Vibration of unsymmetrically laminated thick quadrilateral plates

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The problem of free vibration of arbitrary quadrilateral unsymmetrically laminated plates subject to arbitrary boundary conditions is considered. The Ritz procedures supplemented by the simple polynomial shape functions are employed to derive the governing eigenvalue equation. The displacements are approximated by a set of polynomials which consist of a basic boundary function that impose the various boundary constraints. A first-order shear deformable plate theory is employed to account for the effects of the transverse shear deformation. The numerical accuracy of the solution is verified by studying the convergence characteristics of the vibration frequencies and also by comparison with existing results. The new results of this study include the sensitivity of the vibration responses to variations in the lamination, boundary constraints and thickness effects, and also their interactions. These numerical values are presented for a typical graphite/epoxy material, in tabular and graphical forms. © *1999 Acoustical Society of America.* [S0001-4966(99)04403-3]

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INTRODUCTION

In recent years, we have witnessed an increasing use of laminated composite materials for manufacturing plate components due to its high strength-to-weight ratio and better corrosion resistance as well as the advantages of composite tailoring. Because of the increasing use of fibrous composite materials in flight vehicle structures, an improved understanding of the vibration behavior of composite panels is required.

It is known that the matrix material used in fiber-reinforced composites is of relatively low shearing stiffness when compared to the fibers. In order to achieve a more reliable prediction of the vibration responses for the composite laminates, the effect of transverse shear deformation has to be incorporated in the mathematical modeling. The importance of transverse shear deformation for the analysis of composite laminates was well documented.[1–4] Modeling of this effect via refined plate theories is normally adopted to account for the transverse shear strain distribution through the thickness dimension. Among the researchers, Yang, Norris, and Stavsky (YNS)[5] were the earliest to consider the effects of shear deformation in vibration analysis of composite laminated plates. For vibration analysis, the first-order YNS shear deformation theory is normally sufficient. Although the theory assumes a constant shear stress distribution through the thickness, it gives reasonably accurate vibration frequencies when used in conjunction with a shear correction factor on the shear modulus.

Considerable work has been devoted to the study of free vibration of plates.[6–9] For predicting the vibration responses, the possibility of variation in stacking sequences for laminated panels is an important factor to be considered in the mathematical modeling. This has led to the development of lamination plate theories to incorporate the different fiber orientations and possible stacking sequences. The other important factor to be considered is the variation of boundary conditions which is normally accommodated either by an approximate method or a numerical technique. Researchers[10–13] have attempted to investigate various aspects of the vibration behavior of composite laminates by varying the boundary conditions, thickness, and stacking sequences. These analyses have been carried out using different approximate techniques and shear deformation theories.

This paper describes an analysis method for vibration of unsymmetrically laminated composite plates with the inclusion of transverse shear deformation using the YNS shear deformation theory. The solution employs the Ritz method,[14] in which a set of polynomials is used as the admissible displacement and rotation functions, for determining the vibration frequencies. The analysis method is capable of handling unsymmetric composite laminates of different boundary conditions, an arbitrary quadrilateral geometry, and anisotropic material properties. Thus we believe an analytical tool, with such capabilities as what we propose, is of great value for preliminary design of composite structures. A set of results for the vibration frequency parameters of unsymmetrically laminated plates of quadrilateral shape subject to different boundary conditions and laminations is presented.

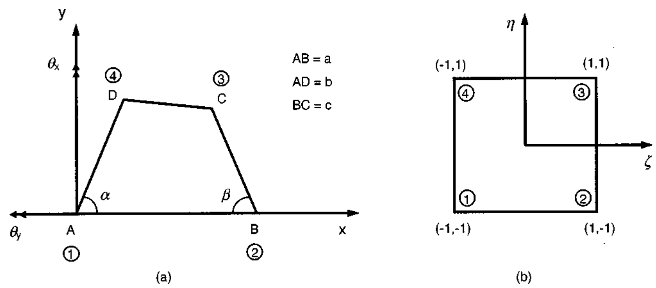

FIG. 1. Geometry and coordinate systems: (a) actual quadrilateral plate and (b) basic square plate.

I. MATHEMATICAL FORMULATIONS

A. Preliminary definition

Consider an arbitrary flat quadrilateral plate of uniform thickness h, composed of any number of anisotropic plies oriented alternately at angles θ and $-\theta$. The Cartesian coordinate system $x-y$ is located at the middle plane of the plate and the geometry of the plate is defined by side lengths a, b, c and two angles α and β, as shown in Fig. 1(a). The material of each ply is assumed to possess a plane of elastic symmetry parallel to the $x-y$ plane.

In this study, the plate under consideration is subjected to different combinations of free, simply supported, and clamped boundary conditions. The plate, as shown in Fig. 1, is described by a symbolism defining the boundary conditions at their four edges, for instance, SCSF means a plate whose edges at AB, BC, CD, and AD are simply supported, clamped, simply supported, and free, respectively. The problem is to determine the natural frequencies of the plate.

B. Energy functional

Employing the first-order shear deformation plate theory,[5] the displacement field for free vibration can be written as

$$\overline{u}=u(x,y,t)+z\grave{e}_x(x,y,t), \tag{1a}$$

$$\overline{v}=v(x,y,t)+z\grave{e}_y(x,y,t), \tag{1b}$$

$$\overline{w}=w(x,y,t), \tag{1c}$$

where $\overline{u}$, $\overline{v}$, and $\overline{w}$ are the displacement components in the x, y, and z directions, respectively, u and v are the in-plane displacements of the mid-plane, and θ_x and θ_y are the rotations about the y and x axes, respectively. The strain–displacement and stress–strain relations for any ply in the (x,y) system are given by

$$\boldsymbol{\epsilon}=\mathbf{LU}, \quad \sigma=\mathbf{Q}\boldsymbol{\epsilon}, \tag{2}$$

where

$$\mathbf{U}^T=\langle \mu \quad v \quad w \quad \theta_x \quad \theta_y\rangle, \tag{3}$$

$$\boldsymbol{\epsilon}^T=\langle \epsilon_{xx} \quad \epsilon_{yy} \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}\rangle, \tag{4}$$

$$\sigma^T=\langle \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{yz} \quad \sigma_{xz} \quad \sigma_{xy}\rangle, \tag{5}$$

$$\mathbf{L}=\begin{bmatrix} \partial_x & 0 & 0 & z\partial_x & 0 \\ 0 & \partial_y & 0 & 0 & z\partial_y \\ 0 & 0 & \partial_y & 0 & 1 \\ 0 & 0 & \partial_x & 1 & 0 \\ \partial_y & \partial_x & 0 & z\partial_y & z\partial_x \end{bmatrix}, \tag{6}$$

$$\mathbf{Q}=\begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{22} & 0 & 0 & Q_{26} \\ 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & Q_{45} & Q_{55} & 0 \\ Q_{16} & Q_{26} & 0 & 0 & Q_{66} \end{bmatrix}, \tag{7}$$

