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On the Minimal Resolution Conjecture for $\mathbf{P}^3$

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Abstract. The Minimal Resolution Conjecture that was formulated by A Lorenzini [2] has been shown to hold true for $\mathbf{P}^2$, $\mathbf{P}^3$ [3] they made use of Quadrics, here we tackle the $\mathbf{P}^3$ case but making use of variant methods i.e. mainly the method of Horace (mèthode d'Horace) to evaluate sections of fibres at given points. This was introduced by A Hirschowitz in 1984 in a letter he wrote to R Hartshorne. For a general set of points $P_1, \ldots, P_m \in \mathbf{P}^3$, for a positive integer m, we show that the map $H^0\big(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^3}(d+1)_{|P_i}$ is of maximal rank.

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1. Introduction

Let $\mathbf{k}$ be an algebraically closed field, $\mathbf{P}^n$ be a projective space over $\mathbf{k}$ and $R = \mathbf{k}[x_0, x_1, ..., x_n]$ be the homogeneous coordinate ring of $\mathbf{P}^n$. If $M = \{P_1, P_2, ..., P_m\}$ is a general set of m points in $\mathbf{P}^n$, then the ideal, I_M of polynomials vanishing at these m points has the minimal resolution attained. In particular when $n = 3$, i.e. for $\mathbf{P}^3$

The consequence is that the homogeneous ideal $I_M \subset k[X_0, X_1, X_2, X_3]$ has the expected number $\left(\frac{1}{2}d(d+2)(d+3) - 3m\right)_+$ of minimal generators of degree $d+1$ and the expected number $\left(\frac{1}{2}d(d+2)(d+3) - 3m\right)_-$ of minimal relations of degree $d+1$,
where $(x)_+ = \max(x, 0)$ and $(x)_- = \max(-x, 0)$.

2. Preliminary Notes

A guide for some of the symbols used

In general we have the sequences below for transformations:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \Omega_{\mathbf{P}^3}(d) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} & \longrightarrow & \Omega_{\mathbf{P}^3}(d+1) & \longrightarrow & \Omega_{\mathbf{P}^2}(d+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(d) & \longrightarrow & \Omega_{\mathbf{P}^3}|_{\mathbf{P}^2}(d+1) & \longrightarrow & \Omega_{\mathbf{P}^2}(d+1) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The sequence for quotients;

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_{\mathbf{P}^2}(d)|_{\overline{T}_l} & \twoheadrightarrow & D'|_{\overline{T}_l} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}|_{W_l} & \twoheadrightarrow & D|_{W_l} \cong \mathcal{O}_{\mathbf{P}^2}|_{W_l} \oplus D'|_{\overline{T}_l} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbf{P}^2}(d)|_{W_l} & = & \mathcal{O}_{\mathbf{P}^2}|_{W_l} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$\boldsymbol{H}'_{\Omega,3}(d+1; r, s, t) \equiv \boldsymbol{H}(\Omega_{\mathbf{P}^3}(d+1), \Omega_{\mathbf{P}^2}(d+1), r, s, t)$, where r represents fibres of dim 3, s fibres of dim 2 and t represents 1 dimensional fibre in a quotient of $\Omega_{\mathbf{P}^2}(d)_{|Q}$ for $Q \in \mathbf{P}^2$

$\boldsymbol{H}'_{\mathcal{O},3}(d-1; u, v, w) \equiv \boldsymbol{H}(\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}, \mathcal{O}_{\mathbf{P}^2}(d), u, v, w)$, where u represents fibres of dim 3, v line bundles and w represents a 2 dimensional fibre in a quotient of $\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}_{|Q}$ for $Q \in \mathbf{P}^2$

$\boldsymbol{H}'_{\Omega,3}(d+1; r, s, t)$ is defined for specific r, s and t as in Lemma 1 while $\boldsymbol{H}'_{\Omega,3}(d+1)$ is the general case i.e. for all non negative integers r, s, t

$\boldsymbol{H}'_{\mathcal{O},3}(d-1; u, v, w)$ is defined for specific u, v and w as in Lemma 2 while $\boldsymbol{H}'_{\mathcal{O},3}(d-1)$ is the general case i.e. for all non negative integers u, v, w

3. Main Results

Hypothesis $\boldsymbol{H}'_{\Omega,3}(d+1;r,s,t)$. *There exists $R_1,\dots,R_r \in \mathbf{P}^3$, $S_1,\dots,S_s \in \mathbf{P}^2$, $T_1,\dots,T_t \in \mathbf{P}^2$ and quotients $\Omega_{\mathbf{P}^2|T_\ell} \twoheadrightarrow D'_{|T_\ell}$ such that the restriction map* (1) *is bijective.*

$$(1)\qquad H^0\big(\mathbf{P}^3,\Omega_{\mathbf{P}^3}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{r}\Omega_{\mathbf{P}^3}(d+1)_{|R_i} \oplus \bigoplus_{j=1}^{s}\Omega_{\mathbf{P}^2}(d+1)_{|S_j} \oplus D'_{|T_\ell}$$

Hypothesis $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,w)$. *There exists $U_1,\dots,U_u \in \mathbf{P}^3$, $V_1,\dots,V_v \in \mathbf{P}^2$, $W_1,\dots,W_w \in \mathbf{P}^2$ and quotients $\mathcal{O}^{\oplus 3}_{\mathbf{P}^3|W_\ell} \twoheadrightarrow D(V)_{|W_\ell} \twoheadrightarrow \mathcal{O}_{\mathbf{P}^2|W_\ell}$ with $\dim D(V)_{|W_\ell} = 2$ such that the restriction map* (2) *is bijective.*

$$(2)\quad H^0\big(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}\big) \longrightarrow \bigoplus_{i=1}^{u}\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}_{|U_i} \oplus \bigoplus_{j=1}^{v}\mathcal{O}_{\mathbf{P}^2}(d)_{|V_j} \oplus D(V)_{|W_\ell}$$

From the above hypotheses we define two lemmas and give their proofs.

