

Testing Bell inequalities with photon-subtracted Gaussian states

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(Received 24 October 2007; revised manuscript received 14 July 2008; published 1 October 2008)

Recently, photon-subtracted Gaussian states (PSGSs), which well approximate superpositions of coherent states (SCSs) with small amplitudes, were generated by several experimental groups. We compare the ideal SCSs and the PSGSs for experimental tests of certain types of Bell inequalities. In particular, we analyze the effects of the key experimental components used to generate PSGSs: mixedness of the Gaussian states, limited transmittivity of the beam splitter, and an avalanche photodetector that cannot resolve photon numbers. As a result of this analysis, the degrees of mixedness and the beam splitter transmittivity that can be allowed for successful tests of Bell inequalities are revealed.

DOI: [10.1103/PhysRevA.78.042101](https://doi.org/10.1103/PhysRevA.78.042101)

PACS number(s): 03.65.Ud, 03.67.Mn, 42.50.Dv

I. INTRODUCTION

Quantum optics enables one to experimentally explore quantum physics and quantum-information processing in the context of continuous variables. Gaussian continuous-variable states with nonclassical properties, such as single-mode and two-mode squeezed states, have been generated and used for various applications in numerous quantum-optics experiments. For example, continuous-variable quantum teleportation has been performed using two-mode squeezed states of light [1,2]. However, it is experimentally more difficult to generate and control non-Gaussian continuous-variable states.

Superpositions of two coherent states (SCSs) in free-traveling optical fields, a well-known class of non-Gaussian continuous-variable states, have attracted remarkable attention. The SCSs, when the component coherent states are well separate in phase space, are often called “Schrödinger cat states” as they show typical properties of macroscopic quantum superposition [3,4]. The SCSs show nonclassical properties such as interference patterns in phase space and negative values in the Wigner functions [4]. Once single-mode SCSs are generated, it is relatively easy to generate two-mode SCSs using a 50:50 beam splitter. These two-mode SCSs (also called entangled coherent states [5]) have been found [6–8] using various types of measurements to violate Bell inequalities [9–11]. It has also been shown that the free-traveling SCSs can be useful for various applications in quantum-information processing (QIP) [12–16]. All the QIP applications and Bell inequality tests using SCSs require free-traveling SCSs with reasonably large amplitudes. For example, Lund *et al.* showed that SCSs with amplitudes $\alpha > 1.2$ allow for fault-tolerant quantum computing with reasonable photon loss [17].

On the other hand, until recently, it has been known to be extremely hard to experimentally generate free-traveling SCSs. There have been schemes to generate such SCSs using strong nonlinear interactions [18] or photon-number-resolving detectors [19,20], neither of which is feasible using current technology. Recently, the generation of free-traveling SCSs has been studied by several authors and more feasible schemes have been suggested [21–26]. Among those, a

simple and useful observation was made that SCSs with small amplitudes, such as $\alpha < 1.2$, are very well approximated by squeezed single photons [21]. It was also pointed out that squeezed single photons are identical to the photon-subtracted Gaussian states (PSGSs) obtained by subtracting one photon from pure squeezed vacuums [27]. Meanwhile, photon-subtracted PSGSs have been generated by several experimental groups [28–31]. Some effects of experimental imperfections including inconclusive photon subtraction and impurity of Gaussian sources have been theoretically analyzed [32,33]. It was pointed out that subtracting a single photon from a Gaussian state is experimentally more efficient than directly squeezing the single-photon state [34].

Very recently, a remarkable breakthrough was made: free-traveling SCSs, called Schrödinger cat states, were generated and detected [35], where the size of the states ($\alpha = 1.6$) was reasonably large for fundamental tests of quantum theory and quantum-information processing. However, the fidelity of the generated states is yet to be improved for practical quantum-information processing. In the meantime, the PSGSs will be useful for small-scale tests of Bell inequalities and quantum-information processing. Experimental efforts and progress are being made to generate SCSs of even larger amplitudes and higher fidelity [36].

In this paper, we study tests of certain types of Bell inequalities [42] with PSGSs and discuss the possibility of experimental realization using current technology. Gaussian squeezed states, a beam splitter, and an avalanche photodetector are experimental components typically used to generate PSGSs [28–31]. We analyze the detrimental effects of these experimental components, i.e., the mixedness of the Gaussian states, limited transmittivity of the beam splitter, and an avalanche photodetector that cannot resolve photon numbers. As a result of this analysis, we reveal the degrees of mixedness and the beam splitter transmittivity that can be allowed for successful tests of Bell inequalities. In particular, we show that quantum nonlocality may be verified using current technology with homodyne detection based on the Clauser and Horn (CH) version of Bell’s inequality [11].

There have been publications on the study of Bell’s inequality utilizing photon subtraction methods, including Refs. [37–41]. In particular, a state generated by subtracting one photon from each mode (total two photons) of a two-

mode squeezed state was considered for Bell inequality tests [37–41]. We note that, while such two-photon subtraction is useful to enhance quantum nonlocality [37–41], the method considered in our paper using the PSGS can be more efficient (i.e., the success probability is larger) because it requires only a single-photon subtraction.

