] does not achieve any change in the effective loss of the channel.

In the next section we will describe a protocol which symmetrizes the distillation protocol from 1. At the same time, this protocol changes the effective loss on the channel so that an improvement in the performance of protocols which utilize CV entanglement is achieved.

VII. SYMMETRIZED CV DISTILLATION

The protocol developed in the references in [5] only considered the fully asymmetric loss scenario which is not optimal for CV teleportation. This section considers a symmetrized version of this protocol.

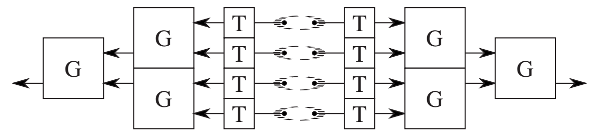

FIG. 4. This figure is similar to Fig. 3. The distillation step is now performed on both sides. Also, the loss profile has changed and it is assumed that the loss affects both modes of the source entanglement equally.

There are two main alterations which will be made to the protocol. First, the channel will have a symmetric loss profile. Second, the distillation procedure involving counting photons will be performed by both parties as shown in Fig. 4. Performing the distillation procedure on both sides will mean that it may be true that the probability of success will be reduced. However, as the loss profile is different comparing the probability of success of the symmetrized scheme and the original scheme will depend on the particular figure of merit used for to describe the application of the distilled entanglement.

After the symmetrized distillation procedure succeeds for both parties the zero- and one-photon components of the output density operator are

$$\frac{\rho_{1100}}{\rho_{0000}} = \sigma_{1100} = \left(\frac{2\eta^2-1}{\eta}\right)^2 \frac{\tau\tanh r}{1-(1-\tau)^2\tanh^2 r}, \tag{61}$$

$$\frac{\rho_{1010}}{\rho_{0000}} = \sigma_{1010} = \left(\frac{2\eta^2-1}{\eta}\right)^2 \frac{\tau(1-\tau)\tanh^2 r}{1-(1-\tau)^2\tanh^2 r} = \sigma_{0101}. \tag{62}$$

These expressions can be simplified by writing

$$\beta = \frac{2\eta^2-1}{\eta}, \tag{63}$$

$$\Lambda = \frac{\tau\tanh r}{1-(1-\tau)^2\tanh^2 r}\beta^2, \tag{64}$$

and

$$\epsilon = (1-\tau)\tanh r, \tag{65}$$

then the matrix elements of the density operation become

$$\sigma_{1100} = \Lambda \tag{66}$$

and

$$\sigma_{0101} = \sigma_{1010} = \epsilon\Lambda. \tag{67}$$

The parameter Λ can be chosen freely by setting β^2. All values of β^2 are possible ranging from zero when $T=1/\sqrt{2}$ to very large values for small T. Note that ϵ is unchanged by any choice of β. The resultant covariance matrix from this state after the Gaussification procedure is of the form given in Eq. (19) with

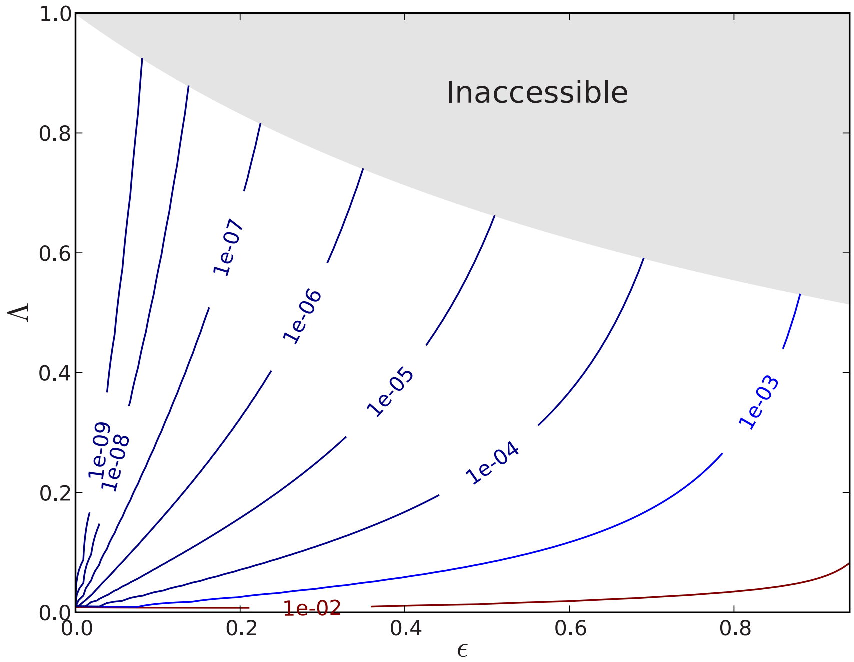

FIG. 5. (Color online) This plot shows the region of parameters ϵ and Λ which allows convergence under Gaussification. The states which are not physical and do not converge are in the region labeled "inaccessible." Contours are shown for a numerically calculated probability of successful operation of the distillation step and one round of Gaussification for a channel with 95% loss.

