Entanglement Renormalization and Topological Order

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The multiscale entanglement renormalization ansatz (MERA) is argued to provide a natural description for topological states of matter. The case of Kitaev's toric code is analyzed in detail and shown to possess a remarkably simple MERA description leading to distillation of the topological degrees of freedom at the top of the tensor network. Kitaev states on an infinite lattice are also shown to be a fixed point of the renormalization group flow associated with entanglement renormalization. All of these results generalize to arbitrary quantum double models.

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Renormalization group (RG) transformations aim to obtain an effective description of the large distance behavior of extended systems [1]. For lattice systems, this is achieved by constructing a sequence of increasingly coarse-grained lattices $\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots\}$, where a single site of lattice $\mathcal{L}_\tau$ effectively describes a block of an increasingly large number $n_\tau \sim \exp(\tau)$ of sites in the original $\mathcal{L}_0$ [2]. Real-space RG methods can, in particular, be applied to study quantum systems at $T = 0$, where each site of $\mathcal{L}_\tau$ is represented by a Hilbert space $\mathcal{K}_\tau$ [3]. There the goal is to identify the local degrees of freedom relevant to the ground state physics and retaining them in $\mathcal{K}_\tau$, whose dimension d_τ must be large enough. A severe problem is that in $D \geq 2$ dimensions, d_τ grows (doubly) exponentially in τ [4] because short-range entanglement accumulates at the block boundary.

Entanglement renormalization [5] is a novel real-space RG transformation, proposed in order to solve the above difficulties. Its defining feature is the use of disentanglers prior to the coarse-graining step. These are unitary operations acting on the interface of the blocks, reducing the amount of entanglement in the system (see Fig. 1.) A major achievement of the approach is that, when applied to a large class of ground states in both one [5] and two [6] spatial dimensions, the dimension d_τ does not grow with τ, which is made possible by the disentangling step and has deep implications [5,6]: In principle, the resulting transformation can be iterated indefinitely at constant computational cost, allowing to explore arbitrarily large length scales. Moreover, the system can be compared with itself at different length scales, so RG flows in the space of ground states or Hamiltonian couplings can be studied. A constant d_τ also leads to an efficient representation of the ground state in terms of a tensor network, the multiscale entanglement renormalization ansatz (MERA) [7].

Strongly correlated quantum systems at $T = 0$ organize in a plethora of phases or orders, e.g., local symmetry breaking and topological orders [8]. The former are described by a symmetry group and a local order parameter, and are associated with the physical mechanism of condensation of pointlike objects. Transitions between two such phases involve a symmetry change, as described by Landau's theory. A simple picture emerges from the perspective of entanglement renormalization [5,6]: under iterations of the RG transformation, ground states with local symmetry breaking order progressively lose entanglement and eventually converge to a trivial fixed point, an unentangled ground state. On the other hand, critical ground states describing transitions between these phases are nontrivial (entangled) fixed points. In either case, the MERA

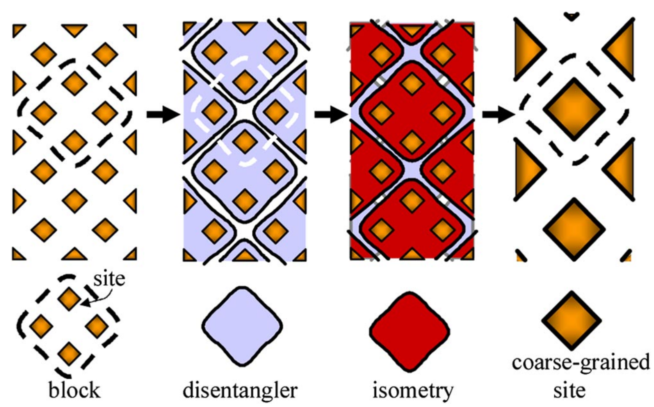

FIG. 1 (color online). RG transformation based on entanglement renormalization. To build an effective site from a 4-site block, disentanglers are first applied between sites of the block and surrounding sites, removing part of the short-ranged entanglement between the block and its surroundings. Then the four sites are coarse-grained into one by an isometry selecting the subspace $\mathcal{K}' \subseteq \mathcal{K}^{\otimes 4}$ to be kept. The case of a tilted square lattice is shown (in the toric code, each site will contain 4 qubits.)

provides an efficient, accurate representation of the ground state.

