Quantum information processing with Schrödinger cats

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ABSTRACT

We investigate the utility of a superposition of coherent states for quantum information processing. We show that quantum computation circuits using coherent states as the logical qubits can be constructed from simple linear networks, conditional measurements and coherent superposition resource states. We can utilize such states and associated quantum circuits for two quantum metrology applications, one for weak force measurements and the second for precise phase estimation. We show in both cases that a sensitivity at the Heisenberg limit is achievable.

Keywords: Quantum Computation, Quantum Optics, Quantum Information, Quantum Metrology, Schrödinger Cats

1. INTRODUCTION

Quantum optics has played a major role in the testing of fundamental properties of quantum mechanics and in more recently implementing simple quantum information protocols.[1,2] This has been made possible because photons are easily produced and manipulated. This is especially true as the electro-magnetic environment at optical frequencies can be regarded as vacuum and is hence are relatively decoherence free.

One of the earliest proposals for implementing quantum computation made by Milburn[3] and was based on encoding each qubit in two optical modes, each containing exactly one photon. This was a very elegant proposal, but unfortunately to entangle these photons required two qubit gates with massive and reversible non-linearities. Such reversible non-linearities well beyond those presently available and hence it was thought quantum optics would not provide a practical path to efficient and scalable quantum computation. Knill, Laflamme and Milburn[4] recently challenged this orthodoxy when they showed that given appropriate single photon sources and detectors, linear optics alone could create a non-deterministic 2 qubit gates (see Fig(1)). Furthermore they showed that

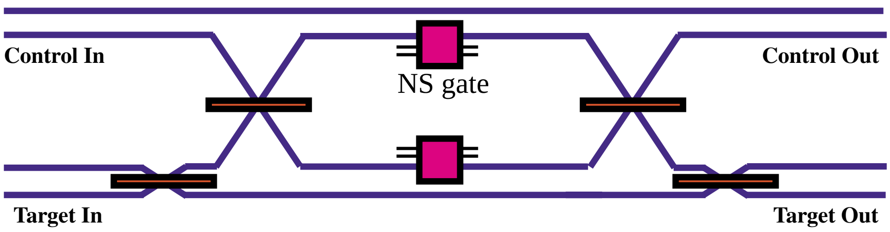

Figure 1. Schematic circuit for the KLM CNOT gate. The boxes in the center of the circuit are nonlinear sign shifts gates that perform the transformation $c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$ to $c_0|0\rangle + c_1|1\rangle - c_2|2\rangle$. Such a transformation can be performed using on linear optical elements, single photon sources and single photon number resolving detectors.

near deterministic gates could be created from these non-deterministic gates through a technique of teleporting gates.[5] This therefore provided a route for efficient and scalable quantum computation with only single photon sources, photon counting and linear optics.

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This does however raise the question where there are other architectures based on different encoding schemes which have similar characteristics. These other architectures may have advantages in that there optical circuit are less complex. We could trade of the complexity of the circuit in the KLM scheme for more complicated initial resources, for instance continuous variable multi-photon fields. The idea of encoding quantum information on continuous variables of multi-photon fields has emerged recently[6] and a number of schemes have been proposed for realizing quantum computation in this way.[7–9] A significant drawback of these proposals is that **hard** non-linear interactions are required *in-line* of the computation and make such proposals difficult to implement in practice. In this paper we investigate an efficient scheme which is elegant in its simplicity and the use of only **easy**, linear in-line interactions. The hard interactions are only required for *off-line* production of resource states. Our proposal involves encoding the quantum information in multi-photon coherent states, rather than single photon states. Simple optical manipulations acquire unexpected power in this situation. However the required resource, which may be produced non-deterministically *off-line*, is a superposition of coherent states.[10] Given this, the scheme is deterministic and requires only simple linear optics. Qubit readout uses homodyne detection which can be highly efficient.[11] This allows us to produce a universal set of gates sufficient for quantum computation. Are there other nature applications for resources and gates. One nature application lies in high precision measurements where we will examine two specific examples: the detection of weak tidal forces due to gravitational radiation[1, 12, 13] and improving the sensitivity of Ramsey fringe interferometry.[14, 15]

This paper is structured as follows: Section (2) describes a scheme for generating the superposition of coherent states while section (3) describes how to achieve a universal set of gates sufficient for quantum information processing. Finally section (4) illustrates two quantum metrology examples.