Lemma 1. (a) *If $\boldsymbol{H}'_{\Omega,3}(d+1;r,s,t)$ is true, then we have*

$$(3a)\qquad 2s+t \le h^0\big(\Omega_{\mathbf{P}^2}(d+1)\big) = d(d+2),$$

$$(3b)\qquad 2s+t \equiv h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) = \frac{1}{2}d(d+2)(d+3) \equiv d(d-1) \pmod 3,$$

$$(3c)\qquad r = \frac{1}{3}\left(h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) - 2s - t\right)$$

(b) *If $d \ge 0$, $s \ge 0$, and $t \ge 0$ are non negative integers verifying* (3a) *and* (3b), *then the r defined by* (3c) *satisfies $r \ge 0$.*

Proof. (a) Suppose $\boldsymbol{H}'_{\Omega,3}(d+1;r,s,t)$ is true then we have the following exact sequences:

$$\begin{array}{ccccccccc} 0\to & H^\circ(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}) & \longrightarrow & H^\circ(\mathbf{P}^3,\Omega_{\mathbf{P}^3}(d+1)) & \xrightarrow{\gamma} & H^\circ(\mathbf{P}^2,\Omega_{\mathbf{P}^2}(d+1)) & \to 0 \\ & \downarrow \text{inj} & & \alpha\downarrow\cong & & \beta\downarrow\text{surj} & \\ 0\to & \bigoplus_{i=1}^r \Omega_{\mathbf{P}^3|R_i} & \to & \bigoplus_{i=1}^r \Omega_{\mathbf{P}^3|R_i} \oplus \bigoplus_{i=1}^s \Omega_{\mathbf{P}^2|S_i} \oplus D'_{|T_l} & \to & \bigoplus_{i=1}^s \Omega_{\mathbf{P}^2|S_i} \oplus D'_{|T_l} & \to 0 \end{array}$$

Bijectivity of α and surjectivity of γ impose surjectivity on β thus

$$2s+t \le h^0\big(\Omega_{\mathbf{P}^2}(d+1)\big) \qquad (3a)$$

From α's bijectivity we have that $3r+2s+t = h^\circ(\Omega_{\mathbf{P}^3}(d+1))$ thus

$$r = \tfrac{1}{3}\left(h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) - 2s - t\right) \qquad (3c)$$

From (3c) and since r is a non negative integer then $h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) - 2s - t \equiv 0$

(mod 3)

$$\text{thus } 2s + t \equiv h^0\big(\Omega_{\mathbf{P}^3}(d+1\big) \pmod 3 \qquad (3b)$$

proof for (b),
r as defined by (3c) is bounded below by $\frac{1}{3}(h^0\big(\Omega_{\mathbf{P}^3}(d+1)\big) - d(d+2))$ i.e. equality on (3a) thus

$$r \geq \tfrac{1}{3}(d+2)(d+1)d \geq 0 \quad \text{for all } d \geq 0 \qquad \square$$

Lemma 2. (a) *If* $\boldsymbol{H}'_{\mathcal{O},3}(d-1; u, v, w)$ *is true, then we have*

$$v{+}w \leq h^0(\mathcal{O}_{\mathbf{P}^2}(d)) \text{ and } w \leq h^0(\Omega_{\mathbf{P}^3}(d)) \qquad (4a)$$

$$v + 2w \equiv 0 \pmod 3 \qquad (4b)$$

$$u = \tfrac{1}{3}(h^0\big(\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} - v - 2w)) \qquad (4c)$$

(b) *If* $d \geq 1$*,* $v \geq 0$*, and* $w \geq 0$ *are non negative integers verifying* (4a) *and* (4b)*, then the* u *defined by* (4c) *satisfies* $u \geq 0$*.*

Proof. (a) Suppose $\boldsymbol{H}'_{\mathcal{O},3}(d-1; u, v, w)$ is true then we have the following exact sequences:

$$0 \longrightarrow H^\circ(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d) \longrightarrow H^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} \overset{\overline{\gamma}}{\longrightarrow} H^\circ(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) \longrightarrow 0$$

$$\downarrow \text{inj} \qquad\qquad \overline{\alpha} \downarrow \cong \qquad\qquad \overline{\beta} \downarrow \text{surj}$$

$$\bigoplus_{i=1}^{u} \mathcal{O}^{\oplus 3}_{\mathbf{P}^3|U_i} \oplus D'_{|W_l} \to \bigoplus_{i=1}^{u} \mathcal{O}^{\oplus 3}_{\mathbf{P}^3|U_i} \oplus \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2|V_i} \oplus D_{|W_l} \to \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2|V_i} \oplus \mathcal{O}_{\mathbf{P}^2|W_l}$$

Bijectivity of $\overline{\alpha}$ and surjectivity of $\overline{\gamma}$ impose surjectivity on $\overline{\beta}$ thus $v + w \leq h^0\big(\mathcal{O}_{\mathbf{P}^2}(d)\big)$ while $w \leq h^0(\Omega_{\mathbf{P}^3}(d))$ is true for all $d \geq 0$ (4a)

From $\overline{\alpha}$'s bijectivity we have that $3u + v + 2w = h^\circ(\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3})$ thus

$$u = \tfrac{1}{3}\left(h^0\big(\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}\big) - v - 2w\right) \qquad (4c)$$