in which σ_{ij} and ϵ_{ij} are the stress and strain components, respectively, and $\tilde{a}_{ij}=2\mathring{a}_{ij}$; Q_{ij} are the elements of the plane-stress reduced constitutive matrix of the ply material which are obtained from the transform matrix considering fiber orientation and the material properties (Young's moduli, Poisson's ratios, and shear moduli), E_1, E_2, v_{12}, v_{21}, G_{12}, G_{13}, G_{23} of each ply; and ∂_x and ∂_y indicate the partial differentiation with respect to x and y, respectively. The energy functional π of the entire plate can be written in terms of the maximum strain energy $U_{\max}$ and maximum kinetic energy $T_{\max}$ as

$$\pi=U_{\max}-T_{\max}, \tag{8}$$

where

$$U_{\max}=\frac{1}{2}\int_V \boldsymbol{\epsilon}^T Q \boldsymbol{\epsilon}\, dV, \tag{9}$$

$$T_{\max}=\frac{1}{2}\Omega^2\int_V \rho U_1^T U_1\, dV, \tag{10}$$

$$U_1^T=\langle \overline{u} \quad \overline{v} \quad \overline{w}\rangle, \tag{11}$$

in which ρ denotes density of the ply material per unit volume, V represents the total volume of the plate, and Ω and U_1 are angular frequency and displacement amplitude for a vibrating plate with harmonic motion, respectively.

C. Geometric mapping

For the convenience in numerical integration and application of boundary conditions, the actual quadrilateral plate in the $x-y$ plane is mapped into a 2×2 basic square plate in the $\zeta-\eta$ plane [as shown in Fig. 1(b)] using the coordinate transformation defined by

$$x=\sum_{i=1}^{4} N_i x_i, \quad y=\sum_{i=1}^{4} N_i y_i, \tag{12}$$

where x_i and y_i are the coordinates of the ith corner of the quadrilateral plate in the $x-y$ plane. The mapping functions N_i are defined by

$$N_i=\tfrac{1}{4}(1+\zeta\zeta_i)(1+\eta\eta_i), \quad i=1,2,3,4, \tag{13}$$

where ζ_i and η_i are the coordinates of the ith corner of the basic square plate in the $\zeta-\eta$ plane.

TABLE I. Values of θ_j in Eq. (21) for different edge boundary conditions.

Supporting condition for jth edge	θ_j				
	ϕ_{x1}	ϕ_{y1}	ϕ_{z1}	φ_{x1}	φ_{y1}
Clamped (C)	1	1	1	1	1
Free (F)	0	0	0	0	0
Simply supported (S*)	0	0	1	0	0
Simply supported (S) with edge ∥ to x axis	0	1	1	1	0
Simply supported (S) with edge ∥ to y axis	1	0	1	0	1

Using the chain rule of differentiation, the first derivatives of any quantity ( ) in the two coordinate systems are related by

$$\begin{Bmatrix} \partial_x(\,) \\ \partial_y(\,) \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \partial_\zeta(\,) \\ \partial_\eta(\,) \end{Bmatrix}, \tag{14}$$

where

$$\mathbf{J} = \begin{bmatrix} \partial_\zeta x & \partial_\zeta y \\ \partial_\eta x & \partial_\eta y \end{bmatrix} \tag{15}$$

in which $\mathbf{J}$ denotes the Jacobian matrix of the geometric mapping. Equations (14) and (15) will be used later to transform the $x-y$ domain integrals in Eqs. (9) and (10) into the $\zeta-\eta$ domain integrals.

D. The polynomial shape functions

For the laminated plate, the displacement and rotation components may be expressed by a set of polynomial shape functions in the $\zeta-\eta$ plane as

$$u(\zeta,\eta) = \sum_{q=0}^{p_1} \sum_{r=0}^{q} a_i \phi_{xi}(\zeta,\eta) = \sum_{i=1}^{m_1} a_i \phi_{xi}(\zeta,\eta) = \mathbf{a}^T \Phi_x, \tag{16a}$$

$$v(\zeta,\eta) = \sum_{q=0}^{p_2} \sum_{r=0}^{q} b_i \phi_{yi}(\zeta,\eta) = \sum_{i=1}^{m_2} b_i \phi_{yi}(\zeta,\eta) = \mathbf{b}^T \Phi_y, \tag{16b}$$

$$w(\zeta,\eta) = \sum_{q=0}^{p_3} \sum_{r=0}^{q} c_i \phi_{zi}(\zeta,\eta) = \sum_{i=1}^{m_3} c_i \phi_{zi}(\zeta,\eta) = \mathbf{c}^T \Phi_z, \tag{16c}$$

$$\theta_x(\zeta,\eta) = \sum_{q=0}^{p_4} \sum_{r=0}^{q} d_i \psi_{xi}(\zeta,\eta) = \sum_{i=1}^{m_4} d_i \psi_{xi}(\zeta,\eta) = \mathbf{d}^T \Psi_x, \tag{16d}$$

$$\theta_y(\zeta,\eta) = \sum_{q=0}^{p_5} \sum_{r=0}^{q} e_i \psi_{yi}(\zeta,\eta) = \sum_{i=1}^{m_5} e_i \psi_{yi}(\zeta,\eta) = \mathbf{e}^T \Psi_y, \tag{16e}$$

where p_s $(s=1,2,3,4,5)$ is the degree of the mathematically complete two-dimensional polynomial space; a_i, b_i, c_i, d_i, and e_i are the unknown coefficients to be varied with the subscript i given by

$$i = \frac{(q+1)(q+2)}{2} - (q-r); \tag{17}$$

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ are the unknown coefficient vectors having a_i, b_i, c_i, d_i, and e_i as respective elements; m_s $(s=1,2,3,4,5)$ are, respectively, the dimensions of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ given by

$$m_s = \tfrac{1}{2}(p_s+1)(p_s+2), \tag{18}$$

and Φ_x, Φ_y, Φ_z, Ψ_x, and Ψ_y are the shape function vectors whose elements are given by

$$\phi_{xi} = f_i \phi_{x1}, \tag{19a}$$

$$\phi_{yi} = f_i \phi_{y1}, \tag{19b}$$

$$\phi_{zi} = f_i \phi_{z1}, \tag{19c}$$

$$\psi_{xi} = f_i \psi_{x1}, \tag{19d}$$

$$\psi_{yi} = f_i \psi_{y1}, \tag{19e}$$

where

$$f_i = \zeta^{q-r} \eta^r, \tag{20}$$

and ϕ_{x1}, ϕ_{y1}, ϕ_{z1}, ψ_{x1}, and ψ_{y1} are the basic functions corresponding to u, v, w, θ_x, and θ_y, respectively. The basic functions consist of the products of boundary expressions of the laminated plate to ensure the automatic satisfaction of geometric boundary conditions.