Topological phases are fundamentally different [8]. They do not stem from (the breakdown of) local group symmetries, but their topological order is linked to more complex mathematical objects, like tensor categories, topological quantum field theory, and quantum groups. Physically, topological phases exhibit gapped ground levels with robust degeneracy dependent only on the topology of the underlying space. This, and the existence of excitations possessing anyonic statistics, boosts the interest of these phases for topological quantum information storage and processing. Condensation of stringlike objects (in the string-net models of [9]) has been proposed as a general mechanism controlling topological phases. As may be expected, such profound differences are also reflected in the way the ground state is entangled. Specifically, topological entanglement entropy [10] (the subleading term in a large-perimeter expansion of the entanglement entropy of a region) has arisen as a quantitative measure of the ground state entanglement due to topological effects. Topologically ordered systems thus provide an unexplored scenario for entanglement renormalization techniques.

In this Letter we establish entanglement renormalization and the MERA as valid tools for the description and investigation of topological phases of matter. For simplicity, we analyze in detail Kitaev's toric code [11], a fourfold degenerate ground level widely discussed in the context of quantum computation and closely related to $\mathbb{Z}_2$ lattice gauge theory [12] and to the simplest of Levin and Wen's string-net models [9]. We show that (i) a MERA with finite, constant d_τ can represent the toric code exactly, (ii) at each RG iteration, entanglement renormalization factors out local degrees of freedom from the lattice, leaving topological degrees of freedom untouched, (iii) the MERA representation of the four ground states is identical except in its top tensor, which stores the topological degrees of freedom, and (iv) in an infinite system, the toric code is the fixed point of this RG transformation. These results also hold for more complicated models, such as quantum double lattice models [13]. We conclude that the MERA is naturally fitted to represent topologically ordered states, and entanglement renormalization offers a new, useful framework for their study.

Following [11], consider a square lattice Λ on the torus, with spin-$1/2$ (qubit) degrees of freedom attached to each link. The Hamiltonian

$$H = -\sum_{+} A_+ - \sum_{\square} B_\square \tag{1}$$

is a sum of constraints associated with vertices "+" and plaquettes "□," namely

$$A_+ = \prod_{i\in +} X_i, \qquad B_\square = \prod_{i\in\square} Z_i. \tag{2}$$

Stabilizers A_+ act as simultaneous spin flips on all four qubits adjacent to a given vertex. Stabilizers $B_\square$ yield the product of $\mathbb{Z}_2$ assignments ± 1 at the four qubits around a plaquette. Stabilizers commute with each other and have eigenvalues ± 1. Hamiltonian (1) is gapped, and states in the ground level (Kitaev states) are simultaneous eigenstates of all A_+, $B_\square$ with eigenvalue $+1$. The ground level degeneracy (number of Kitaev states) depends on the topology of the underlying manifold: If this is a nontrivial Riemann surface, information is encoded in nontrivial cycles, since operators $\prod_{i\in\mathcal{C}_{a,b}} Z_i$, where $\mathcal{C}_{a,b}$ are nontrivial cycles along bonds, commute with all stabilizers. Besides, such operators along homologically equivalent cycles $\mathcal{C}_a$, $\tilde{\mathcal{C}}_a$ act identically on Kitaev states. Hence, on a torus, two logical qubits are encoded.

Kitaev states are efficiently written in terms of their stabilizers. The stabilizer formalism [14] also furnishes a useful language to analyze the action of operators on Kitaev states, and has proved instrumental in finding an exact MERA. The key observation is that there exist "elementary moves" [15], minimal deformations of the lattice and Kitaev states, respecting the topological properties of the code. These moves consist of addition or removal of faces and vertices together with qubits, and can be written in terms of controlled-NOT (CNOT) operators, whose adjoint action on stabilizers is

$$I\otimes Z \leftrightarrow Z\otimes Z, \qquad Z\otimes I \mapsto Z\otimes I, \tag{3}$$

$$I\otimes X \mapsto I\otimes X, \qquad X\otimes I \leftrightarrow X\otimes X. \tag{4}$$