From (4c) we have $3u$, $h^0\big(\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}\big)$ are $\equiv 0 \pmod 3$ then $v + 2w \equiv 0 \pmod 3$ thus

$$v + 2w \equiv 0 \pmod 3 \qquad (4b)$$

Proof (b)
u as defined by (4c) is bounded below by $\frac{1}{6}[(d+2)(d+1)d - d(d+1)(d+2)] = 0$ due to equality in (4a) thus $u \geq 0$ for all $d \geq 0$

$\square$

Hypothesis $\boldsymbol{H}_{\Omega,3}(d+1)$. *For all* $s \geq 0$*, all* $0 \leq t \leq 1$ *and* r *verifying* (3a)*,* (3b)*, and* (3c)*, the hypothesis* $\boldsymbol{H}'_{\Omega,3}(d+1; r, s, t)$ *is true.*

Hypothesis $\boldsymbol{H}_{\mathcal{O},3}(d-1)$. *For all* $v \geq 0$*, all* $0 \leq w \leq 1$ *and* u *verifying* (4a)*,* (4b)*, and* (4c)*, the hypothesis* $\boldsymbol{H}'_{\mathcal{O},3}(d-1; u, v, w)$ *is true.*

Goal. To prove $\boldsymbol{H}_{\Omega,3}(d+1)$ and $\boldsymbol{H}_{\mathcal{O},3}(d-1)$ for $d \geq 1$.

This goal should suffice because we have the following theorem which is the main part dealing with Maximal Rank property that we mentioned we were going to show at the beginning.

Theorem 3. *Suppose $\boldsymbol{H}_{\Omega,3}(d+1)$ is true. Then for any non negative integer m, there exists a set $S = \{P_1, P_2, \ldots, P_m\}$ of m points in $\mathbf{P}^3$ such that the evaluation map*

$$H^0\big(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^3|P_i}$$

is of maximal rank.

Proof. Since $\boldsymbol{H}_{\Omega,3}(d+1)$ is true then there exists a non negative integer r, such that we can find a set $R = \{P_1, P_2, \ldots, P_r\}$ of r points in $\mathbf{P}^3$ such that the evaluation map

$$H^0\big(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)\big) \longrightarrow \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^3|P_i}$$

is bijective. That is r is the critical number of points for truth of $\boldsymbol{H}_{\Omega,3}(d+1)$.

Suppose we have any set $S = \{P_1, P_2, \ldots, P_m\}$ of m points in $\mathbf{P}^3$ where m is an arbitrary non negative integer.
Then there exists 3 possiblities that

(i) If $m < r$, i.e. we have less points than the critical number of points, our map will surject, i.e. since α is bijective and γ surjective then it follows that β is surjective, i.e.

$$\begin{array}{ccc} H^0\big(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)\big) & \overset{\beta}{\twoheadrightarrow} & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^3|P_i} \\ \searrow \alpha & & \uparrow \gamma \\ & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^3|P_i} \oplus \bigoplus_{i=m+1}^{r} \Omega_{\mathbf{P}^3|P_i} & \end{array}$$

(ii) If $m = r$, i.e. we have the same number of points as the critical number, our map will biject, i.e. then since α is bijective and γ the identity then it follows that β is bijective, i.e.

$$\begin{array}{ccc} H^0\big(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)\big) & \overset{\beta}{\longrightarrow} & \bigoplus_{i=1}^{m} \Omega_{\mathbf{P}^3|P_i} \\ \searrow \alpha & & ||\gamma \end{array}$$

$$\bigoplus_{i=1}^{n} \Omega_{\mathbf{P}^3|P_i} \oplus \bigoplus_{i=n+1}^{r} \Omega_{\mathbf{P}^3|P_i}$$

(iii) If $m > r$, i.e. we have more points than the critical number, our map will inject, i.e. then since α is bijective and γ injective then it follows that β is injective.

$$H^0\big(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)\big) \overset{\beta}{\hookrightarrow} \bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^3|P_i} \oplus \bigoplus_{i=r+1}^{m} \Omega_{\mathbf{P}^3|P_i}$$

$$\searrow \alpha \qquad \nearrow \gamma$$

$$\bigoplus_{i=1}^{r} \Omega_{\mathbf{P}^3|P_i}$$

Thus we have proved the maximal rank property (injectivity, surjectivity or bijectivity) for any non negative integer m. $\square$

Lemma 4. *The Initial Cases*
(a)$\boldsymbol{H}_{\Omega,3}(d+1)$ *is true when* $d = 2$ *and*
(b)$\boldsymbol{H}_{\mathcal{O},3}(d-1)$ *is true when* $d = 1$

Proof. (a) $d = 2$ then $h^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)^{\oplus 3}) \leq 3r \leq h^\circ(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(3))$ i.e. $12 \leq 3r \leq 20$ i.e. $4 \leq r \leq 6$
and so $\boldsymbol{H}_{\Omega,3}(d+1)$ follows from truth of; (i)$\boldsymbol{H}'_{\Omega,3}(3; 6, 1, 0)$, (ii)$\boldsymbol{H}'_{\Omega,3}(3; 5, 2, 1)$, (iii)$\boldsymbol{H}'_{\Omega,3}(3; 4, 0, 0)$

(i) $\boldsymbol{H}'_{\mathcal{O},3}(1; 3, 3, 0)$ implies $\boldsymbol{H}'_{\Omega,3}(3; 6, 1, 0)$ by lemma 5 i.e.

