AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **40**, 5264 (1999); doi: 10.1063/1.533029

View online: http://dx.doi.org/10.1063/1.533029

View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/40/10?ver=pdfcov

Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 40, NUMBER 10 OCTOBER 1999

Quasi-Hopf superalgebras and elliptic quantum supergroups

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(Received 6 October 1998; accepted for publication 11 February 1999)

We introduce the quasi-Hopf superalgebras which are $\mathbf{Z}_2$-graded versions of Drinfeld's quasi-Hopf algebras. We describe the realization of elliptic quantum supergroups as quasi-triangular quasi-Hopf superalgebras obtained from twisting the normal quantum supergroups by twistors which satisfy the graded shifted cocycle condition, thus generalizing the quasi-Hopf twisting procedure to the supersymmetric case. Two types of elliptic quantum supergroups are defined, that is, the face type $\mathcal{B}_{q,\lambda}(\mathcal{G})$ and the vertex type $\mathcal{A}_{q,p}[\mathrm{sl}(\widehat{n|n})]$ (and $\mathcal{A}_{q,p}[\mathrm{gl}(\widehat{n|n})]$), where $\mathcal{G}$ is any Kac–Moody superalgebra with symmetrizable generalized Cartan matrix. It appears that the vertex type twistor can be constructed only for $U_q[\mathrm{sl}(\widehat{n|n})$ in a nonstandard system of simple roots, all of which are fermionic. © *1999 American Institute of Physics.* [S0022-2488(99)00210-8]

I. INTRODUCTION

One of the aims of this paper is to introduce $\mathbf{Z}_2$-graded versions of Drinfeld's quasi-Hopf algebras,[1] which are referred to as quasi-Hopf superalgebras. We then introduce elliptic quantum supergroups, which are defined as quasi-triangular quasi-Hopf superalgebras arising from twisting the normal quantum supergroups by twistors which satisfy the graded shifted cocycle condition, thus generalizing Drinfeld's quasi-Hopf twisting procedure[2–6] to the supersymmetric case. We adopt the approach in Ref. 4 and construct two types of twistors, i.e., the face-type twistor associated to any Kac–Moody superalgebra $\mathcal{G}$ with a symmetrizable generalized Cartan matrix and the vertex-type twistor associated to $\mathrm{sl}(\widehat{n|n})$ in a nonstandard simple root system in which all simple roots are odd (or fermionic). It should be pointed out that the face-type twistors for certain classes of *nonaffine* simple superalgebras were also constructed in Ref. 5.

The elliptic quantum groups[7,8] are believed to provide the underlying algebraic structures for integrable models based on elliptic solutions of the (dynamical) Yang–Baxter equation, such as Baxter's eight-vertex model,[9] the ABF (Andrews—Baxter–Forrester) model,[10] and their group theoretical generalizations.[11,12] The elliptic quantum supergroups described in this paper are expected to play a similar role in supersymmetric integrable models based on elliptic solutions[13,14] of the graded (dynamical) Yang–Baxter equation.

II. QUASI-HOPF SUPERALGEBRAS

Definition 1: A $\mathbf{Z}_2$*-graded quasi-bialgebra is a* $\mathbf{Z}_2$*-graded unital associative algebra A over a field K which is equipped with algebra homomorphisms* ϵ*:* $A\to K$ *(counit),* Δ*:* $A\to A\otimes A$ *(coproduct), and an invertible homogeneous element* $\Phi\in A\otimes A\otimes A$ *(coassociator) satisfying*

$$(1\otimes\Delta)\Delta(a)=\Phi^{-1}(\Delta\otimes 1)\Delta(a)\Phi,\quad \forall a\in A, \tag{II.1}$$

$$(\Delta\otimes 1\otimes 1)\Phi\cdot(1\otimes 1\otimes\Delta)\Phi=(\Phi\otimes 1)\cdot(1\otimes\Delta\otimes 1)\Phi\cdot(1\otimes\Phi), \tag{II.2}$$

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$$(\epsilon\otimes 1)\Delta=1=(1\otimes\epsilon)\Delta, \tag{II.3}$$

$$(1\otimes\epsilon\otimes 1)\Phi=1. \tag{II.4}$$

Equations (II.2)–(II.4) imply that Φ also obeys

$$(\epsilon\otimes 1\otimes 1)\Phi=1=(1\otimes 1\otimes\epsilon)\Phi. \tag{II.5}$$

The multiplication rule for the tensor products is $\mathbf{Z}_2$ graded and is defined for homogeneous elements $a,b,a',b'\in A$ by

$$(a\otimes b)(a'\otimes b')=(-1)^{[b][a']}(aa'\otimes bb'), \tag{II.6}$$

where $[a]\in\mathbf{Z}_2$ denotes the grading of the element a.

Definition 2: A quasi-Hopf superalgebra is a $\mathbf{Z}_2$-graded quasi-bialgebra (A,Δ,ϵ,Φ) equipped with a $\mathbf{Z}_2$-graded algebra anti-homomorphism S: $A\rightarrow A$ (anti-pode) and canonical elements α, $\beta\in A$ such that

$$m\cdot(1\otimes\alpha)(S\otimes 1)\Delta(a)=\epsilon(a)\alpha,\quad \forall a\in A, \tag{II.7}$$

$$m\cdot(1\otimes\beta)(1\otimes S)\Delta(a)=\epsilon(a)\beta,\quad \forall a\in A, \tag{II.8}$$

$$m\cdot(m\otimes 1)\cdot(1\otimes\beta\otimes\alpha)(1\otimes S\otimes 1)\Phi^{-1}=1, \tag{II.9}$$

$$m\cdot(m\otimes 1)\cdot(S\otimes 1\otimes 1)(1\otimes\alpha\otimes\beta)(1\otimes 1\otimes S)\Phi=1. \tag{II.10}$$

Here m denotes the usual product map on A: $m\cdot(a\otimes b)=ab$, $\forall a,b\in A$. Note that since A is associative, we have $m\cdot(m\otimes 1)=m\cdot(1\otimes m)$. For the homogeneous elements $a,b\in A$, the anti-pode satisfies

$$S(ab)=(-1)^{[a][b]}S(b)S(a), \tag{II.11}$$

which extends to inhomogeneous elements through linearity.

Applying ϵ to definitions (II.9) and (II.10) we obtain, in view of (II.4), $\epsilon(\alpha)\epsilon(\beta)=1$. It follows that the canonical elements α and β are both even. By applying ϵ to (II.7), we have $\epsilon(S(a))=\epsilon(a)$, $\forall a\in A$.

In the following we show that the category of quasi-Hopf superalgebras is invariant under a kind of gauge transformation. Let (A,Δ,ϵ,Φ) be a qausi-Hopf superalgebra, with α, β, and S satisfying (II.7)–(II.10), and let $F\in A\otimes A$ be an invertible homogeneous element satisfying the counit properties

$$(\epsilon\otimes 1)F=1=(1\otimes\epsilon)F. \tag{II.12}$$

It follows that F is even. Throughout we set

$$\Delta_F(a)=F\Delta(a)F^{-1},\quad \forall a\in A, \tag{II.13}$$

$$\Phi_F=(F\otimes 1)(\Delta\otimes 1)F\cdot\Phi\cdot(1\otimes\Delta)F^{-1}(1\otimes F^{-1}). \tag{II.14}$$

Theorem 1: $(A,\Delta_F,\epsilon,\Phi_F)$, *defined by (II.13) and (II.14), together with* α_F,β_F, and S_F *given by*

$$S_F=S,\quad \alpha_F=m\cdot(1\otimes\alpha)(S\otimes 1)F^{-1},\quad \beta_F=m\cdot(1\otimes\beta)(1\otimes S)F, \tag{II.15}$$

is also a quasi-Hopf superalgebra. The element F is referred to as a twistor, throughout.

The proof of this theorem is elementary. For demonstration we show in some detail the proof of the antipode properties. Care has to be taken of the gradings in tensor product multiplications and also in extending the antipode to the whole algebra. First of all let us state the following lemma.

Lemma 1: For any elements $\eta \in A\otimes A$ *and* $\xi \in A\otimes A\otimes A$,

$$m\cdot(1\otimes\alpha_F)(S\otimes 1)\eta = m\cdot(1\otimes\alpha)(S\otimes 1)(F^{-1}\eta), \tag{II.16}$$

$$m\cdot(1\otimes\beta_F)(1\otimes S)\eta = m\cdot(1\otimes\beta)(1\otimes S)(\eta F), \tag{II.17}$$

$$\begin{aligned} &m\cdot(m\otimes 1)\cdot(1\otimes\beta_F\otimes\alpha_F)(1\otimes S\otimes 1)\xi \\ &\quad = m\cdot(m\otimes 1)\cdot(1\otimes\beta\otimes\alpha)(1\otimes S\otimes 1)[(1\otimes F^{-1})\cdot\xi\cdot(F\otimes 1)], \end{aligned} \tag{II.18}$$

$$\begin{aligned} &m\cdot(m\otimes 1)\cdot(S\otimes 1\otimes 1)(1\otimes\alpha_F\otimes\beta_F)(1\otimes 1\otimes S)\xi \\ &\quad = m\cdot(m\otimes 1)\cdot(S\otimes 1\otimes 1)(1\otimes\alpha\otimes\beta)(1\otimes 1\otimes S)\cdot[(F^{-1}\otimes 1)\cdot\xi\cdot(1\otimes F)]. \end{aligned} \tag{II.19}$$

Proof: Write $F=f_i\otimes f^i$ and $F^{-1}=\overline{f_i}\otimes\overline{f^i}$. Here and throughout, summation convention on repeated indices is assumed. Then (II.15) can be written as

$$\alpha_F = S(\overline{f_i})\alpha\overline{f^i}, \quad \beta_F = f_i\beta S(f^i). \tag{II.20}$$