Figure 2 depicts the construction of elementary moves. A face is created by introducing a new spin in a plaquette. Arrows stand for CNOTs from controls (all qubits in one of the semiplaquettes) to the target n (the new qubit, introduced in state $|0\rangle$). The following transformation of stabilizers holds:

$$Z_1Z_2Z_3Z_4Z_5 \mapsto Z_1Z_2Z_3Z_4Z_5, \tag{5}$$

$$Z_n \mapsto Z_1Z_2Z_n, \tag{6}$$

enforcing plaquette constraints. Similarly, the relevant vertex constraints are extended to the new qubit. A new vertex is created instead by introducing a new qubit in state $|+\rangle$. This now acts as control for CNOTs acting on the qubits adjacent to one of the split vertices. Stabilizers transform as

$$X_1X_2X_3X_4X_5 \mapsto X_1X_2X_3X_4X_5, \tag{7}$$

$$X_n \mapsto X_5X_1X_n, \tag{8}$$

again compatible with the code constraints. Both final sets of operators are the correct stabilizers for the code in the modified lattice (recall $X^2 = Z^2 = I$); the two relevant plaquette constraints are extended to the new qubit.

These operations can be inverted to decouple qubits in states $|0\rangle$ and $|+\rangle$ from the system. Disentanglers and

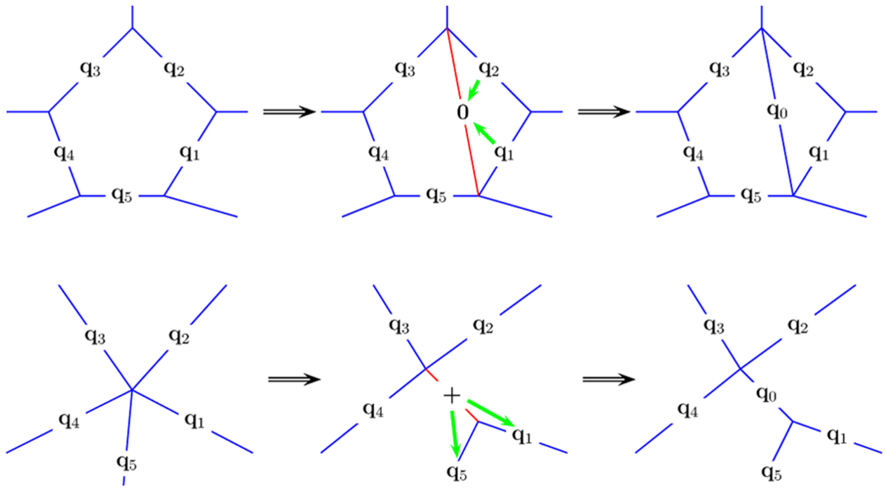

FIG. 2 (color online). Elementary moves adding plaquettes and vertices to a toric code. Arrows stand for CNOTs.

isometries defining the RG transformation and the MERA for Kitaev states are made of several such decoupling moves. We regard the original square lattice Λ, on which the toric code is defined, as a (tilted) square lattice $\mathcal{L}_0$ where each site contains 4 qubits. Then disentanglers and isometries act on 4-site blocks of $\mathcal{L}_0$ as in Fig. 1—i.e., on 16-qubit blocks in Λ. They consist of series of CNOTs, as specified in Figs. 3 and 4.

Upon applying the RG transformation, a coarse-grained lattice $\mathcal{L}_1$ obtains, locally identical to $\mathcal{L}_0$ and where, by construction, the toric code constraints still hold. This is quite remarkable. It is the first nontrivial example, in the context of entanglement renormalization, where the RG transformation is exact [16], leading to the first nontrivial model that can be exactly described with the MERA. On the other hand, on an infinite lattice, the above observation implies that Kitaev states are an explicit fixed point of the RG flow in the space of ground states induced by the present RG transformation [17].