2. GENERATION OF SCHRÖDINGER CATS STATE

Before we focus our attention on logic gates and quantum metrology it is critical to highlight how small amplitude Schrödinger cat states of the form

$$|\Psi_\pm\rangle \quad = \quad \frac{1}{\sqrt{\mathcal{N}_\pm}}\left[|\alpha\rangle \pm |-\alpha\rangle\right], \tag{1}$$

where $\mathcal{N}_\pm = 2 \pm 2e^{-2|\alpha|^2}$, can be realized using technologies currently available or likely in the near future. The amplitude of these cat states need not be large to demonstrate logic gates for instance ($\alpha \approx 2$ is sufficient). An elegant proposal has been made by Dakna et.al[16] who scheme is depicted in Fig (2a). A squeezed state of the

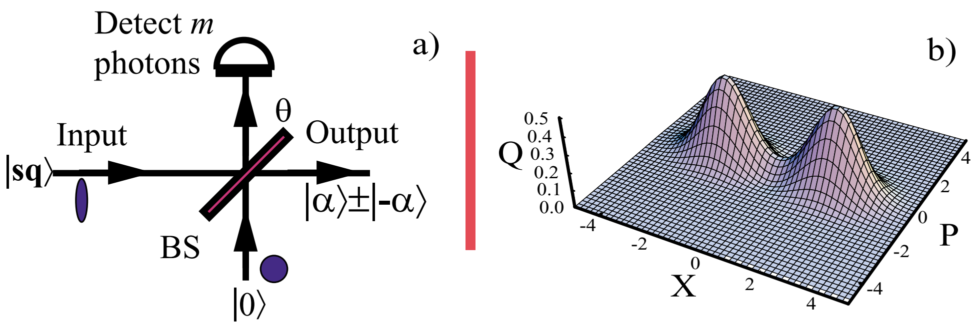

Figure 2. Schematic diagram in a) of the generation of a Schrödinger like cat state by means of a conditional photon number measurement on a beam splitter. A single mode squeezed state is input into one port of a variable reflectivity beam-splitter with the vacuum on the second port. A definite measurement of m photons (with $m > 0$) on one output port of the beam-splitter prepares to a good approximation the required cat state. In b) we plot the Q function $Q = |\langle\beta|\Psi\rangle|^2$ versus the two canonical phase space variables ($X = Re(\beta)$ and $P = Im(\beta)$) for the state (3) with $m = 10$ and a mean photon number of $\langle n\rangle = 4$.

form

$$|\Psi_{sq}\rangle = \left(1-|\lambda|^2\right)^{\frac{1}{4}} \sum_n \frac{\sqrt{(2n)!}}{n!} \left(\frac{\lambda}{2}\right)^n |2n\rangle \tag{2}$$

(with λ being the squeezing parameter) is input into a variable transmitivity beam-splitter (with coefficient $\cos^2\theta$). The second port has only a vacuum input. On one output port from the beam-splitter a definite photon number measurement given by the POVM $|m\rangle\langle m|$ is performed giving the result m. This conditional state at the remaining output port is

$$|\Psi_m\rangle = \frac{1}{\sqrt{\mathcal{N}_m}} \sum_n c_{n,m} \left(\frac{\lambda\cos^2\theta}{2}\right)^{\frac{n+m}{2}} |n\rangle \tag{3}$$

where $\mathcal{N}_m$ is the normalization constant. For m ($m \geq 4$) sufficiently large (3) is a good approximation to the state (1). The overlap between these state is high (in excess of ninety five percent for a wide range of parameters). In Fig (2b) we show the Q function of the state (3) versus the two canonical phase space variables $X = Re(\beta)$ and $P = Im(\beta)$. The two peaks in the Q function are characteristic of the Schrodinger cat nature and indicate that an approximate cat state has been generated. These cat resources are also easily entangled using linear optics using for instance the arrangement depicted in Fig (3). A maximally entangled state of the form

$$|\Psi\rangle = \frac{1}{\sqrt{\bar{\mathcal{N}}}} \left[|\alpha\rangle|\alpha\rangle + |-\alpha\rangle|-\alpha\rangle\right] \tag{4}$$

where $\bar{\mathcal{N}}$ is the normalization constant can be created by combining a single mode cat state of the form $|\sqrt{2}\alpha\rangle + |-\sqrt{2}\alpha\rangle$ with the vacuum state on a 50/50 beam-splitter. This resulting entangled state is a critical resource in creating our universal set of quantum logic gates.

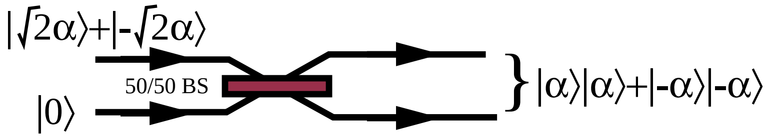

Figure 3. Schematic circuit to produce a two mode maximally entangled cat state

3. UNIVERSAL QUANTUM LOGIC GATES

As we start to consider a universal set of quantum logic gates, it is essential to define what our qubits are. We will consider an encoding of logical qubits in coherent states with the logical 0 state being $|0\rangle_L = |-\alpha\rangle$ and $|1\rangle_L = |\alpha\rangle$ being the logical one state. For convenience and without loss of generality we will choose α to be real. These qubits are not strictly orthogonal, but the approximation is good for α even moderately large as $|\langle\alpha|-\alpha\rangle|^2 = e^{-4\alpha^2}$. For $\alpha \geq 2$ the overlap between the zero and one logic qubit state gives $|\langle\alpha|-\alpha\rangle|^2 \leq 10^{-6}$. Given the definition our qubit basis states what gates operations are necessary to be able to do a general computation. It is well known that to perform universal computation one needs to be able to perform an arbitrary single qubit rotations. Such rotations for our case can be built from four basic single qubit gates. The first two gates are bit and sign flip operations and are given as follows:

- A bit-flip: The logical value of a qubit can be flipped by delaying it with respect to the local oscillator by half a cycle. Thus the "bit-flip" gate X is given by

  $$X = Exp[i\pi\hat{a}^\dagger\hat{a}]. \tag{5}$$

  where $a, a^\dagger$ are the annihilation and creation operators for the single mode of the electromagnetic field. For example, a qubit of the form $\mu|-\alpha\rangle + \nu|\alpha\rangle$ is flipped to $\mu|\alpha\rangle + \nu|-\alpha\rangle$ under the action of the X operator.