$$C_1 = C_2 = \frac{\Lambda^2(1-\epsilon^2)+1}{(1-\epsilon\Lambda)^2-\Lambda^2}, \tag{68}$$

$$S = \frac{2\Lambda}{(1-\epsilon\Lambda)^2-\Lambda^2}. \tag{69}$$

The Gaussification procedure will only converge if the given covariance matrix represents a physical state that is $\Gamma+i\Sigma \geq 0$. For symmetric and isotropic covariance matrices this inequality reduces to $C^2-S^2 \geq 1$ as per Eq. (26). In terms of the ϵ and Λ parameters just introduced, this requirement is

$$\Lambda < \frac{1}{1+\epsilon}. \tag{70}$$

The range of parameters Λ and ϵ parameters which will produce a state that converges under Gaussification are shown in Fig. 5.

It is clear that this procedure does not change the value of ϵ. Both the input and output states must have the same value for the parameter ϵ defined above. However, the value of Λ for the output state (assuming that β is greater than unity) will increase. By inverting the relationship between r', τ' and Λ, ϵ, one finds that

$$\tanh r' = \epsilon + \Lambda^2(1-\epsilon^2). \tag{71}$$

As $\Lambda \to (1+\epsilon)^{-1}$ one finds that $\tanh r' \to 1$. As ϵ is unchanged through the distillation procedure, one finds that the maximum effective throughput of the channel is $\tau' = 1-\epsilon$. As $\tanh r$ of the original state is less than unity the effective channel throughput of that the output state exhibits must increase (i.e., $\tau' > \tau$). Unlike the protocol described in [5] the effective loss on the entanglement is changed. Also, to achieve the highest throughput for a particular given channel a low value of initial squeezing is desired as this would make ϵ very small.

VIII. CONVERGENCE WITH GAUSSIFICATION

The Gaussification procedure involves iterating a sequence of linear optics and photon detection. As the iterations are applied the state converges toward the desired state. The rate of convergence is then an important consideration. Here we analyze the rate of convergence of the Gaussification procedure when used with the distillation procedure.

To quantify the rate of convergence of quantities of interest in CV entanglement, we have chosen to analyze the Duan Inseparability Criterion [8] using only the second-order moment (i.e., the elements of the covariance matrix) even for non-Gaussian states. For protocols involving CV quantum key distribution and reverse reconciliation [9] these moments are all that one is interested in as they saturate important bounds involving security.

Using the expressions for the covariance matrix of the converged state it is possible to show that the Duan inseparability criterion is equal to

$$C - S = \frac{\Lambda^{-1}-(1-\epsilon)}{\Lambda^{-1}+(1-\epsilon)}. \tag{72}$$

As $\Lambda \to (1+\epsilon)^{-1}$ this expression becomes

$$C - S = \epsilon. \tag{73}$$

Figure 6 shows the convergence of the inseparability criterion to these values with weak initial squeezing of $r = \text{arctanh}(0.01)$ and 100% channel throughput. Figure 7 shows convergence from the same initial states when the channel throughput is 0.05%. The convergence rate is slightly slower in the case with channel loss when the desired output entanglement is high.

All of these plots show the general properties of the convergence of the Gaussification procedure. When small entanglement output is desired from the distillation protocol, then convergence is fast. The rate of convergence is slower

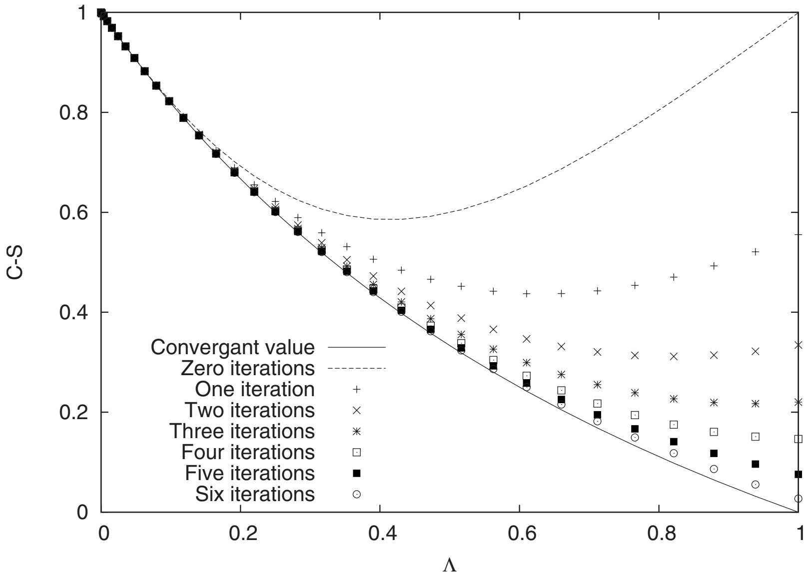

FIG. 6. This plot shows the rate of convergence of the Duan inseparability criterion as a function of the Λ parameter in a channel with no loss. The seed entanglement has squeezing parameter $r=\text{arctanh}(0.01)$. The data are generated by a numerical simulation at the particular points shown. The first ten energy eigenstates of each mode were used in the simulation. The curve is from the theoretical covariance matrix derived in the text.

as the target entanglement is higher. Also note that the introduction of loss changes the rate at which convergence is achieved. Depending on the desired accuracy and the particular application that the entanglement is used for there may be little value in adding more steps to the Gaussification procedure beyond the first few iterations.

