A Comparison of Inequality Measurement Techniques

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There are numerous statistical techniques designed for the measurement of inequality. Each individual index has a unique set of properties which can make the choice of an appropriate measure difficult. This paper reviews the desirable properties for inequality indices to exhibit and proposes an additional characteristic that an effective measure may satisfy. Existing inequality measures are assessed against these criteria and a new technique that satisfies all desirable properties is proposed. An empirical demonstration of the proposed measure is provided.

1. Introduction

The literature on inequality measurement gives an extensive range of indices that may be used to analyze the distribution of an economic resource such as income. The development of an axiomatic framework helps to restrict this eclectic range of measures to a more manageable group by ruling out techniques that fail to meet certain useful criteria. The fundamental axioms of symmetry and scalar invariance (see Fields and Fei, 1978) require an inequality measure be naïve to permutations of incomes (symmetry) and invariant to scalar transformations (scalar invariance). Dalton's principle of population (Dalton, 1920) rules out any index that is affected by a proportional replication of the underlying population, while the Pigou-Dalton transfer principle (see Pigou (1912), Dalton (1920) and Rothschild and Stiglitz (1973)) requires that a measure respond appropriately to any redistribution of income from one individual to another. These axioms alone still allow for a wide variety of possible inequality indices. However the introduction of the diminishing transfer property (Kolm, 1976) and decomposition principle from Theil (1967), Cowell (1977), (Cowell and Kuga, 1981a,b) Foster (1983) Shorrocks (1980) and Bourguignon (1979) has reduced the range of viable indices to a subset of the Generalized Entropy (GE) class of measures.

Despite this thorough axiomatic structure, certain ad hoc measures which satisfy only basic requirements such as the Gini coefficient, Relative Mean Deviation and Coefficient of Variation remain extremely popular for empirical studies, as well as the sophisticated classes of social welfare based measures proposed by Dalton (1920) Aigner and Heins (1967) and Atkinson (1970) (now referred to as the Atkinson-Kolm-Sen or A-K-S class of measure).

For the social welfare classes of measure the reason for their popularity is fairly clear; the indices neatly summarize either the potential social welfare wasted due to inequality or the reduction in incomes we could tolerate for an equivalent welfare should we decide to distribute incomes equally. An additional advantage of these approaches is that they allow the user to explicitly model his or her attitudes to inequality amongst the lower and higher ends of the distribution.

The reasons for the popularity of the more ad hoc measures (Gini coefficient, Logarithmic Variance, Relative Mean Deviation, Coefficient of Variation and others) are not so clear. While sharing the advantage of predating other more coherent classes of measure[1], it seems unlikely that these techniques will ever be completely abandoned in favor of the more axiomatic Generalized Entropy measures. In this paper we review some of the properties of these types of inequality measures and investigate some of mechanics behind them. We are specifically interested in taking inequality measures of the form $I = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$ (where x_i is the income accruing to the *i*th individual and I is the inequality measure) and examining the properties of the evaluation function $f(x_i)$. We suggest that the evaluation functions of certain ad hoc measures are easily reconcilable with basic notions of inequality and that this property has a certain merit. We propose an inequality measure that exhibits this property for the evaluation function and also satisfies the given axioms of measurement. The new measure is demonstrated with an empirical example using income data from Taiwan.

2. Axioms of inequality measurement

Consider the set of distributions $D_n = \{x \in \Re^n \wedge x \geq 0^n\}$ where x is income or some other economic variable of interest. Let $I(x_1, x_2 ... x_n)$ be the inequality measure. The axioms of measurement are given below.

(1) Symmetry

A symmetric measure considers only the incomes of the individuals being measured such that any rearrangements of incomes amongst individuals will leave the measure unchanged. Take the distribution $x \in D$. Inequality measure I is symmetric if $I(x) = I(xP)$ where P is any $n \times n$ permutation matrix[2]. This is analogous to stating $I(x_1, x_2 ... x_n) = I(x_2, x_1, .. x_n)$ for any possible regrouping of x.

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[1] The Gini coefficient first appeared in 1912, while other measures such as the Relative Mean Deviation have been used considerably longer.

[2] A matrix which may be constructed from any permutation of the rows and columns of an $n \times n$ identity matrix.

(2) Scale independence

A measure should be insensitive to proportional changes in the underlying variable including changes in the units of measurement. The inequality measure I satisfies the relativity or scale independence axiom if $I(x) = I(Gx) \ \forall G \in \Re_+$.