This paper is organized as follows. In Sec. II, we briefly review the comparison between the pure PSGS and the SCS by means of optimized fidelity [21]. In Sec. III, we investigate Bell inequality tests suggested by Banaszek and Wódkiewicz (BW) [42] for ideal PSGSs and compare the results with those for ideal SCSs. In Sec. IV, we consider experimental detrimental effects based on the approaches in Refs. [32,33] for our Bell inequality tests and analyze the results. We then conclude the paper in Sec. V with final remarks on prospects for experimental tests of Bell inequalities using PSGSs.

II. SUPERPOSITION OF COHERENT STATES AND PHOTON-SUBTRACTED SQUEEZED VACUUM

A SCS can be represented as

$$|SCS_{\varphi}(\alpha)\rangle = N_{\varphi}(\alpha)(|\alpha\rangle + e^{i\varphi}|- \alpha\rangle), \tag{1}$$

where $N_{\varphi}(\alpha)$ is a normalization factor, $|\pm \alpha\rangle$ is a coherent state of amplitude $\pm \alpha$, and φ is a real local phase factor. The amplitude α is assumed to be real for simplicity without loss of generality. The size of the SCS can be defined by the magnitude of the amplitude α . The SCSs such as $|SCS_{\pm}(\alpha)\rangle = N_{\pm}(\alpha)(|\alpha\rangle \pm |- \alpha\rangle)$ are called even and odd SCSs, respectively, because the even (odd) SCS always contains an even (odd) number of photons. By splitting a SCS at a 50:50 beam splitter, the two-mode entangled coherent state

$$|ECS_{\varphi}(\alpha)\rangle = N_{\varphi}(\alpha)(|\beta\rangle|\beta\rangle + e^{i\varphi}|- \beta\rangle|- \beta\rangle), \tag{2}$$

where $\beta = \alpha/\sqrt{2}$, can be simply generated.

Jeong *et al.* pointed out that a squeezed single photon is identical to a pure PSGS (i.e., an exact single-photon subtraction from a pure Gaussian state) [27]. This can be shown by applying the annihilation operator \hat{a} to a squeezed vacuum state:

$$\hat{a}S(r)|0\rangle = -\sinh rS(r)|1\rangle, \tag{3}$$

where $S(r) = e^{(r/2)(\hat{a}^2 - \hat{a}^{\dagger 2})}$ is the single-mode squeezing operator and the squeezing parameter r is supposed to be real throughout the paper. We note that a squeezed single photon can also be obtained by adding a photon to a squeezed vacuum as $\hat{a}^{\dagger}S(r)|0\rangle = \cosh rS(r)|1\rangle$.

Lund *et al.* showed that a small odd SCS with $\alpha \leq 1.2$ is well approximated by a squeezed single photon (i.e., a pure PSGS) as follows [21]. When the squeezing operator is applied to a single photon the resultant state can be expanded in terms of photon number states as

$$S(r)|1\rangle = \sum_{n=0}^{\infty} \frac{(-\tanh r)^n \sqrt{(2n+1)!}}{(\cosh r)^{3/2} 2^n n!} |2n+1\rangle. \tag{4}$$

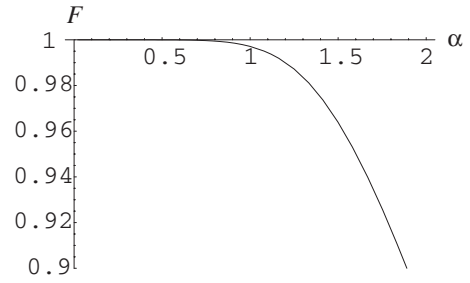


FIG. 1. Maximized fidelity F between an ideal odd SCS of amplitude α and a pure photon-subtracted Gaussian state.

The fidelity of this state to an odd SCS is

$$F(r, \alpha) = |\langle SCS_{-}(\alpha) | S(r) | 1 \rangle|^2 = \frac{2\alpha^2 \exp[-\alpha^2(\tanh r + 1)]}{(\cosh r)^3 [1 - \exp(-2\alpha^2)]}.$$

Figure 1 shows the maximized fidelity on the y axis plotted against a range of possible values for α for the desired odd SCS. It shows that the fidelity approaches unity for α very close to zero, while it decreases as α gets larger. The fidelity is maximized when r satisfies

$$\cosh r = \sqrt{\frac{1}{2} + \frac{1}{6}\sqrt{9 + 4\alpha^4}}. \tag{5}$$

The fidelity is $F > 0.99$ when $\alpha < 1.2$. Some example values are $F = 0.9998$ for $\alpha = 1/\sqrt{2}$ and $F = 0.997$ for $\alpha = 1$, where the maximizing squeezing parameters are $r = -0.164$ and -0.313 , respectively.