Further, write $\eta=\eta_k\otimes\eta^k$ and $\xi=\Sigma_i x_i\otimes y_i\otimes z_i$. Then

$$\begin{aligned} \text{lhs of (II.16)} &= m\cdot(1\otimes S(\overline{f_i})\alpha\overline{f^i})(S(\eta_k)\otimes\eta^k) \\ &= m\cdot(S(\eta_k)\otimes S(\overline{f_i})\alpha\overline{f^i}\eta^k) \\ &= S(\eta_k)S(\overline{f_i})\alpha\overline{f^i}\eta^k \\ &= S(\overline{f_i}\eta_k)\alpha\overline{f^i}\eta^k\times(-1)^{[\eta_k][\overline{f_i}]}, \end{aligned}$$

$$\begin{aligned} \text{rhs of (II.16)} &= m\cdot(1\otimes\alpha)(S\otimes 1)(\overline{f_i}\eta_k\otimes\overline{f^i}\eta^k)\times(-1)^{[\overline{f^i}][\eta_k]} \\ &= S(\overline{f_i}\eta_k)\alpha\overline{f^i}\eta^k\times(-1)^{[\overline{f_i}][\eta_k]}, \end{aligned}$$

thus proving (II.16). Equation (II.17) can be proved similarly. As for (II.18) we have

$$\begin{aligned} \text{lhs of (II.18)} &= \sum_i x_i\beta_F S(y_i)\alpha_F z_i \\ &= \sum_i x_i f_j\beta S(f^j)S(y_i)S(\overline{f_k})\alpha\overline{f^k}z_i \\ &= \sum_i x_i f_j\beta S(\overline{f_k}y_i f^j)\alpha\overline{f^k}z_i\times(-1)^{[y_i][f_j]+[\overline{f_k}])+[\overline{f_k}][f_j]}, \end{aligned}$$

$$\begin{aligned} \text{rhs of (II.18)} &= m\cdot(m\otimes 1)\cdot(1\otimes\beta\otimes\alpha)(1\otimes S\otimes 1) \\ &\quad \cdot\sum_i [x_i f_j\otimes\overline{f_k}y_i f^j\otimes\overline{f^k}z_i]\times(-1)^{[y_i]([f_j]+[\overline{f_k}])+[\overline{f_k}][f_j]}, \end{aligned}$$

$$=\sum_i x_i f_j \beta S(\overline{f_k} y_i f^j)\alpha \overline{f^k} z_i \times (-1)^{[y_i]([f_j]+[\overline{f_k}])+[\overline{f_k}][f_j]},$$

where we have used the fact that the element F is even. Equation (II.19) is proved similarly.

Now let us prove the property (II.7) for α_F and Δ_F. We write, following Sweedler,

$$\Delta(a)=\sum_{(a)} a_{(1)}\otimes a_{(2)}. \tag{II.21}$$

Then, in view of Lemma 1,

$$\begin{aligned}
m\cdot(1\otimes\alpha_F)(S\otimes 1)\Delta_F(a) &= m\cdot(1\otimes\alpha)(S\otimes 1)(F^{-1}\Delta_F(a)\\
&= m\cdot(1\otimes\alpha)(S\otimes 1)(\Delta(a)F^{-1})\\
&= m\cdot(1\otimes\alpha)\sum_{(a)} (S(a_{(1)}\overline{f_i})\otimes a_{(2)}\overline{f^i})\times(-1)^{[\overline{f_i}][a_{(2)}]}\\
&= S(\overline{f_i})\sum_{(a)} S(a_{(1)})\alpha a_{(2)}\overline{f^i}\times(-1)^{[f_i]([a_{(1)}]+[a_{(2)}])}\\
&= S(\overline{f_i})\sum_{(a)} S(a_{(1)})\alpha a_{(2)}\overline{f^i}\times(-1)^{[f_i][a]}\\
&= (-1)^{[\overline{f_i}][a]}S(\overline{f_i})\sum_{(a)} S(a_{(1)})\alpha a_{(2)}\overline{f^i} \overset{\text{(II.17)}}{=} S(\overline{f_i})\epsilon(a)\alpha\overline{f^i}\times(-1)^{[\overline{f_i}][a]}\\
&= S(\overline{f_i})\epsilon(a)\alpha\overline{f^i} \overset{\text{(II.20)}}{=} \epsilon(a)\alpha_F,
\end{aligned} \tag{II.22}$$

where we have used the fact that

$$\epsilon(a)=0 \quad \text{if } [a]=1. \tag{II.23}$$

The property (II.8) for β_F and Δ_F is proved similarly. We then prove property (II.9), which reads in terms of the twisted objects

$$m\cdot(m\otimes 1)\cdot(1\otimes\beta_F\otimes\alpha_F)(1\otimes S\otimes 1)\Phi_F^{-1}=1. \tag{II.24}$$

Let us write

$$\Phi^{-1}=\sum_\nu \overline{X}_\nu\otimes\overline{Y}_\nu\otimes\overline{Z}_\nu. \tag{II.25}$$

Then, in view of (II.18),

$$\begin{aligned}
&\text{lhs of (II.24)}\\
&= m\cdot(m\otimes 1)\cdot(1\otimes\beta\otimes\alpha)(1\otimes S\otimes 1)[(1\otimes F^{-1})\Phi_F^{-1}(F\otimes 1)]\\
&= m\cdot(m\otimes 1)\cdot(1\otimes\beta\otimes\alpha)(1\otimes S\otimes 1)[(1\otimes\Delta)F\cdot\Phi^{-1}\cdot(\Delta\otimes 1)F^{-1}]\\
&= m\cdot(m\otimes 1)\cdot(1\otimes\beta\otimes\alpha)(1\otimes S\otimes 1)\\
&\quad\cdot\sum_{\nu,(f),(\bar{f})}[f_i\overline{X}_\nu\overline{f}_{j(1)}\otimes f^i_{(1)}\overline{Y}_\nu\overline{f}_{j(2)}\otimes f^i_{(2)}\overline{Z}_\nu\overline{f^j}]\\
&\quad\times(-1)^{([\overline{X}_\nu]+[\overline{f}_{j(1)}])([f^i_{(1)}]+[f^i_{(2)}])+[\overline{Z}_\nu]([\overline{f}_{j(1)}]+[\overline{f}_{j(2)}])+[\overline{Y}_\nu]([\overline{f}_{j(1)}]+[f^i_{(2)}])+[f^i_{(2)}][\overline{f}_{j(2)}]}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu} f_i \overline{X_\nu} \sum_{(\bar{f})} f_{j(1)} \beta S(\overline{f_{j(2)}}) S(\overline{Y_\nu}) \sum_{(f)} S(f^i_{(1)}) \alpha f^i_{(2)} \overline{Z_\nu} \overline{f^j} \\
&\quad \cdot (-1)^{([\overline{X_\nu}]+[\overline{Y_\nu}])([f^i_{(1)}]+[f^i_{(2)}])+([\overline{Y_\nu}]+[\overline{Z_\nu}])([\overline{f_{j(1)}}]+[\overline{f_{j(2)}}])+([f^i_{(1)}]+[f^i_{(2)}])([\overline{f_{j(1)}}]+[\overline{f_{j(2)}}])} \\
&= \sum_{\nu} f_i \overline{X_\nu} \sum_{(f)} f_{j(1)} \beta S(\overline{f_j}(2)) S(\overline{Y_\nu}) \sum_{(f)} S(f^i_{(1)}) \alpha f^i_{(2)} \overline{Z_\nu} \overline{f^j} \\
&\quad \cdot (-1)^{([\overline{X_\nu}]+[Y_\nu])[f^i]+([Y_\nu]+[\overline{Z_\nu}])[\overline{f_j}]+[f^i][f_j]} \\
&= \sum_{\nu} f_i \overline{X_\nu} \cdot (-1)^{([X_\nu]+[Y_\nu])[f^i]+([Y_\nu]+[\overline{Z_\nu}])[f_j]+[f^i][f_j]} \\
&\quad \cdot \sum_{(f)} f_{j(1)} \beta S(f_{j(2)}) S(\overline{Y_\nu}) \sum_{(f)} S(f^i_{(1)}) \alpha f^i_{(2)} \overline{Z_\nu} \overline{f^j} \\
&\overset{\text{(II.7),(II.8)}}{=} \sum_{\nu} f_i \overline{X_\nu} \epsilon(\overline{f_j}) \beta S(\overline{Y_\nu}) \epsilon(f^i) \alpha Z_\nu f^j \\
&\quad \cdot (-1)^{([X_\nu]+[Y_\nu])[f^i]+([Y_\nu]+[Z_\nu])[f_j])+[f^i][f_j]} \\
&\overset{\text{(II.23)}}{=} \sum_{\nu} f_i \overline{X_\nu} \epsilon(\overline{f_j}) \beta S(\overline{Y_\nu}) \epsilon(f^i) \alpha \overline{Z_\nu} \overline{f^j} = m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1) \\
&\quad \cdot [((1 \otimes \epsilon) F \otimes 1) \cdot \Phi^{-1} \cdot ((\epsilon \otimes 1) F^{-1} \otimes 1)] \\
&\overset{\text{(II.12)}}{=} m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1) \Phi^{-1} \overset{\text{(II.9)}}{=} 1.
\end{aligned}
$$

The property (II.10) for the twisted objects, which reads

$$m \cdot (m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha_F \otimes \beta_F)(1 \otimes 1 \otimes S) \Phi_F = 1, \tag{II.26}$$

is proved in a similar way.

