All Bipartite Entangled States Display Some Hidden Nonlocality

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One of the most significant and well-known properties of entangled states is that they may lead to violations of Bell inequalities and are thus inconsistent with any local-realistic theory. However, there are entangled states that cannot violate any Bell inequality, and in general the precise relationship between entanglement and observable nonlocality is not well understood. We demonstrate that a violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality can be demonstrated in a certain kind of Bell experiment for *all* entangled states. Our proof of the result consists of two main steps. We first provide a simple characterization of the set of states that do not violate the CHSH inequality even after general local operations and classical communication. Second, we prove that for each entangled state σ, there exists another state ρ not violating the CHSH inequality, such that $\rho \otimes \sigma$ violates the CHSH inequality.

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In a local-realistic theory the outcomes of local measurements are determined in advance—either stochastically or deterministically—only by unknown (or hidden) variables, and the local measurements performed.

In 1964 Bell ruled out the possibility that a local-realistic theory could reproduce all the experimental predictions given by quantum mechanics [1]. In a local-realistic theory the outcomes of local measurements are determined in advance—either stochastically or deterministically—by unknown (or hidden) variables. Bell's theorem states that the quantum mechanical probabilities for outcomes of measurements distributed in space cannot, in general, be replicated in *any* local-realistic theory. This fact is demonstrated for particular states and measurements by the violation of a Bell inequality. Entanglement is in some way responsible for this phenomenon since entangled states are required to demonstrate the violation of Bell inequalities. However, since it has been known for some time that there are entangled states that do not violate any Bell inequality [2,3], the precise relationship between entanglement and Bell inequality violation has remained poorly understood.

The definition of entangled state is made in terms of the physical resources needed for the *preparation* of the state: a multipartite state is said to be entangled if it cannot be prepared from classical correlations using local quantum operations [2]. But this definition tells us nothing about the "behavior" of the state. For example, does the state violate a Bell inequality, or is it useful in some quantum protocol such as teleportation?

It is known that every pure entangled state violates a Bell inequality [4] and that no separable state does [2], but the situation gets more complicated for mixed entangled states. There are bipartite mixed states that, though being entangled, possess a local hidden variable model (LHVM) whenever measurements are made on a single copy of the state (see, for, example [2,3]). But some of these states do violate Bell inequalities if, prior to the measurement, the state is processed by local operations and classical communication (LOCC) [5], this phenomenon has been termed *hidden nonlocality.* These protocols often involve *local filtering*, that is local measurements that if successful are followed by the Bell inequality experiment, but if unsuccessful result in the state being discarded. Moreover, by allowing joint measurements on several copies of the state in addition to the local preprocessing, it was shown that an even larger set of entangled states could be detected through their violation of a Bell inequality [6]. The question of whether all entangled states might display some kind of hidden nonlocality has remained open.

Generalizing this idea one can get a strong test of the nonlocality "hidden" in a state by combining local filtering operations and collective measurements: Perform joint local filtering operations on an arbitrarily large number of copies of the state and then a Bell inequality test on the resulting state. If the resulting probabilities violate a Bell inequality, we say that the original state violates this inequality asymptotically. In Ref. [7] it is shown that a bipartite state violates the Clauser-Horne-Shimony-Holt (CHSH) inequality [8] asymptotically if, and only if, it is distillable. This result suggests that undistillable entangled states may admit a LHVM description even when experiments are performed on many copies of the state.

Given these negative results, it seems necessary to allow still more general protocols for the nonlocality hidden in arbitrary entangled states to manifest itself. One natural possibility is to allow joint processing with auxiliary states (that do not themselves violate the Bell inequality) rather than just with more copies of the state in question. This idea has been fruitful to show that useful entanglement can be extracted from all nonseparable states [9], and in this Letter we use it to show that there is indeed some hidden nonlocality in all entangled states. This gives a conclusive answer to the long-standing question of whether or not all entangled states have hidden nonlocality [2–6]. It is important to remark that, as shown in Ref. [10], Bell inequality violation is, in general, unrelated to the teleportation power of a state.

In order to investigate the possibilities of this more general kind of hidden nonlocality we introduce the concept of a *simulable* state. We say that a bipartite state σ is simulable by classical correlations, or just *simulable*, if in any protocol (possibly involving other resources such as shared quantum states) two separated parties sharing classical correlations instead of σ can obtain the same statistics for the outcomes of the protocol. In this sense, simulable states have a completely classical behavior. Of course, our interest in this Letter is in the case where the protocol concerned is a test of nonlocality. Clearly, all separable states are simulable. A possible way to simulate a separable state is by just preparing it from classical correlations [2].