III. VIOLATIONS OF BELL'S INEQUALITY WITH PURE STATES

A. Bell–Clauser–Horne–Shimony–Holt inequality formalism with the Wigner functions

BW studied the Clauser, Horne, Shimony, and Holt (CHSH)'s version of Bell's inequality based upon photon number parity measurements [42]. The measurement operator is defined as

$$\begin{aligned} \Pi(z) &= \Pi^+(z) - \Pi^-(z) \\ &= D(z) \sum_{n=0}^{\infty} (|2n\rangle\langle 2n| - |2n+1\rangle\langle 2n+1|) D^{\dagger}(z), \end{aligned} \tag{6}$$

where $D(z)$ is the displacement operator $D(z) = \exp(z\hat{a}^{\dagger} - z^*\hat{a})$ for bosonic operators \hat{a} and \hat{a}^{\dagger} . Such a measurement should be able to discriminate between even and odd numbers of photons. It is known that the displacement operation can be effectively performed using a beam splitter with transmittivity close to 1 and a strong coherent state injected into the other input port. It was pointed out that in order to maximize the violation of the Bell-CHSH inequality for two-mode squeezed states and entangled coherent states, the BW formalism needs to be generalized to write the Bell operator as [7]

$$\begin{aligned} \mathcal{B}_{\text{BW}} = & \Pi_1(z_1)\Pi_2(z_2) + \Pi_1(z'_1)\Pi_2(z_2) + \Pi_1(z_1)\Pi_2(z'_2) \\ & - \Pi_1(z'_1)\Pi_2(z'_2), \end{aligned} \quad (7)$$

while BW assumed two of the four parameters equal to zero as $z_1=z_2=0$. The Bell-CHSH inequality can then be represented by the Wigner function as

$$\begin{aligned} |B_{\text{CHSH}}| = |\langle \mathcal{B}_{\text{BW}} \rangle| = & \frac{\pi^2}{4} |W(z_1, z_2) + W(z_1, z'_2) + W(z'_1, z_2) \\ & - W(z'_1, z'_2)| \leq 2, \end{aligned} \quad (8)$$

where $W(z_1, z_2)$ represents the Wigner function of a given state. In this paper, we refer to B_{CHSH} as the Bell-CHSH function. Using $\Pi_1(z_1)\Pi_1(z_1) = \Pi_2(z_2)\Pi_2(z_2) = 1$, it is straightforward to check the Cirel'son bound $|\langle \mathcal{B}_{\text{BW}} \rangle| \leq 2\sqrt{2}$ [43] in the generalized BW formalism [7].

The Wigner function of a PSGS can be obtained from its characteristic function

$$\begin{aligned} \chi_s(\eta) = & \text{Tr}[S(r)|1\rangle\langle 1|S^\dagger(r)e^{\eta\hat{a}^\dagger - \eta^*\hat{a}}] \\ = & \exp\left(-\frac{1}{2}(e^{2r}\eta_r^2 + e^{-2r}\eta_i^2)\right)(1 - e^{2r}\eta_r^2 + e^{-2r}\eta_i^2). \end{aligned} \quad (9)$$

The Wigner function is then

$$\begin{aligned} W_s(z) = & \frac{1}{\pi^2} \int e^{\eta^*z - \eta z^*} \chi_s(\eta) d^2\eta \\ = & \frac{2}{\pi} \exp[-2(e^{2r}z_r^2 + e^{-2r}z_i^2)] \\ & \times (4e^{2r}z_r^2 + 4e^{-2r}z_i^2 - 1). \end{aligned} \quad (10)$$

In order to perform a Bell inequality test, the single-mode PSGS should be divided by a beam splitter to generate a two-mode state shared by distant parties. The beam-splitter operator \hat{O}_{BS} acting on modes \hat{a} and \hat{b} is represented as

$$\hat{O}_{\text{BS}}(\theta) = \exp\left(\frac{\theta}{2}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})\right), \quad (11)$$

where the reflectivity and transmittivity are defined as $R = \sin^2(\theta/2)$ and $T = 1 - R$, respectively. When the PSGS passes through a 50:50 beam splitter, the resulting state is

$$W_{\text{tot}}(z_1, z_2) = W_s\left(\frac{z_1 + z_2}{\sqrt{2}}\right)W_v\left(\frac{-z_1 + z_2}{\sqrt{2}}\right), \quad (12)$$

where $W_v(z)$ is the Wigner function of the vacuum,

$$W_v(z) = \frac{2}{\pi} \exp(-2|z|^2). \quad (13)$$

The two-mode state $W_{\text{tot}}(z_1, z_2)$ can be used to calculate the Bell-CHSH function in Eq. (8). The Wigner function of the SCS is obtained by the same method as

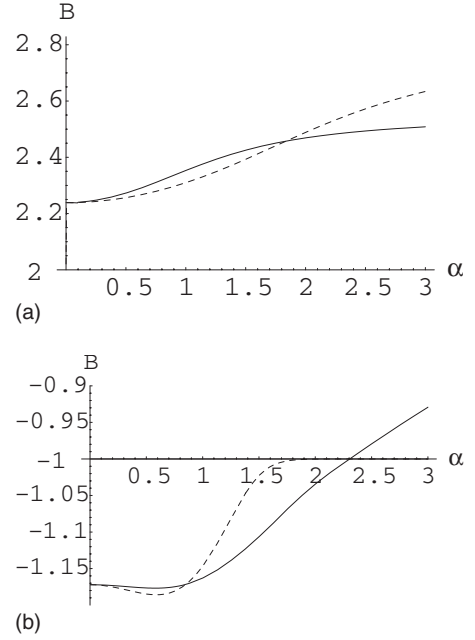


FIG. 2. (a) Optimized Bell-CHSH function $B = |B_{\text{CHSH}}|_{\text{max}}$ for a PSGS (solid curve) and a SCS (dashed curve), each of them divided at a 50:50 beam splitter, with photon number parity measurements and displacement operations. (b) The optimized Bell-CH function $B = |B_{\text{CH}}|_{\text{max}}$ for a PSGS (solid curve) and a SCS (dashed curve), each of them divided at a 50:50 beam splitter, with photon measurements and displacement operations.