Consider now a finite lattice $\mathcal{L}_0$ on the torus. The coarse-grained state carries exactly the same topological information ($\prod Z$ along nontrivial cycles) as the original state, since elementary moves preserve it at each intermediate step: Kitaev states are not mixed by the RG transformation. Iteration yields a sequence of increasingly coarse-grained

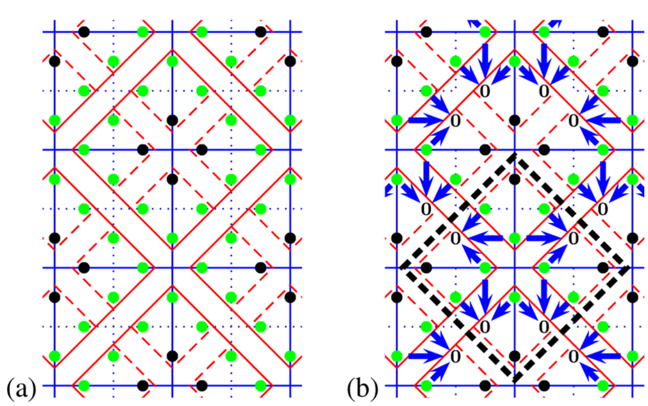

FIG. 3 (color online). (a) The square lattice Λ for the toric code, with qubits (dots) on the links, is reorganized into a tilted square lattice $\mathcal{L}_0$ where each site is made of four qubits. The lattice constant is doubled (dotted lines disappear) after the RG transformation, producing a new 4-qubit site for lattice $\mathcal{L}_1$ from every block of 16 qubits (the 12 light qubits in the block are decoupled in known product states). (b) First step of the RG transformation: Disentanglers. Arrows stand for simultaneous CNOT operators from control to target qubits. Disentanglers act on 16-qubit domains overlapping with four blocks each (thick dashed line, cf. Fig. 1.) 4 qubits per block decouple as $|0\rangle$.

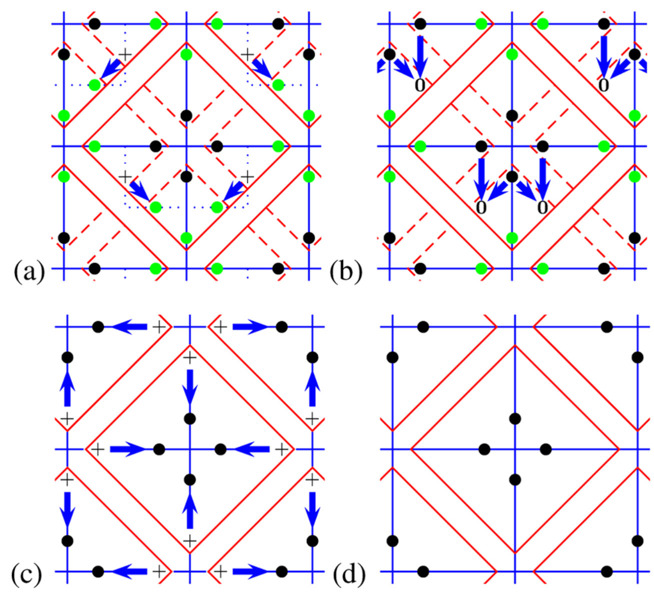

FIG. 4 (color online). (a)–(c) Second step of the RG transformation: Isometries. (a) 2 qubits per block decouple as $|+\rangle$. (b) 2 more qubits per block decouple as $|0\rangle$. (c) One qubit per edge (4 per block), decouple as $|+\rangle$. The isometry also traces out all 12 decoupled qubits. (d) State after the RG transformation.

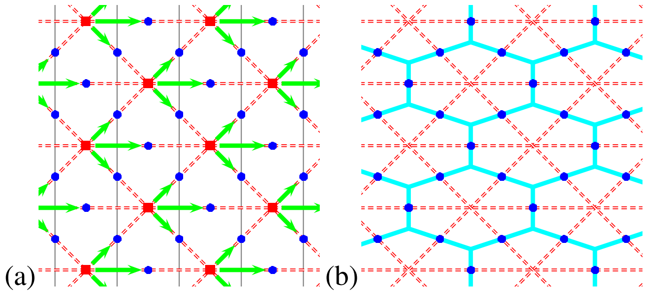

FIG. 5 (color online). Local mapping between the toric code on a square lattice (a) and on a triangular lattice (b). The dual model in a honeycomb lattice (displayed for reference) is Levin and Wen's loop model.