Definition 3: A quasi-Hopf superalgebra (A,Δ,ϵ,Φ) *is called quasi-triangular if there exists an invertible homogeneous element* $\mathcal{R} \in A \otimes A$ *such that*

$$\Delta^T(a) \mathcal{R} = \mathcal{R} \Delta(a), \quad \forall a \in A, \tag{II.27}$$

$$(\Delta \otimes 1) \mathcal{R} = \Phi^{-1}_{231} \mathcal{R}_{13} \Phi_{132} \mathcal{R}_{23} \Phi^{-1}_{123}, \tag{II.28}$$

$$(1 \otimes \Delta) \mathcal{R} = \Phi_{312} \mathcal{R}_{13} \Phi^{-1}_{213} \mathcal{R}_{12} \Phi_{123}. \tag{II.29}$$

Throughout, $\Delta^T = T \cdot \Delta$ with T being the graded twist map which is defined, for homogeneous elements $a, b \in A$, by

$$T(a \otimes b) = (-1)^{[a][b]} b \otimes a; \tag{II.30}$$

and Φ_{132}, etc. are derived from $\Phi \equiv \Phi_{123}$ with the help of T:

$$\Phi_{132} = (1 \otimes T) \Phi_{123},$$

$$\Phi_{312} = (T \otimes 1) \Phi_{132} = (T \otimes 1)(1 \otimes T) \Phi_{123},$$

$$\Phi^{-1}_{231} = (1 \otimes T) \Phi^{-1}_{213} = (1 \otimes T)(T \otimes 1) \Phi^{-1}_{123},$$

and so on. We remark that our convention differs from the usual one which employs the inverse permutation on the positions (cf. Ref. 4).

It is easily shown that the properties (II.27)–(II.29) imply the graded Yang–Baxter-type equation,

$$\mathcal{R}_{12}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}=\Phi_{321}^{-1}\mathcal{R}_{23}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}, \tag{II.31}$$

which is referred to as the graded quasi-Yang–Baxter equation, and the counit properties of $\mathcal{R}$:

$$(\epsilon\otimes 1)\mathcal{R}=1=(1\otimes\epsilon)\mathcal{R}. \tag{II.32}$$

Theorem 2: *Denoting by the set $(A,\Delta,\epsilon,\Phi,\mathcal{R})$ a quasi-triangular quasi-Hopf superalgebra, then $(A,\Delta_F,\epsilon,\Phi_F,\mathcal{R}_F)$ is also a quasi-triangular quasi-Hopf superalgebra, with the choice of $\mathcal{R}_F$ given by*

$$\mathcal{R}_F=F^T\mathcal{R}F^{-1}, \tag{II.33}$$

where $F^T=T\cdot F\equiv F_{21}$. Here Δ_F and Φ_F are given by (II.13) and (II.14), respectively.

The proof of this theorem is elementary computation. As an example, let us illustrate the proof of the property (II.28) for Δ_F, $\mathcal{R}_F$, and Φ_F. Applying the homomorphism $T\otimes 1$ to $(\Phi_F^{-1})_{123}$, one obtains

$$\begin{aligned}(\Phi_F^{-1})_{213}&=F_{13}(T\otimes 1)(1\otimes\Delta)F\cdot\Phi_{213}^{-1}\cdot(\Delta^T\otimes 1)F^{-1}\cdot(F^T)_{12}^{-1}\\ &=F_{13}\sum_{(f)}(-1)^{[f^i_{(1)}][f_i]}(f^i_{(1)}\otimes f_i\otimes f^i_{(2)})\Phi_{213}^{-1}(\Delta^T\otimes 1)F^{-1}\cdot(F^T)_{12}^{-1},\end{aligned} \tag{II.34}$$

which gives rise to, by applying the homomorphism $1\otimes T$ to both sides,

$$\begin{aligned}(\Phi_F^{-1})_{231}&=F_{12}\sum_{(f)}(-1)^{([f^i_{(1)}]+[f^i_{(2)}])[f_i]}(f^i_{(1)}\otimes f^i_{(2)}\otimes f_i)\Phi_{231}^{-1}(1\otimes T)(\Delta^T\otimes 1)F^{-1}\cdot(F^T)_{13}^{-1}\\ &=F_{12}(\Delta\otimes 1)F^T\cdot\Phi_{231}^{-1}(1\otimes T)(\Delta^T\otimes 1)F^{-1}\cdot(F^T)_{13}^{-1}.\end{aligned} \tag{II.35}$$

Then,

$$\begin{aligned}(\Delta_F\otimes 1)\mathcal{R}_F&=(F\otimes 1)(\Delta\otimes 1)\mathcal{R}_F\cdot(F^{-1}\otimes 1)\\ &=F_{12}(\Delta\otimes 1)(F^T\mathcal{R}F^{-1})\cdot F_{12}^{-1}\\ &=F_{12}(\Delta\otimes 1)F^T(\Delta\otimes 1)\mathcal{R}(\Delta\otimes 1)F^{-1}\cdot F_{12}^{-1}\\ &\overset{(II.28)}{=}F_{12}(\Delta\otimes 1)F^T\cdot\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}(\Delta\otimes 1)F^{-1}\cdot F_{12}^{-1}\\ &\overset{(II.35)}{=}(\Phi_F^{-1})_{231}(F^T)_{13}(1\otimes T)(\Delta^T\otimes 1)F\cdot\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}(\Delta\otimes 1)F^{-1}\cdot F_{12}^{-1}\\ &\overset{(II.14)}{=}(\Phi_F^{-1})_{231}(F^T)_{13}(1\otimes T)(\Delta^T\otimes 1)F\cdot\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}(1\otimes\Delta)F^{-1}\cdot F_{23}^{-1}(\Phi_F^{-1})_{123}\end{aligned}$$

$$=(\Phi_F^{-1})_{231}(F^T)_{13}(1\otimes T)[(\Delta^T\otimes 1)F\cdot\mathcal{R}_{12}]\cdot\Phi_{132}\mathcal{R}_{23}(1\otimes\Delta)F^{-1}$$
$$\cdot F_{23}^{-1}(\Phi_F^{-1})_{123}$$

$$\overset{(II.27)}{=}(\Phi_F^{-1})_{231}(F^T)_{13}(1\otimes T)[\mathcal{R}_{12}(\Delta\otimes 1)F]\cdot\Phi_{132}(1\otimes\Delta^T)F^{-1}$$
$$\cdot\mathcal{R}_{23}F_{23}^{-1}(\Phi_F^{-1})_{123}$$
$$=(\Phi_F^{-1})_{231}(F^T)_{13}\mathcal{R}_{13}(1\otimes T)[(\Delta\otimes 1)F]\cdot\Phi_{132}(1\otimes\Delta^T)F^{-1}$$
$$\cdot\mathcal{R}_{23}F_{23}^{-1}(\Phi_F^{-1})_{123}$$

$$\overset{(II.33)}{=}(\Phi_F^{-1})_{231}(\mathcal{R}_F)_{13}F_{13}^{-1}(1\otimes T)[(\Delta\otimes 1)F]$$
$$\cdot\Phi_{132}(1\otimes\Delta^T)F^{-1}(F^T)_{23}^{-1}(\mathcal{R}_F)_{23}(\Phi_F^{-1})_{123}$$
$$=(\Phi_F^{-1})_{231}(\mathcal{R}_F)_{13}(1\otimes T)[F_{12}^{-1}(\Delta\otimes 1)F\Phi_{123}(1\otimes\Delta)F^{-1}\cdot F_{23}^{-1}]$$
$$\cdot(\mathcal{R}_F)_{23}(\Phi_F^{-1})_{123}$$

$$\overset{(II.14)}{=}(\Phi_F^{-1})_{231}(\mathcal{R}_F)_{13}(1\otimes T)(\Phi_F)_{123}\cdot(\mathcal{R}_F)_{23}(\Phi_F^{-1})_{123}$$
$$=(\Phi_F^{-1})_{231}(\mathcal{R}_F)_{13}(\Phi_F)_{132}(\mathcal{R}_F)_{23}(\Phi_F^{-1})_{123}. \tag{II.36}$$

Let us now consider the special case that A arises from a normal quasi-triangular Hopf superalgebra via twisting with F. A quasi-triangular Hopf superalgebra is a quasi-triangular quasi-Hopf superalgebra with $\alpha=\beta=1$ and $\Phi=1\otimes 1\otimes 1$. Hence A has the following $\mathbf{Z}_2$ graded quasi-Hopf algebra structure:

$$\begin{gathered}\Delta_F(a)=F\Delta(a)F^{-1},\quad \forall a\in A,\\ \Phi_F=F_{12}\cdot(\Delta\otimes 1)F\cdot(1\otimes\Delta)F^{-1}\cdot F_{23}^{-1},\\ \alpha_F=m\cdot(S\otimes 1)F^{-1},\quad \beta_F=m\cdot(1\otimes S)F,\\ \mathcal{R}_F=F^T\mathcal{R}F^{-1}.\end{gathered}\tag{II.37}$$

The twisting procedure is particularly interesting when the twistor $F\in A\otimes A$ depends on an element $\lambda\in A$, i.e., $F=F(\lambda)$, and is a shifted cocycle in the following sense. Here λ is assumed to depend on one (or possible several) parameters.

Definition 4: A twistor $F(\lambda)$ depending on $\lambda\in A$ is a shifted cocycle if it satisfies the graded shifted cocycle condition:

$$F_{12}(\lambda)\cdot(\Delta\otimes 1)F(\lambda)=F_{23}(\lambda+h^{(1)})\cdot(1\otimes\Delta)F(\lambda), \tag{II.38}$$

where $h^{(1)}=h\otimes 1\otimes 1$ and $h\in A$ is fixed.

Let $(A,\Delta_\lambda,\epsilon,\Phi(\lambda),\mathcal{R}(\lambda))$ be the quasi-triangular quasi-Hopf superalgebra obtained from twisting the quasi-triangular Hopf superalgebra by the twistor $F(\lambda)$. Then we have the following.