$$0 \longrightarrow H^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1)^{\oplus 3}) \longrightarrow H^\circ(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(3)) \longrightarrow H^\circ(\mathbf{P}^2, \Omega_{\mathbf{P}^2}(3)) \longrightarrow 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbf{P}^3}(1)^{\oplus 3}_{|R_i} \oplus \bigoplus_{i=4}^{6} \mathcal{O}_{\mathbf{P}^2}(2)_{|R_i} \to \bigoplus_{i=1}^{6} \Omega_{\mathbf{P}^3|R_i} \oplus \Omega_{\mathbf{P}^2|S} \to \bigoplus_{i=4}^{6} \Omega_{\mathbf{P}^2|R_i} \oplus \Omega_{\mathbf{P}^2|S}$$

and $\boldsymbol{H}'_{\mathcal{O},3}(1; 0, 3, 0)$ implies $\boldsymbol{H}'_{\mathcal{O},3}(1; 3, 3, 0)$as follows;

$$0 \longrightarrow \underset{(0,3,0)}{\overset{3}{\mathcal{O}_{\mathbf{P}^3}^{\oplus 3}}} \longrightarrow \underset{(3,3,0)}{\overset{12}{\mathcal{O}_{\mathbf{P}^3}(1)^{\oplus 3}}} \longrightarrow \underset{(3,0,0)}{\overset{9}{\mathcal{O}_H(1)^{\oplus 3}}} \longrightarrow 0$$

and is true while $\boldsymbol{H}'_{\mathcal{O},3}(1; 0, 3, 0)$ is proved below (b).

(ii) $\boldsymbol{H}'_{\mathcal{O},3}(1; 3, 1, 1)$ implies $\boldsymbol{H}'_{\Omega,3}(3; 5, 2, 1)$ just like in (i)above and again as we have done above,$\boldsymbol{H}'_{\mathcal{O},3}(1; 3, 1, 1)$ can be reduced to $\boldsymbol{H}'_{\mathcal{O},0}(1; 0, 1, 1)$ is proved below (b).

(iii) $\boldsymbol{H}'_{\mathcal{O},3}(1; 4, 0, 0)$ implies $\boldsymbol{H}'_{\Omega,3}(3; 4, 4, 0)$ just like in (i)above and again as we have done above,$\boldsymbol{H}'_{\mathcal{O},3}(1; 3, 1, 1)$ can be reduced to $\boldsymbol{H}'_{\mathcal{O},0}(1; 1, 0, 0)$ which is true.

(b) For $\boldsymbol{H}_{\mathcal{O},3}(0)$ it follows from 3 cases;

(i)$\boldsymbol{H}'_{\mathcal{O},3}(0;1,0,0)$ is true since we have triples of constants being evaluated at a point,and so the map $\rho: H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}^{\oplus 3}) \longrightarrow \mathcal{O}_{\mathbf{P}^3|P}^{\oplus 3}$ is bijective.

(ii)For $\boldsymbol{H}'_{\mathcal{O},3}(0;0,3,0)$ consider the following:

$$\begin{array}{ccccccc} 0 \longrightarrow H^0(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(1)) & \longrightarrow & H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}^{\oplus 3}) & \overset{\pi}{\longrightarrow} & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \longrightarrow 0 \\ \| & & & & \\ 0 & & \downarrow \phi & & \downarrow \rho \\ & & \bigoplus_{i=1}^3 \mathcal{O}_{\mathbf{P}^2}(1)_{|P_i} & \overset{\iota}{\longrightarrow} & \bigoplus_{i=1}^3 \mathcal{O}_{\mathbf{P}^2}(1)_{|P_i} \end{array}$$

Since ι is an identity, ρ is bijetive and π is bijective, and $\phi = \rho \circ \pi$ thus bijective, and $\boldsymbol{H}'_{\mathcal{O},3}(0;0,3,0)$ true for 3 general points P_1, P_2, P_3 in $\mathbf{P}^2$ i.e. they generate it.

(iii)$\boldsymbol{H}'_{\mathcal{O},3}(0;0,1,1)$ follows from:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \ker \varphi_{|W} & \longrightarrow & H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}^{\oplus 3}) & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(1)_{|W} \oplus D'_{|W} & \longrightarrow & 0 \\ & \downarrow \pi & & \downarrow \phi & & \| \rho & & \\ 0 \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(1)_{|V} & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(1)_{|V} \oplus \mathcal{O}_{\mathbf{P}^2}(1)_{|W} \oplus D'_{|W} & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(1)_{|W} \oplus D'_{|W} & \longrightarrow & 0 \end{array}$$

π is bijective and ρ an identity hence ϕ is bijective. □

Lemma 5. *First Vectorial Method*
Suppose d, r, s, t satisfy (3a), (3b)*, and* (3c)*. Write $h^0(\Omega_{\mathbf{P}^2}(d+1)) - 2s - t = 2v + w$ with v, w non negative integers and $0 \le w \le 1$. Set $u = r - v - w$. If u is a non negative integer, then $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,w)$ implies $\boldsymbol{H}'_{\Omega,3}(d+1;r,s,t)$.*

Proof. If $h^0(\Omega_{\mathbf{P}^2}(d+1)) - 2s - t = 0 = 2v + w$ i.e. $2s + t = h^0(\Omega_{\mathbf{P}^2}(d+1)) - 2s - t$ and thus we have

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ H^0(\mathbf{P}^2, \Omega_{\mathbf{P}^2}(d+1)) & \overset{\alpha}{\longrightarrow} & \bigoplus_{i=1}^s \Omega_{\mathbf{P}^2}(d+1)_{|S_i} \oplus D'_{|T_l} \\ \uparrow & & \uparrow \\ H^\circ(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1) & \overset{\beta}{\longrightarrow} & \bigoplus_{i=1}^r \Omega_{\mathbf{P}^3}(d+1)_{|R_i} \oplus \bigoplus_{i=1}^s \Omega_{\mathbf{P}^2|S_i} \oplus D'_{|T_l} \\ \uparrow & & \uparrow \\ H^\circ(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}) & \overset{\gamma}{\longrightarrow} & \bigoplus_{i=1}^u \mathcal{O}_{\mathbf{P}^3}(d-1)_{|U_i}^{\oplus 3} \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

suppose γ is bijective (it is Lemma 8) and since α is bijective by proposition 13 then β is bijective.

