Quantum Metrology: Dynamics versus Entanglement

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A parameter whose coupling to a quantum probe of n constituents includes all two-body interactions between the constituents can be measured with an uncertainty that scales as $1/n^{3/2}$, even when the constituents are initially unentangled. We devise a protocol that achieves the $1/n^{3/2}$ scaling without generating any entanglement among the constituents, and we suggest that the protocol might be implemented in a two-component Bose-Einstein condensate.

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Quantum mechanics determines the fundamental limits on measurement precision. In the prototypal quantum metrology scheme, the value of a parameter is imprinted on a quantum probe through an interaction in which the parameter appears as a coupling constant [1]. The number n of constituents in the probe is often considered to be the most important resource for such schemes. We denote the parameter to be estimated by γ, and we write the interaction Hamiltonian as $\mathcal{H} = \hbar\gamma H$, where H is a dimensionless coupling Hamiltonian. The measurement precision is quantified by the units-corrected root-mean-square deviation $\delta\gamma$ of the estimate γ_{est} from its true value [2].

The scaling of $\delta\gamma$ with n depends on the probe dynamics as expressed in H [3–10]. For an interaction that acts independently on the probe constituents, the optimal measurement precision scales as $1/n$, a scaling often called the "Heisenberg limit." In contrast, a nonlinear Hamiltonian that includes all possible k-body couplings gives an optimal sensitivity that scales as $1/n^k$. To achieve this requires that the initial probe state be entangled. If practical considerations preclude initializing the probe in an entangled state, sensitivity that scales as $1/n^{k-1/2}$ is possible using a probe that is initially in a product state [3,4,6,8,10]. Both of these scalings can be achieved with separable measurements.

Practical interest in using nonlinear interactions for quantum metrology comes from the fact that, even with two-body couplings and initial product states, it is possible to obtain a $1/n^{3/2}$ scaling for $\delta\gamma$ [3,4,6,8–10]. In all such schemes proposed until now, (particle) entanglement is generated during the protocol that leads to better than $1/n$ scaling. We formulate here a protocol that generates no entanglement among the probe constituents yet still achieves the $1/n^{3/2}$ scaling; in this protocol, it is clearly the *dynamics* alone that leads to improvement over the $1/n$ scaling. The dynamics has the same quantum-mechanical character as linear ($k = 1$) metrology schemes but is n times faster. Even though this Letter is mainly about improving on the Heisenberg scaling, any experimental demonstration of a scaling better than $1/n^{1/2}$ would be of considerable interest to the metrology community.

A typical $k = 2$ choice for a probe made of qubits is $H = J_z^2$, where $J_z = \frac{1}{2}\sum_{j=1}^{n} Z_j$ is the z component of the "total angular momentum," with Z_j being the Pauli Z operator for the jth qubit. We denote the eigenvectors of Z by $|0\rangle$ and $|1\rangle$. An optimal initial state, evolving for a time t, gives a signal that oscillates in γ with frequency $tn^2/4$. Sampling from this signal over ν trials leads to a measurement precision $\delta\gamma = 4/tn^2\sqrt{\nu}$ [5].

If the initial probe state is required to be a product state, an optimal input state is of the form $e^{-iJ_y\beta}|0\rangle^{\otimes n} = [\cos(\beta/2)|0\rangle + \sin(\beta/2)|1\rangle]^{\otimes n}$, where $0 < \beta \leq \pi/2$. A measurement of J_y after the probe has evolved for a time t under the J_z^2 Hamiltonian leads to a measurement precision that scales as $1/tn^{3/2}\sqrt{\nu}$, provided γt is small [8]. This scaling applies for all values of $\beta \neq \pi/2$, but the optimal sensitivity occurs for $\beta = \pi/4$. The restriction to small times arises because J_z eigenstates accumulate phase shifts quadratic in n, leading to a "phase dispersion" that after a short time renders it impossible to determine γ optimally from a separable measurement such as that of J_y.

For the J_z^2 Hamiltonian, the entanglement generated during evolution from an initial product state and the phase dispersion are two aspects of the same phenomenon. One might think that the generated entanglement and associated phase dispersion somehow play a role in the enhanced $1/n^{3/2}$ scaling, but it would normally be expected that the phase dispersion is best avoided [8].