$$W_c^\pm(z) = \frac{e^{-2|z|^2}}{\pi(1 \pm e^{-2\alpha^2})} [e^{-2\alpha^2}(e^{-4\alpha z_r} + e^{4\alpha z_r}) \pm 2 \cos 4\alpha z_i], \quad (14)$$

where $W_c^+(z)$ [$W_c^-(z)$] is the Wigner function of the even (odd) SCS. The Bell-CHSH function can be obtained in the same manner.

The optimized Bell-CHSH functions $|B_{\text{CHSH}}|_{\text{max}}$ for the cases of the PSGS and the SCS are plotted in Fig. 2(a). In this figure, the SCS of amplitude α and the PSGS that best approximates the SCS for the given amplitude are compared. The PSGS is found to show larger violations for $\alpha < 1.84$, i.e., for the range where the fidelity between the PSGS and the SCS is $F > 0.91$. Note that the fidelity between the two states is high when α is small enough, and in this limit (i.e., $\alpha \rightarrow 0$) the optimized Bell-CHSH functions for the two states become the same as shown in Fig. 2(a).

B. Bell-CH inequality formalism with the Q functions

BW used the Q function for a test of a Bell-CH inequality violation using photon presence (i.e., on-off) measurements [42]. Note that in this paper, we refer to a dichotomic measurement discriminating between “no photon” and “any photon(s)” as a photon presence measurement. This is obviously more realistic for an experimental Bell inequality test since it is difficult to measure the parity of photon numbers using currently available photodetectors. The Q function for a two-mode state ρ_{12} is defined as

$$Q_{12}(z_1, z_2) = \frac{{}_2\langle z_2 | \langle z_1 | \rho_{12} | z_1 \rangle | z_2 \rangle_2}{\pi^2}, \quad (15)$$

where $|z_1\rangle$ and $|z_2\rangle$ are coherent states of amplitude z_1 and z_2 , respectively. The Bell-CH function in terms of Q representation is

$$B_{\text{CH}} = \pi^2 [Q_{12}(z_1, z_2) + Q_{12}(z_1, z'_2) + Q_{12}(z'_1, z_2) - Q_{12}(z'_1, z'_2)] - \pi [Q_1(z_1) + Q_2(z_2)], \quad (16)$$

where $Q_1(z_1)$ and $Q_2(z_2)$ are the marginal Q functions of modes 1 and 2. Then the local theory imposes the Bell-CH inequality

$$-1 \leq B_{\text{CH}} \leq 0. \quad (17)$$

Equation (16) is a generalized version of the BW formalism [7]. The Q function for the entangled coherent state is

$$\begin{aligned} Q_{\text{ECS}}(z_1, z_2) &= N_-(\alpha)^2 \{ \exp(-|z_1 - \beta|^2 - |z_2 + \beta|^2) \\ &\quad + \exp(-|z_1 + \beta|^2 - |z_2 - \beta|^2) \\ &\quad - \exp[-(z_1 - \beta)(z_1^* + \beta) - (z_2 + \beta)(z_2^* - \beta) - 4\beta^2] \\ &\quad - \exp[-(z_1^* - \beta)(z_1 + \beta) - (z_2^* + \beta)(z_2 - \beta) - 4\beta^2] \}. \end{aligned} \quad (18)$$

The marginal Q function can also be calculated from (18) so that the Bell-CH function in Eq. (16) can be obtained. The optimized Bell-CH functions $|B_{\text{CH}}|_{\text{max}}$ for the cases of the SCS of amplitude α and the PSGS optimized to approximate the SCS are plotted in Fig. 2(b). Interestingly, the SCS shows large Bell violations when $\alpha < 0.85$ ($F > 0.999$) but the PSGS outperforms the SCS when $\alpha > 0.85$. This result is obviously different from the one for the Bell-CHSH inequality using photon parity measurements shown in Fig. 2(a).

IV. ANALYSIS WITH IMPURE GAUSSIAN STATES, A BEAM SPLITTER, AND A REALISTIC AVALANCHE PHOTODETECTOR

In this section, we consider some detrimental factors that can affect Bell inequality tests with PSGSs in real experiments. First, experimental Gaussian states are not pure states but have a small amount of mixing. Therefore, we take the impurity of the squeezed states into consideration in our calculation. Second, when one actually subtracts a photon from a Gaussian state, a beam splitter with a high transmittivity and an avalanche photodetector, which cannot resolve photon numbers, are typically used. In such an experimental setup, if the transmittivity of the beam splitter approaches 1, the probability of subtracting only one photon becomes close to 100%, provided the detector clicked. However, in this limit the success probability (i.e., the probability of the ‘‘click’’ event) approaches zero. Therefore we need to consider cases where the transmittivity of the beam splitter is not unity. In this case, the generated state will be a mixed state and that may affect the results of the Bell inequality tests. Furthermore, we need to assess the effects of the nonunit

efficiency and dark counts of a realistic avalanche photodetector used for photon subtraction. Note that we shall consider the PSGSs of $r > 0$ in this section without losing generality.