Next If $h^0(\Omega_{\mathbf{P}^2}(d+1)) - 2s - t = 2v + w \neq 0$ then we have two possibilities i.e. that $2v + w$ is an even or odd positive integer.

Suppose $2v+w$ is even then $v=\frac{1}{2}[h^0(\Omega_{\mathbf{P}^2}(d+1))-2s-t]$ and w=0 and thus

$$\begin{array}{ccc}
0 & & 0\\
\uparrow & \alpha & \uparrow\\
H^0(\mathbf{P}^2,\Omega_{\mathbf{P}^2}(d+1)) & \longrightarrow & \bigoplus_{i=1}^{s}\Omega_{\mathbf{P}^2}(d+1)_{|S_i}\oplus\bigoplus_{i=1}^{v}\Omega_{\mathbf{P}^2}(d+1)_{|V_i}\oplus D'_{|T_l}\\
\uparrow & \beta & \uparrow\\
H^\circ(\mathbf{P}^3,\Omega_{\mathbf{P}^3}(d+1) & \longrightarrow & \bigoplus_{i=1}^{r}\Omega_{\mathbf{P}^3}(d+1)_{|R_i}\oplus\bigoplus_{i=1}^{s}\Omega_{\mathbf{P}^2|S_i}\oplus D'_{|T_l}\\
\uparrow & \gamma & \uparrow\\
H^\circ(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}) & \longrightarrow & \bigoplus_{i=1}^{u}\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}_{|U_i}\oplus\bigoplus_{i=1}^{v}\mathcal{O}_{\mathbf{P}^2}(d)_{|V_i}\\
\uparrow & & \uparrow\\
0 & & 0
\end{array}$$

suppose γ is bijective and since α is bijective by proposition 13 (we get bijectivity by specializing the required number of points,v from $\mathbf{P}^3$ to $\mathbf{P}^2$) then β is bijective.

So far we have used the méthode d'Horace simple only, see [4], méthode simple.

If $2v+w$ is odd then we have a quotient so we use lemma 12 and we have

$\lambda: H^0(\mathbf{P}^2,\Omega_{\mathbf{P}^2}(d+1)) \twoheadrightarrow \bigoplus_{i=1}^{s}\Omega_{\mathbf{P}^2|S_i}\oplus\bigoplus_{i=1}^{v}\Omega_{\mathbf{P}^2|V_i}\oplus D'_{|T_l}$ setting

$L=\bigoplus_{i=1}^{s}\Omega_{\mathbf{P}^2|S_i}\oplus\bigoplus_{i=1}^{v}\Omega_{\mathbf{P}^2|V_i}\oplus D'_{|T_l}$ then $H^0(\mathbf{P}^2,\Omega_{\mathbf{P}^2}(d+1))\hookrightarrow \text{Ł}\oplus\Omega_{\mathbf{P}^2|W}$ thus

$$\begin{array}{ccc}
0 & & 0\\
\uparrow & \alpha & \uparrow\\
H^0(\mathbf{P}^2,\Omega_{\mathbf{P}^2}(d+1)) & \longrightarrow & \bigoplus_{i=1}^{s}\Omega_{\mathbf{P}^2}(d+1)_{|S_i}\oplus\bigoplus_{i=1}^{v}\Omega_{\mathbf{P}^2}(d+1)_{|V_i}\oplus D'_{|T_l}\oplus\overline{D}_{|W_l}\\
\uparrow & \beta & \uparrow\\
H^\circ(\mathbf{P}^3,\Omega_{\mathbf{P}^3}(d+1) & \longrightarrow & \bigoplus_{i=1}^{r}\Omega_{\mathbf{P}^3}(d+1)_{|R_i}\oplus\bigoplus_{i=1}^{s}\Omega_{\mathbf{P}^2|S_i}\oplus D'_{|T_l}\\
\uparrow & \gamma & \uparrow\\
H^\circ(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}) & \longrightarrow & \bigoplus_{i=1}^{u}\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}_{|U_i}\oplus\bigoplus_{i=1}^{v}\mathcal{O}_{\mathbf{P}^2}(d)_{|V_i}\oplus\overline{D}_{|W_l}\\
\uparrow & & \uparrow\\
0 & & 0
\end{array}$$

suppose γ is bijective and since α is by proposition 13 (we get bijectivity by specializing the required number of points,v+1 from $\mathbf{P}^3$ to $\mathbf{P}^2$) then β is bijective.

Other subcases are: If s=t=0 then $h^0(\Omega_{\mathbf{P}^2}(d+1))=2v+w$ and

If $2v+w$ is even then $w=0$ and $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,0)$ implies $\boldsymbol{H}'_{\Omega,3}(d;r,0,0)$.

If $2v+w$ is odd then $w=1$ and $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,1)$ implies $\boldsymbol{H}'_{\Omega,3}(d;r,0,0)$.