- A sign-flip: The sign flip gate Z can be achieved using a teleportation protocol and the maximally entangled resource (4). Consider that we wish to sign flip the qubit state $\mu|-\alpha\rangle + \nu|\alpha\rangle)$. A Bell state measurement is performed between one half of the resource (4) and the qubit of interest. Depending on which of the four possible outcomes are found the other half of the Bell cat is projected into one of the following four states with equal probability:

$$\begin{array}{ll} \mu|-\alpha\rangle + \nu|\alpha\rangle \ (1) & \mu|-\alpha\rangle - \nu|\alpha\rangle \ (2) \\ \mu|\alpha\rangle + \nu|-\alpha\rangle \ (3) & \mu|\alpha\rangle - \nu|-\alpha\rangle \ (4) \end{array} \quad (6)$$

  The bit flips in results three and four can be corrected using the X gate. After X correction the gate has two possible outcomes: either the identity has been applied, in which case we repeat the process, or else the required transformation

$$Z(\mu|-\alpha\rangle + \nu|\alpha\rangle) = \mu|-\alpha\rangle - \nu|\alpha\rangle \quad (7)$$

  has been implemented. On average this will take two attempts and thus consume on average two Bell-cats.

The remaining two operations are arbitrary rotations about the z and x axis and like the sign flip operation X use a teleportation protocol to achieve the gate. These operations are given by

- An arbitrary rotation ϕ about the Z axis, schematically depicted in Fig (4) can be implemented by first by displacing our arbitrary input qubit $\mu|-\alpha\rangle + \nu|\alpha\rangle$ by a small amount $\beta = \alpha\theta$ in the imaginary direction. This results in the state

$$D(i\beta)\,(\mu|-\alpha\rangle + \nu|\alpha\rangle) \quad = \quad \mu|-\alpha(1-i\theta)\rangle + \nu|\alpha(1+i\theta)\rangle \quad (8)$$

  which is a small distance outside the computational space. The teleportation then projects us back into the qubit space resulting in the state

$$T_X D(i\beta)\,(\mu|-\alpha\rangle + \nu|\alpha\rangle) = e^{-\theta^2\alpha^2/2}(e^{-i\theta\alpha^2}\mu|-\alpha\rangle + e^{i\theta\alpha^2}\nu|\alpha\rangle) \quad (9)$$

  Up to the global phase the transformation is then $R(Z, 2\theta\alpha^2)$. This gate is near deterministic for a

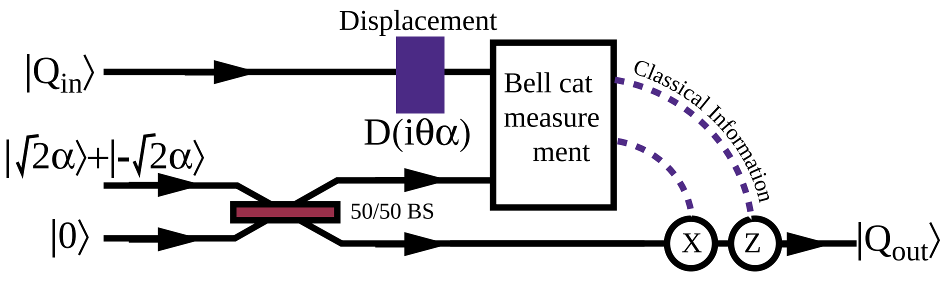

Figure 4. Schematics for implementing the $R(Z, \phi)$ gate. We begin by first shifting our qubit a small distance out of the computational basis and then using teleportation to project back into the qubit space.

  sufficiently small values of $\theta^2\alpha^2$. Repeated application of this gate can build up a finite rotation with high probability.

