Optimal states for Bell-inequality violations using quadrature-phase homodyne measurements

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We identify what ideal correlated photon number states are required to maximize the discrepancy between local realism and quantum mechanics when a quadrature homodyne phase measurement is used. Various Bell-inequality tests are considered. [S1050-2947(99)05606-1]

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I. INTRODUCTION

There has recently been active interest in tests of quantum mechanics [1] versus local realism in a high efficiency detection limit. Several authors [2–4] including ourselves have considered detection schemes involving quadrature-phase homodyne measurements. Such schemes use strong local oscillators and hence have very high detection efficiency [5]. This removes one of the current loopholes [6–9] and potentially allows a strong test of quantum mechanics [10] to be performed.

The original idea of Gilchrist *et al.* [2] was to use a *circle* or pair coherent state [11–13] produced by nondegenerate parametric oscillation with the pump mode adiabatically eliminated. Using highly efficient quadrature-phase homodyne measurements, the Clauser-Horne strong Bell inequality [14–16] could be tested in all optical regimes. A small (approximately 1.5%) but significant theoretical violation was found for this extremely ideal system. While the mean photon number for the system may be low (approximately 1.12), the use of homodyne measurements allows a macroscopic current to be detected.

In this article, we take an unphysical but interesting approach and answer the following questions.

(1) Given that your detection scheme is a quadrature-phase homodyne measurement, what is the optimal input or correlated photon number state to maximize the potential violation?

(2) What is the optimal Bell inequality to test?

To begin we will restrict our attention to correlated photon number states of the form

$$|\Psi\rangle=\sum_{n=0}^{\infty} c_n|n\rangle|n\rangle. \tag{1}$$

Two main sources of correlated photon number currently exist, each having its own particular form of c_n. The most well known is simply the nondegenerate parametric amplifier specified by an ideal Hamiltonian of the form [17]

$$H=-\hbar\chi\epsilon(ab+a^{\dagger}b^{\dagger}), \tag{2}$$

where ϵ is the field amplitude of a nondepleting classical pump and χ is proportional to the susceptibility of the medium. a,b are the boson operators for the orthogonal signal and idler modes. After a time τ, the state of the system is given by Eq. (1) with c_n specified by

$$c_n=\frac{\tanh^n[\chi\epsilon\tau]}{\cosh[\chi\epsilon\tau]}. \tag{3}$$

In the quadrature-phase-amplitude basis this state has a positive Wigner function. Hence it can be described as a local hidden variable theory and thus cannot violate a Bell inequality.

The other source of highly correlated photon number states exists in nondegenerate parametric oscillation. In the limit of very large parametric nonlinearity and high Q cavities, a state of the form [13]

$$|\Psi\rangle=\frac{e^{r^2}}{\sqrt{4\pi^2 I_0(2r^2)}}\int_0^{2\pi}d\theta|re^{i\theta}\rangle|re^{-i\theta}\rangle \tag{4}$$

can be generated. Here r is the size of the circle of the coherent states and I_0 is the zeroth order modified Bessel function. Equivalently this state can be written in the form of Eq. (1) with c_n given by

$$c_n=\frac{r^{2n}}{n!I_0(2r^2)}. \tag{5}$$

This was the state considered by Gilchrist *et al.* [2].

Given the general form of known correlated number states (1), the next fundamental question that should be initially addressed is what we mean by the Bell inequality. A number of Bell inequalities exist, and the particular one used depends heavily on your application and experimental setup. The Bell inequalities to be considered in this article are the Clauser-Horne [15], the *spin* [14], and the information-theoretic [18] Bell inequality. A detailed derivation of the various inequalities will not be given; the reader is referred to Refs. [15,14,18]. Here we will consider only strong inequalities, that is, inequalities where auxiliary assumptions (not based on local realism) are not required. In Fig. 1 we depict a very idealized setup for a general Bell-inequality experiment.