(3) Population Replication

The inequality measure should be invariant to proportional changes in the underlying population. Consider distribution x' which is a proportional replicate of distribution x where each element occurs with frequency f i.e. $x' = \{x_{11},...x_{1f}, x_{21,}...x_{2f}, x_{n1},...x_{nf}\}$. I will satisfy the principle of population replication if $I(x; f) = I(x) \ \forall f \in N_+$.

(4) Pigou-Dalton transfer principle

The Pigou-Dalton transfer principle requires that any transfer of income from a higher to a lower income earner must reduce the inequality measure. The definition given here is taken from Cowell and Kuga (1981). $I(x)$ will satisfy the transfer principle if $T_{ji}(x) < 0$ for $x_i < x_j$ where it exists and $T_{ji}(x) \equiv \lim_{\varepsilon \to 0} \frac{\{I(x_1, x_2,...x_i + \varepsilon) - I(x_1, x_2,...x_j - \varepsilon)\}}{\varepsilon}$.

(5) Decomposability

Consider a partition of x into s subgroups such that each subgroup x^k has $n^k \geq 1$ elements, $x^k \cap x^t = \varnothing$ for $x^k \neq x^t$ and $\sum_{i=1}^{s} n^i = n$. The inequality index I is decomposable if it may be written as a function of the *within group* and *between group* inequalities. $I = f(I^1, I^2,...I^s, \mu^1, \mu^2,...\mu^s, n^1, n^2,...n^s)$ where μ^k is the mean income level of partition x^k and I^k is the inequality measured within x^k.

(6) Principle of diminishing transfers

The principle of diminishing transfers requires that a progressive transfer (from a higher to a lower income earner) should have a diminishing effect at higher income levels. The example given by Kolm (1976) says a small transfer from an individual with an income of 900 units to an individual with 500 should reduce inequality less than an equivalent transfer from the individual with the income of 500 units to another earning 100 units. Using the definition of the transfer principle from (3) the diminishing transfer principle may be formally stated as $T_{ji}(x) < T_{lk}(x)$ where $x_j - x_i = x_k - x_l$ and $x_i < x_k$.

3. The mechanics of some inequality measures

Consider an inequality measure of the form $I = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$ with mean income level μ. Our objective here is to investigate the manner in which f could assign individual components to each income x_i. One method for assignment used by most ad hoc measures is for $f(x_i)$ to reflect the 'extremeness' of income x_i relative to the mean of the distribution. These measures typically assign positive components to all $x_i \neq \mu$, with the assigned values increasing monotonically as x_i diverges from μ. The inequality measure may then be calculated as the expected value of these components. An advantage of this property is that we can divide the population into any arbitrary subgroups and observe the extent to which total inequality is driven by the incomes within each subgroup, while the original measure may be reconstructed as the population weighted average of these group contributions. Thus the inequality measure may be written as

(1) $$I = n^1 \frac{1}{n}\sum_{i=1}^{n^1} f\left(x_i^1\right) + n^2 \frac{1}{n}\sum_{i=1}^{n^2} f\left(x_i^2\right) ... + n^s \frac{1}{n}\sum_{i=1}^{n^s} f\left(x_i^s\right) \text{ for } k = 1...s$$

where each $\frac{1}{n^k}\sum_{i=1}^{k} f\left(x_i^k\right) \geq 0$ and gives an indication of the 'extremeness' of the incomes within subgroup x^k. In this paper we suggest the ability for an inequality measure to be dissected in this manner such that we can observe the proportion of total inequality that may be attributed to any individual income or group has some appeal.

Three simple measures that exemplify this notion are the Relative Mean Deviation, the Logarithmic Variance and the Variance of the Logarithms. These are

(2) $$RMD = \frac{1}{n}\sum_{i=1}^{n} \frac{|x_i - \mu|}{\mu}$$

(3) $$LV = \frac{1}{n}\sum_{i=1}^{n} \left[\ln\left(\frac{x_i}{\mu}\right)\right]^2$$

(4) $$VL = \frac{1}{n}\sum_{i=1}^{n} \left[\ln x_i - \mu_L\right]^2$$

where μ_L is the mean of the logarithm of income.

Each measure assigns a value of zero to mean incomes and increasingly positive components as incomes diverge from a point of central reference[3]. This characteristic can be defined with the conditions

$$f(x_i) \geq 0 \text{ for all } x_i,\ f'(x_i) < 0 \text{ for } x_i < \mu \text{ and } f'(x_i) > 0 \text{ for } x_i < \mu$$

where μ is the point of central reference. For the Variance of Logarithms measure this reference point is μ_L.