A. Effects of impure Gaussian states, a beam splitter, and an ideal avalanche photodetector

Kim *et al.* derived the characteristic function and the Wigner function of the PSGS under experimentally realistic assumptions as follows [32]. We are interested in Gaussian states with the characteristic functions in the form of

$$C(\xi) = \exp\left(-\frac{A}{2}\xi_r^2 - \frac{B}{2}\xi_i^2\right), \quad (19)$$

where A and B are determined by the quadrature variances of the field. When the squeezed Gaussian field in Eq. (19) is divided at a beam splitter, the two-mode characteristic function for the output field of modes 1 and 2 is [44]

$$C_{\text{out}}(\eta, \xi) = \exp\left(-\frac{1}{2}\mathbf{x}\mathbf{V}\mathbf{x}^T\right), \quad (20)$$

where $\mathbf{x} = (\eta_r, \eta_i, \xi_r, \xi_i)$ and the correlation matrix

$$\mathbf{V} = \begin{pmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & m_1 & 0 \\ 0 & c_2 & 0 & m_2 \end{pmatrix} \quad (21)$$

with

$$\begin{aligned} n_1 &= TA + R, & n_2 &= TB + R, & c_1 &= \sqrt{TR}(A - 1), \\ c_2 &= \sqrt{TR}(B - 1), & m_1 &= RA + T, & m_2 &= RB + T. \end{aligned} \quad (22)$$

When the avalanche photodetector, which does not resolve photon numbers, clicks at mode 2, the state generated at mode 1 is

$$\hat{\rho}_a = \mathcal{N} \sum_{n=1}^{\infty} {}_2\langle n | \hat{\rho}_{\text{out}} | n \rangle_2, \quad (23)$$

where \mathcal{N} is the normalization factor. Consider the unnormalized density operator for mode 1 of the output field,

$$\hat{\rho}_t = \text{Tr}_2(\hat{\rho}_{\text{out}}) = \sum_{n=0}^{\infty} {}_2\langle n | \hat{\rho}_{\text{out}} | n \rangle_2. \quad (24)$$

It is then clear from Eqs. (23) and (24) that

$$\hat{\rho}_a = \mathcal{N}(\hat{\rho}_t - {}_2\langle 0 | \hat{\rho}_{\text{out}} | 0 \rangle_2), \quad (25)$$

where

$$\hat{\rho}_t = \frac{1}{\pi} \int C_{\text{out}}(\eta, 0) \hat{D}_1(-\eta) d^2\eta \quad (26)$$

and

$${}_2\langle 0|\hat{\rho}_{\text{out}}|0\rangle_2 = \frac{1}{\pi^2} \int C_{\text{out}}(\eta, \xi) e^{-|\xi|^2/2} \hat{D}_1(-\eta) d^2\eta d^2\xi. \quad (27)$$

Using $C_{\text{out}}(\eta, \xi)$ in Eqs. (20)–(22) and Eqs. (25)–(27), the characteristic function $C_a(\zeta)$ for $\hat{\rho}_a$ is found [32]:

$$C_a(\zeta) = \text{Tr}[\hat{\rho}_a \hat{D}(\zeta)] = \mathcal{N} e^{-(n_1 \zeta_r^2 + n_2 \zeta_i^2)/2} \left[1 - \frac{2}{\sqrt{(m_1+1)(m_2+1)}} \times \exp\left(\frac{c_1^2}{2(m_1+1)} \zeta_r^2 + \frac{c_2^2}{2(m_2+1)} \zeta_i^2 \right) \right]. \quad (28)$$

The normalization factor is calculated as

$$\mathcal{N} = \frac{\sqrt{(m_1+1)(m_2+1)}}{\sqrt{(m_1+1)(m_2+1)} - 2}.$$

It is then straightforward to derive the Wigner function from Eq. (28) by the Fourier transform as performed in Eq. (10). The Q function is obtained in the same way but using the normally ordered characteristic function $C_a^Q(\zeta) = C_a(\zeta) \exp[-(1/2)|\zeta|^2]$. We also obtain the success probability of the click event for the avalanche photodetector as

$$P_s = 1 - {}_2\langle 0|\rho'_t|0\rangle_2 = \frac{2}{\sqrt{(1+m_1)(1+m_2)}}, \quad (29)$$

where

$$\hat{\rho}'_t = \frac{1}{\pi} \int C_{\text{out}}(0, \xi) \hat{D}_2(-\xi) d^2\xi. \quad (30)$$

The numerically optimized Bell-CHSH functions are plotted and compared with the corresponding success probabilities in Fig. 3(a). The violation for the pure PSGS ($r=0.3$) decreases as T gets smaller. We also consider the total variance $A \times B$, which is strictly related to the purity in the case of Gaussian states, as a measure of mixedness for the input Gaussian state. The total variance is 1 for a pure Gaussian state and it increases as the level of mixedness becomes larger. We can observe that the violation is quite sensitive to the degree of mixedness. We note that only a small amount of mixing (total variance about 1.02) decreases the Bell violation significantly, as shown in Fig. 3(a). The success probabilities for the corresponding cases are plotted in Fig. 3(b). When we equally increase A and B , the violation disappears when the variance is around $A=1.024$.