Probably the most well known inequality is the Clauser-Horne strong Bell inequality [15] given by

$$|\mathbf{B}_{\rm CH}|\leqslant 1, \tag{6}$$

where

$$B_{\rm CH}=\frac{P_{11}(\theta,\phi)-P_{11}(\theta',\phi)+P_{11}(\theta,\phi')+P_{11}(\theta',\phi')}{P_1(\theta')+P_1(\phi)}. \tag{7}$$

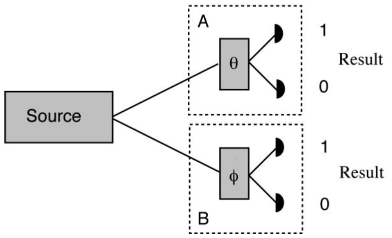

FIG. 1. Schematic of a very generalized Bell experiment setup. After a source prepares two particles, these particles are directed out to the locations A and B. At each location there is an analyzer with adjustable parameters θ,ϕ. The particles are then detected, resulting in a binary result ''1'' or ''0'' individually. These results can then be used to build up the statistics necessary to test the various Bell inequalities.

Here P_{11} is the probability that a ''1'' result occurs at each analyzer A,B given θ,ϕ. Similarly P_1 is the probability that a ''1'' occurs at a detector while having no information about the second. For many of the actual experimental considerations an angle factorization occurs so that $P_{11}(\theta,\phi)$ depends only on $\theta+\phi$. Also $P_1(\theta)$ and $P_1(\phi)$ are independent of θ,ϕ. In this case $B_{\rm CH}$ can be simplified to

$$B_{\rm CH}=\frac{3P_{11}(\psi)-P_{11}(3\psi)}{2P_1}, \tag{8}$$

where $\psi=\theta+\phi=-\theta'-\phi'=\theta+\phi'$ and $3\psi=\theta'+\phi$.

The second form of the Bell inequality (sometimes referred to as the *spin* or original Bell inequality) is [14]

$$B_s=|E(\theta,\phi)-E(\theta',\phi)+E(\theta,\phi')+E(\theta',\phi')|\leqslant 2, \tag{9}$$

where the correlation function $E(\theta,\phi)$ is given by

$$E(\theta,\phi)=P_{11}(\theta,\phi)+P_{00}(\theta,\phi)-P_{10}(\theta,\phi)-P_{01}(\theta,\phi). \tag{10}$$

Here, as discussed above, P_{11} is the probability that a ''1'' result occurs at each analyzer A,B given θ,ϕ. P_{00} is the probability that a ''0'' result occurs at each analyzer A,B, while $P_{10}(P_{01})$ is the probability that a ''1'' (''0'') result occurs at the analyzer A and a ''0'' (''1'') at B. With the angle factorization given above, the inequality (9) can be rewritten as

$$B_s=|3E(\psi)-E(3\psi)|\leqslant 2. \tag{11}$$

Our final form of the Bell inequality to be considered in this article was developed by Braunstein and Caves [18]. This classical information-theoretic Bell inequality has the form

$$B_{\rm info}\geqslant 0, \tag{12}$$

where

$$B_{\rm info}=-H(\theta|\phi)+H(\theta|\phi')+H(\phi'|\theta')+H(\theta'|\phi). \tag{13}$$

Here $H(\theta|\phi)$ is given by

$$H(\theta|\phi)=-\sum_{a,b}P(a,b)\log\left(\frac{P(a,b)}{P(a)}\right), \tag{14}$$

with $\log[P(a,b)/P(a)]$ being the information gained at B given the result at A is known. The conditional information is then given by $H(\theta|\phi)$. The base of the logarithm determines the units of the information (base 2 for bits, base e for nats). For quantum computing purposes, this inequality should prove highly useful as it directly deals with information content. Several other Bell inequalities do exist, such as the Clauser-Horne-Shimony-Holt inequality [6], but these are not considered here due to their weaker nature. Auxiliary assumptions are necessary in their derivation which open up several loopholes [7–9].

II. CORRELATED STATES

From Eq. (1) we need to find the optimal c_n which gives the largest Bell-inequality violation. Before determining the c_n we need to briefly focus our attention on the quadrature-phase homodyne measurement.

A quadrature-phase-amplitude homodyne measurement $X(\theta)$ at A can be achieved by combining a signal field (say $\hat{a}$) with a strong local oscillator field (say ϵ) to form two new fields given by $\hat{c}_\pm=[\hat{a}\pm\epsilon\exp(i\theta)]/\sqrt{2}$. Here θ is a phase shift which allows the choice of particular observable to be measured, for instance, choosing θ as 0 or $\pi/2$ allows the measurement of the conjugate phase variables $X(0)$ and $X(\pi/2)$, respectively. The homodyne measurement gives the photocurrent difference as

$$I_d=c_+^\dagger c_+-c_-^\dagger c_-=\epsilon(\hat{a}e^{-i\theta}+\hat{a}^\dagger e^{-i\theta})=\epsilon X(\theta). \tag{15}$$