Four types of boundary conditions can be considered when analyzing plates using a first-order shear deformation plate theory. These are free edge (F), clamped edge (C), the first kind of simply supported edge (S), and the second kind of simply supported edge (S*). The S condition requires the transverse and lateral displacement and the tangential rotation to be zero while the S* condition requires only the transverse displacement along the support to be zero. A detailed definition of the four types of boundary conditions for isotropic plates is available elsewhere.[14]

The basic functions for the displacements and bending slopes can be expressed by a single expression

$$\prod_{j=1}^{4} [\Gamma_j(\zeta,\eta)]^{\theta_j}, \tag{21}$$

where Γ_j is the boundary expression of the jth supporting edge in the $\zeta-\eta$ plane and Θ_j, depending on the support edge conditions, is determined from Table I.

TABLE II. Comparison of frequency parameter $\lambda=\omega a^2\sqrt{\rho/(Eh^2)}$ of a square simply supported isotropic plate ($\nu=0.3$, $h/a=0.1$).

Sources	Mode sequence No.				
	1	2	3	4	5
$p_s=4$	5.770	13.804	27.561	33.205	46.581
$p_s=6$	5.769	13.764	26.040	32.689	43.567
$p_s=8$	5.769	13.764	25.738	32.295	43.169
$p_s=9$	5.769	13.764	25.734	32.294	42.421
$p_s=10$	5.769	13.764	25.734	32.284	42.420
3D elasticity solution (Ref. 10)	5.780	13.805	25.867	32.491	42.724
Reddy's FEM solution (Ref. 11)	5.793	14.081	27.545	35.050	49.693
CPT solution (Ref. 15)	5.973	14.934	29.867	38.829	53.868

E. Eigenvalue problem

Transforming the integration domain of the integrals in Eqs. (9) and (10) to the $\zeta-\eta$ plane and substituting Eq. (16) in the resulting expressions, the energy function π in Eq. (8) can be written as

$$\pi=\tfrac{1}{2}\mathbf{q}^T(\mathbf{K}-\Omega^2\mathbf{M})\mathbf{q} \quad (22)$$

in which

$$\mathbf{q}^T=\langle \mathbf{a}^T \quad \mathbf{b}^T \quad \mathbf{c}^T \quad \mathbf{d}^T \quad \mathbf{e}^T\rangle \quad (23)$$

$$\mathbf{K}=\int_{\bar{A}}\begin{bmatrix} K_{aa} & K_{ab} & K_{ac} & K_{ad} & K_{ae} \\ & K_{bb} & K_{bc} & K_{bd} & K_{be} \\ & & K_{cc} & K_{cd} & K_{ce} \\ \text{symmetric} & & & K_{dd} & K_{de} \\ & & & & K_{ee} \end{bmatrix}|\mathbf{J}|d\bar{A}, \quad (24)$$

$$\mathbf{M}=\int_{\bar{A}}\begin{bmatrix} M_{aa} & M_{ab} & M_{ac} & M_{ad} & M_{ae} \\ & M_{bb} & M_{bc} & M_{bd} & M_{be} \\ & & M_{cc} & M_{cd} & M_{ce} \\ \text{symmetric} & & & M_{dd} & M_{de} \\ & & & & M_{ee} \end{bmatrix}|\mathbf{J}|d\bar{A}. \quad (25)$$

Here **K** and **M** represent the stiffness and mass matrices of the plate, respectively, and $\bar{A}$ denotes the area of the basic square plate. Details of the submatrices of **K** and **M** are given in the Appendix.

Setting the first variation of the energy functional in Eq. (22) to zero gives the governing eigenvalue problem

$$(\mathbf{K}-\Omega^2\mathbf{M})\mathbf{q}=0. \quad (26)$$

In the numerical analysis, the Gauss quadrature method is employed to evaluate the integrals appearing in Eqs. (24) and (25) after making use of Eq. (14) to calculate the derivatives with respect to x and y. Standard eigenvalue solvers may be used to compute the natural frequencies of laminated quadrilateral plates by solving the general eigenvalue problem defined in Eq. (26).

II. NUMERICAL RESULTS AND DISCUSSIONS

The laminated plates under consideration are composed of any number of plies that have the same geometric and material properties, and lie in an unsymmetric stacking sequence. For example, a four-ply laminate is arranged in a manner of ($\theta/-\theta/\theta/-\theta$). A high-modulus graphite/epoxy is used to study the vibration behavior of the unsymmetrically laminated composite plates. Each ply is a unidirectional fiber reinforced composite possessing the dimensionless material properties of: $E_1/E_2=40$; $G_{12}/E_2=G_{13}/E_2=0.6$; $G_{23}/E_2=0.5$; and $v_{12}=0.25$ (for all examples considered here, otherwise the material properties will be specified).

A. Convergence and comparison studies

As a check on the numerical accuracy of the analysis method, natural frequency parameters were first obtained for a moderately thick isotropic plate with relative thickness ratio $h/a=0.10$. The present results are compared with the values from 3-D linear elasticity analysis[10] and the Reddy's FEM solution[11] in Table II. The present solutions have been computed using values of p_s increasing from 4 to 10 which is equivalent to a variation in matrix size from 75×75 to 330×330. Clearly the eigenvalues of lower modes converge relatively faster than the higher modes, and $p_s=10$ is needed to give convergent eigenvalues. In general, the present results are in close agreement with those values given by Srinivas *et al.*[10] and Reddy[11] for isotropic plates.

A further verification on the numerical accuracy of the present method is shown in Table III. The fundamental frequency parameters for a simply supported three-ply orthotropic square laminated plate are given together with the values of 3-D linear elasticity analysis.[10] A convergence study was again carried out by varying the number of degree of polynomials p_s for the laminated plate with various degree of orthotropy E_1/E_2. It is obvious that a degree p_s of less than 4 (the determinant size is 75×75) is more than enough to furnish the convergent results since only the fundamental frequency parameters are of interest in this comparison. The results obtained from the present analysis are seen to be in close agreement with the 3-D elasticity solutions.[10]

Table IV presents nondimensional frequency parameters for a four-ply unsymmetrically laminated square plate with stacking sequence (45 deg/−45 deg/45 deg/−45 deg) of relative thickness ratio $h/a=0.10$. A degree $p_s=10$ (the determinant size is 330×330) is used for this computation. The results of the first ten frequency parameters for six different combinations of boundary conditions (SSSS, SSCS, CSCS, FSFS, FSSS, FSCS) of laminated plates from various

TABLE III. Comparison of nondimensional fundamental frequency $\omega h\sqrt{\rho_2/(E_x)_2}$ of a simply supported three-ply orthotropic square plate ($h_1:h_2:h_3=0.1:0.8:1$, $h/a=0.1$).