If $s\neq 0$ but $t=0$ then we have $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,w)$ implies $\boldsymbol{H}'_{\Omega,3}(d;r,s,0)$. □

Remark 1. Now u,v, and w defined above have to satisfy

(i) $v + w \leq h^0(\mathcal{O}_{\mathbf{P}^2}(d))$ and

(ii) u is a non negative integer.

(i) we know that $2v+w = d(d+2)-2s-t$ i.e. $2v+w+w = d(d+2)-2s-t+w$ $v+w = \frac{1}{2}(d(d+2)-2s-t+w) = \frac{1}{2}(d^2+2d+w-2s-t) \leq \frac{1}{2}(d^2+2d+d+2) = \frac{1}{2}(d+1)(d+2) = h^0(\mathcal{O}_{\mathbf{P}^2}(d))$ i.e. $v+w \leq h^0(\mathcal{O}_{\mathbf{P}^2}(d))$ since $d+2 \geq w-2s-t$ for $\leq w \leq 0, s \geq 0, 0 \geq t$ and $d \geq -1$

(ii) u is defined thus $u = r - v - w$.

$u \geq r_{\min} - (v+w)_{\max}$

i.e. $r_{\min} = \frac{1}{3}(\frac{1}{2}(d+3)(d+2)d - 2s - t) = \frac{1}{3}(\frac{1}{2}(d+3)(d+2)d - d(d+2)) = \frac{1}{6}(d+2)(d+1)d$ $(v+w)_{\max} = \frac{1}{2}(d+1)(d+2)$

Thus $u \geq \frac{1}{6}(d+2)(d+1)d - \frac{1}{2}(d+1)(d+2) = \frac{1}{6}(d+2)(d+1)(d-3) \geq 0$ for all $d \geq 3$

For $d = 0$ and $d = 1$, we get less than 3 points in $\mathbf{P}^3$ for which the MRC does not hold thus we do not need these cases. When $d = 2$ then we have the initial cases, i.e. lemma 4

Lemma 6. *Second Vectorial Method*

Suppose d, u, v, w *satisfy* (4*a*), (4*b*), *and* (4*c*). *Write* $h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - v - w = \overline{s}$, *where* $\overline{s}$ *is a non negative integer. Set* $\overline{r} = u - \overline{s}$ *and* $\overline{t} = w$. *If* $\overline{r}$ *is a non negative integer, then* $\boldsymbol{H}'_{\Omega,3}(d;\overline{r},\overline{s},\overline{t})$ *implies* $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,w)$.

Proof. From the sequences below

The map $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) \longrightarrow \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2}(d)_{|V_i} \oplus \mathcal{O}_{\mathbf{P}^2}(d)_{|W_l}$ is surjective by Lemma 2. To make it bijective, we specilize $\overline{s}$ points from $\mathbf{P}^3$ to $\mathbf{P}^2$ and thus we have α being bijective (by Proposition 14) (If $\overline{s} = 0$, then we already have bijectivity)

If γ is bijective and α (can always be as we have seen) then by the 3 Lemma β us bijective thus $\boldsymbol{H}'_{\Omega,3}(d;\overline{r},\overline{s},\overline{t})$ implies $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,w)$.

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) & \xrightarrow{\alpha} & \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2}(d)_{|V_i} \oplus \bigoplus_{i=1}^{\overline{s}} \mathcal{O}_{\mathbf{P}^2}(d)_{|S_i} \oplus \mathcal{O}_{\mathbf{P}^2}(d)_{|W_l} \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} & \xrightarrow{\beta} & \bigoplus_{i=1}^{u} \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}_{|U_i} \oplus \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2}(d)_{|V_i} \oplus D_{|W_l} \\
\uparrow & & \uparrow \\
H^0(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d)) & \xrightarrow{\gamma} & \bigoplus_{i=1}^{\overline{r}} \Omega_{\mathbf{P}^3}(d)_{|U_i} \oplus \bigoplus_{i=1}^{\overline{s}} \Omega_{\mathbf{P}^2}(d)_{|S_i} \oplus \overline{D'}_{|T_l} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

□

Remark 2. For $\boldsymbol{H}'_{\Omega,3}(d;\overline{r},\overline{s},\overline{t})$ to hold the $\overline{r}, \overline{s}, \overline{t}$ defined should satisfy

(i) $\overline{r} \geq 0$ and

(ii) $2\overline{s} + \overline{t} \leq h^0(\Omega_{\mathbf{P}^2}(d)) = (d-1)(d+1)$

(i) $\overline{s}$ is bounded above by $h^(\mathcal{O}_{\mathbf{P}^2}(d)) = \frac{1}{2}(d+1)(d+2)$ i.e. $v+w = 0$ and u is bounded below by $\frac{1}{3}(\frac{1}{2}(d+2)(d+1)d - (v+2w))$ $v+w = 0$ implies $v+2w = 0$