These properties alone however are not sufficient to ensure that the transfer principle (axiom 4) is satisfied. To ensure this we need to add the requirement[4] $f''(x_i) > 0$ for all x_i, a condition not satisfied by these three measures. The Relative Mean Deviation fails the transfer principle as it is insensitive to transfers that do not cross the mean while *VL* and *LV* can record an increase in inequality from progressive transfers at high income levels. As these measures fail to satisfy this crucial axiom they remain as peripheral techniques for the measurement of inequality.

The square of the Coefficient of Variation[5] is a more effective index as it satisfies the transfer principle and is another measure that exhibits this property for *f*. The measure may be calculated as

(5) $$CV^2 = \frac{1}{n}\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\mu^2}$$

It is simple to verify that $f(x_i)$ increases as x_i diverges from μ, that these assigned components are non negative, and that $f''(x_i) > 0$ for all x_i. As this measure is a special case of the Generalized Entropy family, it may be additively decomposed into between and within group inequalities (axiom 5). However this measure places an equal weighting on equivalent transfers regardless of the positioning in the distribution of the incomes concerned. For this reason the measure fails the diminishing transfer principle (axiom 6), leading to criticism of this measure from Atkinson (1970) and Love and Wolfson (1976) who suggest that this characteristic alone may be enough to disqualify the (square root of this) measure from consideration.

The Gini coefficient possess a similar property in assigning individual components based on the extremeness of that income, but with respect to all other incomes rather than just the mean income level. This measure may be calculated as

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[3] In the case of the VL this is the mean of the logarithm of income.
[4] See section 4 for more detail on this property.
[5] It is not possible to extract the individual elements of the Coefficient of Variation, however to do so for the square of this measure is straightforward.

(6) $$G=\frac{1}{n}\sum_{i=1}^{n}\frac{\sum_{j=1}^{n}\left|x_i-x_j\right|}{2n\mu}$$

where $\sum_{j=1}^{n}\frac{\left|x_i-x_j\right|}{2n\mu}$ is a non negative measure of the relative distance of x_i from the other incomes. As such we can still use the Gini coefficient to examine the extent that a particular income drives inequality by looking at each individual component, although there is no clear-cut point of central reference. Despite this property, the Gini coefficient is unable to be exhaustively decomposed into between and within subgroup inequalities (see Pyatt, 1976) and any attempt to do so leaves an awkward residual component (axiom 5). The measure also fails to satisfy the diminishing transfer principle (axiom 6) as the weighting of a transfer depends only the rankings of incomes involved (Cowell, 1977).

It is difficult to apply the same concept to the social welfare based A-K-S class of measures as they are not easily written in the form $I=\frac{1}{n}\sum_{i=1}^{n}f(x_i)$ such that the individual elements may be analyzed. Dalton's measure however may be written as

(7) $$I=\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{W(x_i)}{W(\mu)}\right)$$

where $W(x_i)$ is the welfare of individual i and is a concave and symmetric function of income. For these measures $f(x_i)$ is simply a measure of the relative welfare of individual i, normalized such that lower income earners have negative relative welfare and higher positive income earners have positive relative welfare. Thus the expected value of $f(x_i)$ for some arbitrary subgroup of x gives an indication of the relative welfare of the subgroup rather than the extremeness of the incomes concerned. A subgroup may have a mean relative welfare of zero, which could be due to all the included incomes being equal to μ, or just as easily, be the result of some very high incomes being offset by the inclusion of very low incomes. Hence the average inequality component of group *k*, $I_c^k=\frac{1}{n^k}\sum_{i=1}^{k}f(x_i^k)$ carries some information about the total magnitude of incomes in subgroup x^k (a negative value implies low welfare which implies low incomes), but contains no information about how far these incomes deviate from μ. Conversely the 'extremeness' property for $f(x_i)$ ensures that I_c^k carries information about the extent to which the constitute incomes differ from μ (or other measure of central tendency) but not as to the manner in which they deviate i.e. by being higher or lower than the mean.

The social welfare based measures satisfy axioms 1-4 and 6 from above but are unable to be additively decomposed. They do however possess a weaker 'aggregative' property as described by Bourguignon (1979).