Proposition 1: We have

$$\Phi(\lambda)\equiv\Phi_F=F_{23}(\lambda+h^{(1)})F_{23}(\lambda)^{-1}, \tag{II.39}$$

$$\Delta_\lambda(a)^T\mathcal{R}(\lambda)=\mathcal{R}(\lambda)\Delta_\lambda(a), \quad \forall a\in A, \tag{II.40}$$

$$(\Delta_\lambda\otimes 1)\mathcal{R}(\lambda)=\Phi_{231}(\lambda)^{-1}\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda+h^{(1)}), \tag{II.41}$$

$$(1\otimes\Delta_\lambda)\mathcal{R}(\lambda)=\mathcal{R}_{13}(\lambda+h^{(2)})\mathcal{R}_{12}(\lambda)\Phi_{123}(\lambda). \tag{II.42}$$

As a corollary, $\mathcal{R}(\lambda)$ *satisfies the graded dynamical Yang–Baxter equation*

$$\mathcal{R}_{12}(\lambda+h^{(3)})\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda+h^{(1)})=\mathcal{R}_{23}(\lambda)\mathcal{R}_{13}(\lambda+h^{(2)})\mathcal{R}_{12}(\lambda). \tag{II.43}$$

III. QUANTUM SUPERGROUPS

Let $\mathcal{G}$ be a Kac–Moody superalgebra[15,16] with a symmetrizable generalized Cartan matrix $A=(a_{ij})_{i,j,\in I}$. As is well known, a given Kac–Moody superalgebra allows many inequivalent systems of simple roots. A system of simple roots is called distinguished if it has minimal odd roots. Let $\{\alpha_i, i\in I\}$ denote a chosen set of simple roots. Let (,) be a fixed invariant bilinear form on the root space of $\mathcal{G}$. Let $\mathcal{H}$ be the Cartan subalgebra and throughout we identify the dual $\mathcal{H}^*$ with $\mathcal{H}$ via (,). The generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$ is defined from the simple roots by

$$a_{ij}=\begin{cases}\dfrac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}, & \text{if } (\alpha_i,\alpha_i)\neq 0,\\ (\alpha_i,\alpha_j), & \text{if } (\alpha_i,\alpha_i)=0.\end{cases} \tag{III.1}$$

As we mentioned in the previous section, quantum Kac–Moody superalgebras are quasi-triangular quasi-Hopf superalgebras with $\alpha=\beta=1$ and $\Phi=1\otimes1\otimes1$. We shall not give the standard relations obeyed by the simple generators (or Chevalley generators) $\{h_i, e_i, f_i, i\in I\}$ of $U_q(\mathcal{G})$, but mention that for certain types of Dynkin diagrams extra q-Serre relations are needed in the defining relations. We adopt the following graded Hopf algebra structure,

$$\begin{gathered}\Delta(h)=h\otimes1+1\otimes h,\\ \Delta(e_i)=e_i\otimes1+t_i\otimes e_i, \quad \Delta(f_i)=f_i\otimes t_i^{-1}+1\otimes f_i,\\ \epsilon(e_i)=\epsilon(f_i)=\epsilon(h)=0,\\ S(e_i)=-t_i^{-1}e_i, \quad S(f_i)=-f_it_i, \quad S(h)=-h,\end{gathered} \tag{III.2}$$

where $i\in I$, $t_i=q^{h_i}$ and $h\in\mathcal{H}$.

The canonical element $\mathcal{R}$ is called the universal R-matrix of $U_q(\mathcal{G})$, which satisfies the basic properties [e.g., (II.27)–(II.29) with $\Phi=1\otimes1\otimes1$ and (II.32)]

$$\begin{aligned}&\Delta^T(a)\mathcal{R}=\mathcal{R}\Delta(a), \quad \forall a\in U_q(\mathcal{G}),\\ &(\Delta\otimes1)\mathcal{R}=\mathcal{R}_{13}\mathcal{R}_{23},\\ &(1\otimes\Delta)\mathcal{R}=\mathcal{R}_{13}\mathcal{R}_{12},\\ &(\epsilon\otimes1)\mathcal{R}=(1\otimes\epsilon)\mathcal{R}=1,\end{aligned} \tag{III.3}$$

and the graded Yang–Baxter equation [cf. (II.31) with $\Phi=1\otimes1\otimes1$]

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{III.4}$$

The Hopf superalgebra $U_q(\mathcal{G})$ contains two important Hopf subalgebras U_q^+ and U_q^- which are generated by e_i and f_i, respectively. By Drinfeld's quantum double construction, the universal R-matrix $\mathcal{R}$ can be written in the form

$$\mathcal{R}=\left(1\otimes 1+\sum_t a^t\otimes a_t\right)\cdot q^{-\mathcal{T}}, \tag{III.5}$$

where $\{a^t\}\in U_q^+$ and $\{a_t\}\in U_q^-$. The element $\mathcal{T}$ is defined as follows. If the symmetrical Cartan matrix is nondegenerate, then $\mathcal{T}$ is the usual canonical element of $\mathcal{H}\otimes\mathcal{H}$. Let $\{h_l\}$ be a basis of $\mathcal{H}$ and $\{h^l\}$ be its dual basis. Then $\mathcal{T}$ can be written as

$$\mathcal{T}=\sum_l h_l\otimes h^l. \tag{III.6}$$

In the case of a degenerate symmetrical Cartan matrix, we extend the Cartan subalgebra $\mathcal{H}$ by adding some elements to it in such a way that the extended symmetrical Cartan matrix is nondegenerate.[17] Then $\mathcal{T}$ stands for the canonical element of the extended Cartan subalgebra. It still takes the form (III.6), but now $\{h_l\}$ ($\{h^l\}$) is understood to be the (dual) basis of the extended Cartan subalgebra. After such enlargement, one has $h=\Sigma_l(h^l,h)h_l=\Sigma_l(h_l,h)h^l$ for any given h in the enlarged Cartan subalgebra.

For later use, we work out the explicit form of the universal R-matrix for the simplest quantum affine superalgebra $U_q[\widehat{sl(1|1)}]$. This algebra is generated by Chevalley generators $\{e_i,f_i,h_i,d,i=0,1\}$ with e_i, f_i odd, and h_i, d even. Here and throughout d stands for the derivation operator. Let us write $h_i=\alpha_i$. Then we have $h_0=\delta-\varepsilon_1+\delta_1$ and $h_1=\varepsilon_1-\delta_1$, where $\{\varepsilon_1,\delta_1,\delta\}$ satisfy $(\varepsilon_1,\varepsilon_1)=1=-(\delta_1,\delta_1)$, $(\varepsilon_1,\delta_1)=(\delta,\delta)=(\delta,\varepsilon_1)=(\delta,\delta_1)=0$. We extend the Cartan subalgebra by adding to it the element $h_{\rm ex}=\varepsilon_1+\delta_1$. A basis for the enlarged Cartan subalgebra is thus $\{h_{\rm ex},h_0,h_1,d\}$. It is easily shown that the dual basis is $\{h^{\rm ex},h^0,h^1,c\}$, where $h^{\rm ex}=\frac{1}{2}(\varepsilon_1-\delta_1)=\frac{1}{2}h_1$, $h^0=d$, and $h^1=\varepsilon_1+d-\frac{1}{2}(\varepsilon_1-\delta_1)=d+\frac{1}{2}h_{\rm ex}$. As is well known, $U_q[\widehat{sl(1|1)}]$ can also be realized in terms of the Drinfeld generators[18] $\{X_n^\pm,H_n,H_n^{\rm ex},n\in\mathbf{Z},c,d\}$, where $X_n^\pm$ are odd and all other generators are even. The relations satisfied by the Drinfeld generators read[19]

$$\begin{gathered}
[c,a]=[H_0,a]=[d,d]=[H_n,H_m]=[H_n^{\rm ex},H_m^{\rm ex}]=0,\quad \forall a\in U_q[\widehat{sl(1|1)}],\\
q^{H_0^{\rm ex}}X_n^\pm q^{-H_0^{\rm ex}}=q^{\pm 2}X_n^\pm,\\
[d,X_n^\pm]=nX_n^\pm,\quad [d,H_n]=nH_n,\quad [d,H_n^{\rm ex}]=nH_n^{\rm ex},\\
[H_n,H_m^{\rm ex}]=\delta_{n+m,0}\frac{[2n]_q[nc]_q}{n},\\
[H_n^{\rm ex},X_m^\pm]=\pm\frac{[2n]_q}{n}X_{n+m}^\pm q^{\mp|n|c/2},\\
[H_n,X_m^\pm]=0=[X_n^\pm,X_m^\pm],\\
[X_n^+,X_m^-]=\frac{1}{q-q^{-1}}\left(q^{(c/2)(n-m)}\psi_{n+m}^+-q^{-(c/2)(n-m)}\psi_{n+m}^-\right),
\end{gathered} \tag{III.7}$$

where $[x]_q=(q^x-q^{-x})/(q-q^{-1})$, $[a,b]\equiv ab-(-1)^{[a][b]}ba$ denotes the supercommutator, and $\psi_{\pm n}^\pm$ are related to $H_{\pm n}$ by relations

$$\sum_{n\geqslant 0}\psi^{\pm}_{\pm n}z^{\mp n}=q^{\pm H_0}\exp\left(\pm(q-q^{-1})\sum_{n>0}H_{\pm n}z^{\mp n}\right). \tag{III.8}$$