whence $\overline{r} \geq \frac{1}{3}(\frac{1}{2}(d+2)(d+1)d) - \frac{1}{2}(d+1)(d+2) = \frac{(d+2)(d+1)(d-3)}{6}$ i.e $\overline{r} \geq 0$ for $d \geq 3$
For cases where $d \leq 2$ see the following:
For $d = 1$ we have $\boldsymbol{H}'_{\mathcal{O},3}(0;1,0,0)$,$\boldsymbol{H}'_{\mathcal{O},3}(0;0,3,0)$ and $\boldsymbol{H}'_{\mathcal{O},3}(0;0,1,1)$ the initial cases.
For $d = 2$ we can use the initial cases to prove bijectivity i.e.
In $\boldsymbol{H}'_{\mathcal{O},3}(1;u,v,w)$,there are only 5 possibilities for (u,v,w) i.e.
(a) $(4,0,0)$, $(3,3,0)$, and $(3,1,1)$ are special cases of corollary 11, by lemmas 5, 6 or 7.
(b) $(2,4,1)$ and $(2,6,0)$ and the initial case $\boldsymbol{H}'_{\Omega,3}(2;2,0,0)$ implies $\boldsymbol{H}'_{\mathcal{O},3}(1;u,v,w)$ when $v = 4$, $w = 1$ and when $v = 6$, $w = 0$
(ii) we know $\overline{s} = \frac{1}{2}(d+1)(d+2) - v - w$ so $2\overline{s} = (d+1)(d+2) - 2v - 2w$ but $\overline{t} = w$ thus we have $2\overline{s} + \overline{t} = (d+1)(d+2) - 2v - w$.
Question: When is $2\overline{s} + \overline{t} \leq h^0(\Omega_{\mathbf{P}^2}(d)$?
Answer: When $2\overline{s} + \overline{t} = (d+1)(d+2) - 2v - w$ is less than $(d-1)(d+1)$ i.e. when $3d+3 \leq 2v + w$ i.e. $w \in \{0,1\}$, $d \geq 0$ and $v + \frac{1}{2}w \geq \frac{3}{2}(d+1)$

Lemma 7. *Divisorial Method*
Suppose d,u,v,w are non negative integers satisfying the conditions of Lemma 2. Set $u' = u - h^0(\mathcal{O}_{\mathbf{P}^2}(d-1)) = u - \frac{1}{2}d(d+1)$. If one has $u' \geq 0$, then $\boldsymbol{H}'_{\mathcal{O},3}(d-2;u',v,w)$ implies $\boldsymbol{H}'_{\mathcal{O},3}(d-1;u,v,w)$.

Proof. Consider the following strict exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3}(d-2)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} \longrightarrow \mathcal{O}_H(d-1)^{\oplus 3} \longrightarrow 0$$

The map $\rho : H^0(H, \mathcal{O}_H(d-1)^{\oplus 3}) \longrightarrow \bigoplus_{i=1}^{y} \mathcal{O}_H(d-1)^{\oplus 3},_{Y_i}$ is bijective if $y = \frac{1}{2}d(d+2)$. Thus if γ is bijective then β is bijective see the sequence;

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & \alpha & \uparrow \\
H^0(H, \mathcal{O}_H(d-1))^{\oplus 3} & \longrightarrow & \bigoplus_{i=1}^{y} \mathcal{O}_H(d-1)^{\oplus 3}_{|Y_i} \\
\uparrow & \beta & \uparrow \\
H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} & \longrightarrow & \bigoplus_{i=1}^{u} \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}_{|U_i} \oplus \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2}(d)_{|V_i} \oplus D_{|W_l} \\
\uparrow & \gamma & \uparrow \\
H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-2)^{\oplus 3} & \longrightarrow & \bigoplus_{i=1}^{u'} \mathcal{O}_{\mathbf{P}^3}(d-2)^{\oplus 3}_{|U_i} \oplus \bigoplus_{i=1}^{v} \mathcal{O}_{\mathbf{P}^2}(d-1)_{|V_i} \oplus D_{|W_l} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Here we specialize y points from $\mathbf{P}^3$ to a $H \cong \mathbf{P}^2$ but not containing v and w then $u' = u - y$ □

Remark 3. Consider the following sequence

$$\begin{array}{ccccccc}
 & & \frac{1}{2}(d+1)d(d-1) & & \frac{1}{2}(d+2)(d+1)d & & \frac{3}{2}d(d+1) \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(d-2)^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} & \longrightarrow & \mathcal{O}_H(d-1)^{\oplus 3} \longrightarrow 0 \\
 & & (u',v,w) & & (u,v,w) & & (\frac{1}{2}d(d+1),0,0)
\end{array}$$

u', v, w defined must be such that
(i) $v + w \leq h^0(\mathcal{O}_{\mathbf{P}^2}(d-1)$ and so $d \geq 2$
(ii)$u' \geq 0$ with $0 \leq w \leq 1$ and $d \geq 3$
i.e. $u' = u - \frac{1}{2}d(d+1) = \frac{1}{3}(\frac{1}{2}(d+2)(d+1)d - v - 2w) - \frac{1}{2}d(d+1) = \frac{1}{6}(d-1)(d+1)d - 2(v+2w)$
Using $v + w \leq \frac{1}{2}d(d+1)$ i.e. the upper bound we take equality we end up with $u' \geq \frac{1}{6}(d-2)(d)(d+1) - w$ thus for $u' \geq 0$, we must have $0 \leq w \leq 1$ and $d \geq 3$. For $d = 2$ is an initial case.

Theorem 8. (a) *For* $d \geq 1, \boldsymbol{H}_{\mathcal{O},3}(d-1)$ *implies* $\boldsymbol{H}_{\Omega,3}(d+1)$
(b) *For* $d \geq 2, \boldsymbol{H}_{\Omega,3}(d)$ *and* $\boldsymbol{H}_{\mathcal{O},3}(d-2)$ *imply* $\boldsymbol{H}_{\mathcal{O},3}(d-1)$