The essential observation we make here is that, if the J_z^2 Hamiltonian were replaced with one of the form $H = nJ_z$, there would be no phase dispersion and no generated entanglement. An nJ_z Hamiltonian acts as a linear coupling whose strength is proportional to n. Physically, an nJ_z coupling cannot arise from a fundamentally linear coupling, as that would require the coupling strength to be a function of the number of constituents in the probe. We show here that it can arise *naturally* from quadratic couplings to the parameter.

With a pure nJ_z interaction, the optimal initial product state is $e^{-iJ_y\pi/2}|0\rangle^{\otimes n} = [(|0\rangle + |1\rangle)/\sqrt{2}]^{\otimes n}$. The state remains unentangled at all times, evolving to $[(e^{-i\gamma tn/2}|0\rangle + e^{i\gamma tn/2}|1\rangle)/\sqrt{2}]^{\otimes n}$. A measurement of J_x at time t has expectation value $\langle J_x\rangle = \frac{1}{2}n\cos(\gamma tn)$ and uncertainty $\Delta J_x = \frac{1}{2}\sqrt{n}|\sin(\gamma tn)|$, leading to measurement precision $\delta\gamma = \Delta J_x/(\sqrt{\nu}|d\langle J_x\rangle/d\gamma|) = 1/tn^{3/2}\sqrt{\nu}$ after ν trials. A measurement of any other equatorial component of $\boldsymbol{J}$ achieves the same sensitivity. The enhanced scaling in this protocol is clearly due to the dynamics alone, not to entanglement of the constituent qubits. These results indicate that, in quantum metrology, entanglement is important only in providing an optimal initial state, which leads to an improvement by a factor of $1/n^{1/2}$ over initial product states.

We are interested in investigating measurement protocols that use both J_z^2 and nJ_z interactions in systems of bosons that can occupy two modes with creation operators $a_1^\dagger$ and $a_2^\dagger$. In the Schwinger representation, we have $J_z = \frac{1}{2}(n_1 - n_2)$ and $n = n_1 + n_2$, where $n_1 = a_1^\dagger a_1$ and $n_2 = a_2^\dagger a_2$ are the numbers of particles in the two modes. The bosons that we consider interact with one another, but the interactions conserve particle number. Our measurement protocols, for both types of coupling, can be represented in terms of the interferometer with nonlinear phase shifters depicted in Fig. 1. In practical implementations, the interferometer might be an optical or Ramsey interferometer or an interferometer made up of coupled nanomechanical resonators [10]. Optical interferometers using coherent states as inputs will have phase dispersion even with the nJ_z interaction because of the uncertainty in n. To avoid phase dispersion, we focus on bosonic systems with nonzero chemical potential in which number states are easy to create.

An nJ_z coupling acts as a linear coupling with a coupling strength proportional to n. Thus the effect of decoherence on our measurement protocol is the same as that on a linear protocol with a product-state input. In particular, decoherence that acts independently on the probe particles preserves the $1/n^{3/2}$ scaling at all times [8,10].

We turn now to the problem of implementing the nonlinear interferometer of Fig. 1 in a laboratory system of considerable interest. For this purpose [7,9] we consider a two-mode Bose-Einstein condensate (BEC) in which the n atoms can occupy two internal states (modes) labeled $|1\rangle$ and $|2\rangle$, which are typically hyperfine levels. The atoms that form the initial BEC are all in the internal state $|1\rangle$. In the mean-field approximation, they all share the same spatial wave function $\psi_n(\boldsymbol{r})$, which is the n-dependent ground-state solution of the Gross-Pitaevskii equation for a trapping potential $V(\boldsymbol{r})$ and a scattering term characteristic of internal state $|1\rangle$. An external field (the first beam splitter in Fig. 1) drives transitions between the two internal states [11], resulting in every atom being in the same superposition of the two internal states. We assume that the