The relationship between the Drinfeld generators and the Chevalley generators is

$$\begin{aligned}&e_1=X_0^+,\quad f_1=X_0^-,\quad h_1=H_0,\quad h_{\rm ex}=H_0^{\rm ex},\\&e_0=X_1^-q^{-H_0},\quad f_0=-q^{H_0}X_{-1}^+,\quad h_0=c-H_0.\end{aligned} \tag{III.9}$$

With the help of the Drinfeld generators, we find the following universal R-matrix,

$$\mathcal{R}=\mathcal{R}'\cdot q^{-\mathcal{T}}, \tag{III.10}$$

where

$$\begin{aligned}\mathcal{T}&=h_{\rm ex}\otimes h^{\rm ex}+h_0\otimes h^0+h_1\otimes h^1+d\otimes c\\&=\tfrac{1}{2}(H_0\otimes h_0^{\rm ex}+H_0^{\rm ex}\otimes H_0)+c\otimes d+d\otimes c,\\&\mathcal{R}'=\mathcal{R}^<\mathcal{R}^0\mathcal{R}^>,\\\mathcal{R}^<&=\prod_{n\geqslant 0}^{\rightarrow}\exp[(q-q^{-1})(q^{-nc/2}X_n^+\otimes q^{nc/2}X_{-n}^-)],\\\mathcal{R}^0&=\exp\left[-(q-q^{-1})\sum_{n=1}^{\infty}\frac{n}{[2n]_q}(H_n\otimes H_{-n}^{\rm ex}+H_n^{\rm ex}\otimes H_{-n})\right],\\\mathcal{R}^>&=\prod_{n\geqslant 0}^{\leftarrow}\exp[-(q-q^{-1})(X_{n+1}^-q^{nc/2-H_0}\otimes q^{-nc/2+H_0}X_{-n-1}^+)].\end{aligned} \tag{III.11}$$

Here and throughout,

$$\prod_{k\geqslant 0}^{\rightarrow}A_k=A_0A_1A_2\cdots,\quad \prod_{k\geqslant 0}^{\leftarrow}A_k=\cdots A_2A_1A_0. \tag{III.12}$$

It seems to us that even for this simplest quantum affine superalgebra $U_q[\widehat{sl(1|1)}]$ the universal R-matrix has not been written down in its explicit form before.

Let us compute the image of $\mathcal{R}$ in the two-dimensional evaluation representation (π,V) of $U_q[\widehat{sl(1|1)}]$, where $V=\mathbf{C}^{1|1}=\mathbf{C}v_1\otimes\mathbf{C}v_2$ with v_1 even and v_2 odd. Let e_{ij} be the 2×2 matrix whose (i,j)-element is unity and zero otherwise. In the homogeneous gradation, the simple generators are represented by

$$\begin{aligned}&e_1=\sqrt{[\theta]_q}e_{12},\quad f_1=\sqrt{[\theta]_q}e_{21},\quad h_1=\theta(e_{11}+e_{22}),\quad h_{\rm ex}=2e_{11}+c_0(e_{11}+e_{22}),\\&e_0=z\sqrt{[\theta]_q}e_{21},\quad f_0=-z^{-1}\sqrt{[\theta]_q}e_{12},\quad h_0=-\theta(e_{11}+e_{22}),\end{aligned} \tag{III.13}$$

where θ and c_0 are arbitrary constants. Then it can be shown that the Drinfeld generators are represented by

$$\begin{aligned}&H_n=z^n\frac{[n\theta]_q}{n}(e_{11}+e_{22}),\quad H_n^{\rm ex}=z^n\frac{[2n]_q}{n}q^{n\theta}e_{11}+z^nc_n(e_{11}+e_{22}),\\&X_n^+=z^nq^{n\theta}\sqrt{[\theta]_q}e_{12},\quad X_n^-=z^nq^{n\theta}\sqrt{[\theta]_q}e_{21},\end{aligned} \tag{III.14}$$

where again c_n are arbitrary constants. In the following we set c_n to be zero. Then the image $R_{VV}(z;\theta,\theta')=(\pi_\theta\otimes\pi_{\theta'})\mathcal{R}$ depends on two extra nonadditive parameters θ, θ', and is given by

$$R_{VV}(z;\theta,\theta')=\frac{q^{-\theta-\theta'}-z}{1-zq^{-\theta-\theta'}}e_{11}\otimes e_{11}+e_{22}\otimes e_{22}+\frac{q^{-\theta'}-zq^{-\theta}}{1-zq^{-\theta-\theta'}}e_{11}\otimes e_{22}+\frac{q^{-\theta}-zq^{-\theta'}}{1-zq^{-\theta-\theta'}}e_{22}\otimes e_{11}$$
$$+\sqrt{[\theta]_q[\theta']_q}q^{-\theta}\frac{q-q^{-1}}{1-zq^{-\theta-\theta'}}e_{12}\otimes e_{21}-\sqrt{[\theta]_q[\theta']_q}q^{-\theta'}\frac{z(q-q^{-1})}{1-zq^{-\theta-\theta'}}e_{21}\otimes e_{12}. \tag{III.15}$$

Equation (III.15) is nothing but the R-matrix obtained in Ref. 20 by solving the Jimbo equation.

IV. ELLIPTIC QUANTUM SUPERGROUPS

Following Jimbo *et al.*,[4] we define elliptic quantum supergroups to be quasi-triangular quasi-Hopf superalgebras obtained from twisting the normal quantum supergroups (which are quasi-triangular quasi-Hopf superalgebras with $\alpha=\beta=1$ and $\Phi=1\otimes1\otimes1$) by twistors which satisfy the graded shifted cocycle condition.

A. Elliptic quantum supergroups of face type

Let ρ be an element in the (extended) Cartan subalgebra such that $(\rho,\alpha_i)=(\alpha_i,\alpha_i)/2$ for all $i\in I$, and

$$\phi=\mathrm{Ad}\,(q^{(1/2)\Sigma_l h_l h^l-\rho}), \tag{IV.1}$$

be an automorphism of $U_q(\mathcal{G})$. Here $\{h_l\}$ and $\{h^l\}$ are as in (III.6) and are the dual basis of the (extended) Cartan subalgebra. Namely,

$$\phi(e_i)=e_it_i,\quad \phi(f_i)=t_i^{-1}f_i,\quad \phi(q^h)=q^h. \tag{IV.2}$$

In the following we consider the special case in which the element λ introduced before belongs to the (extended) Cartan subalgebra. Let

$$\phi_\lambda=\phi^2\cdot\mathrm{Ad}\,(q^{2\lambda})=\mathrm{Ad}\,(q^{\Sigma_l h_l h^l-2\rho+2\lambda}) \tag{IV.3}$$

be an automorphism depending on the element λ and $\mathcal{R}$ be the universal R-matrix of $U_q(\mathcal{G})$. Following Jimbo *et al.*,[4] we define a twistor $F(\lambda)$ by the infinite product

$$F(\lambda)=\overleftarrow{\prod_{k\geq1}}(\phi_\lambda^k\otimes1)(q^{\mathcal{T}}\mathcal{R})^{-1}. \tag{IV.4}$$

It is easily seen that $F(\lambda)$ is a formal series in parameter(s) in λ with leading term 1. Therefore the infinite product makes sense. The twistor $F(\lambda)$ is referred to as a face-type twistor. It can be shown that $F(\lambda)$ satisfies the graded shifted cocycle condition

$$F_{12}(\lambda)(\Delta\otimes1)F(\lambda)=F_{23}(\lambda+h^{(1)})(1\otimes\Delta)F(\lambda), \tag{IV.5}$$

where, if $\lambda=\Sigma_l\lambda_lh^l$, then $\lambda+h^{(1)}=\Sigma_l(\lambda_l+h_l^{(1)})h^l$. The proof of (IV.5) is identical to the non-super case given by Jimbo *et al.*,[4] apart from the use of the graded tensor products. Moreover, it is easily seen that $F(\lambda)$ obeys the counit property

$$(\epsilon\otimes1)F(\lambda)=(1\otimes\epsilon)F(\lambda)=1. \tag{IV.6}$$

We have the following definition.

Definition 5 (Face-type elliptic quantum supergroup): We define elliptic quantum supergroup $\mathcal{B}_{q,\lambda}(\mathcal{G})$ *of face type to be the quasi-triangular quasi-Hopf superalgebra* $(\mathrm{U}_q(\mathcal{G}),\Delta_\lambda,\epsilon,\Phi(\lambda),\mathcal{R}(\lambda))$ *together with the graded algebra anti-homomorphism S defined by (III.2) and* $\alpha_\lambda = m\cdot(S\otimes 1)F(\lambda)^{-1}$, $\beta_\lambda = m\cdot(1\otimes S)F(\lambda)$. *Here* ϵ *is defined by (III.2), and*

$$\begin{aligned}\Delta_\lambda(a)&=F(\lambda)\Delta(a)F(\lambda)^{-1},\quad \forall a\in \mathrm{U}_q(\mathcal{G}),\\ \mathcal{R}(\lambda)&=F(\lambda)^T\mathcal{R}F(\lambda)^{-1},\\ \Phi(\lambda)&=F_{23}(\lambda+h^{(1)})F_{23}(\lambda)^{-1}.\end{aligned}\tag{IV.7}$$