- The fourth gate to consider is an arbitrary rotation ϕ about the X axis. The gate is shown schematically in Fig.(5). For an arbitrary input state $\mu|-\alpha\rangle + \nu|\alpha\rangle$ this circuit and interaction produces the output state

$$C_a C_b U_{BS}(\mu|-\alpha\rangle + \nu|\alpha\rangle) = \{(e^{i\theta\alpha^2}\mu \pm e^{-i\theta\alpha^2}\nu)|-\alpha\rangle + (\pm e^{-i\theta\alpha^2}\mu \pm e^{i\theta\alpha^2}\nu)|\alpha\rangle\} \quad (10)$$

  where C_a and C_b represent cat state projections onto either the even or odd parity cat. The $\pm$ signs depend on the outcome of the cat state measurements. However using X and Z gates we can correct all the $\pm$'s to

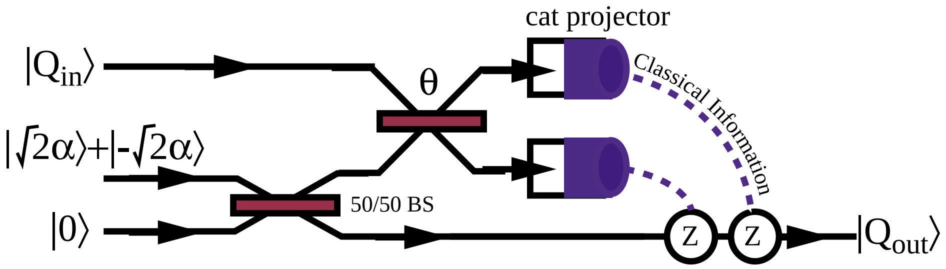

Figure 5. Schematics for implementing the $R(X,\phi)$ gate.

+'s. The gate $R(X, 2\theta\alpha^2)$ then implements a rotation $2\theta\alpha^2$ about the X. This gate is near deterministic for a sufficiently small values of $\theta^2\alpha^2$. Repeated application of this gate can build up a finite rotation. As a specific example, if we started with the initial state $|\alpha\rangle$ then for an appropriate valve of $\theta\alpha^2$ the output state from this gate is $|\alpha\rangle - |-\alpha\rangle$. Hence the gate has implemented an Hadamard transformation.

By combining these gates it is possible to achieve an arbitrary single qubit rotation. If we can supplement these with a single two qubit entangling operation between the qubits, then we have a universal set. We will show two very simple and related examples

- The beam-splitter as a phase gate. We depict the gate circuit schematically in Figure (6).Consider initially that we have our control qubit in the state $|\psi_c\rangle = \mu_1|-\alpha\rangle + \nu_1|\alpha\rangle$ and the target state as $|\psi_t\rangle = \mu_2|-\alpha\rangle + \nu_2|\alpha\rangle$. We begin displacing both by an amount $D(\alpha)$. The resulting total state is

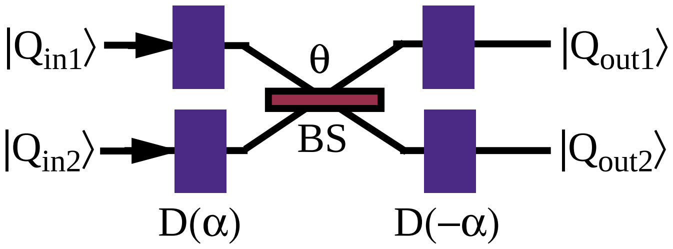

Figure 6. Schematics of implementing a two qubit phase gate gate.

$$|\Psi_c\rangle = [\mu_1|0\rangle + \nu_1|2\alpha\rangle]_c \otimes [\mu_2|0\rangle + \nu_2|2\alpha\rangle]_t \tag{11}$$

Now combining both modes on a beam-splitter with reflectivity θ and displacing the resulting modes by an amount $D(-\alpha)$ each, we get

$$|\Psi_c\rangle = \mu_1\mu_2|-\alpha\rangle|-\alpha\rangle + \mu_1\nu_2|-\alpha\rangle|\alpha\rangle + \nu_1\mu_2|\alpha\rangle|-\alpha\rangle + e^{4\theta\alpha^2}\nu_1\nu_2|\alpha\rangle|\alpha\rangle \tag{12}$$

For $4\theta\alpha^2 = \pi$ we have the $|\alpha\rangle|\alpha\rangle$ term flipping sign while leaving all the other terms unaffected. This is a controlled phase gate. There is an implicit assumption that $\theta^2\alpha^2 \ll 1$ which means that the gate may need to be applied a number of time to build up the appropriate phase shift. A teleportation circuit could be added to both the control and target modes to project the state back into the qubit space.

- The second example we will consider is the 2-qubit $R(Z \otimes Z, -\phi)$ gate which can be implemented in a similar way to the single qubit Z rotation. This explicitly uses a teleportation circuit. A schematic circuit for the gate is depicted in Fig (7). Consider the action of two two coherent state qubits $|\gamma\rangle_a$ and $|\beta\rangle_b$ input