Performing a measurement on the quadrature-phase amplitude $X(\theta)$ at A yields a result $x_1(\theta)$ which ranges in size and sign. Similarly a measurement on the quadrature-phase amplitude $X(\phi)$ at B yields a result $x_2(\phi)$. For our state given by Eq. (1), the probability of obtaining the result $x_1(\theta),x_2(\phi)$ is simply

$$P_{x_1x_2}(\theta,\phi)=|\langle x_1(\theta)|\langle x_2(\phi)|\Psi\rangle|^2, \tag{16}$$

where

$$\langle x(\varphi)|n\rangle=\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-in\varphi}e^{-x_i^2/2}H_n(x_i). \tag{17}$$

Here $H_n(x_i)$ is the Hermite polynomial and φ is the phase of the local oscillator. Equation (16) can be explicitly written as

$$P_{x_1x_2}(\psi)=\sum_{n,m}\frac{c_nc_m^*e^{-i(n-m)\psi}}{2^{n+m}n!m!\pi}\prod_{i=1}^{2}e^{-x_i^2}H_n(x_i)H_m(x_i), \tag{18}$$

where $\psi=\theta+\phi$, that is, our expression depends only on the sum of the individual local angles.

The probability given by Eq. (18) is for continuous variables. The majority of the tests of quantum mechanics versus local realism require a binary result. Hence for a given quadrature measurement x_i we classify the result as ''1'' if $x_i\geqslant 0$ and the mutually exclusive ''0'' if $x_i<0$. Here we have set the binning window about $x_i=0$. Where this binning window is located is quite arbitrary, but the maximum violation occurs for the value we have selected.

The probability of obtaining both particles in the "1" bin is

$$P_{11}(\psi)=\int_0^\infty\int_0^\infty dx_1dx_2P_{x_1x_2}(\psi) \tag{19}$$

while the probability of obtaining both particles in the "0" bin is

$$P_{00}(\psi)=\int_{-\infty}^0\int_{-\infty}^0 dx_1dx_2P_{x_1x_2}(\psi). \tag{20}$$

The other probabilities such as $P_{10}(\psi),P_{01}(\psi)$ can be calculated in a similar fashion. The probabilities formulated above are joint probabilities. Various of the strong Bell inequalities also require marginal probabilities of the form

$$P_1(\psi)=\int_0^\infty\int_{-\infty}^\infty dx_1dx_2P_{x_1x_2}(\psi). \tag{21}$$

The above integrals can be easily evaluated using the results [19]

$$\int_0^\infty e^{-x^2}H_n(x)H_m(x)=\frac{\pi 2^{n+m}}{n-m}[\mathcal{F}(n,m)-\mathcal{F}(m,n)] \quad (\text{for } n\neq m), \tag{22}$$

$$\int_{-\infty}^\infty e^{-x^2}H_n(x)H_m(x)=2^n n!\sqrt{\pi}\,\delta_{n,m},$$

where $\mathcal{F}(n,m)$ is given by

$$\mathcal{F}^{-1}(n,m)=\Gamma\left(\frac{1}{2}-\frac{1}{2}n\right)\Gamma\left(-\frac{1}{2}m\right), \tag{23}$$

with Γ being the gamma function. Performing the integrals for Eqs. (19) and (20) we find

$$\begin{aligned}P_{11}(\psi)&=P_{00}(\psi)\\&=\frac{1}{4}+\sum_{n>m}\frac{2^{n+m+1}\pi c_nc_m^*}{n!m!(n-m)^2}\\&\quad\times[\mathcal{F}(n,m)-\mathcal{F}(m,n)]^2\cos[(n-m)\psi].\end{aligned} \tag{24}$$

Similarly Eq. (21) simplifies to

$$P_1=1/2, \tag{25}$$

which is independent of the sum of the local oscillator angle ψ. It is also simple to calculate the correlation function $E(\psi)$,

$$E(\psi)=\sum_{n>m}\frac{2^{n+m+3}\pi c_nc_m^*}{n!m!(n-m)^2}[\mathcal{F}(n,m)-\mathcal{F}(m,n)]^2\times\cos[(n-m)\psi]. \tag{26}$$