The scenario that we consider is the typical Bell-like experiment, where two parties share a bipartite system and perform local measurements on it. Alice chooses between the observables $x = 0, 1$ and obtains the outcomes $a = 0, 1$, and analogously for Bob, y and b. All the relevant experimental information is contained in the joint probability distribution for the outcomes conditioned on the choice of observables $P(a, b|x, y)$. It is convenient to define the correlation functions

$$C_{xy} \equiv P(a = b|x, y) - P(a \neq b|x, y). \tag{1}$$

By local relabeling of (a, b, x, y) it is always possible to make $C_{00}, C_{01}, C_{10} \geq 0$. With this convention, the distribution $P(a, b|x, y)$ admits a LHVM if [11], and only if, it satisfies the CHSH inequality [8]

$$C_{00} + C_{01} + C_{10} - C_{11} \leq 2. \tag{2}$$

Let us characterize the set of bipartite states that do not violate the this inequality after preprocessing.

Definition. Denote by $\mathcal{C}$ the set of bipartite states that do not violate the CHSH inequality with a single copy, even after stochastic local operations without communication.

By stochastic we mean that the operation can fail, and we do not care about the probability of failure, as long as it is strictly smaller than 1. Up to normalization, these operations allow the transformations

$$\rho \rightarrow \Omega(\rho) = \sum_i (A_i \otimes B_i)\rho(A_i \otimes B_i)^\dagger, \tag{3}$$

where A_i and B_i are, respectively, Kraus operators acting on the first and second system. This class of maps is known as the *separable maps*.

In Ref. [7] it is shown that the states in $\mathcal{C}$ do not violate CHSH even after stochastic local operations *with* communication. So the exact nature of the local operations allowed in the definition of $\mathcal{C}$ is not important. Clearly, states that do not violate the CHSH inequality asymptotically are in $\mathcal{C}$. Thus, $\mathcal{C}$ contains all undistillable states [7]. We are now able to state precisely the central result of this Letter.

Theorem. A bipartite state σ is entangled if, and only if, there exists a state $\rho \in \mathcal{C}$ such that $\rho \otimes \sigma$ is not in $\mathcal{C}$.

The consequences of this theorem are dramatic. If ρ belongs to $\mathcal{C}$, no matter how much additional classical correlation (which can always be represented by a separable state η_{sep}) we supply to it, the result $\rho \otimes \eta_{\text{sep}}$ is still in $\mathcal{C}$. Contrary, the state $\rho \otimes \sigma$ is not in $\mathcal{C}$ even if both ρ and σ are in $\mathcal{C}$ (in this sense, the result bears some resemblance with existing result on activation of distillation using infinitesimal amount of bound entanglement [12]).

The violation of CHSH manifests the qualitatively different behavior between $\rho \otimes \sigma$ and $\rho \otimes \eta_{\text{sep}}$, where η_{sep} is any separable state, and σ is any entangled state. Summarizing, for each entangled state σ there exists a protocol (which also involves the auxiliary state ρ associated with the theorem) in which σ cannot be substituted by an arbitrarily large amount of classical correlations without changing the result: Entangled states are the ones that cannot be simulated by classical correlations.

A remark is in order. The theorem requires that ρ does not violate the CHSH inequality, but it says nothing about other inequalities, like the ones presented in Ref. [13]. However, even if ρ violates another inequality, we know by definition that ρ, and thus $\rho \otimes \eta_{\text{sep}}$, does not violate the CHSH inequality. Hence, in the protocol that we are considering, σ cannot be simulated by any classical correlations η_{sep}.

The proof of the above theorem has two main ingredients. First, we note that $\mathcal{C}$ is a convex set and provide a characterization of $\mathcal{C}$ in terms of witnesslike operators that detect CHSH violation. Second, we use convexity arguments similar to those in Ref. [9] to prove by contradiction that there exists some $\rho \in \mathcal{C}$ such that one of these witnesses may be constructed for $\rho \otimes \sigma$ whenever σ is entangled. To carry this argument through we require a characterization of the separable completely positive maps between Bell diagonal states that can be found in Ref. [14]. First, we describe the witnesses for CHSH violation.