lattices $\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_T\}$ for ever smaller toruses. The top lattice $\mathcal{L}_T$ contains only a few qubits. Recall that the MERA is made of all disentanglers and isometries used in the RG transformations, together with a top tensor describing the state of $\mathcal{L}_T$ [7]. Hence, MERAs for different toric code states have identical disentanglers and isometries, differing only in their top tensor, where all topological information is stored.

These results extend directly to the simplest of Levin-Wen string-net models [9], a loop model. Indeed, the toric code on a square lattice can be locally transformed, using the decoupling moves of Fig. 5, into a toric code on a triangular lattice, equivalent to the ground state of the loop model on the dual (hexagonal) lattice. This local transformation shows that the topological order of both models is identical, as pointed out in [18]. This can also be understood in terms of the projected entangled-pair state ansatz (PEPS) [19].

Finally, our construction generalizes almost verbatim to quantum double models (see, e.g., [11]) for Abelian and non-Abelian groups. This is achieved by replacing CNOTs with controlled group multiplication operators and paying due attention to the order of the operations [13].

In conclusion, we have given an exact MERA representation for several topologically ordered models, where topological degrees of freedom are naturally isolated in its top tensor. We have also seen that such models are fixed points of the RG flow induced by entanglement renormalization. Our results are an unambiguous sign that entanglement renormalization and the MERA, originally developed to simulate efficiently systems with local symmetry breaking phases, provide also a natural framework to study topological phases.

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[1] M. E. Fisher, Rev. Mod. Phys. **70**, 653 (1998).
[2] L. P. Kadanoff, Physics **2**, 263 (1966).
[3] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975); S. R. White, Phys. Rev. Lett. **69**, 2863 (1992); Phys. Rev. B **48**, 10 345 (1993).
[4] In a D dimensional lattice the ground state typically obeys a boundary law $S_l \sim l^{D-1}$ for the entanglement entropy of a block of l^D sites. In this case the dimension d_τ for a site of $\mathcal{L}_\tau$ must at least scale doubly exponentially in τ, $d_\tau \sim \exp(\exp(\tau))$. Indeed, the dimension of an effective space for a block of l^D sites must be at least $d \sim \exp(S_l) = \exp(l^{D-1})$. On the other hand, $l_\tau \sim \exp(\tau)$ after τ iterations of the RG transformation, where $n_\tau = l_\tau^D$ counts the sites of $\mathcal{L}_0$ effectively described by a single site of $\mathcal{L}_\tau$.
[5] G. Vidal, Phys. Rev. Lett. **99**, 220405 (2007).
[6] G. Evenbly and G. Vidal, arXiv:0710.0692; arXiv:0801.2449
[7] G. Vidal, arXiv:quant-ph/0610099.
[8] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, New York, 2004).
[9] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[10] A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Rev. A **71**, 022315 (2005); A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006); M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[11] A. Yu. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[12] F. J. Wegner, J. Math. Phys. (N.Y.) **12**, 2259 (1971).
[13] See the appendix of the arXiv version of this Letter: M. Aguado and G. Vidal, arXiv:0712.0348.
[14] D. Gottesman, Ph.D. thesis, Caltech, 1997.
[15] E. Dennis *et al.*, J. Math. Phys. (N.Y.) **43**, 4452 (2002).
[16] The qubits removed from the lattice during the RG transformation are exactly in a product state. In all previous examples [5,6] it was only possible to approximately disentangle the subsystems before their removal.
[17] This points to a more general result, to be discussed elsewhere: under the present RG based on entanglement renormalization, topologically ordered systems flow toward Levin and Wen's string-net models [9], substantiating ideas already advocated by these authors.
[18] Z. Nussinov and G. Ortiz, arXiv:cond-mat/0702377.
[19] F. Verstraete, M. M. Wolf, D. Pérez-García, and J. I. Cirac, Phys. Rev. Lett. **96**, 220601 (2006).