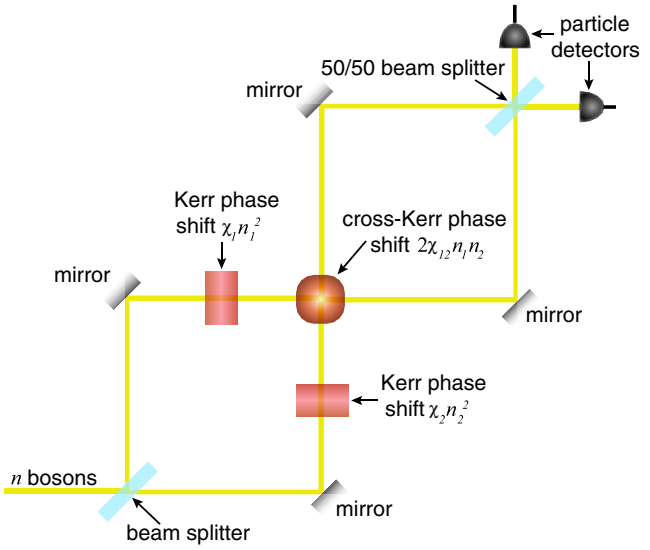

FIG. 1 (color online). Nonlinear interferometer giving J_z^2 and nJ_z couplings. An incoming beam of n bosons is split at a beam splitter, which puts each boson into an appropriate superposition of being in the two arms (modes). The two initial nonlinear phase shifters produce Kerr phase shifts $\chi_1 n_1^2$ and $\chi_2 n_2^2$. The phase shifter at the intersection of the beams produces a cross-Kerr phase shift $2\chi_{12}n_1n_2$. The final 50/50 beam splitter converts the required measurement of an equatorial component of $\boldsymbol{J}$ into a measurement of J_z, i.e., a counting of the difference of the numbers of particles in the two output beams. The net effect of the nonlinear phase shifters is the same as a probe Hamiltonian $\mathcal{H}$ acting for a time t, with $\mathcal{H}t/\hbar = \chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12}n_1n_2 = (\chi + \chi_{12})n^2/2 + (\chi_1 - \chi_2)nJ_z + 2(\chi - \chi_{12})J_z^2$, where $\chi = \frac{1}{2}(\chi_1 + \chi_2)$ is the average Kerr phase shift. The term proportional to n^2 produces an overall phase shift and can be ignored. The nJ_z coupling comes from having different Kerr phase shifters in the two arms; to eliminate the J_z^2 interaction requires a cross-Kerr coupling $\chi_{12} = \chi$. Under these circumstances, we have $\mathcal{H} = \hbar\gamma nJ_z$, with $\gamma t = \chi_1 - \chi_2$.

atomic collisions are elastic, so the only scattering channels are $|1\rangle|1\rangle \to |1\rangle|1\rangle$, $|2\rangle|2\rangle \to |2\rangle|2\rangle$, and $|1\rangle|2\rangle \to |1\rangle|2\rangle$. These have amplitudes g_{11}, g_{22}, and g_{12}, respectively, where $g_{ij} = 4\pi\hbar^2 a_{ij}/m$, with a_{ij} being the s-wave scattering length. The effect that we seek is the differential phase shift between the two internal states due to their different scattering properties. After some period of evolution, a second external field (the second beam splitter in Fig. 1) drives a $\pi/2$ pulse between the internal states. A final measurement then determines the population difference between the two internal states. In the following, we are interested in the BEC dynamics that occurs between application of the external fields.

We assume, as can be achieved in the laboratory, that the two internal states are chosen so that both see the same trapping potential V. Nonetheless, the spatial wave functions of the two internal states will diverge because they experience different scattering interactions. The effect of the scattering terms on the spatial wave functions becomes important at the atom number n_c, where the scattering

energy becomes comparable to the total atomic kinetic energy. For n small compared to n_c, the two spatial wave functions remain essentially the same; for n larger than n_c, we neglect changes in the spatial wave functions to keep our analysis simple. We address the effect of the breakdown of this approximation below.

With these assumptions, the Hamiltonian for the two-mode BEC [12,13] takes the form

$$\mathcal{H} = \mathcal{H}_0 + \gamma_1 \eta (n-1) J_z + \gamma_2 \eta J_z^2, \qquad (1)$$

where $\eta = \int d\boldsymbol{r} |\psi_n(\boldsymbol{r})|^4$, $\gamma_1 = \frac{1}{2}(g_{11} - g_{22})$, $\gamma_2 = g - g_{12}$, and $g = \frac{1}{2}(g_{11} + g_{22})$ (notice that γ_1 and γ_2 do not have units of frequency). The only effect of the Hamiltonian $\mathcal{H}_0 = nE_0 + \frac{1}{4}(g + g_{12})\eta n^2 - \frac{1}{2} g \eta n$, where E_0 is the single-particle kinetic plus trap potential energy corresponding to ψ_n, is to introduce an overall phase, and thus $\mathcal{H}_0$ can be ignored. We assume that n is large enough that we can replace $n - 1$ with n in $\mathcal{H}$.