Lemma 1. A bipartite state ρ acting on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ belongs to $\mathcal{C}$ if, and only if, it satisfies

$$\text{tr}[\rho(A \otimes B)H_\theta(A \otimes B)^\dagger] \geq 0, \tag{4}$$

for all matrices of the form $A:\mathbb{C}^2 \rightarrow \mathcal{H}_{\mathcal{A}}$, $B:\mathbb{C}^2 \rightarrow \mathcal{H}_{\mathcal{B}}$ and all numbers $\theta \in [0, \pi/4]$, where

$$H_\theta \equiv \mathbb{I} \otimes \mathbb{I} - \cos\theta\sigma_x \otimes \sigma_x - \sin\theta\sigma_z \otimes \sigma_z, \tag{5}$$

$\mathbb{I}$ being the 2×2 identity matrix and $\{\sigma_i\}_{i=x,y,z}$ the Pauli matrices.

Proof. To start off, we recall that, without preprocessing, a two-qubit state ϱ violates the CHSH inequality if and only if $\mu_1^2 + \mu_2^2 > 1$, where μ_1 and μ_2 are the two largest singular values of the 3×3 real matrix $R_{ij} = \mathrm{tr}[\varrho \sigma_i \otimes \sigma_j]$, with indices $i, j = x, y, z$ [15]. Equivalently, (μ_1, μ_2) derived from ϱ must lie outside the unit circle $\mu_1^2 + \mu_2^2 = 1$, which is true if and only if there exists $\theta \in [0, 2\pi]$ such that

$$\mu_1 \cos\theta + \mu_2 \sin\theta > 1. \tag{6}$$

Now, it is also well known that by appropriate local unitary transformations U, V, it is always possible to arrive at a local basis such that R is diagonal with $\mu_1 = R_{xx}$ and $\mu_2 = R_{zz}$. From the definition of R it follows that

$$\mu_1 \cos\theta = \mathrm{tr}[(U \otimes V)\varrho(U \otimes V)^\dagger(\cos\theta\sigma_x \otimes \sigma_x)], \tag{7}$$

with the expression for $\mu_2 \sin\theta$ involving obvious modifications. Since singular values are non-negative, it thus follows that if ϱ violates the CHSH inequality then there exist $U, V \in \mathrm{SU}(2)$, $\theta \in [0, \pi/4]$ such that

$$\mathrm{tr}[\varrho(U \otimes V)^\dagger H_\theta(U \otimes V)] < 0. \tag{8}$$

On the other hand suppose that there exists some (U, V, θ) satisfying (8). Thus we have $R_{xx}\cos\theta + R_{zz}\sin\theta > 1$. If we assume $\mu_1 \geq \mu_2$, the inequalities $|R_{xx}|, |R_{zz}| \leq \mu_1 \leq 1$ follow from the definition of singular values and the well-known fact that all singular values of R are less than one. Since $0 \leq \theta \leq \pi/4$ both R_{xx} and R_{zz} must be positive and since $\cos\theta \geq \sin\theta$ we may assume without loss of generality that $R_{xx} \geq R_{zz}$. The singular values of R obey the inequality $|R_{xx} + R_{zz}| \leq \mu_1 + \mu_2$ [16] and as a result we find $\mu_1 \cos\theta + \mu_2 \sin\theta > 1$ so ϱ violates the CHSH inequality. Thus ϱ violates the CHSH inequality if and only if (8) holds.

Let us come back to the question of CHSH violation after local filtering operations. Assume that ρ violates the CHSH inequality after stochastic local operations. Let us show that it must violate (4) for some (A, B, θ). In Ref. [7] it is proven that, if a state violates the CHSH inequality then it can be transformed by stochastic local operations into a two-qubit state which also violates the CHSH inequality. Therefore, there must exist a separable map Ω with two-qubit output, such that the state $\Omega(\rho)$ satisfies condition (8) for some (U, V, θ), denote them by (U_0, V_0, θ_0). Clearly, if $\Omega(\rho)$ satisfies (8) there must exist at least one value of i in the Kraus decomposition of (3) such that $(A_i \otimes B_i)\rho(A_i \otimes B_i)^\dagger$ also satisfies (8). This implies that ρ violates (4) for $A = A_i^\dagger U_0^\dagger$, $B = B_i^\dagger V_0^\dagger$, and $\theta = \theta_0$. This proves one direction of the lemma, let us show the other.