It is more difficult to place an interpretation on the individual components that make up the Generalized Entropy class of measure. These are typically given as

(8) $$I_{GE} = \frac{1}{\alpha^2 - \alpha}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\mu}\right)^{\alpha} - 1\right]$$

where the parameter α dictates the sensitivity of the measure at higher and lower ends of the distribution. A problem arises due to the existence of several different methods for the calculation of various GE measures, each of which assigns different values to individual incomes for the same index. Calculated as is, the GE index assigns positive components to high incomes and negative components to low incomes for positive values for α. These are reversed when α changes sign, making it difficult to assign a direct economic interpretation upon the individual components. While these measures have a solid grounding in information theory, the normalization implicit in equation (8) dilutes any interpretation that may be placed on the individual components. Cowell (1977) derives this class of measure from first principles, which is useful for determining an interpretation for $f(x_i)$. The measures may be written as

(9) $$I = \sum_{i=1}^{n}\left[\left(\frac{1}{n}h\left(\frac{1}{n}\right) - \frac{x_i}{n\mu}h\left(\frac{x_i}{n\mu}\right)\right)N(\beta; n)\right]$$

where the function h assigns information content to the proportional share of income x_i and is usually assumed to be of the form

(10) $$h = \left[1 - \left(\frac{x_i}{n\mu}\right)^{\beta}\right]\beta^{-1}$$

The parameter β determines the shape of the information theoretic function h and is used to control the emphasis placed on higher or lower incomes, while $N(\beta, n)$ is used to normalize the measure chosen such that $I = I_{GE}$. The advantage of equation (9) is that each $f(x_i)$ is proportional to the difference between the information content of each income and the information content of an income corresponding to perfect equality (i.e. $x_i = \mu$). As these individual elements may take negative or positive values depending on the size and sign of the individual incomes, the average of elements in any subgroup will have a similar property to that of a social welfare based measure.

4. Determining a new measure

Given that none of the established measures satisfy both the extremeness property for $f(x_i)$ and the given axioms, we propose a new measure here for this purpose. Consider the index

(11) $$I = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i - \mu}{\mu}\ln\left(\frac{x_i}{\mu}\right)$$

In continuous form

(12) $$I = \int_0^{\infty}\frac{x}{\mu}\ln\left(\frac{x}{\mu}\right)f(x)dx + \int_0^{\infty}\ln\left(\frac{\mu}{x}\right)f(x)dx$$

With grouped data

(13) $$I = \sum_{k=1}^{s}\left(s^k - p^k\right)\ln\left(\frac{s^k}{p^k}\right)$$

where s^k is the income share of x^k, and p^k is the population share.

The given index is equal to the sum of Theil's T and L inequality measures, special cases of the GE class of measure as $\alpha \to 0{,}1$ respectively. It is straightforward to verify that $f(x_i) \geq 0$ for $i = 1...n$, $f'(x_i) < 0$ for $x_i < \mu$ and $f'(x_i) > 0$ for $x_i < \mu$ and thus exhibits the desired properties for $f(x_i)$. The measure is shown to satisfy axioms 1-6 below.

Axiom (1) Symmetry: The measure is symmetric as $f(x_i) = f(x_j)\ \forall\ x_i = x_j$

Axiom (2) Scale independence: The measure is insensitive to income scaling factor G since

(13) $$I(Gx) = \frac{1}{n}\sum_{i=1}^{n}\frac{Gx_i - G\mu}{G\mu}\ln\left(\frac{Gx_i}{G\mu}\right) = \frac{1}{n}\sum_{i=n}^{n}\frac{x_i - \mu}{\mu}\ln\left(\frac{x}{\mu}\right) = I(x)\ \forall\ G \in \Re_+$$

Axiom (3) Population invariance: Consider a replication of underlying distribution x where each x_i occurs with frequency $f \in N_+$. The inequality measure is

(14) $$I(x; f) = \sum_{t=1}^{f}\sum_{i=1}^{n}\frac{1}{nf}\frac{x_i - u}{\mu}\ln\left(\frac{x}{\mu}\right)$$

which equals

(15) $$I(x; f) = f\sum_{i=1}^{n}\frac{1}{nf}\frac{x_i - \mu}{\mu}\ln\left(\frac{x}{\mu}\right) = I(x)$$

Axiom (4) Pigou-Dalton transfer principle: The transfer principle will be satisfied if $f''(x_i) > 0 \;\; i = 1...n$. To see this, consider a progressive transfer of size dx from higher income earner *j* to lower income earner *i*. The effect of the transfer on the inequality index *I* with evaluation function $f(x)$ is

(16) $$T_{ji} = -\frac{1}{n}\frac{df(x)}{dx_j} + \frac{1}{n}\frac{df(x)}{dx_i} < 0 \qquad \text{if} \quad \frac{df(x)}{dx_j} > \frac{df(x)}{dx_i} \text{ for } j > i$$

We can ensure $\frac{df(x)}{dx_j} > \frac{df(x)}{dx_i}$ by requiring $\frac{d^2 f(x)}{dx^2} > 0$ for all x_i.