In a harmonically trapped BEC, the repulsive scattering interactions cause the single-particle ground-state wave function ψ_n to spread as the number of particles increases. This effect appears in the BEC Hamiltonian in the factor η, which is inversely proportional to the effective volume occupied by the ground-state wave function. The n dependence of η gives the coupling strength a dependence on n that must be included in our analysis of the precision in estimating $\gamma_{1,2}$.

When the number of atoms is small compared to n_c, the total kinetic energy far exceeds the scattering energy, resulting in a ground-state wave function that is independent of n. In a three-dimensional harmonic trap with ground-state half-width s, the total kinetic energy is $\sim n(\hbar^2/ms^2)$, and the scattering energy is $\sim n^2(g_{11}/s^3)$ (for atoms in internal state $|1\rangle$), giving $n_c \sim s/a_{11}$. Typical values of $a_{11} \sim 10$ nm and $s \sim 10$ μm give $n_c \sim 1000$. Hence, for a condensate composed of tens to a few hundred or so atoms, η does not depend significantly on n, implying a scaling of $1/n^{3/2}$ in such small BECs.

In large harmonically trapped BECs, with $n \gg n_c$, η acquires an n dependence that defeats the desire to improve on $1/n$ scaling. Strategies for dealing with this include using traps with harder walls than a harmonic trap and working with BECs confined to fewer than three dimensions. To assess these strategies, we compute the n dependence of η when the BEC is trapped in d longitudinal dimensions by a spherically symmetric potential $V = \frac{1}{2} k r^q$ and is tightly confined in the remaining $D = 3 - d$ transverse dimensions by a harmonic potential. The longitudinal trap is characterized by the hardness parameter q and the half-width of its (bare) ground-state wave function $R_0 = (\hbar^2/mk)^{1/(q+2)}$, for which a typical value might be $R_0 \sim 10$ μm. The tight transverse potential is characterized by its resonant frequency ω_0 and the half-width $s = (\hbar/2m\omega_0)^{1/2}$ of its ground-state wave function, for which a typical value would be $s \sim 100$ nm.

There are now two critical atom numbers. The first, $n_L = (R_0/a_{11})(s/R_0)^D$, occurs when the scattering energy is comparable to the longitudinal kinetic energy. As n increases from n_L, the ground-state wave function spreads in the longitudinal dimensions, its size growing as $R \sim R_0(n/n_L)^{1/(q+d)}$. The second critical atom number, $n_T = (s/a_{11})(R_0/s)^{d(q+2)/q}$, arises when the scattering energy becomes as large as the transverse kinetic energy (so does not apply when $d = 3$), at which point the longitudinal extent of the wave function is R_T. The corresponding atomic number density $n_T/s^D R_T^d \sim 1/a_{11}s^2 \sim 10^{16}$ cm^{-3} is right at the upper limit on number density set by three-body scattering losses, so we consider only atom numbers smaller than n_T.

For atom numbers between n_L and n_T, a reasonable approximation to the ground-state solution of the Gross-Pitaevskii equation is obtained by using the Gaussian ground state of width s in the transverse dimensions and using the Thomas-Fermi approximation for the longitudinal wave function [14]. In this approximation, we find

$$\eta = \frac{2q}{2q+d}\frac{\lambda}{n g_{11}} = \frac{\alpha_{q,d}}{s^D R_0^d}\left(\frac{n_L}{n}\right)^{d/(d+q)}, \qquad (2)$$

where $\lambda = \mu - \frac{1}{2} D\hbar\omega_0$ is the longitudinal part of the chemical potential μ and $\alpha_{q,d}$ is a geometric factor of order unity that depends on q and d but not on n. The n dependence of η implies an effective coupling strength that scales as $n^{\xi - 1/2}$, where $\xi = (d + 3q)/2(d + q)$. The precision of estimating γ_1 or γ_2 thus scales as $1/n^{\xi}$.

For a three-dimensional BEC trapped in a harmonic potential, the measurement precision scales as $1/n^{9/10}$, worse than the Heisenberg scaling but still better than $1/n^{1/2}$. To achieve super-Heisenberg scalings requires a trapping potential that is harder than a harmonic potential or else working with a one- or two-dimensional BEC. For $d = 2$, a BEC trapped in a harmonic potential matches the $1/n$ scaling, and a one-dimensional harmonic BEC betters it, achieving a $1/n^{7/6}$ scaling. A d-dimensional BEC achieves super-Heisenberg scaling when the hardness parameter q exceeds d. The limit of large q corresponds to a trap with hard walls and extent $2R_0$ and has $\xi = 3/2$ regardless of d. For a one-dimensional BEC, an alternative to hard caps is to use a ring geometry.