Assume that ρ violates (4) for (A_0, B_0, θ_0). It is straightforward to see that ρ violates the CHSH inequality after stochastic LOCC. Consider operation that transforms ρ into $(A_0 \otimes B_0)^\dagger \rho (A_0 \otimes B_0)$. By assumption, the final state satisfies (8) with $U = V = \mathbb{1}$ and $\theta = \theta_0$, which implies that it violates the CHSH inequality. $\square$

The above characterization is interesting on its own. Here we use it to prove our main result.

Proof of the theorem. If σ is separable, then $\rho \in \mathcal{C}$ implies $\rho \otimes \sigma \in \mathcal{C}$. This is so because the preprocessing by LOCC on ρ, before the Bell experiment, can include the preparation of the state σ. Let us prove the other direction of the theorem.

From now on σ is an arbitrary entangled state acting on $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Let us show that there always exists an ancilla state $\rho \in \mathcal{C}$ such that $\rho \otimes \sigma \notin \mathcal{C}$. Fix ρ to act on the bipartite Hilbert space $[\mathcal{H}_{\mathcal{A}'} \otimes \mathcal{H}_{\mathcal{A}''}] \otimes [\mathcal{H}_{\mathcal{B}'} \otimes \mathcal{H}_{\mathcal{B}''}]$, where $\mathcal{H}_{\mathcal{A}'} = \mathcal{H}_\mathcal{A}$, $\mathcal{H}_{\mathcal{B}'} = \mathcal{H}_\mathcal{B}$, and $\mathcal{H}_{\mathcal{A}''} = \mathcal{H}_{\mathcal{B}''} = \mathbb{C}^2$ (see Fig. 1).

Our aim is to prove that the state $\rho \otimes \sigma$ violates (4) for some choice of A, B, and θ. In particular, let

$$\tilde{A} = |\Phi_{\mathcal{A}\mathcal{A}'}\rangle \otimes \mathbb{1}_{\mathcal{A}''}, \qquad \tilde{B} = |\Phi_{\mathcal{B}\mathcal{B}'}\rangle \otimes \mathbb{1}_{\mathcal{B}''}, \qquad \theta = \pi/4,$$

where $|\Phi_{\mathcal{A}\mathcal{A}'}\rangle$ is the maximally entangled state between the spaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_{\mathcal{A}'}$ (which have the same dimension), and $\mathbb{1}_{\mathcal{A}''}$ is the identity matrix acting on $\mathbb{C}^2$ (analogously for Bob). One can show that for any ρ

$$\mathrm{tr}[\rho \otimes \sigma(\tilde{A} \otimes \tilde{B})H_{\pi/4}(\tilde{A} \otimes \tilde{B})^\dagger] = \nu\, \mathrm{tr}[\rho(\sigma^T \otimes H_{\pi/4})],$$

where ν is a positive constant and σ^T stands for the transpose of σ. The requirement that inequality (4) is violated with $\theta = \pi/4$, $A = \tilde{A}$, $B = \tilde{B}$ becomes

$$\mathrm{tr}[\rho(\sigma^T \otimes H_{\pi/4})] < 0. \tag{9}$$

For convenience, in the rest of the proof we allow ρ to be unnormalized. The only constraints on the matrices $\rho \in \mathcal{C}$ are positive semidefiniteness ($\rho \in \mathcal{S}^+$), and satisfiability of all the inequalities (4) in Lemma 1. $\mathcal{C}$ is now a convex cone, and its dual cone is defined as

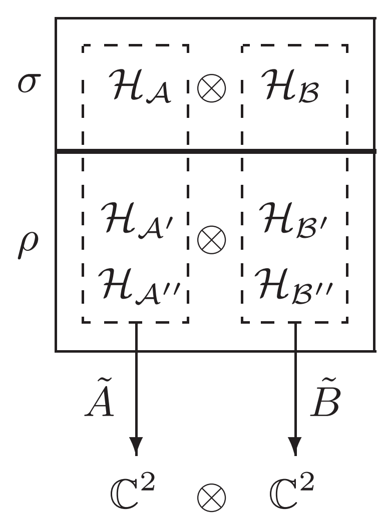

FIG. 1. Schematic diagram illustrating the local filtering operations $\tilde{A}$ and $\tilde{B}$ involved in our protocol. The solid box on top is a schematic representation of the state σ, whereas that on the bottom is for the ancilla state ρ. Left and right dashed boxes, respectively, enclose the subsystems possessed by the two experimenters $\mathcal{A}$ and $\mathcal{B}$.