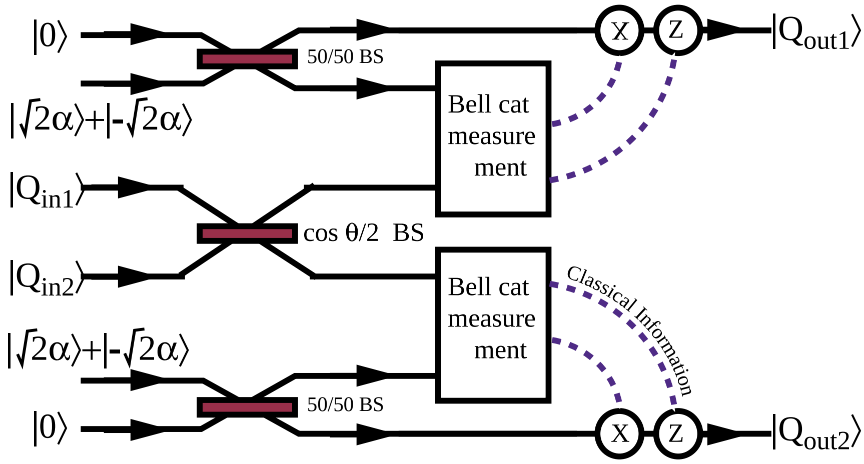

Figure 7. Schematics of implementing the $R(Z \otimes Z, -\pi/2)$ gate. For a sufficiently small value of $\theta^2\alpha^2$ this gate is near deterministic. Repeated application of this gate can build up to a $\pi/2$ rotation with high probability.

on a beam-splitter whose interaction is given by

$$U_{ab} = \exp[i\theta(ab^\dagger + a^\dagger b)] \tag{13}$$

The output state from the beam-splitter is

$$U_{ab}|\gamma\rangle_a|\beta\rangle_b = |\cos\theta\gamma + i\sin\theta\beta\rangle_a|\cos\theta\beta + i\sin\theta\gamma\rangle_b \tag{14}$$

If both output beams are now projected back into the qubit space of $|\pm\alpha\rangle$ using teleportation we find for an arbitrary input state $\nu|-\alpha\rangle_a|-\alpha\rangle_b + \mu|\alpha\rangle_a|-\alpha\rangle_b + \tau|-\alpha\rangle_a|\alpha\rangle_b + \gamma|\alpha\rangle_a|\alpha\rangle_b$ that the resultant state is

$$|\Psi\rangle_{out} = e^{i\theta\alpha^2}\nu|-\alpha\rangle_a|-\alpha\rangle_b + e^{-i\theta\alpha^2}\mu|\alpha\rangle_a|-\alpha\rangle_b + e^{-i\theta\alpha^2}\tau|-\alpha\rangle_a|\alpha\rangle_b + e^{i\theta\alpha^2}\gamma|\alpha\rangle_a|\alpha\rangle_b \tag{15}$$

where as before we have assumed orthogonality of the qubit basis state and $\theta^2\alpha^2 << 1$. If we choose $\phi = 2\theta\alpha^2 = \pi/2$ then $R(Z \otimes Z, -\pi/2)$ operation implements a controlled phase gate. The output state is of the form

$$\nu|-\alpha\rangle_a|-\alpha\rangle_b - i\mu|\alpha\rangle_a|-\alpha\rangle_b - i\tau|-\alpha\rangle_a|\alpha\rangle_b + \gamma|\alpha\rangle_a|\alpha\rangle_b \tag{16}$$

which up to single qubit rotations is a simply a two qubit controlled phase gate.

Both of these two qubit phase gates can use to entangle input qubits and hence with our set of single qubit gates form a universal set of operations for quantum logic. Such gates hence can be used for both quantum computation and communication. They also can be used in various quantum metro-logical applications which we will discuss next.

4. QUANTUM METROLOGY

In this section we illustrate the utility of the Schrodinger cat states for these two specific metrology applications. We begin with the detection of weak forces.

4.1. The detection of weak forces

Before we begin our discussion of the application of Schrodinger cats states to weak force detection it is essential to establish the best classical limit. It is well known that when a classical force given by $F(t)$ acts for a fixed time on a simple harmonic oscillator, it displaces the complex amplitude of this oscillator in phase space. The resulting amplitude and phase of the displacement are determined by the time dependence of the force.[17] This displacement can be represented by the unitary operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{17}$$

where α is a complex amplitude which determines the average field amplitude and $a, a^\dagger$ are the annihilation and creation operators for the single mode of the electromagnetic field. If the oscillator begins in a coherent state $|\alpha_0\rangle$ (with α_0 is real) then a displacement $D(i\epsilon)$ (assumed for simplicity to be orthogonal to the coherent amplitude of the initial state) causes the coherent state to evolve to $e^{i\epsilon\alpha_0}|\alpha_0 + i\epsilon\rangle$. When can this state be The maximum signal to noise ratio is then $SNR = S/\sqrt{V} = 2\epsilon$. This must be greater than unity for the displacement to be resolved and hence establishes the standard quantum limit (SQL)[1] of $\epsilon_{SQL} \geq 1/2$.