The evaluation function $f(x)$ and first and second derivatives are:

(17) $$f(x) = \frac{x-\mu}{\mu}\ln\left(\frac{x}{\mu}\right)$$

(18) $$\frac{df(x)}{dx} = \frac{1}{\mu}\left(\ln\left(\frac{x}{\mu}\right) + 1\right) - \frac{1}{x}$$

(19) $$\frac{d^2 f(x)}{dx^2} = \frac{1}{x^2} + \frac{1}{\mu x} > 0 \qquad x > 0$$

Axiom (5) Decomposability: Various proofs and demonstrations of the decomposition of Theil's T and L measures are given by Theil (1967), Cowell and Kuga (1980), Foster (1983), Shorrocks (1980) and Bourguignon (1979) and are not reproduced here. The decomposition of the proposed index requires that the measure of inequality within subgroup k can be written as $I^k = T^k + L^k$ where T^k and L^k are Theil's inequality indices. That is, we must be able to separate each *within group* inequality estimate I^k into contributions T^k and L^k as each receives a different weighting for the decomposition of the index. Then the index may be decomposed as

(20) $$I = \sum_{k=1}^{s}\left(s^k T^k + p^k L^k\right) + \left(T^B + L^B\right)$$

where T^B and L^B are the *between group* inequality indices. It should be noted that this decomposition is somewhat messier than the standard GE decomposition due to the required separation of T and L and a more complex weighting system required to reconstruct the original measure. This messiness may be seen as a disadvantage of this technique.

Axiom (6) Diminishing Transfers: I will satisfy the diminishing transfer principle if $T_{ji}(x) < T_{lk}(x)$ where $x_j - x_i = x_k - x_l$ and $x_i < x_k$. We can write

(21) $$T_{ji} - T_{lk} = \left(-\frac{1}{n}\frac{df(x)}{dx_i} + \frac{1}{n}\frac{df(x)}{dx_j} \right) - \left(-\frac{1}{n}\frac{df(x)}{dx_k} + \frac{1}{n}\frac{df(x)}{dx_j} \right)$$

which will be positive if $\frac{d^3 f(x)}{dx^3} < 0$ for all *x*. It is simple to verify that

(22) $$\frac{d^3 f(x)}{dx^3} = \frac{-2}{x^3} - \frac{1}{\mu x^2} < 0 \qquad x > 0$$

thus the measure satisfies the diminishing transfer axiom.

5. An empirical example

In this section a brief example of the proposed measure is given. The data analyzed is decile data for Taiwan from 1993 and is taken from a study by Chotikapanich, Griffiths and Rao (2007). The decile shares refer to the proportion of total income accruing to income-ordered population groups of size 10%, while the individual components are given by $\frac{1}{n} f(x_i)$. The aggregates of these are presented in the bottom row in bold type.

Table 1. Decile shares and inequality components for Taiwan 1993.

Decile share	$\frac{1}{n} f(x_i)$
0.037	0.062637893
0.051	0.032993883
0.061	0.019277557
0.070	0.010700248
0.079	0.004950169
0.089	0.001281872
0.103	8.86764E-05
0.12	0.003646431
0.148	0.018818020
0.242	0.125494991
1	**0.279889741**

As the given measure exhibits the discussed 'extremeness' property, each individual component $\frac{1}{n} f(x_i)$ reflects the extent to which the income of each decile share differs from the mean income level. Using this concept we may take each individual component to determine the amount of total inequality attributable to each decile share. As expected, the highest and lowest decile shares give the largest contributions to total inequality as these shares represent the most extreme incomes. According to the measure, slightly over 44% (0.125494991/0.279889741) of all the measured inequality comes from the highest 10% of incomes, while the lowest earning 10% contributes over 22% of total inequality. We also see that the decile shares with more moderate incomes (where income shares become roughly proportional to population shares) contribute diminishing amounts to total inequality. This dissection of the measure says nothing about the inequality within the subgroups however, only on the inequality between the subgroups and how much of the *between group* inequality is driven by each decile share. If information on the distribution within each subgroup is available however, this information may be included in the analysis using decomposition equation (20).

6. Conclusion

The paper establishes the properties and demonstrates the empirical viability of a new inequality measure. The measure works by assigning positive values to each income or income share based on the extent to which they deviate from a point of central reference, the average of which may be used as an inequality summary statistic. Of the many inequality indices that have this characteristic, the proposed measure is the only one that also satisfies the axioms of decomposability and diminishing transfers. An empirical application to Taiwanese data showcases a desirable aspect of this property.

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