Sources	$(E_x)_1/(E_x)_2$				
	1	2	3	4	5
$p_s=2$	0.0501	0.0605	0.0839	0.1122	0.1345
$p_s=4$	0.0474	0.0573	0.0796	0.1065	0.1278
$p_s=6$	0.0474	0.0573	0.0796	0.1065	0.1278
3D elasticity solution (Ref. 10)	0.0474	0.0570	0.0772	0.0981	0.1120
CPT (Ref. 16)	0.0497	0.0606	0.0853	0.1153	0.1390

sources are presented. The table includes the closed form solutions of Bert and Chen,[12] results of classical plate theory by Jones *et al.*,[16] and finite element solutions of Reddy.[11] The present results are in good agreement with those closed form[12] and finite element solutions.[11] However, they do not correlate well with those values from the classical plate theory since the effects of shear deformation were not considered in their formulation.[16]

B. Parametric studies

Tables V–IX present results for the first six nondimensional frequency parameters for a four-ply unsymmetrically laminated quadrilateral plate with different boundary conditions. Five sets of boundary conditions are considered, namely CFFF, SCFF, CCFF, CCCF, and CCCC, respectively. The plate geometry is defined by parameters given as follows: $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, and $\beta=75$ deg as shown in Fig. 1. The results for the plates are given with the relative thickness ratio h/a ranging from 0.01 (a thin laminate) to 0.20 (a moderately thick laminate) and fiber orientation angle θ ranging from 0° to 90°.

The effect of boundary conditions on vibration responses of laminated plates has also been examined. Tables V–IX show that an increasing boundary constraint induces a higher vibration response. In Table IX, close agreement is seen between the solutions from the present paper and those from finite element package ANSYS using $20\times20\times10$ mesh and SOLID46 element. A higher constraint has higher plate stiffness and thus results in a higher physical frequency of vibration. For instance, it is clearly evident that the CFFF plate possesses a lower vibration frequency than the CCFF plate and the CCCC plate, e.g., compare Tables V, VII, and IX. The frequency parameters for the CCCC plate are the highest among all the plates investigated. Since the CCCC plate has more boundary kinematic constraints than the CCFF and CFFF plates, it is expected that imposing more boundary kinematic constraints tends to induce a higher frequency of vibration.

It is of interest to investigate the effects of transverse shear deformation by considering plates with different relative thickness ratios h/a. It is obvious that as the plate thickness increases, a significant decrease in vibration frequency occurs due to shear flexibility. These results (Tables V–IX) clearly indicate that the effect of transverse shear deformation becomes more pronounced for higher modes of vibration. For example, we compare this to the cantilevered four-ply quadrilateral laminated plate with $\theta=90°$ as tabulated in

TABLE IV. Comparison of frequency parameter $\lambda=\omega a^2\sqrt{\rho/(E_2h^2)}$ of four-ply laminated square plates [(45 deg/−45 deg/45 deg/−45 deg), $h/a=0.1$].

Mode No.	Edge boundary conditions					
	SSSS	SSCS	CSCS	FSFS	FSSS	FSCS
1	18.4633[a] 18.46[b] 18.609[c] 23.53[d]	19.4146	20.4821	6.2645	10.0492	10.3573
2	34.4144[a]	34.9489	35.0258	15.4105	24.1490	25.3702
3	34.8739[a] 34.87[b] 35.405[c]	36.4707	38.0732	23.7974	26.3117	26.3228
4	34.8739[a]	50.8870	51.2703	29.5499	32.4374	41.1690
5	50.5204[a] 50.52[b]	54.2742	54.2743	30.9739	40.9123	42.4812
6	54.2741[a] 54.27[b] 54.360[c]	55.6697	56.9964	32.3327	40.9990	47.1965
7	54.2741[a] 54.27[b]	61.2240	67.2011	34.4144	47.1970	51.8856
8	67.1731[a] 67.17[b] 67.637[c]	67.1871	68.5613	45.9121	51.8856	57.4100
9	67.1731 67.17[b]	67.8609	75.5939	47.1525	57.1046	57.6118
10	68.8290	75.5874	77.4117	47.7045	58.6215	58.6218

[a]Present solution.
[b]Navier solution (Ref. 12).
[c]FEM solution (Ref. 11).
[d]CPT solution (Ref. 16).

TABLE V. Variation of frequency parameter $\lambda=\omega a^2\sqrt{\rho/(E_2h^2)}$ with nondimensional thickness h/a for a four-ply laminated quadrilateral cantilever (CFFF) plate $(\theta/-\theta/\theta/-\theta)$ having $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

		Mode sequence No.					
θ	h/a	1	2	3	4	5	6
0°	0.01	2.2563	8.1953	11.8461	23.3086	31.0204	45.8713
	0.05	2.2418	7.9392	11.5733	22.1499	26.3208	29.5124
	0.10	2.2122	7.4888	10.9171	13.1604	20.0256	23.7195
	0.15	2.1712	6.9560	8.7736	10.0738	15.8130	17.7714
	0.20	2.1209	6.3990	6.5802	9.1861	11.8597	15.4305
30°	0.01	3.5351	17.4442	22.3211	47.1132	58.7759	78.9316
	0.05	3.3895	16.2671	20.1569	41.0384	41.6310	49.8356
	0.10	3.2226	14.4623	17.1527	20.4950	31.1102	33.7266
	0.15	3.0700	12.6339	13.6496	14.5047	20.7402	27.3062
	0.20	2.9257	10.2263	11.0133	12.3504	15.5628	22.6110
45°	0.01	6.1402	26.2006	30.5065	67.4438	72.9078	86.5239
	0.05	5.7917	22.9487	26.8472	54.8328	59.0032	63.7771
	0.10	5.3312	18.6221	21.5916	31.8562	40.0761	43.2858
	0.15	4.8904	15.2228	17.3656	21.2290	30.6122	32.4749
	0.20	4.4804	12.7182	14.2885	15.9558	24.2648	24.6639
60°	0.01	9.1317	26.1052	42.9302	62.2516	72.1352	106.8625
	0.05	8.6596	22.8392	36.1535	50.5121	56.6187	79.4564
	0.10	7.7657	18.1898	26.9628	37.0135	40.5321	50.5277
	0.15	6.8271	14.7235	20.7487	28.8910	30.9310	33.5532
	0.20	6.8271	14.7235	20.7487	28.8910	30.9310	33.5532
90°	0.01	12.5558	19.3126	33.8502	55.3454	69.9477	78.9487
	0.05	11.7455	17.4554	30.1719	36.8467	45.8967	52.4128
	0.10	10.0151	14.2899	18.4233	24.4072	32.6693	35.8728
	0.15	8.3122	11.5068	12.2822	19.8149	24.6878	26.9986
	0.20	6.9440	9.2117	9.4085	16.5385	19.8701	21.5538

TABLE VI. Variation of frequency parameter $\lambda=\omega a^2\sqrt{\rho/(E_2h^2)}$ with nondimensional thickness h/a for a four-ply laminated quadrilateral SCFF plate $(\theta/-\theta/\theta/-\theta)$ having $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