It is well known that this limit may be overcome if the oscillator is prepared in a non-classical state. What however is the sensitivity achieved by (1) and does this reach the ultimate limit, the Heisenberg limit. When a weak classical force acts on the even photon number cat state $|\alpha\rangle + |-\alpha\rangle$ with alpha real (see Fig (8a)) it is displaced to

$$|\phi\rangle_{out} \quad \approx \quad \frac{1}{\sqrt{2}}\left(e^{i\epsilon\alpha}|\alpha\rangle + e^{-i\epsilon\alpha}|-\alpha\rangle\right) \tag{18}$$

Our problem is thus reduced to finding the optimal readout to be able to distinguish (18) from $|\alpha\rangle + |-\alpha\rangle$. The theory of optimal parameter estimation[18] indicates that the limit on the precision with which the parameter $\epsilon\alpha$ can be determined is $(\delta\theta)^2 \geq 1/Var(\hat{\sigma}_x)_{in}$ where $Var(\hat{\sigma}_x)_{in}$ is the variance in the generator of the rotation in the input state $|\alpha\rangle + |-\alpha\rangle$. In this case the variance is simply unity. It thus follows that the minimum detectable force is $\epsilon \geq 1/2\sqrt{\bar{n}}$ where $\bar{n}$ is the mean photon number given by $\bar{n} = |\alpha|^2$. It is straight forward to show this 'measurement' is the Heisenberg limit for a displacement measurement. An interesting question what type of measurement is required to achieve this limit. In effect we need to be able to distinguish the even parity cat state from the odd parity cat state. Currently this experimentally challenging. However by performing a Hadamard operation (one of the single qubit gates discussed previously), the even and odd schrodinger cats are transformed to the coherent states $|\alpha\rangle$ from $|-\alpha\rangle$ which can be easily distinguished via a standard homodyne measurement.

If the weak force acts over a reason spatial range it would be possible to have a number of spatial mode of light. Could this help us exceed the limit above even if we constraint the total mean photon number of the entire multimode system. We depict in Fig (8b) a schematic for the setup of a proposed experiment. A single mode cat state $|\alpha\rangle + |-\alpha\rangle$ is input into one mode of an N port symmetric beam-splitter with the remaining input ports empty. The output state from this beam-splitter is then the massively entangled GHZ like state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\frac{\alpha}{\sqrt{N}}, \frac{\alpha}{\sqrt{N}}, \ldots, \frac{\alpha}{\sqrt{N}}\rangle + |-\frac{\alpha}{\sqrt{N}}, -\frac{\alpha}{\sqrt{N}}, \ldots, -\frac{\alpha}{\sqrt{N}}\rangle\right] \tag{19}$$

which has a total mean photon number of $n_{tot} = |\alpha|^2$. We now assume that the weak force acts simultaneously on all modes of this N party entangled state displacing them each by an amount $D(i\epsilon)$ (for $\epsilon \ll 1$). The resulting state after the action of the force is

$$|\psi(\theta)\rangle \quad = \quad \frac{1}{\sqrt{2}}\left[e^{i\sqrt{N}\epsilon\alpha}|\frac{\alpha}{\sqrt{N}}, \ldots, \frac{\alpha}{\sqrt{N}}\rangle + e^{-i\sqrt{N}\epsilon\alpha}|-\frac{\alpha}{\sqrt{N}}, \ldots, -\frac{\alpha}{\sqrt{N}}\rangle\right]. \tag{20}$$

The theory of optimal parameter estimation indicates that the limit on the precision with which the displacement parameter ϵ be estimated is bounded by

$$\epsilon_{min} \quad = \quad \frac{1}{\sqrt{N\left[1 + 4n_{tot}\right]}} \sim \frac{1}{2\sqrt{Nn_{tot}}}$$

for $n_{tot} \gg 1$. If however we had used N independent cat state each with a photon number n_{tot}/N then then ϵ_{min} for the entire system would have scaled as $\epsilon_{min} \sim 1/\sqrt{4n_{tot}}$ which is the same result we obtained for the