$$\mathcal{C}^* = \{X:\mathrm{tr}[\rho X] \geq 0, \quad \forall\ \rho \in \mathcal{C}\}, \tag{10}$$

where X are Hermitian matrices. Farkas' lemma [17] states that all matrices in $\mathcal{C}^*$ can be written as non-negative linear combinations of matrices $P \in \mathcal{S}^+$ and matrices $(A \otimes B)H_\theta(A \otimes B)^\dagger$ with $A:\mathbb{C}^2 \to \mathcal{H}_{\mathcal{A}'} \otimes \mathcal{H}_{\mathcal{A}''}$ and $B:\mathbb{C}^2 \to \mathcal{H}_{\mathcal{B}'} \otimes \mathcal{H}_{\mathcal{B}''}$.

We now show that there always exists $\rho \in \mathcal{C}$ satisfying (9) by supposing otherwise and arriving at a contradiction. Suppose that for all $\rho \in \mathcal{C}$ the inequality $\mathrm{tr}[\rho(\sigma^T \otimes H_{\pi/4})] \geq 0$ holds, and thus the matrix $\sigma^T \otimes H_{\pi/4}$ belongs to $\mathcal{C}^*$. Applying Farkas' lemma [17] we can write

$$\sigma^T \otimes H_{\pi/4} = \int dx (A_x \otimes B_x) H_{\theta_x} (A_x \otimes B_x)^\dagger + \int dy P_y,$$

which is equivalent to

$$\sigma^T \otimes H_{\pi/4} - \int dx \Omega_x (H_{\theta_x}) \geq 0, \tag{11}$$

where each Ω_x is a separable map (3). We prove in Lemma 2 that (11) requires that σ is separable, which gives the desired contradiction. Thus the result is proven. □

In order to arrive at a contradiction from (11) it is necessary to use the constraint that the maps Ω_x are separable. The problem of characterizing the separable maps is hard in general since it maps onto the separability problem for bipartite states. However, it turns out only to be necessary to determine the set of separable maps that take Bell diagonal states to Bell diagonal states and this can be done exactly [14]. This characterization may be used to prove the following lemma and thereby our theorem.

Lemma 2. Let $\Omega_\theta:[\mathbb{C}^2] \otimes [\mathbb{C}^2] \to [\mathcal{H}_{\mathcal{A}} \otimes \mathbb{C}^2] \otimes [\mathcal{H}_{\mathcal{B}} \otimes \mathbb{C}^2]$ be a family of maps, separable with respect to the partition denoted by the brackets. Let μ be a unit-trace, positive semidefinite matrix acting on $[\mathcal{H}_{\mathcal{A}}] \otimes [\mathcal{H}_{\mathcal{B}}]$ such that

$$\mu^T \otimes H_{\pi/4} - \int dx \Omega_x (H_{\theta_x}) \geq 0, \tag{12}$$

where H_θ is defined in (5), then μ has to be separable. This lemma is proven in the online version of this paper [18].

In summary, for each entangled state σ there exists a protocol in which σ cannot be substituted by an arbitrarily large amount of classical correlations, without changing the result: Entangled states are the ones that cannot be simulated by classical correlations.

This provides us with a new interpretation of entanglement in terms of the behavior of the states, in contrast with the usual definition in terms of the preparation of the states. It also gives a conclusive answer to the long-standing question of whether all entangled states display some hidden nonlocality [2–6].

Differently, one can be interested in the set of bipartite states σ which cannot be simulated by classical correlations in scenarios where no other kind of entanglement is present. That is, σ may be processed with more copies of itself $\sigma^{\otimes n}$ but never with different entangled states ρ. Following [7] we say that a state σ violates a Bell inequality asymptotically, if after jointly processing by LOCC a sufficiently large number of copies of σ, the result violates the Bell inequality. In Ref. [7] it is proven that the states violating the CHSH inequality asymptotically are the distillable ones. This, together with the results of this Letter, establishes an appealing picture:

entangled $\Longleftrightarrow$ nonsimulable distillable $\Longleftrightarrow$ asymptotic violation of the CHSH inequality.

Entangled states are, by definition, the ones that cannot be generated from classical correlations and local quantum operations. We have shown that in the bipartite case one can equivalently define entangled states as the ones that cannot be simulated by classical correlations.

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