We now consider the particularly interesting case where $\mathcal{G}$ is of affine type. Then ρ contains two parts,

$$\rho=\bar{\rho}+gd,\tag{IV.8}$$

where $g=(\psi,\psi+2\bar{\rho})/2$, $\bar{\rho}$ is the graded half-sum of positive roots of the nonaffine part $\bar{\mathcal{G}}$, and ψ is highest root of $\mathcal{G}$; d is the derivation operator which gives the homogeneous gradation

$$[d,e_i]=\delta_{i0}e_i,\quad [d,f_i]=-\delta_{i0}f_i,\quad i\in I.\tag{IV.9}$$

We also set

$$\lambda=\bar{\lambda}+(r+g)d+s'c,\quad r,s'\in \mathbf{C},\tag{IV.10}$$

where $\bar{\lambda}$ stands for the projection of λ onto the (extended) Cartan subalgebra of $\bar{\mathcal{G}}$. Denoting by $\{\bar{h}_j\}$ and $\{\bar{h}^j\}$ the dual basis of the (extended) Cartan subalgebra of $\bar{\mathcal{G}}$ and setting $p=q^{2r}$, we can decompose ϕ_λ into two parts,

$$\phi_\lambda=\mathrm{Ad}\,(p^dq^{2cd})\cdot\bar{\phi}_\lambda,\quad \bar{\phi}_\lambda=\mathrm{Ad}\,(q^{\Sigma_j\bar{h}_j\bar{h}^j+2(\bar{\lambda}-\bar{\rho})}).\tag{IV.11}$$

Introduce a formal parameter z (which will be identified with spectral parameter) into $\mathcal{R}$ and $F(\lambda)$ by setting

$$\begin{aligned}\mathcal{R}(z)&=\mathrm{Ad}\,(z^d\otimes 1)\mathcal{R},\\ F(z,\lambda)&=\mathrm{Ad}\,(z^d\otimes 1)F(\lambda),\\ \mathcal{R}(z,\lambda)&=\mathrm{Ad}\,(z^d\otimes 1)\mathcal{R}(\lambda)=F(z^{-1},\lambda)^T\mathcal{R}(z)F(z,\lambda)^{-1}.\end{aligned}\tag{IV.12}$$

Then it can be shown from the definition of $F(\lambda)$ that $F(z,\lambda)$ satisfies the difference equation

$$\begin{aligned}F(pq^{2c^{(1)}}z,\lambda)&=(\bar{\phi}_\lambda\otimes 1)^{-1}(F(z,\lambda))\cdot q^{\mathcal{T}}\mathcal{R}(pq^{2c^{(1)}}z),\\ F(0,\lambda)&=F_{\bar{\mathcal{G}}}(\bar{\lambda}).\end{aligned}\tag{IV.13}$$

The initial condition follows from the fact that $\mathcal{R}(z)q^{d\otimes c+c\otimes d}|_{z=0}$ reduces to the universal R-matrix of $\mathrm{U}_q(\bar{\mathcal{G}})$.

Let us give some examples.

1. The case $\mathcal{B}q,\lambda[\mathrm{sl}(1|1)]$

In this case the universal R-matrix is given simply by

$$\begin{aligned}\mathcal{R}&=\exp[(q-q^{-1})e\otimes f]q^{-\mathcal{T}}=[1+(q-q^{-1})e\otimes f]q^{-\mathcal{T}},\\ \mathcal{T}&=\tfrac{1}{2}(h\otimes h_{\mathrm{ex}}+h_{\mathrm{ex}}\otimes h).\end{aligned}\tag{IV.14}$$

Let us write

$$\lambda=(s'+1)\tfrac{1}{2}h+s\tfrac{1}{2}h_{\mathrm{ex}},\quad s',s\in\mathbf{C}. \tag{IV.15}$$

Since h commutes with everything, ϕ_λ is independent of s'. Setting $w=q^{2(s+h)}$, we have

$$\phi_\lambda=\mathrm{Ad}(w^{1/2h_{\mathrm{ex}}}). \tag{IV.16}$$

The formula for the twistor becomes

$$\begin{aligned}F(w)&=\prod_{k\geqslant 1}(1-(q-q^{-1})w^k q^{-h}e\otimes fq^h)\\&=1-(q-q^{-1})\sum_{k=1}^{\infty}w^k q^{-h}e\otimes fq^h\\&=1-(q-q^{-1})\frac{w}{1-w}q^{-h}e\otimes fq^h.\end{aligned} \tag{IV.17}$$

2. The case $\mathcal{B}_{q,\lambda}[\mathrm{sl}(\hat{1}|1)$

Taking a basis $\{c,d,h,h_{\mathrm{ex}}\}$ of the enlarged Cartan subalgebra of $\mathrm{sl}(\widehat{1}|1)$, we write

$$\lambda=rd+s'c+(s''+1)\tfrac{1}{2}h+s\tfrac{1}{2}h_{\mathrm{ex}},\quad r,s',s'',s\in\mathbf{C}. \tag{IV.18}$$

Then ϕ_λ is independent of s' and s''. Set

$$p=q^{2r},\quad w=q^{2(s+h)}. \tag{IV.19}$$

Set $F(z;p,w)\equiv F(z,\lambda)$. Then (IV.13) take the form

$$F(pq^{2c^{(1)}}z;p,w)=(\overline{\phi_w}^{-1}\otimes 1)(F(z;p,w))\cdot q^{\mathcal{T}}\mathcal{R}(pq^{2c^{(1)}}z), \tag{IV.20}$$

$$F(0;p,w)=F_{\mathrm{sl}(1|1)}(w), \tag{IV.21}$$

where $\overline{\phi_w}=\mathrm{Ad}(w^{(1/2h_{\mathrm{ex}}})$.

The image of (IV.20) in the two-dimensional representation (π,V) given by (III.13) (by setting $\theta=1$) yields a difference equation for $F_{VV}(z;p,w)=(\pi\otimes\pi)F(z;p,w)$. Noting that $\pi\cdot\overline{\phi_w}=\mathrm{Ad}\,(D_w^{-1})\cdot\pi$, where $D_w=e_{11}+we_{22}$, we find

$$F_{VV}(pz;p,w)=\mathrm{Ad}\,(D_w\otimes 1)(F_{VV}(z;p,w))\cdot KR_{VV}(pz), \tag{IV.22}$$

where $K=(\pi\otimes\pi)q^{\mathcal{T}}=q^2e_{11}\otimes e_{11}+qe_{11}\otimes e_{22}+qe_{22}\otimes e_{11}+e_{22}\otimes e_{22}$ and $R_{VV}(pz)$ is given by (III.15) (with $\theta=\theta'=1$). Equation (IV.22) is a system of difference equations of q-KZ (Kaizhnik–Zamolodchikov) equation type,[21] and can be solved with the help of the q-hypergeometric series. The solution with the initial condition (IV.21) is given by

$$\begin{aligned}F_{VV}(z;p,w)=&{}_1\phi_0(z;p,w)e_{11}\otimes e_{11}+e_{22}\otimes e_{22}+f_{11}(z;p,w)e_{11}\otimes e_{22}+f_{22}(z;p,w)e_{22}\otimes e_{11}\\&+f_{12}(z;p,w)e_{12}\otimes e_{21}+f_{21}(z;p,w)e_{21}\otimes e_{12},\end{aligned} \tag{IV.23}$$

where

$${}_1\phi_0(z;p,w)=\frac{(pq^{-2}z;p)_\infty}{(pq^2z;p)_\infty},$$

$$
\begin{aligned}
f_{11}(z;p,w) &= {}_2\phi_1\left(\begin{matrix} wq^{-2} & q^{-2} \\ & w \end{matrix} ;p,pq^2 z\right), \\
f_{12}(z;p,w) &= -\frac{w(q-q^{-1})}{1-w}\,{}_2\phi_1\left(\begin{matrix} wq^{-2} & pq^{-2} \\ & pw \end{matrix} ;p,pq^2 z\right), \\
f_{21}(z;p,w) &= \frac{zpw^{-1}(q-q^{-1})}{1-pw^{-1}}\,{}_2\phi_1\left(\begin{matrix} pw^{-1}q^{-2} & pq^{-2} \\ & p^2 w^{-1} \end{matrix} ;p,pq^2 z\right), \\
f_{22}(z;p,w) &= {}_2\phi_1\left(\begin{matrix} pw^{-1}q^{-2} & q^{-2} \\ & pw^{-1} \end{matrix} ;p,pq^2 z\right).
\end{aligned}
\tag{IV.24}
$$

Here

$$
\begin{aligned}
2\phi 1\left(\begin{matrix} q^a & q^b \\ & q^c \end{matrix} ;p,x\right) &= \sum_{n=0}^{\infty} \frac{(q^a;p)_n (q^b;p)_n}{(p;p)_n (q^c;p)_n} x^n, \\
(a;p)_n &= \prod_{k=0}^{n-1} (1-ap^k), \quad (a;p)_0 = 1.
\end{aligned}
\tag{IV.25}
$$

B. Elliptic quantum supergroups of vertex type

As we mentioned before, a given Kac–Moody superalgebras $\mathcal{G}$ allows many inequivalent simple root systems. By means of the ''extended'' Weyl transformation method introduced in Ref. 22, one can transform from one simple root system to another inequivalent one.[23] For $\mathcal{G} = \mathrm{sl}(\widehat{n|n})$, there exists a simple root system in which all simple roots are odd (or fermionic). This system can be constructed from the distinguished simple root system by using the ''extended'' Weyl operation repeatedly. We find the following simple roots, all of which are odd (or fermionic),

$$
\begin{aligned}
\alpha_0 &= \delta - \varepsilon_1 + \delta_n, \\
\alpha_{2j} &= \delta_j - \varepsilon_{j+1}, \quad j = 1,2,\ldots,n-1, \\
\alpha_{2i-1} &= \varepsilon_i - \delta_i, \quad i = 1,2,\ldots,n
\end{aligned}
\tag{IV.26}
$$

with δ, $\{\varepsilon_i\}_{i=1}^n$ and $\{\delta_i\}_{i=1}^n$ satisfying

$$
\begin{aligned}
(\delta,\delta) = (\delta,\varepsilon_i) = (\delta,\delta_i) = 0, &\quad (\varepsilon_i,\varepsilon_j) = \delta_{ij}, \\
(\delta_i,\delta_j) = -\delta_{ij}, &\quad (\varepsilon_i,\delta_j) = 0.
\end{aligned}
\tag{IV.27}
$$

Such a simple root system is usually called nonstandard. It seems to us that $\mathrm{sl}(\widehat{n|n})$ is the only nontwisted affine superalgebra which has a nonstandard system of simple roots, all of which are fermionic.