Given the probabilities $P_{11},P_{00},\ \ldots$ it is also possible to calculate the conditional information $H(\theta|\phi)$,

$$\begin{aligned}H(\theta|\phi)=&-P_{11}\log_2[2P_{11}]-P_{00}\log_2[2P_{00}]\\&-P_{10}\log_2[2P_{10}]-P_{01}\log_2[2P_{01}].\end{aligned} \tag{27}$$

It is now possible to calculate the Clauser-Horne (6) and spin (9) and information-theoretic (12) Bell inequalities. Some insight into the problem can be achieved by a careful examination of the term

$$\frac{2^{n+m}\pi}{n!m!(n-m)^2}[\mathcal{F}(n,m)-\mathcal{F}(m,n)]^2, \tag{28}$$

which is present in all the joint probability distributions. This expression has several interesting features. First, as the difference between n and m becomes large, the smaller the probability that the above expression contributes to any of the probability distributions. The main contribution for the expression comes from the case $m=n\pm1$. Second, when $n-m$ is even, the above expression is zero. Finally, as n becomes large, the difference between the $n,m=n-1$ and $n+1,m=n$ elements for fixed large n vanishes and they reach an asymptotic limit which is smaller than the $n=1,m=0$ case. If these higher order n terms dominate due to the choice of the c_n in the probability formula, then the various Bell inequalities cannot be violated. This also has the implication that the mean photon number cannot be high if a violation is to occur and hence it is not a macroscopic test of quantum mechanics.

III. A SIMPLE CASE

To begin our investigations of the Bell inequalities, consider the case where we have only two photon pair states, that is,

$$|\Psi\rangle=c_0|0\rangle|0\rangle+c_1|1\rangle|1\rangle, \tag{29}$$

where for convenience we choose c_n real. We also require $c_0^2+c_1^2=1$. The joint probability distributions are readily calculated and in fact

$$P_{11}(\psi)=P_{00}(\psi)=\frac{1}{4}+\frac{c_0c_1}{\pi}\cos[\psi], \tag{30}$$

$$P_{10}(\psi)=P_{01}(\psi)=\frac{1}{4}-\frac{c_0c_1}{\pi}\cos[\psi]. \tag{31}$$

Calculating $B_{\rm CH}$ and B_s from Eqs. (6) and (9) we find

$$B_{\rm CH}=\frac{1}{2}+\frac{c_0c_1}{\pi}\{3\cos[\psi_0]-\cos[3\psi]\}, \tag{32}$$

$$B_s=\frac{4c_0c_1}{\pi}\{3\cos[\psi_0]-\cos[3\psi]\}. \tag{33}$$

Optimizing for the angle ψ we find

$$B_{\rm CH}=\frac{1}{2}+\frac{2\sqrt{2}c_0c_1}{\pi}, \tag{34}$$

$$B_s=\frac{8\sqrt{2}c_0c_1}{\pi}, \tag{35}$$

TABLE I. The optimal c_n parameters to maximize the violation of the Clauser-Horne and spin Bell inequalities. The c_n values for the circle state of Gilchrist *et al.* are also given.

n	c_n	Eq. (5) with $r \sim 1.12$
0	0.4990	0.5495
1	0.6355	0.6893
2	0.4760	0.4323
3	0.3135	0.1808
4	0.1465	0.0567
5	0.0235	0.0142
6	0.0075	0.0029
7	0.0024	0.0005
B_{CH}	1.019	1.016
B_s	2.076	2.064

that is, $|B_{\mathrm{CH}}| \leqslant 1$ and $|B_s| \leqslant 2$ for all c_0. No violation of the strong Clauser-Horne or spin Bell inequality is possible.

For the information-theoretic case we find

$$H(\psi) = -\frac{1}{2}\log_2\left[\frac{1}{4} - \lambda^2\right] - \lambda \log_2\left[\frac{1+2\lambda}{1-2\lambda}\right], \tag{36}$$

where

$$\lambda = \frac{2c_0c_1}{\pi}\cos[\psi]. \tag{37}$$

The information-theoretic Bell inequality is given by $B_{\mathrm{info}} = 3H(\psi) - H(3\psi) \geqslant 0$. A violation of this inequality is possible if $B_{\mathrm{info}} < 0$. Unfortunately for all c_0c_1 and ψ we have $B_{\mathrm{info}} > 0$.

No violation is possible for any of the Bell inequalities considered for the ideal state (29) when the detection scheme is based on homodyne quadrature-phase measurements. If more correlated photon pairs are present can a violation be achieved? The obvious answer is yes, because of the recent work of Gilchrist *et al.* [2]. The real question is how large this violation is.