		Mode sequence No.					
θ	h/a	1	2	3	4	5	6
0°	0.01	13.8460	17.8158	30.0756	53.1808	61.7054	83.9846
	0.05	12.9312	15.8825	27.7375	46.9823	49.8065	58.4074
	0.10	10.9234	13.0366	24.1572	29.2037	33.2656	41.2432
	0.15	8.9012	10.9968	19.4691	21.0072	25.4842	32.7146
	0.20	7.3089	9.5930	14.6019	18.3458	20.7853	26.4601
30°	0.01	15.0008	28.9744	51.1799	68.2917	91.0909	107.0619
	0.05	13.6157	23.8238	42.2318	54.1084	73.9337	78.3244
	0.10	11.2917	18.5858	33.4827	37.9725	44.6419	52.0581
	0.15	9.3255	15.2421	27.0967	28.6788	29.9363	38.6305
	0.20	7.8639	12.9677	21.7057	22.7869	23.5750	30.5150
45°	0.01	14.2253	33.1021	59.3108	74.3919	103.3488	132.5401
	0.05	12.8510	27.3612	49.1983	60.1734	77.4813	77.9251
	0.10	10.7888	21.7209	36.1281	38.7829	44.7335	53.1769
	0.15	9.0808	17.7841	25.7833	27.6090	34.4751	39.3143
	0.20	7.7783	14.9569	19.3219	22.0684	27.6496	30.9389
60°	0.01	11.3138	33.3563	49.0644	77.8675	96.3661	107.8581
	0.05	10.3479	29.5224	42.3633	57.8389	63.5807	76.5789
	0.10	8.9845	24.3936	28.8783	32.7126	46.5415	54.0519
	0.15	7.8309	19.2256	20.1305	25.6278	35.3447	40.2143
	0.20	6.9017	14.3983	16.8525	20.7428	28.0896	31.4680
90°	0.01	4.3641	18.6107	38.8560	49.8370	64.2270	68.5829
	0.05	4.2726	17.7055	34.8301	41.1501	43.5695	55.2468
	0.10	4.1131	16.0223	20.5750	28.7493	33.6621	38.0649
	0.15	3.9199	13.7167	14.2353	23.3082	25.3766	26.6132
	0.20	3.7096	10.2875	12.6024	19.0324	19.0545	21.9154

TABLE VII. Variation of frequency parameter $\lambda=\omega a^2\sqrt{\rho/(E_2h^2)}$ with nondimensional thickness h/a for a four-ply laminated quadrilateral CCFF plate $(\theta/-\theta/\theta/-\theta)$ having $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

		Mode sequence No.					
θ	h/a	1	2	3	4	5	6
0°	0.01	16.2181	19.3708	35.4926	60.9359	68.0647	92.2907
	0.05	14.7060	17.6086	32.5252	51.9527	55.5002	58.3557
	0.10	11.8966	15.0074	27.8684	29.1779	36.1104	44.4848
	0.15	9.5337	12.8859	19.4519	23.7358	27.4953	33.7148
	0.20	7.8395	11.2333	14.5889	20.3530	22.2436	27.1379
30°	0.01	17.5281	33.1535	59.3769	73.5924	103.0377	114.5025
	0.05	15.5668	26.8011	48.7010	56.7692	79.3287	84.2004
	0.10	12.6714	20.8756	37.0807	39.5471	45.1375	53.4121
	0.15	10.3826	17.0056	28.2004	30.0708	30.7386	39.3700
	0.20	8.7069	14.2808	21.9574	23.1596	24.9881	30.9649
45°	0.01	17.9685	40.8303	65.5348	89.9501	112.2603	141.7445
	0.05	15.5734	32.8326	52.2516	68.9487	80.9073	93.7450
	0.10	12.6329	25.0122	37.0189	46.7172	48.4525	53.9696
	0.15	10.4315	19.7239	27.7908	31.1349	36.2706	39.5705
	0.20	8.8049	16.1139	22.0856	23.3369	28.7354	31.0545
60°	0.01	16.5152	44.0349	58.1094	91.5423	110.6112	124.3205
	0.05	14.4972	36.3630	46.9534	68.7853	82.1525	88.5825
	0.10	11.9760	27.3179	33.9424	48.0579	55.2884	58.2165
	0.15	10.0206	21.2974	25.9053	36.2377	40.4972	41.3746
	0.20	8.5348	17.3436	20.8220	28.8297	30.5240	32.1720
90°	0.01	14.2579	29.5475	54.3926	70.2148	79.2786	91.5496
	0.05	13.2857	26.7113	45.2776	53.0904	60.3084	61.0317
	0.10	11.3989	21.9543	30.1542	32.3052	37.4387	43.1905
	0.15	9.5711	17.9172	20.1028	24.4090	28.8181	33.4063
	0.20	8.0963	14.8976	15.0771	19.5923	23.3279	25.5129

TABLE VIII. Variation of frequency parameter $\lambda=\omega a^2\sqrt{\rho/(E_2h^2)}$ with nondimensional thickness h/a for a four-ply laminated quadrilateral CCCF plate $(\theta/-\theta/\theta/-\theta)$ having $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

		Mode sequence No.					
θ	h/a	1	2	3	4	5	6
0°	0.01	17.6973	34.1609	59.7968	68.0236	91.7802	96.6534
	0.05	16.5700	31.5583	51.7301	54.6482	69.3039	83.0134
	0.10	14.3393	26.8563	36.0348	42.3832	44.2622	48.2187
	0.15	12.2501	22.5476	27.4426	28.2555	35.8438	37.0344
	0.20	10.5648	19.0786	21.1916	22.1921	29.5662	30.0301
30°	0.01	26.3791	59.7396	75.8855	107.1104	129.2748	159.1530
	0.05	23.1584	49.2694	58.4055	82.7382	91.7070	106.8977
	0.10	18.6825	36.6229	40.3126	57.7555	60.1870	67.1212
	0.15	15.1416	28.1949	30.0146	43.2144	43.7882	47.8056
	0.20	12.5351	22.6918	23.7432	34.1207	34.3637	36.8416
45°	0.01	35.2908	76.0432	87.7724	143.2751	149.1184	163.7034
	0.05	29.5709	58.0057	65.3167	98.0825	102.1905	108.4718
	0.10	22.1908	39.9198	43.6071	62.3605	65.0708	67.8583
	0.15	17.0677	29.6423	31.8339	44.7680	46.6648	48.5036
	0.20	13.6710	23.4241	24.9368	34.6805	36.1925	37.5879
60°	0.01	42.9179	72.3946	107.8570	120.3575	154.8198	187.3991
	0.05	34.1284	53.6813	74.2079	84.6497	100.4231	120.5027
	0.10	23.8396	36.6540	46.9332	56.2842	62.9365	72.7234
	0.15	17.7255	27.5221	33.6805	41.7413	45.1242	51.1581
	0.20	13.9939	22.0168	26.1735	33.0744	35.0002	39.2784
90°	0.01	43.4300	68.7582	82.5915	101.8683	127.5584	143.7919
	0.05	32.7566	47.4887	59.1411	72.5697	75.7385	84.2147
	0.10	22.4246	30.5186	37.8693	40.9431	47.5062	54.3330
	0.15	16.5342	22.5906	25.2462	31.0756	35.0580	39.2727
	0.20	13.0084	18.0414	18.9346	24.8554	27.8639	30.7196

TABLE IX. Variation of frequency parameter $\lambda = \omega a^2 \sqrt{\rho/(E_2 h^2)}$ with nondimensional thickness h/a for a four-ply laminated quadrilateral CCCC plate $(\theta/-\theta/\theta/-\theta)$ having $b/a = 0.9$, $c/a = 0.7$, $\alpha = 65$ deg, $\beta = 75$ deg.