The numerically optimized Bell-CH functions using photon presence measurements are plotted in Fig. 4. In this case, the violation is obviously less sensitive to the mixedness of the input Gaussian state. We have considered a recent experimental achievement, where the squeezing degree is -3.57 dB below and 4.26 dB above the vacuum level, and the Bell-CH inequality is still violated [31,45]. The total variance in this case is about 1.17. This confirms that the Bell-CH inequality test using photon presence measurements is experimentally more feasible than the Bell-CHSH inequality test using photon number parity measurements.

Here we briefly compare our results with previous ones [38]. In our numerical study, the CH inequality using on-off

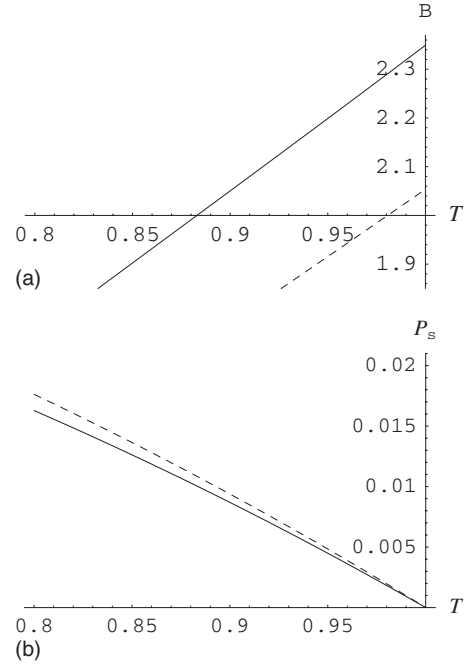


FIG. 3. (a) Numerically optimized Bell-CHSH function $B = |B_{\text{CHSH}}|_{\text{max}}$ for a PSGS using a pure Gaussian state of $r=0.3$ (2.61 dB, solid curve) and mixed Gaussian states (dashed curve). The mixed Gaussian state corresponds to -2.56 dB below and 2.65 dB above the vacuum level (the two dimensional variance $A \times B$ is about 1.02). The horizontal axis represents the transmittivity T of the beam splitter used to generate the PSGS. (b) Success probability of the pure (solid curve) and mixed (dashed curve) cases shown in (a).

photon detection is violated for $T > 0.64$ when $r=0.39$ and $\eta=1$. This is consistent with the results in Ref. [38], where the CHSH inequality based on photon on-off measurements is analyzed. However, in our study, the Bell-CHSH inequality using photon parity measurements shows Bell-CHSH violations for $T > 0.89$, while $T > 0.8$ is sufficient for the Bell-CHSH violation based on the photon on-off measurements.

B. Effects of nonunit quantum efficiency and dark counts for a realistic avalanche photodetector

We need to consider the effects of nonunit quantum efficiency of the avalanche photodetector used for photon subtraction. The postselection to generate the PSGS is made only when the detector clicks, regardless of the number of photons. As we pointed out, when the transmittivity T of the beam splitter is large, the probability of subtracting only one photon becomes dominant compared to the probability of subtracting more than one. This essential nature does not change even when the detection efficiency is low. Therefore, one may predict that the Bell inequality tests with the PSGS will be insensitive to the detection inefficiency when T is sufficiently large. This is confirmed by the following analysis.

Olivares and Paris obtained the generalized quasiprobability function $W_j(z)$ of the photon-subtracted squeezed state with the detection efficiency ϵ as [33]

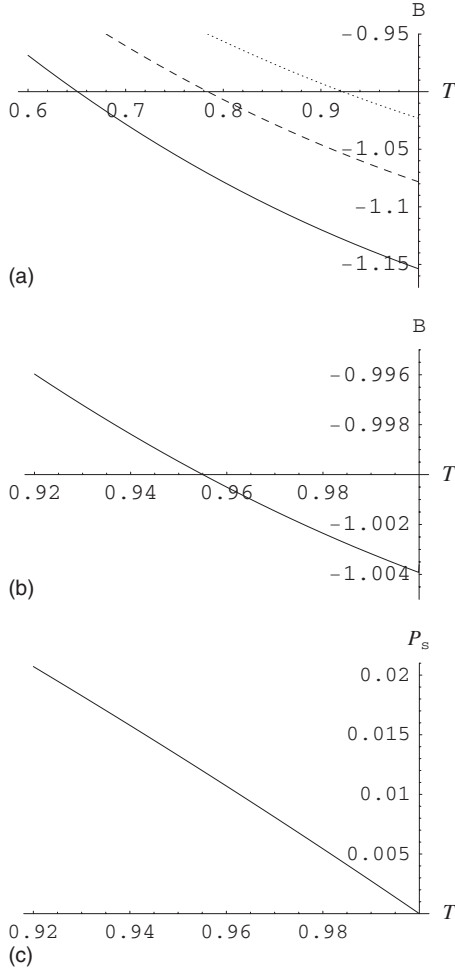


FIG. 4. (a) Numerically optimized Bell-CH function $B = |B_{\text{CH}}|_{\text{max}}$ for a PSGS using a pure Gaussian state of $r=0.3$ (2.61 dB, solid curve) and mixed Gaussian states. The mixed state case represented by the dashed curve corresponds to -2.52 dB below and 2.69 dB above the vacuum level (the variance $A \times B$ is about 1.04). The case represented by the dotted curve corresponds to -2.43 dB below and 2.78 dB above the vacuum level (the variance $A \times B$ is about 1.08). (b) Numerically optimized Bell-CH function for a PSGS using an experimentally feasible Gaussian state. The squeezing degree is -3.57 dB below and 4.26 dB above the vacuum level. The two-dimensional variance $A \times B$ in this case is about 1.17. (c) Success probability P_s of the case shown in (b).