As will be shown below, for $\mathcal{G}=\mathrm{sl}(\widehat{n|n})$ with the above fermionic simple roots, one can construct a different type of twistor. Following Jimbo *et al.*,[4] we say this twistor is of *vertex type*.

Let us write $h_i=\alpha_i\,(i=0,1,...,2n-1)$ with α_i given by (IV.26). We extend the Cartan subalgebra of $\mathrm{sl}(\widehat{n|n})$ by adding to it the element $h_{\mathrm{ex}}=\Sigma_{i=1}^{n}(\varepsilon_i+\delta_i)$. A basis of the extended Cartan subalgebra is $\{h_{\mathrm{ex}},h_0,h_1,\dots,h_{2n-1},d\}$. Denote by $\{h^{\mathrm{ex}},h^0,h^1,\dots,h^{2n-1},c\}$ the dual basis. We have

$$h^{\mathrm{ex}}=\frac{1}{2n}\sum_{i=1}^{n}(\varepsilon_i-\delta_i),$$

$$h^{2k}=d+\sum_{i=1}^{k}(\varepsilon_i-\delta_i)-\frac{k}{n}\sum_{i=1}^{n}(\varepsilon_i-\delta_i), \tag{IV.28}$$

$$h^{2k+1}=d+\sum_{i=1}^{k+1}\varepsilon_i-\sum_{i=1}^{k}\delta_i-\frac{2k+1}{2n}\sum_{i=1}^{k}(\varepsilon_i-\delta_i),$$

where $k=0,1,...,n-1$. The canonical element $\mathcal{T}$ in the extended Cartan subalgebra reads

$$\mathcal{T}=h_{\mathrm{ex}}\otimes h^{\mathrm{ex}}+\sum_{i=0}^{2n-1}(h_i\otimes h^i)+d\otimes c. \tag{IV.29}$$

Let τ be the diagram automorphism of $U_q[\mathrm{sl}(\widehat{n|n})]$ such that

$$\tau(e_i)=e_{i+1 \bmod 2n},\quad \tau(f_i)=f_{i+1 \bmod 2n},\quad \tau(h_i)=h_{i+1 \bmod 2n}. \tag{IV.30}$$

Obviously, the automorphism τ is nongraded since it preserves the grading of the generators and, moreover, $\tau^{2n}=1$. Then we can show

$$\tau(h_{\mathrm{ex}})=-h_{\mathrm{ex}}+\xi c,\quad \tau(c)=c,\quad \tau(h^{\mathrm{ex}})=-h^{\mathrm{ex}}+\frac{1}{2n}c,$$

$$\tau(h^{2k})=h^{2k+1 \bmod 2n}+\frac{\xi}{2n}\sum_{i=1}^{n}(\varepsilon_i-\delta_i)-\frac{\xi+n-2k-1}{2n}c, \tag{IV.31}$$

$$\tau(h^{2k+1})=h^{2k+2 \bmod 2n}+\frac{\xi}{2n}\sum_{i=1}^{n}(\varepsilon_i-\delta_i)-\frac{n-2k-1}{2n}c,$$

where $k=0,1,...,n-1$ and ξ is an arbitrary constant. Introduce element

$$\tilde{\rho}=\sum_{i=0}^{2n-1}h^i+\xi n h^{\mathrm{ex}}, \tag{IV.32}$$

which gives the principal gradation

$$[\tilde{\rho},e_i]=e_i,\quad [\tilde{\rho},f_i]=-f_i,\quad i=0,1,...,2n-1. \tag{IV.33}$$

It is easily shown that

$$\tau(\tilde{\rho})=\tilde{\rho},\quad (\tau\otimes\tau)\mathcal{T}=\mathcal{T}. \tag{IV.34}$$

Notice also that

$$(\tau\otimes\tau)\cdot\Delta=\Delta\cdot\tau, \tag{IV.35}$$

$$(\tau\otimes\tau)\mathcal{R}=\mathcal{R}.$$

Here the second relation is deduced from the uniqueness of the universal R-matrix of $U_q[\widehat{sl(n|n)}]$. It can be shown that

$$\sum_{k=1}^{2n}(\tau^k\otimes 1)\mathcal{T}=\tilde{\rho}\otimes c+c\otimes\tilde{\rho}-\frac{2(n^2-1)-3\xi}{6}c\otimes c. \tag{IV.36}$$

Therefore, if we set

$$\tilde{\mathcal{T}}=\frac{1}{2n}\left(\tilde{\rho}\otimes c+c\otimes\tilde{\rho}-\frac{2(n^2-1)-3\xi}{6}c\otimes c\right), \tag{IV.37}$$

then we have

$$\sum_{k=1}^{2n}(\tau^k\otimes 1)(\mathcal{T}-\tilde{\mathcal{T}})=0. \tag{IV.38}$$

Introduce an automorphism

$$\tilde{\phi}_r=\tau\cdot \mathrm{Ad}(q^{[(r+c)/n]\tilde{\rho}}), \tag{IV.39}$$

which depends on a parameter $r\in\mathbf{C}$. Then the $2n$-fold product

$$\overleftarrow{\prod_{2n\geqslant k\geqslant 1}}(\tilde{\phi}_r^k\otimes 1)(q^{\tilde{\mathcal{T}}}\mathcal{R})^{-1}. \tag{IV.40}$$

is a formal power series in $p^{1/2n}$ where $p=q^{2r}$. Moreover, it has leading term 1 thanks to the relation (IV.38). Following Jimbo *et al.*,[4] we define the vertex-type twistor

$$E(r)=\lim_{N\to\infty}\overleftarrow{\prod_{2nN\geqslant k\geqslant 1}}(\tilde{\phi}_r^k\otimes 1)(q^{\tilde{\mathcal{T}}}\mathcal{R})^{-1}. \tag{IV.41}$$

Then one can show that $E(r)$ satisfies the graded shifted cocycle condition

$$E_{12}(r)(\Delta\otimes 1)E(r)=E_{23}(r+c^{(1)})(1\otimes\Delta)E(r). \tag{IV.42}$$

Moreover, $E(r)$ obeys the counit property

$$(\epsilon\otimes 1)E(r)=(1\otimes\epsilon)E(r)=1. \tag{IV.43}$$

We have the following.

Definition 6 (Vertex-type elliptic quantum supergroup): We define elliptic quantum supergroup $\mathcal{A}_{q,p}[\widehat{sl(n|n)}]$ *of vertex type to be the quasi-triangular quasi-Hopf superalgebra* $(U_q[\widehat{sl(n|n)}],\Delta_r,\epsilon,\Phi(r),\mathcal{R}(r))$ *together with the graded algebra anti-homomorphism S defined by (III.2) and* $\alpha_r=m\cdot(S\otimes 1)E(r)^{-1}$, $\beta_r=m\cdot(1\otimes S)E(r)$. *Here* ϵ *is defined by (III.2), and*

$$\begin{aligned}&\Delta_r(a)=E(r)\Delta(a)E(r)^{-1},\quad \forall a\in U_q[\widehat{sl(n|n)}],\\ &\mathcal{R}(r)=E(r)^T\mathcal{R}E(r)^{-1},\\ &\Phi(r)=E_{23}(r+c^{(1)})E_{23}(r)^{-1}.\end{aligned} \tag{IV.44}$$

Similar to the face-type case, introduce a formal parameter ζ (or spectral parameter) into $\mathcal{R}$ and $E(r)$ by the formulas

$$\begin{aligned}\tilde{\mathcal{R}}(\zeta)&=\mathrm{Ad}\,(\zeta^{\tilde{\rho}}\otimes 1)\mathcal{R},\\ E(\zeta,r)&=\mathrm{Ad}\,(\zeta^{\tilde{\rho}}\otimes 1)E(r),\\ \tilde{\mathcal{R}}(\zeta,r)&=\mathrm{Ad}\,(\zeta^{\tilde{\rho}}\otimes 1)\mathcal{R}(r)=E(\zeta^{-1},r)^T\tilde{\mathcal{R}}(\zeta)E(\zeta,r)^{-1}.\end{aligned}\tag{IV.45}$$

Then it can be shown from the definition of $E(r)$ that $E(\zeta,r)$ satisfies the difference equation

$$E(p^{1/2n}q^{(1/n)c^{(1)}}\zeta,r)=(\tau\otimes 1)^{-1}(E(\zeta,r))\cdot q^{\tilde{T}}\tilde{\mathcal{R}}(p^{1/2n}q^{(1/n)c^{(1)}}\zeta),\tag{IV.46}$$

$$E(0,r)=1.\tag{IV.47}$$

The initial condition follows from (IV.38) and the fact that we are working in the principal gradation. Equation (IV.46) implies that

$$E((p^{1/2n}q^{(1/n)c^{(1)}})^{2n}\zeta,r)=E(\zeta,r))\cdot\overleftarrow{\prod_{2n-1\geqslant k\geqslant 0}}q^{\tilde{T}}(\tau\otimes 1)^{2n-k}\tilde{\mathcal{R}}((p^{1/2n}q^{(1/n)c^{(1)}})^{2n-k}\zeta).\tag{IV.48}$$

Some remarks are in order. In nonsuper case,[4] π and τ are commutable in the sense that $\pi\cdot\tau=\mathrm{Ad}(h)\cdot\pi$ with h obeying $hv_i=v_{i+1 \bmod m}$, where $\{v_i\}$ are basis of the vector module $V=\mathbf{C}^m=\mathbf{C}v_1\oplus\cdots\oplus\mathbf{C}c_m$ of $\mathcal{A}_{q,p}(\hat{\mathrm{sl}}_m)$ and τ is the cyclic diagram automorphism of $\hat{\mathrm{sl}}_m$. In the super (or $\mathbf{Z}_2$ graded) case, however, π and τ are not ''commutable'' in the above sense. This is because τ is grading preserving while the $2n$-dimensional defining representation space $V=\mathbf{C}^{n|n}=\mathbf{C}v_1\oplus\cdots\oplus\mathbf{C}v_{2n}$ is graded. So to compute the image, one has to work out the action of τ at the universal level and then apply the representation π. Therefore, the knowledge of the universal R-matrix in its explicit form is required. This makes the image computation of the twistor more involved in the supersymmetric case.