		Mode sequence No.					
θ	h/a	1	2	3	4	5	6
0°	0.01	66.3495	90.0324	116.4425	148.5115	154.1021	191.4730
	0.05	47.1823	61.9749	81.0273	91.1540	107.5743	109.9071
	0.10	30.4523	41.2417	54.8355	54.9536	57.0321	67.0205
	0.15	22.3313	31.3377	36.6357	39.2099	43.9170	48.2761
	0.20	17.7211	25.3115	27.4768	30.5112	35.4955	37.7389
30°	0.01	69.9690	113.4161	153.8890	169.3880	219.6646	240.6992
	0.05	51.6977	78.8926	96.9603	113.0431	128.9760	145.9206
	0.10	34.2667	51.5109	58.6595	72.5893	76.6762	84.1318
	0.10[a]	33.7214	48.2861	59.7392	⋯	76.4849	⋯
	0.15	24.9896	37.5799	41.3285	52.5940	53.7613	58.2197
	0.20	19.5157	29.4261	31.7441	41.0000	41.2449	44.3261
	0.20[a]	19.3151	27.8423	32.7428	37.9514	41.7162	⋯
45°	0.01	71.2040	134.6393	142.0045	219.0485	224.3553	236.8315
	0.05	53.3544	90.6412	93.5954	131.3998	136.5774	138.6236
	0.10	35.4820	56.3467	57.7407	77.7898	81.1901	81.7483
	0.10[a]	34.3553	52.3320	58.4081	73.1138	77.8335	⋯
	0.15	25.6655	39.9508	40.8639	54.2171	56.6539	57.2051
	0.20	19.8951	30.7762	31.4564	41.3947	43.2748	43.8418
	0.20[a]	19.5410	29.1283	32.2272	39.8901	42.2981	⋯
60°	0.01	70.1182	117.1055	151.1298	184.9894	206.8792	256.9287
	0.05	51.6900	80.3975	96.2806	116.1955	126.4063	146.0168
	0.10	34.2223	51.9518	58.4403	72.4923	76.5185	84.7863
	0.10[a]	32.5711	50.7552	55.5471	70.1423	77.1372	⋯
	0.15	24.9457	37.8020	41.1987	51.9979	54.1098	58.7674
	0.20	19.4832	29.5817	31.6604	40.2859	41.7437	44.7770
	0.20[a]	18.9789	28.9711	31.0564	39.2440	42.5898	⋯
90°	0.01	71.6008	87.0326	112.9384	148.7590	182.1487	194.0421
	0.05	48.3066	60.9233	81.4380	98.1799	103.3688	109.9078
	0.10	30.4750	41.4449	54.8299	54.9539	59.1779	65.1740
	0.15	22.2888	31.5762	36.6359	39.4922	44.2076	47.5391
	0.20	17.6919	25.4841	27.4769	30.6821	35.4071	37.4117

[a]FEM solution from ANSYS.

Table V. The fundamental frequency decreases by about half as the thickness ratio increases from 0.01 to 0.20. However, the frequency for the sixth mode (a higher mode) decreases to about a quarter subject to the same thickness condition.

It is also worthwhile to examine the variation of frequency parameters of laminates by increasing the number of plies. Figures 2–5 present results for laminates with various fiber orientation angles and stacking in an unsymmetric manner. Laminates subjected to CCCC, CCFF, S*S*S*S*, and CFCF boundary conditions with stacking sequence $(\theta/-\theta/\theta/-\theta/...)$ are considered. The graphs in Figs. 2–5 show that the fundamental frequency parameter increases initially and reaches a maximum value as the number of plies increases, except when θ is 0 or 90 deg. It is easy to demonstrate this phenomenon by referring to Fig. 2. It can be seen that the fundamental frequency parameter increases and reaches a maximum value of about 32 when number of plies is at about 7 (or 8) for $\theta = 15$ deg. When the number of plies reaches 7 (or 8), the analysis of this laminate can be treated as a single layer anisotropic plate.

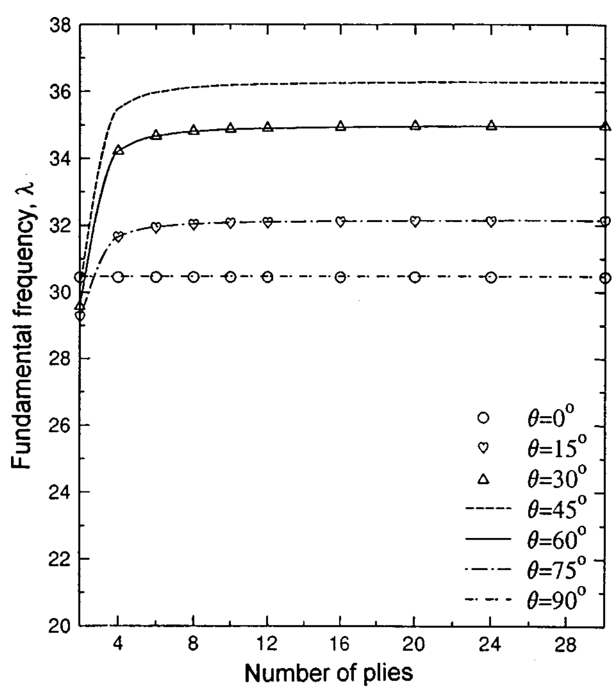

FIG. 2. Effect of number of plies on the fundamental frequency $\lambda = \omega a^2 \sqrt{\rho/(E_2 h^2)}$ of the laminated CCCC quadrilateral plate $(\theta/-\theta/\theta/-\theta/.../\theta/-\theta)$ having $h/a = 0.1$, $b/a = 0.9$, $c/a = 0.7$, $\alpha = 65$ deg, $\beta = 75$ deg.

III. CONCLUDING REMARKS

This paper considers the vibration analysis of arbitrary quadrilateral unsymmetrically laminated composite plates

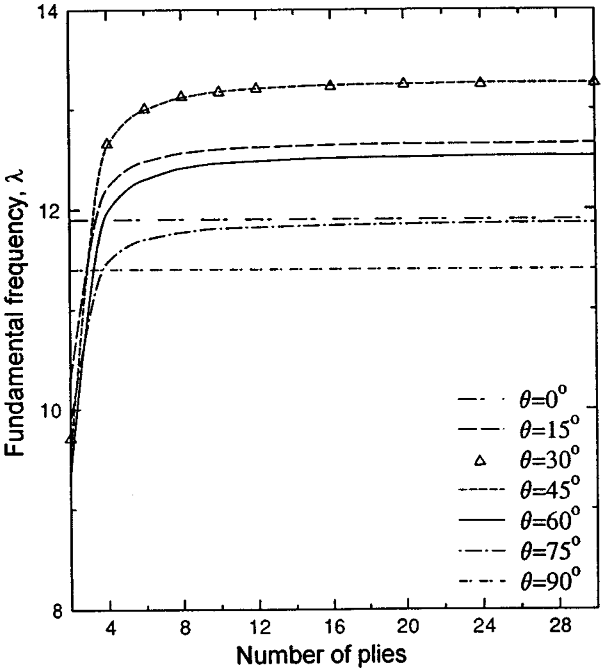

FIG. 3. Effect of number of plies on the fundamental frequency $\lambda = \omega a^2 \sqrt{\rho/(E_2 h^2)}$ of the laminated CCFF quadrilateral plate $(\theta/-\theta/\theta/-\theta/\ldots/\theta/-\theta)$ having $h/a=0.1$, $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

with various boundary conditions. The analysis has been performed using the Ritz method with polynomials as its admissible displacement and rotation functions. This analysis method is extremely versatile because a basic function is employed which easily accommodates various boundary conditions. The significance of the transverse shear deformation has been investigated. This effect was incorporated in the mathematical model using the YNS first-order shear deformation theory.