A good candidate for implementing the generalized metrology protocol is a BEC made of ^{87}Rb atoms. Atoms in the hyperfine level $|F = 1; M_F = -1\rangle = |1\rangle$ are trapped and cooled to form a BEC, and then a Raman or microwave-driven transition is used to create a superposition of $|1\rangle$ and the hyperfine level [11,15] $|F = 2; M_F = 1\rangle = |2\rangle$. The s-wave scattering lengths a_{11}, a_{22}, and a_{12} are nearly degenerate for ^{87}Rb, with ratios $\{a_{22}{:}a_{12}{:}a_{11}\} = \{0.97{:}1{:}1.03\}$. These values imply that $\gamma_2 = \frac{1}{2} \times (g_{11} + g_{22}) - g_{12}$ is essentially zero for this scheme, meaning that a ^{87}Rb BEC can realize the generalized quantum metrology protocol with a pure $n^{\xi - 1/2} J_z$ cou-

pling. The optimal initial state for this protocol has all atoms in an equally weighted superposition of $|1\rangle$ and $|2\rangle$. The quantity that is estimated is proportional to $\gamma_1 = \frac{1}{2}(g_{11} - g_{22})$, which, though small, is nonzero for the scattering lengths in ^{87}Rb.

Effects neglected in our analysis impose important limitations. The difference in evolution of the spatial wave functions of the two internal states is likely to be the most severe restriction on measurement time. If the spatial wave functions are not identical, there is a position-dependent relative phase between $|1\rangle$ and $|2\rangle$ in addition to the overall relative phase in which we are interested. The sensitivity scaling is unaffected for short times, and preliminary results suggest that this effect can be further suppressed by using a hard trap or a lower-dimensional BEC.

Loss of atoms from the trap is an important decoherence mechanism, mainly due in our protocol to inelastic spin-exchange collisions (exchange of atoms with the thermal cloud that is present around any realistic BEC is negligible). A chief advantage of using protocols that do not rely on entanglement is that loss of atoms does not affect the sensitivity scaling, although it does generally degrade the sensitivity. In the case of spin-exchange collisions, the decoherence can be modeled in terms of a parameter $\Gamma\eta/2$, which we can estimate using data from Ref. [16] and the assumption that $|1\rangle$ and $|2\rangle$ have the same spatial wave function. This estimate gives $\Gamma/2\gamma_1 \sim 1/26$, implying that we can perform a measurement of γ_1 before inelastic collisions have a significant impact.

A final issue is that the number of atoms in a BEC is not known to arbitrary precision, as we have assumed until now. We propose to determine n by counting the number of atoms in both internal states at the output of our protocol. A determination of n with a fractional error of $\Delta n/n \sim 0.01$, which is within current capabilities, would be sufficient for the purpose of demonstrating an enhanced scaling with n, provided that the measurement time is kept short enough that the nonlinear phase shift is much smaller than $n/\Delta n$. We note that, if Δn is bigger than $\sqrt{n}$, the chief practical advantage of the nJ_z interaction is obviated, since the requirement on measurement time is as strict as or stricter than that set by phase dispersion in a J_z^2 protocol. Even so, the ability of the nJ_z coupling to achieve enhanced scalings with no generated entanglement remains an important objective.

We have shown that it is possible to achieve measurement precision that scales better than $1/n$ by using the dynamics generated by nonlinear Hamiltonians. The pure nJ_z scheme introduced here does not use quantum entanglement at any stage to achieve the enhanced scaling. Early experiments to test our scheme in BECs are likely to focus on demonstrating enhanced scaling in the estimation of some combination of atomic scattering lengths. To be useful, however, our scheme must be adapted to measuring external fields that modulate the atomic scattering properties. One possibility is to use a ^{133}Cs BEC with optical trapping of the $|F = 3; M_F = 3\rangle$ state [17], which has a very broad Feshbach resonance at 8 G, which makes the scattering lengths very sensitive to the strength of an external magnetic field [18,19]. This suggests that our scheme might be used for ultrahigh precision magnetometry.

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