IV. NUMERICAL STUDIES

Considering the expression (1) for the correlated photon pairs, what are the optimal c_n coefficients to maximize the violation? Because of the results indicated by Gilchrist *et al.* and our previous discussion we anticipate that the mean photon number per mode must be low to obtain a violation. Hence we will truncate the number state basis at 10 [20] photons per mode. Performing a numerical optimization over all the c_n, the optimal set is found to maximize the Clauser-Horne and spin Bell inequality (Table I). A plot of c_n versus n is depicted in Fig. 2.

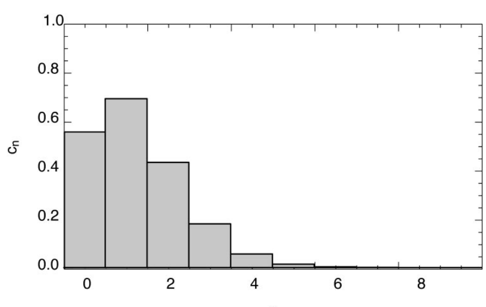

FIG. 2. Plot of c_n versus n.

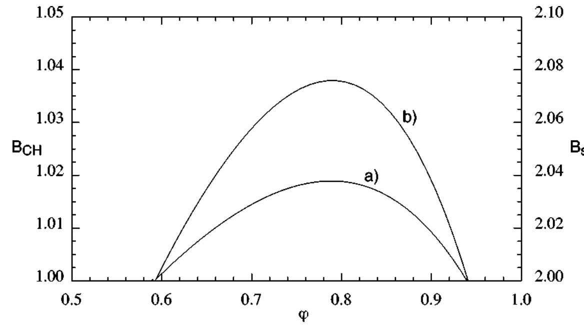

FIG. 3. Plot of the Clauser-Horne (a) and spin (b) Bell inequality versus ψ. A violation occurs for the Clauser-Horne Bell inequality if $B_{\mathrm{CH}} > 1$. A violation of the spin Bell inequality occurs for $B_s > 2$.

It is interesting to now discuss some properties of these optimal c_n. First the general shape of the c_n versus n curve shown in Eq. (2) is similar to that considered in the circle state by Gilchrist *et al.* [2]. It is, however, not exactly the same (see Table I). Given this optimal parameter set, what is the maximum violation of the Bell inequalities we are considering? In Fig. 3 we plot both the Clauser-Horne and spin Bell inequalities versus ψ.

For the Clauser-Horne Bell inequality the maximum violation corresponds to $B_{\mathrm{CH}} = 1.019$, while the maximum violation for the spin Bell inequality corresponds to $B_s = 2.076$. What is interesting here is that the percentage violation of the spin inequality is approximately 3.8% compared with the 1.9% for the Clauser-Horne case. This significantly increases the potential for an experiment to be performed provided such an experiment was not significantly more difficult. Also the results for the optimal c_n set give a Clauser-Horne Bell-inequality violation that is approximately 20% greater than the circle state results of Gilchrist *et al.* [2].

It is interesting to consider whether a greater violation of the Bell inequality can be achieved with the state given by Eq. (5). To this end we show the effect of the variation of both r and ψ (sum of the local oscillator angles) for both the Clauser-Horne and spin Bell inequalities in Fig. 4. As can be seen, the spin Bell inequality can be violated far more significantly than the similar Clauser-Horne case. In fact, as occurred previously, the percentage maximum violation in the spin inequality is twice that of the Clauser-Horne result.

In any of the analysis considered above we have not dis-

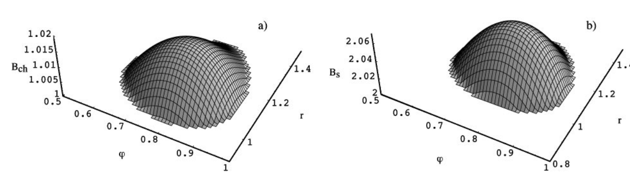

FIG. 4. Plot of the Clauser-Horne (a) and spin (b) Bell inequality versus r and ψ. A violation occurs for the CH Bell inequality if $B_{\mathrm{CH}} > 1$. A violation of the spin Bell inequality occurs for $B_s > 2$.