$$W_j(z) = N_\epsilon \left(\mathcal{G}_1(z) - \frac{\mathcal{G}_2(z)}{\epsilon \sqrt{\text{Det}(H + \sigma_M)}} \right), \quad (31)$$

where N_ϵ is the normalization factor and

$$\mathcal{G}_k(z) = \frac{2 \exp\left(-\frac{2(2\mathcal{A}_k - j)|z|^2 + 4\mathcal{B}_k(z^2 + z^{*2})}{(2\mathcal{A}_k - j)^2 - 16\mathcal{B}_k^2}\right)}{\pi \sqrt{(2\mathcal{A}_k - j)^2 - 16\mathcal{B}_k^2}},$$

$$H = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix},$$

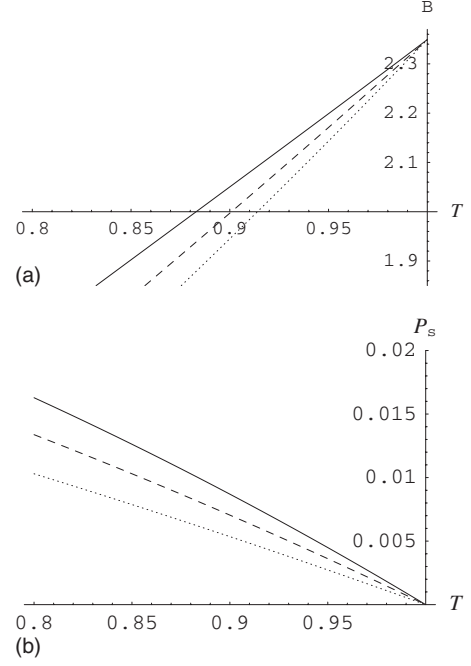


FIG. 5. (a) Numerically optimized Bell-CHSH function $B = |B_{\text{CHSH}}|_{\text{max}}$ for a PSGS using a pure Gaussian state of $r=0.3$ with a beam splitter of transmittivity T and a realistic detector of efficiency ϵ . The quantum efficiency of the detector is considered to be $\epsilon=1$ (solid line), 0.8 (dashed line), and 0.6 (dotted line). (b) Success probability P_s for the cases shown in (a).

$$\sigma_M = \frac{2 - \epsilon}{2\epsilon} \mathbb{1}, \quad (32)$$

with

$$\mathcal{A}_k = \frac{1}{2}(a_k^+ + a_k^-),$$

$$\mathcal{B}_k = \frac{1}{4}(a_k^- - a_k^+),$$

$$a_1^\pm = \frac{1}{2}[1 + (e^{\pm 2r} - 1)t],$$

$$a_2^\pm = \frac{1}{2} \pm \frac{t \sinh r}{\cosh r \mp [1 - \epsilon(1 - t)] \sinh r},$$

$$h^\pm = \frac{1}{2}[e^{\pm 2r}(1 - t) + t]. \quad (33)$$

Here, the function $W_j(z)$ becomes the Wigner function when $j=0$, while it becomes the Q function when $j=-1$. It is then straightforward to obtain the Bell-CHSH and the Bell-CH functions using the Wigner and Q functions, respectively, using Eqs. (8) and (16). We have plotted the numerically optimized Bell-CHSH and Bell-CH functions in Figs. 5 and 6, respectively. It shows again that the Bell-CH inequality is

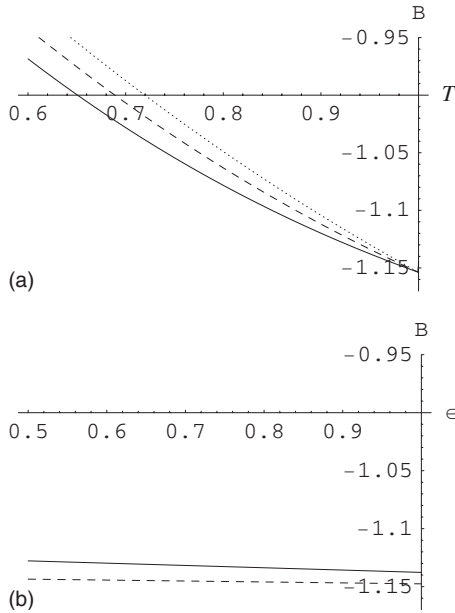


FIG. 6. (a) Numerically optimized Bell-CH function $B = |B_{CH}|_{\max}$ for a PSGS using a pure Gaussian state of $r=0.3$ with a beam splitter of transmittivity T and a realistic detector of efficiency ϵ . The quantum efficiency of the detector is considered to be $\epsilon=1$ (solid line), 0.8 (dashed line), and 0.6 (dotted line). (b) Numerically optimized Bell-CH function B against the detection efficiency ϵ for $T=0.95$ (solid line) and 0.98 (dashed line).

more robust against detection inefficiency. In Fig. 6(b), the numerically optimized Bell-CH function is plotted against the detection efficiency ϵ for $T=0.95$ (solid line) and 0.98 (dashed line). It seems obvious that, when the transmittivity T is as large as $T \geq 0.95$, detrimental effects of the detection inefficiency are very small.