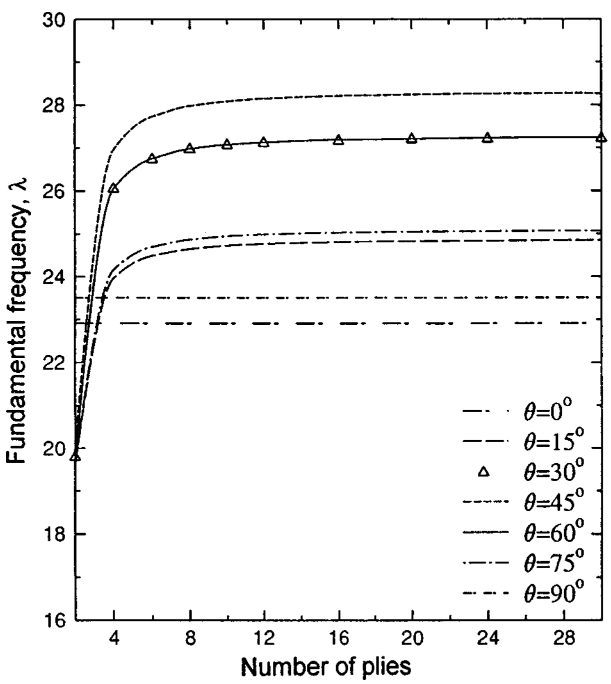

FIG. 4. Effect of number of plies on the fundamental frequency $\lambda = \omega a^2 \sqrt{\rho/(E_2 h^2)}$ of the laminated S*S*S*S* quadrilateral plate $(\theta/-\theta/\theta/-\theta/\ldots/\theta/-\theta)$ having $h/a=0.1$, $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

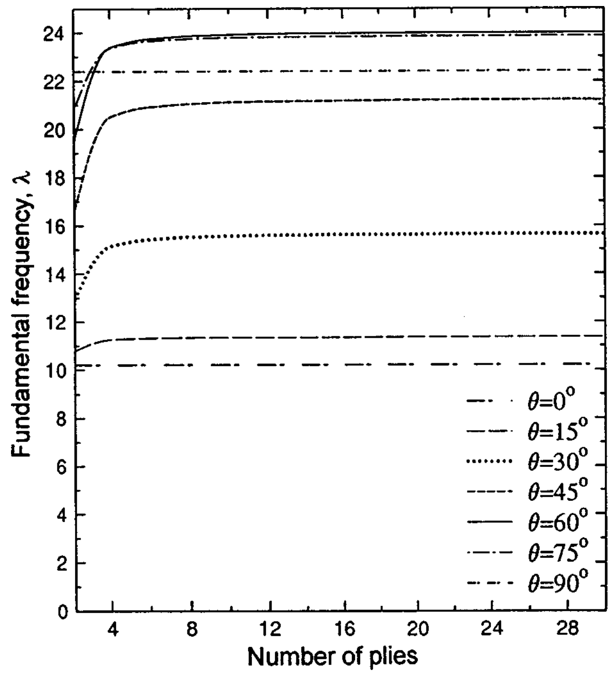

FIG. 5. Effect of number of plies on the fundamental frequency $\lambda = \omega a^2 \sqrt{\rho/(E_2 h^2)}$ of the laminated CFCF quadrilateral plate $(\theta/-\theta/\theta/-\theta/\ldots/\theta/-\theta)$ having $h/a=0.1$, $b/a=0.9$, $c/a=0.7$, $\alpha=65$ deg, $\beta=75$ deg.

The vibration analysis of several plate problems has been presented and the results compared with established solutions from available literature. Close agreement has been obtained for the comparison studies, verifying the accuracy and applicability of the present analysis method. As a closing remark, it is important to point out that, although only a few plate examples has been presented in this study, the method is readily applicable to solving a wide range of quadrilateral unsymmetrically laminated plates of arbitrary shape. As an approximate technique, the method has the advantage of not exhibiting any mesh distortion sensitivity, therefore modeling accurately the quadrilateral geometry without boundary losses.

APPENDIX: SUBMATRICES OF STIFFNESS AND MASS MATRICES

The elements in the stiffness matrices **K** and mass matrices **M**, respectively, can be further expanded as

$$K_{aa} = \partial_x \phi_x (A_{11}\, \partial_x \phi_x^T + A_{16}\, \partial_y \phi_x^T) + \partial_y \phi_x (A_{16}\, \partial_x \phi_x^T + A_{66}\, \partial_y \phi_x^T), \tag{A1a}$$

$$K_{ab} = \partial_x \phi_x (A_{12}\, \partial_y \phi_y^T + A_{16}\, \partial_x \phi_y^T) + \partial_y \phi_x (A_{26}\, \partial_y \phi_y^T + A_{66}\, \partial_x \phi_y^T), \tag{A1b}$$

$$K_{ac} = K_{bc} = 0, \tag{A1c}$$

$$K_{ad} = \partial_x \phi_x (B_{11}\, \partial_x \psi_x^T + B_{16}\, \partial_y \psi_x^T) + \partial_y \phi_x (B_{16}\, \partial_x \psi_x^T + B_{66}\, \partial_y \psi_x^T), \tag{A1d}$$

$$K_{ae} = \partial_x \phi_x (B_{12}\, \partial_y \psi_y^T + B_{16}\, \partial_x \psi_y^T) + \partial_y \phi_x (B_{26}\, \partial_y \psi_y^T + B_{66}\, \partial_x \psi_y^T), \tag{A1e}$$