Proof. (a) follows from lemma 4 and remark 1
(b) we shall prove it by 4 arguments
(i) If $\boldsymbol{H}_{\Omega,3}(d)$ and $\boldsymbol{H}_{\mathcal{O},3}(d-2)$ are both false then (b) is an initial case and thus is true.
(ii) If $\boldsymbol{H}_{\Omega,3}(d)$ is true but $\boldsymbol{H}_{\mathcal{O},3}(d-2)$ false then (b) holds by lemma 6 and remark 2 for $d \geq 1$.
(iii) If $\boldsymbol{H}_{\mathcal{O},3}(d-2)$ is true but $\boldsymbol{H}_{\Omega,3}(d)$ false then (b) holds by lemma 7 and remark 3 with $d \geq 2$
(iv)If $\boldsymbol{H}_{\Omega,3}(d)$ and $\boldsymbol{H}_{\mathcal{O},3}(d-2)$ are both true then (b) is true is true by lemma 6 and remark 2 for $d \geq 1$ or by lemma 7 and remark 3 for $d \geq 2$

We have now attained the Goal we set ourselves before Theorem 3. □

Lemma 9. *For any integer* $d \geq 1$*, the hypothesis* $\boldsymbol{H}'_{\mathcal{O},3}(d-1; h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)), 0, 0)$ *is true.*

Proof. $\boldsymbol{H}'_{\mathcal{O},3}(d-1; h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)), 0, 0)$ is a special case of $\boldsymbol{H}_{\mathcal{O},3}(d-1)$ and thus is true as long as the number of points u in $\mathbf{P}^3$ that we require for the truth of $\boldsymbol{H}'_{\mathcal{O},3}(d-1; h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)), 0, 0)$ are $u = \frac{1}{3}h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3} = h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)$ □

Lemma 10. *The hypothesis* $\boldsymbol{H}'_{\mathcal{O},3}(0; 0, 3, 0)$ *and* $\boldsymbol{H}'_{\mathcal{O},3}(0; 0, 1, 1)$ *are true.*

Proof. Follow from lemma 4 □

A consequence of these last two lemmas is the following statement.

Corollary 11. *Let* $d \geq 1$ *be an integer. Then* $\boldsymbol{H}'_{\mathcal{O},3}(d-1; u, v, w)$ *holds in the following cases:*

a. $u = h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$, $v = 0$, $w = 0$.
b. $u = h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)) - 1$, $v = 3$, $w = 0$.
c. $u = h^0(\mathcal{O}_{\mathbf{P}^3}(d-1)) - 1$, $v = 1$, $w = 1$.

Proof. we use lemmas 7,8 and 9 i.e.
(a) follows from Lemma 9, set $u = h^0(\mathcal{O}_{\mathbf{P}^3}(d-1))$
(b) If d=1 then it is true by lemma 10, if $d \geq 1$ and use lemma 7 to reduce it to lemma 9

(c) If d=1 then it is true by lemma 10, if $d \geq 1$ and use lemma 7 to reduce it to lemma 9.

□

Lemma 12. *The differential méthode d'Horace* [1]
Suppose we are given a surjective morphism of vector spaces,

$$\lambda : H^{\circ}(\mathbf{P}^2, \Omega_{\mathbf{P}^2}(d+1)) \twoheadrightarrow L$$

and suppose there exists a point Z' in $\mathbf{P}^2$ such that

$$H^{\circ}(\mathbf{P}^2, \Omega_{\mathbf{P}^2}(d+1)) \hookrightarrow L \oplus \Omega_{\mathbf{P}^2}(d+1)_{|Z'} \text{ and}$$

Suppose also that $H^1(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}) = 0$. Then there exists a quotient $\mathcal{O}_{\mathbf{P}^3}(d-1)^3_{|Z'} \longrightarrow D$ with kernel contained in $\Omega_{\mathbf{P}^2}(d)_{|Z'}$ of dimension $\dim(D) = \mathrm{rank}(\Omega_{\mathbf{P}^3}(\mathrm{d}+1)) - \dim(\ker \lambda)$ *having the following property.*

$$\text{Let } \mu : H^{\circ}(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)) \longrightarrow M$$

be a morphism of vector spaces then there exists Z in $\mathbf{P}^3$ such that if
$H^{\circ}(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(d-1)^{\oplus 3}) \longrightarrow M \oplus D$ is of maximal rank then
$H^{\circ}(\mathbf{P}^3, \Omega_{\mathbf{P}^3}(d+1)) \longrightarrow M \oplus L \oplus \Omega_{\mathbf{P}^3}(d+1)|_Z$ is also of maximal rank.

Proposition 13. *For all $d \geq 1$, there exists $P_1, \ldots, P_p, Q \in \mathbf{P}^2$ and a quotient $\Omega_{\mathbf{P}^2,Q} \twoheadrightarrow D'_{,Q}$ such that the map $H^0(\mathbf{P}^2, \Omega_{\mathbf{P}^2}(d+1)) \longrightarrow \bigoplus_{i=1}^{p} \Omega_{\mathbf{P}^2|P_i} \oplus D|_Q$ is of maximal rank.*

Proof. Follows from truth of MRC for $\mathbf{P}^2$ for p general points in $\mathbf{P}^2$ and a point Q in $\mathbf{P}^2$ for which we have a quotient.

□

Proposition 14. *For all $d \geq 1$, there exists $M_1, \ldots, M_m, N \in \mathbf{P}^2$ and a quotient $\mathcal{O}_{\mathbf{P}^2,N} \oplus D'_{,N} \twoheadrightarrow D'_{,N}$ such that the map $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) \longrightarrow \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^2|M_i} \oplus \mathcal{O}_{\mathbf{P}^2|_N}$ is of bijective.*

Proof. Follows from truth of MRC for $\mathbf{P}^2$ m general set of points in $\mathbf{P}^2$ and a point N in $\mathbf{P}^2$ for which we have a quotient. □

Remark 4. The Geometric methods used always work for $\mathbf{P}^3$ but for higher dimensional projective spaces one would have to use other methods in addition.

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