If the dark count rate of the photodetector used to subtract a photon is non-negligible, the resulting state will be in a mixture of the photon-subtracted squeezed state and the squeezed vacuum. Such a mixed state can be represented as [28,32]

$$P_m W(z) + (1 - P_m) W_s(z), \tag{34}$$

where $W(z)$ is the Wigner function of the photon-subtracted squeezed state, $W_s(\alpha)$ is the Wigner function of the squeezed vacuum, and P_m is called the modal purity factor. The numerically optimized Bell-CH function is plotted against the modal purity factor in Fig. 7. If $P_m \leq 0.78$, the Bell violation cannot be observed even though T is as large as $T=0.99$ and the initial squeezed state was pure.

V. REMARKS

We have studied tests of Bell inequalities with PSGSs. It has been found that the PSGSs largely violate the Bell-CHSH inequality using photon number parity measurements and the Bell-CH inequality using photon presence measurements. The PSGSs and the SCSs violate Bell inequalities in different ways.

We have analyzed the effects of the key experimental components used to generate the PSGSs, the mixedness of

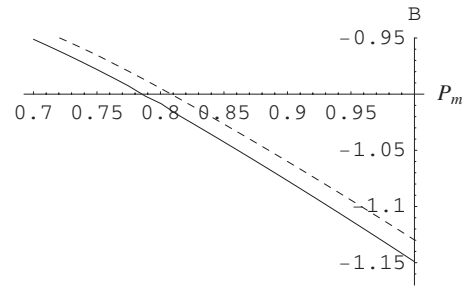


FIG. 7. Numerically optimized Bell-CH function $B = |B_{CH}|_{\max}$ for a PSGS using a pure Gaussian state of $r=0.3$ against the modal purity factor P_m with a beam splitter of transmittivity $T=0.99$ (solid line) and 0.95 (dashed line). The detection efficiency was supposed to be $\epsilon=0.6$.

the Gaussian states, the limited transmittivity of the beam splitter, and an avalanche photodetector that cannot resolve photon numbers, have been taken into consideration in our analysis. As a result of this analysis, the degrees of mixedness and the beam splitter transmittivity that can be allowed for successful tests of Bell's inequality have been revealed. The tests of the Bell-CH inequality using photon presence measurements have been found to be less sensitive to the mixedness of the Gaussian state used to generate the PSGS. We have also analyzed the effect of nonunit quantum efficiency and dark counts of the photon detector used for photon subtraction. We have pointed out that when the transmittivity T is large (e.g., $T \geq 0.95$) the detrimental effects of detection inefficiency are very small.

We now address the experimental feasibility of the Bell inequality tests discussed in this paper. The photon presence measurements are obviously easier to realize using current technology, and therefore the Bell-CH inequality is a better candidate for nonlocality tests of the PSGSs. Of course, the Bell-CH inequality tests using photon presence measurements may still suffer from the inefficiency of the photon detectors used for Bell inequality tests (not the one used for photon subtraction) in real experiments. This is a nontrivial problem to overcome.

As a less challenging experimental task, one may consider the tomography approach of verification of quantum nonlocality using homodyne detection used in Ref. [46]. Our study of the Bell inequality tests is based on the quasiprobability functions [42], and it is possible to reconstruct the quasiprobability functions using homodyne detection. Homodyne detection can be highly efficient using current technology [31]. Furthermore, even when the homodyne efficiency is not satisfactory, one may always correct losses at the detectors to reconstruct the quasiprobability functions of the generated state. The reconstruction of the two-mode and single-mode Q functions would be the most efficient method to verify quantum nonlocality of the generated PSGS. If the currently available Gaussian state is used and the final homodyne inefficiency is corrected, one can experimentally show that the generated state is the one that violates the Bell-CH inequality, as predicted in Fig. 4. Even though such a method is not a direct test of Bell's inequality using homodyne detection

described in Refs. [8,40,41], which are experimentally more demanding, it is enough to show that quantum-mechanically nonlocal states have been generated [46].

Of course, it should be noted that the violation of the Bell-CH inequality shown in Fig. 4(b), based on a recent experimental achievement, is small. Such small values may disappear because of other experimental factors such as the dark count rate of the avalanche photodetector. This indicates that it is important to improve the purity of the Gaussian

state to clearly verify quantum nonlocality of the two-mode PSGS in real experiments.

ACKNOWLEDGMENTS

This work was supported by the DTO-funded U.S. Army Research Office Contract No. W911NF-05-0397, the Australian Research Council, and the Queensland State Government. The author thanks Kentaro Wakui for his useful comments on his recent experiment [31,45].

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