$$K_{bb}=\partial_y\phi_y(A_{22}\,\partial_y\phi_y^T+A_{26}\,\partial_x\phi_y^T)$$
$$+\partial_x\phi_y(A_{26}\,\partial_y\phi_y^T+A_{66}\,\partial_x\phi_y), \quad \text{(A1f)}$$
$$K_{bd}=\partial_y\phi_y(B_{12}\,\partial_x\psi_x^T+B_{26}\,\partial_y\psi_x^T)$$
$$+\partial_x\phi_y(B_{16}\,\partial_x\psi_x^T+B_{66}\,\partial_y\psi_x^T), \quad \text{(A1g)}$$
$$K_{be}=\partial_y\phi_y(B_{22}\,\partial_y\psi_y^T+B_{26}\,\partial_x\psi_y^T)$$
$$+\partial_x\phi_y(B_{26}\,\partial_y\psi_y^T+B_{66}\,\partial_x\psi_y^T), \quad \text{(A1h)}$$
$$K_{cc}=\partial_y\phi_x(A_{44}\,\partial_y\phi_z^T+A_{45}\,\partial_x\phi_z^T)$$
$$+\partial_x\phi_z(A_{45}\,\partial_y\phi_z^T+A_{55}\,\partial_x\phi_z^T), \quad \text{(A1i)}$$
$$K_{cd}=(A_{45}\,\partial_y\phi_z+A_{55}\,\partial_x\phi_z)\,\psi_x^T, \quad \text{(A1j)}$$
$$K_{ce}=(A_{44}\,\partial_y\phi_z+A_{45}\,\partial_x\phi_z)\,\psi_y^T, \quad \text{(A1k)}$$
$$K_{dd}=\partial_x\psi_x(D_1\,\partial_x\psi_x^T+D_{16}\,\partial_y\psi_x^T)$$
$$+\partial_y\psi_x(D_{16}\,\partial_x\psi_x^T+D_{66}\,\partial_y\psi_x^T)+A_{55}\psi_x\psi_x^T, \quad \text{(A1l)}$$
$$K_{de}=\partial_x\psi_x(D_{12}\,\partial_y\psi_y^T+D_{16}\,\partial_x\psi_y^T)$$
$$+\partial_y\psi_x(D_{26}\,\partial_y\psi_y^T+D_{66}\,\partial_x\psi_y^T)+A_{45}\psi_x\psi_y^T, \quad \text{(A1m)}$$
$$K_{ee}=\partial_y\psi_y(D_{22}\,\partial_y\psi_y^T+D_{26}\,\partial_x\psi_y^T)$$
$$+\partial_x\psi_y(D_{26}\,\partial_y\psi_y^T+D_{66}\,\partial_x\psi_y^T)+A_{44}\psi_y\psi_y^T, \quad \text{(A1n)}$$
$$M_{aa}=E_1\phi_x\phi_x^T, \quad \text{(A2a)}$$
$$M_{ad}=E_2\phi_x\psi_x^T, \quad \text{(A2b)}$$
$$M_{bb}=E_1\phi_y\phi_y^T, \quad \text{(A2c)}$$
$$M_{be}=E_2\phi_y\psi_y^T, \quad \text{(A2d)}$$
$$M_{cc}=E_1\phi_z\phi_z^T, \quad \text{(A2e)}$$
$$M_{dd}=E_3\psi_x\psi_x^T, \quad \text{(A2f)}$$
$$M_{ee}=E_3\psi_y\psi_y^T, \quad \text{(A2g)}$$
$$M_{ab}=M_{ac}=M_{ae}=M_{bc}=M_{bd}=M_{cd}=M_{ce}=M_{de}=0. \quad \text{(A2h)}$$

In the above, A_{ij}, B_{ij}, and C_{ij} are effective laminate stiffness coefficients given by

$$(A_{ij},B_{ij},D_{ij})=\int_{-h/2}^{h/2}(1,z,z^2)Q_{ij}\,dz, \quad i,j=1,2,6, \quad \text{(A3a)}$$

$$A_{ij}=k_{ij}^2\int_{-h/2}^{h/2}Q_{ij}\,dz, \quad i,j=4,5, \quad \text{(A3b)}$$

and E_i are the effective inertia coefficients given by

$$(E_1,E_2,E_3)=\int_{-h/2}^{h/2}(1,z,z^2)\rho\,dz, \quad \text{(A4)}$$

where $k_{ij}^2=5/6$ are the transverse shear correction coefficients. Note that for antisymmetric angle-ply laminates A_{16}, A_{26}, A_{45}, B_{11}, B_{12}, B_{22}, D_{16}, B_{26}, and E_2, are identically zero.

[1] J. N. Reddy, ''A review of the literature on finite-element modelling of laminated composite plates and shells,'' Shock Vib. Digest **17**(4), 3–8 (1985).

[2] C. W. Bert, ''Research on dynamic behaviour of composite and sandwich plates, V: Part I,'' Shock Vib. Digest **23**(6), 3–14 (1991).

[3] C. W. Bert, ''Research on dynamic behaviour of composite and sandwich plates, V: Part II,'' Shock Vib. Digest **23**(7), 9–21 (1991).

[4] K. M. Liew, Y. Xiang, and S. Kitipornchai, ''Research on thick plate vibration: a literature survey,'' J. Sound Vib. **180**(1), 163–176 (1995).

[5] P. C. Yang, C. H. Norris, and Y. Stavsky, ''Elastic wave propagation in heterogeneous plates,'' Int. J. Solid Struct. **2**(4), 665–684 (1966).

[6] A. W. Leissa, ''Vibration of plates,'' NASA SP-160, Washington (1969).

[7] A. W. Leissa, ''Recent research in plate vibrations, 1973–76: complicating effects,'' Shock Vib. Digest **10**(12), 21–35 (1978).

[8] A. W. Leissa, ''Plate vibration and research, 1976–1980: complicating effects,'' Shock Vib. Digest **13**(10), 19–36 (1981).

[9] A. W. Leissa, ''Recent studies in plate vibrations, 1981–1985, part II: complicating effects,'' Shock Vib. Digest **19**(3), 10–24 (1987).

[10] S. Srinivas, C. V. J. Rao, and A. K. Rao, ''An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates,'' J. Sound Vib. **12**(2), 187–199 (1970).

[11] J. N. Reddy, ''Free vibration of antisymmetric, angle-ply laminated plates including transverse shear deformation by the finite element method,'' J. Sound Vib. **66**(4), 565–576 (1979).

[12] C. W. Bert and T. L. C. Chen, ''Effect of shear deformation on vibration of antisymmetric angle-ply laminated rectangular plates,'' Int. J. Solid Struct. **14**(3), 465–473 (1978).

[13] Y. Xiang, S. Kitipornchai, and K. M. Liew, ''Buckling and vibration of thick laminates on pasternak foundations,'' J. Engng Mech. **122**(1), 54–63 (1996).

[14] K. M. Liew, Y. Xiang, S. Kitipornchai, and C. M. Wang, ''Vibration of thick skew plates based on Mindlin shear deformation plate theory,'' J. Sound Vib. **168**(1), 39–69 (1993).

[15] K. M. Liew, K. Y. Lam, and S. T. Chow, ''Free vibration analysis of rectangular plates using orthogonal plate function,'' Comput. Struct. **34**(1), 79–85 (1990).

[16] R. M. Jones, H. S. Morgan, and J. M. Whitney, ''Buckling and vibration of antisymmetrically laminated angle-ply rectangular plates,'' J. Appl. Mech. **40**(2), 1143–1144 (1973).