As an example, consider the simplest case of elliptic quantum affine superalgebra $\mathcal{A}_{q,p}[\mathrm{sl}\widehat{(1}|1)]$. Let us calculate the image in the two-dimensional representation (π,V), $V=\mathbf{C}^{1|1}$. As remarked above, we have to work at the universal level first and then apply the representation. We have the following.

Lemma 2: In the principal gradation, the action of τ on the Drinfeld generators is represented on V by

$$\begin{aligned}&\tau(X_n^+)=(-1)^n z^{2n+1}q^{-n}e_{12},\quad \tau(X_n^-)=(-1)^{n+1}z^{2n-1}q^{-n}e_{21},\\ &\tau(H_n)=(-1)^{n+1}z^{2n}\frac{[n]_q}{n}(e_{11}+e_{22}),\\ &\tau(H_n^{\mathrm{ex}})=(-1)^{n+1}z^{2n}\frac{[2n]_q}{n}\left(q^{-n}e_{11}+\frac{q-q^{-1}}{2}[n]_q(e_{11}+e_{22})\right).\end{aligned}\tag{IV.49}$$

Applying $\pi\otimes\pi$ to the both side of (IV.48) and writing $E_{VV}(\zeta;p)\equiv(\pi\otimes\pi)E(\zeta,r)$, where $p=q^{2r}$, we get

$$E_{VV}(p\zeta;p)=E_{VV}(\zeta;p)\cdot(\pi\otimes\pi)((\tau\otimes 1)\tilde{\mathcal{R}}(p^{1/2}\zeta))\cdot\tilde{\mathcal{R}}_{VV}(p\zeta),\tag{IV.50}$$

where $\tilde{R}_{VV}(\zeta)=(\pi\otimes\pi)\tilde{R}(\zeta)$. In view of (IV.49) and the explicit formula (III.11) of the universal R-matrix, (IV.50) is a system of eight difference equations.

We can also proceed directly. We have, with the help of Lemma 2,

$$(\pi\otimes\pi)(\tau^{2k}\otimes 1)(\mathrm{Ad}\,(p^k\zeta)^{\tilde{\rho}}\otimes 1)\mathcal{R}^{-1}q^{-\tilde{\mathcal{T}}}=K\cdot\overline{E}_{2k},$$
$$(\pi\otimes\pi)(\tau^{2k-1}\otimes 1)(\mathrm{Ad}\,(p^{k-1/2}\zeta)^{\tilde{\rho}}\otimes 1)\mathcal{R}^{-1}q^{-\tilde{\mathcal{T}}}=\rho_{2k-1}\cdot K^{-1}\cdot\overline{E}_{2k-1}, \tag{IV.51}$$

where $K=(\pi\otimes\pi)q^{\mathcal{T}}$ and

$$\rho_{2k-1}=\frac{(1+q^2p^{2k-1}\zeta^2)(1+q^{-2}p^{2k-1}\zeta^2)}{(1+p^{2k-1}\zeta^2)^2},$$

$$\begin{aligned}\overline{E}_{2k}=\frac{1}{1-q^2p^{2k}\zeta^2}((1-q^{-2}p^{2k}\zeta^2)e_{11}\otimes e_{11}+(1-q^2p^{2k}\zeta^2)e_{22}\otimes e_{22}\\ +(1-p^{2k}\zeta^2)e_{11}\otimes e_{22}+(1-p^{2k}\zeta^2)e_{22}\otimes e_{11}\\ -(q-q^{-1})p^k\zeta e_{12}\otimes e_{21}+(q-q^{-1})p^k\zeta e_{21}\otimes e_{12}),\end{aligned} \tag{IV.52}$$

$$\begin{aligned}\overline{E}_{2k-1}=\frac{1}{1+q^{-2}p^{2k-1}\zeta^2}((1+q^2p^{2k-1}\zeta^2)e_{11}\otimes e_{11}+(1+q^{-2}p^{2k-1}\zeta^2)e_{22}\otimes e_{22}\\ +(1+p^{2k-1}\zeta^2)e_{11}\otimes e_{22}+(1+p^{2k-1}\zeta^2)e_{22}\otimes e_{11}\\ +(q-q^{-1})p^{k-1/2}\zeta e_{12}\otimes e_{21}-(q-q^{-1})p^{k-1/2}\zeta e_{21}\otimes e_{12}).\end{aligned} \tag{IV.53}$$

Then

$$E_{VV}(\zeta;p)=\overleftarrow{\prod_{k\geqslant 1}}\rho_{2k-1}K\overline{E}_{2k}K^{-1}\overline{E}_{2k-1}=\rho(\zeta;p)(E^1_{VV}(\zeta;p)+E^2_{VV}(\zeta;p)), \tag{IV.54}$$

where

$$\rho(\zeta;p)=\frac{(-pq^2\zeta^2;p^2)_\infty}{(pq\zeta;p)_\infty(-pq\zeta;p)_\infty}, \tag{IV.55}$$

$$\begin{aligned}E^1_{VV}(\zeta;p)=\overleftarrow{\prod_{k\geqslant 1}}\frac{1}{(1+p^{2k-1}\zeta^2)^2}((1-q^{-2}p^{2k}\zeta^2)(1+q^2p^{2k-1}\zeta^2)e_{11}\otimes e_{11}\\ +(1-q^2p^{2k}\zeta^2)(1+q^{-2}p^{2k-1}\zeta^2)e_{22}\otimes e_{22}\\ +(q-q^{-1})p^{k-1/2}\zeta(1-q^{-2}p^{2k}\zeta^2)e_{12}\otimes e_{12}\\ -(q-q^{-1})p^{k-1/2}\zeta(1-q^2p^{2k}\zeta^2)e_{21}\otimes e_{21}),\end{aligned} \tag{IV.56}$$

$$\begin{aligned}E^2_{VV}(\zeta;p)=\overleftarrow{\prod_{k\geqslant 1}}\frac{1}{1+p^{2k-1}\zeta^2}((1-p^{2k}\zeta^2)e_{11}\otimes e_{22}+(1-p^{2k}\zeta^2)e_{22}\otimes e_{11}\\ -(q-q^{-1})p^k\zeta e_{12}\otimes e_{21}+(q-q^{-1})p^k\zeta e_{21}\otimes e_{12}).\end{aligned} \tag{IV.57}$$

The infinite product in $E^2_{VV}(\zeta;p)$ can be calculated directly and we find

$$E^2_{VV}(\zeta;p)=b_E(\zeta)(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})+c_E(\zeta)(e_{12}\otimes e_{21}-e_{21}\otimes e_{12}), \tag{IV.58}$$

where

$$b_E(\zeta)\pm c_E(\zeta)=\frac{(pq^{\pm 1}\zeta;p)_\infty(-pq^{\mp 1}\zeta;p)_\infty}{(-p\zeta^2;p^2)_\infty}. \tag{IV.59}$$

As for $E^1_{VV}(\zeta;p)$, it can be written as

$$E^1_{VV}(\zeta;p)=X_{11}(\zeta;p)e_{11}\otimes e_{11}+X_{22}(\zeta;p)e_{22}\otimes e_{22} \\ +X_{12}(\zeta;p)e_{12}\otimes e_{12}+X_{21}(\zeta;p)e_{21}\otimes e_{21}, \tag{IV.60}$$

where $X_{ij}(\zeta;p)$ are the solution to the following system of four difference equations:

$$\begin{aligned}
X_{11}(p\zeta;p)&=\frac{1}{1-q^{-2}p^2\zeta^2}((1+q^{-2}p\zeta^2)X_{11}(\zeta;p)-p^{1/2}\zeta(q-q^{-1})X_{12}(\zeta;p)),\\
X_{12}(p\zeta;p)&=\frac{1}{1-q^{2}p^2\zeta^2}(-p^{1/2}\zeta(q-q^{-1})X_{11}(\zeta;p)+(1+q^{2}p\zeta^2)X_{12}(\zeta;p)),\\
X_{21}(p\zeta;p)&=\frac{1}{1-q^{-2}p^2\zeta^2}(p^{1/2}\zeta(q-q^{-1})X_{22}(\zeta;p)+(1+q^{-2}p\zeta^2)X_{21}(\zeta;p)),\\
X_{22}(p\zeta;p)&=\frac{1}{1-q^{2}p^2\zeta^2}((1+q^{2}p\zeta^2)X_{22}(\zeta;p)+p^{1/2}\zeta(q-q^{-1})X_{21}(\zeta;p)).
\end{aligned} \tag{IV.61}$$

ACKNOWLEDGMENTS

Our interest in quasi-Hopf algebras was ignited by Bo-Yu Hou's beautiful lecture on the Jimbo *et al.* paper[4] when YZZ was visiting Northwest University, Xi'an, in December 1997. YZZ thanks Bo-Yu Hou for interesting him in the subject and for helpful discussions. The financial support from the Australian Research Council through a Queen Elizabeth II Fellowship Grant for YZZ is also gratefully acknowledged.

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