IX. EXCESS NOISE

Loss is not the only noise process that may occur when the source entangled states are distributed. A thermal noise source may modulate the beam containing the entanglement increasing the variances of the noises that the end points see without any increase in their covariances. That is C may increase while S stays constant. The state will still remain physical under the requirement of Eq. (26) but may not satisfy the requirement that $C \pm S \leq 1$ which ensures that the effective squeeze parameter r' remain real.

When both $C+S>1$ and $C-S>1$ the state of the incoming entanglement cannot be written down as an equivalent state involving pure two-mode squeezing and loss. In fact, the condition that $C \pm S < 1$ is exactly the same as the inseparability criterion presented in [8] for this type of Gaussian state. Hence when both plus and minus conditions are not met then the states cannot have any entanglement as the condition is necessary and sufficient for Gaussian states. This puts a limit on the amount of excess noise that the protocol can be used for which corresponds exactly with the disappearance of entanglement in the initial state.

X. CONCLUSION

In this paper we have generalized the Gaussian distillation protocol presented in [5] and have presented a way of

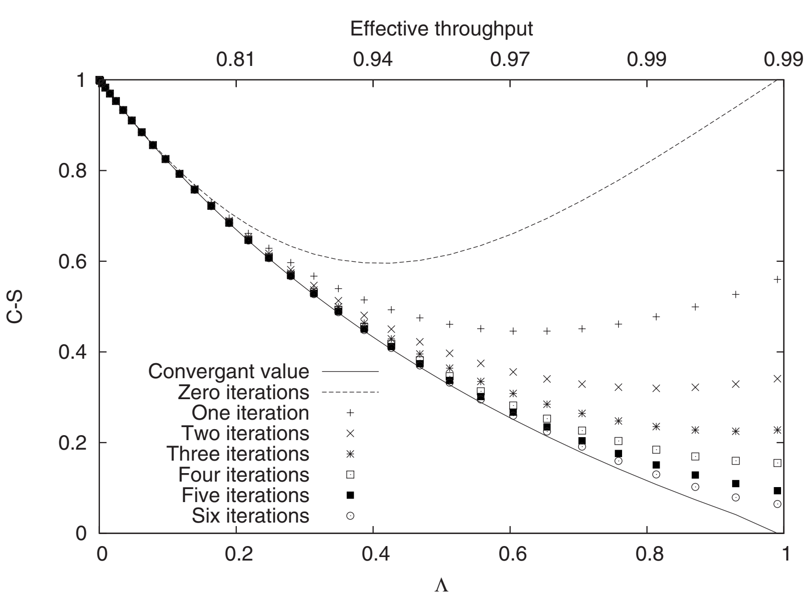

FIG. 7. The plot is the same style as Fig. 6 but the throughput of the two channels which feed the protocol is $\tau=0.05$. The top axis shows the equivalent channel loss for the converged state.

quantifying its performance through the effective squeezing and loss of the distilled output state. It was shown that the original scheme did not improve the loss on the output state compared with the loss under which the states were sent, severely limiting the effectiveness of the protocol. A symmetrized version of this protocol was then presented, and it was shown that this does reduce the loss on the output state compared with the channel loss. It was shown that the protocol is in principle able to purify greatly provided the input states were weakly squeezed.

The original distillation protocols only considered the limit of an infinite number of iterations to produce the output. Here we have shown that if the strength of the entanglement desired is not too high then only a small number of iterations is required to achieve a state close to the fully converged state.

ACKNOWLEDGMENTS

This research was supported by the Australian Research Council and the Defence Science and Technology Organization.

[1] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[2] G. Giedke and J. I. Cirac, Phys. Rev. A **66**, 032316 (2002).
[3] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. A **62**, 032304 (2000).
[4] B. Hage *et al.*, Nat. Phys. **4**, 915 (2008); R. Dong *et al.*, *ibid.* **4**, 919 (2008).
[5] D. E. Browne, J. Eisert, S. Scheel, and M. B. Plenio, Phys. Rev. A **67**, 062320 (2003); D. E. Browne, Ph.D. thesis, Imperial College London, 2004); J. Eisert, D. E. Browne, S. Scheel, and M. B. Plenio, Ann. Phys. **311**, 431 (2004).
[6] T. C. Ralph and P. K. Lam, Phys. Rev. Lett. **81**, 5668 (1998).
[7] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1998).
[8] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[9] F. Grosshans, G. V. Assche, J. Wenger, R. Brouri, N. J. Cerf, and P. Grangier, Nature (London) **421**, 238 (2003).