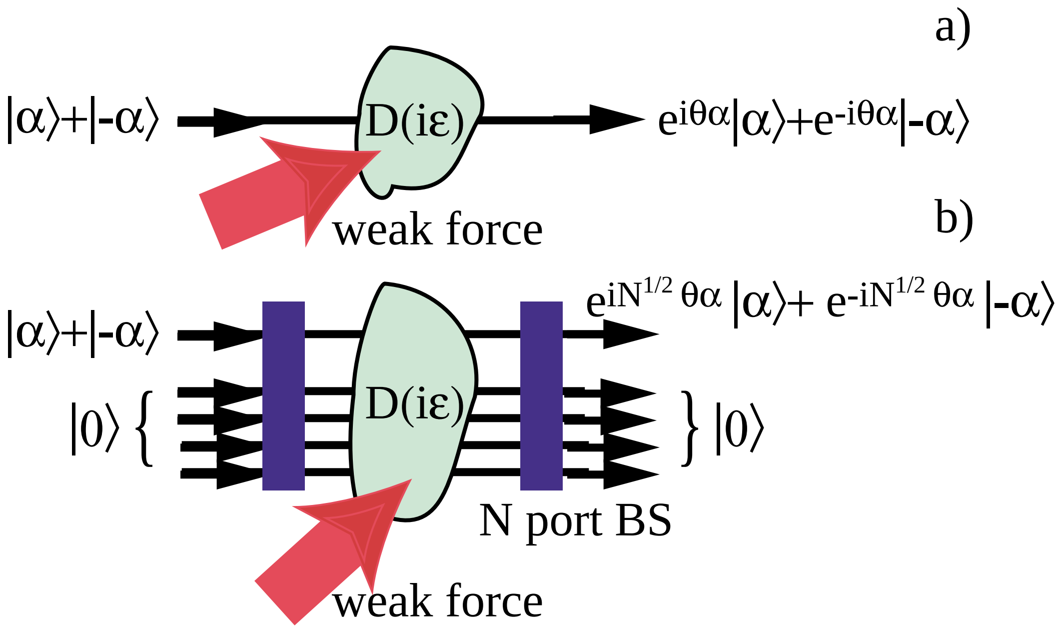

Figure 8. Schematic diagram of the action of a weak force causing a displacement $D(i\epsilon)$ on a Schrodinger cats state $|\alpha\rangle + |-\alpha\rangle$. In a) a single mode case is illustrated while in b) an N mode situation is considered.

single mode case. For large n_{tot}, the preferred regime to work in, we find that the N mode entangled situation gives an extra $\sqrt{N}$ improvement over the single mode cat situation for the same total mean photon number. Now how do we interpret such results? The effect that we are seeing is due to the weak force acting equally on all N modes and the state between the N port beam-splitters being highly entangled. Does this result however violate the Heisenberg limit of $1/\sqrt{n_{tot}}$ which we previously mentioned. The answer is no! A careful analysis using parameter estimation of this multimode situation indicates that our result is at the Heisenberg limit. For displacement measurements the Heisenberg limit does depend on the number of modes.

Our results indicate that subject to the spatial bandwidth of the weak classical force it seems optimal for a cat state with fixed mean photon n_{tot} to split it and entangle it over as many modes as feasible. This in the absence of loss give the best sensitivity. Such techniques are likely to work for other non classical continuous variable states.

4.2. High precision phase measurements

The second metro-logical example we are going to investigate is the estimation of phase. The classic situation to consider is Ramsey fringe interferometry which was first introduced by Bollinger et al.[14] in the mid ninety's. In Ramsey fringe interferometry the objective is to detect the relative phase difference between two superposed qubit basis states $|0\rangle$ and $|1\rangle$. This phase difference problem reduces to a quantum parameter estimation situation in which a unitary transformation $U(\theta) = \exp[i\theta\hat{Z}]$ (with $\hat{Z} = |1\rangle\langle 1| - |0\rangle\langle 0|$) induces a relative phase in the specified basis. For example, an initial state of the form $c_0|0\rangle + c_1|1\rangle$ evolves to $c_0 e^{-i\theta}|0\rangle + c_1 e^{i\theta}|1\rangle$ under the above unitary operation. When can we distinguish these two states. Is there an optimal choice of initial state? The theory of quantum parameter estimation[18] indicates for this situation that we should choose the initial state as $|\psi\rangle_i = (|0\rangle + |1\rangle)/\sqrt{2}$ and that the optimal measurement is a projective measurement in the basis $|\pm\rangle = |0\rangle \pm |1\rangle$. The probability to obtain the result $+$ is $P(+|\theta) = \cos^2\theta$. For N repetitions of this measurement the uncertainty in the inferred parameter θ is $\delta\theta = 1/\sqrt{N}$. This is known as the standard quantum limit. It was noted by Bollinger et al.[14] that a more effective way to use the N two level systems is to first prepare them in the maximally entangled state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2 \ldots |0\rangle_N + |1\rangle_1|1\rangle_2 \ldots |1\rangle_N) \quad (21)$$

and then subject the entire state to the unitary transformation $U(\theta) = \prod_{i=1}^{N} \exp(-i\theta\hat{Z}_i)$. After the unitary transformation the state (21) evolves to

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\exp(-iN\theta)|0\rangle_1|0\rangle_2 \ldots |0\rangle_N + \exp(iN\theta)|1\rangle_1|1\rangle_2 \ldots |1\rangle_N) \quad (22)$$

The uncertainty in the parameter estimation of θ then achieves the Heisenberg lower limit of

$$\delta\theta = \frac{1}{N} \quad (23)$$

This would seem to indicate like in the weak force case that entanglement is a critical requirement to achieve the improved sensitivity. Let us examine this point a little further for the phase estimation situation. The Hilbert space of N two level systems is a tensor product space of dimension 2^N. The entangled state given in Eq.(21) however resides in a much smaller $N+1$ dimensional irreducible lower dimensional subspace of permutation symmetric states.[19] We may use an SU(2) representation to write the entangled state $0\rangle_1|0\rangle_2 \ldots |0\rangle_N + |1\rangle_1|1\rangle_2 \ldots |1\rangle_N$ in the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|-N/2\rangle_{N/2} + |N/2\rangle_{N/2}). \quad (24)$$

This is just an SU(2) 'cat state' for N two-level atoms. Hence a single N level atom can achieve the same phase sensitivity as a maximally entangled GHZ state as it can be written in the form $|-N/2\rangle_{N/2} + |N/2\rangle_{N/2}$. This would also seem to indicate that a superposition of coherent states (a cat state) can attend the same phase resolution. In Fig (9) a schematic diagram is shown for the cat state obtaining the Heisenberg limited phase