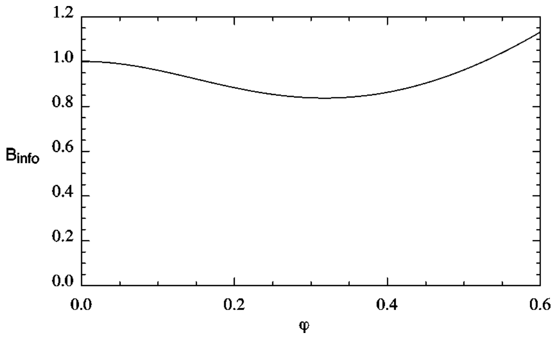

FIG. 5. Plot of the information-theoretic Bell inequality versus ψ. A violation is possible for $B_{\mathrm{info}}<0$.

cussed errors, their sources, and how they affect the potential violation. We will not present any significant details here in this article but refer the reader to [2] for such a decision.

Our final Bell inequality to be considered is the Braunstein and Caves [18] information-theoretic case. In Fig. 5 we plot B_{info} versus ψ. No violation of the information-theoretic inequality is possible for any ψ.

A question to be addressed here is why two of the strong inequalities can be violated while this information-theoretic Bell inequality is far from being violated. In the binning process to give a binary result for a quadrature measurement, information must be discarded. The information-theoretic inequality is much more sensitive to this information loss than the Clauser-Horne inequality. Also why would we fundamentally expect all three inequalities to be violated? A violation of any of the inequalities indicates a discrepancy between quantum mechanics and local realism.

V. CONCLUSION

In this article we have placed strict bounds on the optimal c_n coefficients for the state (1) which maximizes the Clauser-Horne and spin Bell inequalities when a homodyne quadrature-phase measurement is performed. The spin Bell inequality is violated by approximately 3.6% while the Clauser-Horne inequality is violated by approximately 1.9%. The violation is small, however, due to the fact that we are discarding information in the binning process. In fact, due to the information loss in the binning process the information-theoretic Bell inequality is not violated in any regime. A larger violation cannot be obtained using homodyne measurements with the strong inequalities we have considered.

While our optimal c_n coefficient gives a slightly better violation than the pair coherent state, it is difficult to see how such a state could be generated. Closely examining the spin Bell inequality with the pair coherent state still indicates that a greater violation (approximately twice the size) is possible than for the other inequalities. This would make the test much more feasible provided the pair coherent state could be generated. In such a system the mean photon number is small, so this is not strictly a macroscopic test of quantum mechanics. It does, however, have a macroscopic nature due to the strong local oscillator, which means large photodetector currents are obtained.

To conclude, quadrature-phase homodyne measurements provide a mechanism for performing tests of the Bell inequality with highly efficient detection. This allows one of the loopholes in current experiments to be closed. However, due to the inherent information loss in the binning process, the violations are small but should be achievable.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] A. Gilchrist, P. Deuar, and M.D. Reid, Phys. Rev. Lett. **80**, 3169 (1998).
[3] W.J. Munro and G.J. Milburn, Phys. Rev. Lett. **81**, 4285 (1998).
[4] B. Yurke and D. Stoler, Phys. Rev. Lett. **79**, 4941 (1997).
[5] E.S. Polzik, J. Carri, and H.J. Kimble, Phys. Rev. Lett. **68**, 3020 (1992).
[6] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[7] P.G. Kwiat, P.H. Eberhard, A.M. Steinberg, and R.Y. Chiao, Phys. Rev. A **49**, 3209 (1994).
[8] M. Freyberger, P.K. Aravind, M.A. Horne, and A. Shimony, Phys. Rev. A **53**, 1232 (1995).
[9] E.S. Fry, T. Walther, and S. Li, Phys. Rev. A **52**, 4381 (1996).
[10] By a strong test of quantum mechanics versus local realism, we mean a test in which no auxiliary assumptions are necessary.
[11] G.S. Agarwal, Phys. Rev. Lett. **57**, 827 (1986).
[12] K. Tara and G.S. Agarwal, Phys. Rev. A **50**, 2870 (1994).
[13] M.D. Reid and L. Krippner, Phys. Rev. A **47**, 552 (1993).
[14] J.S. Bell, Physics **1**, 195 (1965).
[15] J.F. Clauser and M.A. Horne, Phys. Rev. D **10**, 526 (1974).
[16] J.F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).
[17] M.D. Reid, Phys. Rev. A **40**, 913 (1989).
[18] S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. **61**, 662 (1988).
[19] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publications Inc., New York, 1965).
[20] Checks were performed to ensure that truncation errors were minimal (less than 10^{-6}).