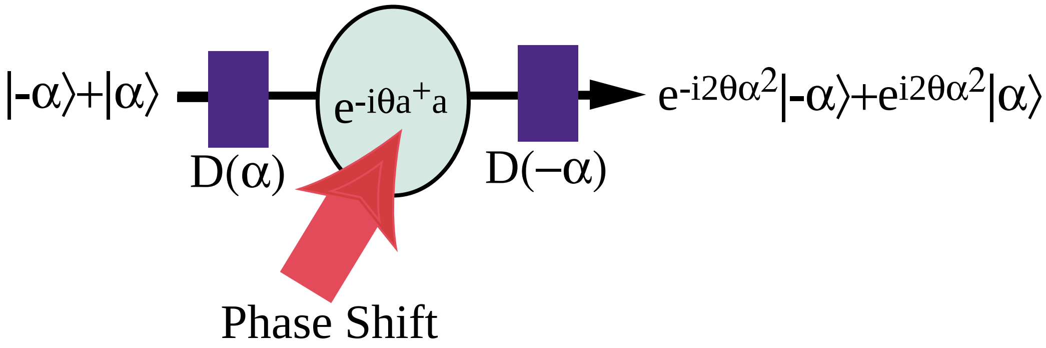

Figure 9. Schematics of quantum circuit illustrating how a phase shift can be seen on an input state of the form

resolution. Such phase shift could be used resolve precisely very small length intervals, a *quantum ruler*[20] in effect. As α increases, a number of high visibility, narrowly spaced fringes emerge, which could enable very short length intervals to be accurately measured. As an example suppose our laser wavelength is $10\mu m$. In a standard interferometer this would enable length intervals of $5\mu m$ to be stepped off. However using the cat-state technique with an α of 10 leads to the fringe separation being reduced to $1\mu m$.

Finally let us return to the question about whether entanglement is necessary to achieve a Heisenberg limited phase measurement. Obviously entanglement is not necessary but what entanglement allows is for one to create an effective cat state without the need of resorting to create a superposition between the ground state and a highly excited one.

5. CONCLUDING REMARKS

In the paper we have investigated the utility of superpositions of coherent states. We have presented a quantum computation scheme based on encoding qubits as coherent states, and their superposition. The optical networks required are conceptually simple and require only linear interactions, homodyne measurements and photon counting. We have concentrated on the simplest implementation which unfortunately requires large α. However with a modest increase in complexity the non-deterministic operation of the gates at low α can form the basis of a scalable system which will be detailed in a future publication.[21] We have then shown how superpositions of coherent states can be used to achieve extremely sensitive force detection and phase measurements.

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REFERENCES

1. See for instance: D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).
2. M. Nielsen and I. Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, UK 2000).
3. G. J. Milburn, Phys.Rev.Lett. **62**, 2124 (1988).
4. E. Knill and L. Laflamme and G. J. Milburn, Nature **409**, 46 (2001).
5. D. Gottesman, A. Kitaev, J. Preskill, Phys Rev A **64**, 012310 (2001).
6. A. Furasawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble and E. S. Polzik, Science **282**, 706 (1998).
7. S. Lloyd, S. L. Braunstein, Phys Rev Lett **82**, 1784 (1999).
8. S. D. Bartlett, Hubert de Guise, B. C. Sanders, Phys. Rev. A **65**, 052316 (2002).
9. H. Jeong and M. S. Kim, Phys. Rev. A **65**, 042305 (2002).
10. T.C.Ralph, W.J.Munro and G.J.Milburn, Quantum Computation with Coherent States, Linear Interactions and Superposed Resources, Hewlett Packard Labs Tech Report 2001-266.
11. E. S. Polzik, J. Carry, and H. J. Kimble, Phys. Rev. Lett **68**, 3020 (1992).
12. C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg and M. Zimmermann, Rev. Mod. Phys. **52**, 341, (1980).
13. W.J.Munro, K.Nemoto, G.J.Milburn and S.L.Braunstein, Phys. Rev. A **66**, 023819 (2002).
14. J. J . Bollinger, Wayne M. Itano, D. J. Wineland and D. J. Heinzen, Phys. Rev. A. **54**, R4649, (1996).
15. S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio and J. I. Cirac, Phys.Rev.Lett. **79**, 3865, (1997).
16. M. Dakna, T. Anhut, T. Opatrny, L. Knll and D. -G. Welsch, Phys. Rev. A **55**, 3184 (1997).
17. V. Braginsky and F. Ya. Khalili, *Quantum measurement*, (Cambridge University press, Cambridge, 1992).
18. S. L. Braunstein , C. M. Caves and G. J. Milburn, Annals of Physics, **247**, 135, (1996).
19. R. H. Dicke, Phys. Rev. **93**, 99 (1954).
20. T. C. Ralph, Phys. Rev. A **65**, 042313 (2002).
21. A. Gilchrist, T. C. Ralph, W. J. Munro and G. J. Milburn, in preparation (2002).