Bell's inequality test with entangled atoms

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Previous work on Bell's inequality realized in the laboratory has used entangled photons. Here we describe how entangled atoms can violate Bell's inequality, and how these violations can be measured with a very high detection efficiency. We first discuss a simple scheme based on two-level atoms inside a cavity to prepare the entangled state. We then discuss a scheme using three-level atoms, which requires a parameter regime much easier to access experimentally using current technology. As opposed to other schemes, our proposal relies on the presence of finite decay rates and its implementation should therefore be much less demanding.

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I. INTRODUCTION

Bell's inequalities have a central role in tests of quantum mechanics and relate to the degree of entanglement between subsystems, an essential resource in quantum information processing. There are a number of Bell inequalities for two subsystems where each subsystem contains a qubit of information. For example, there exist the original *spin* [1], Clauser-Horne (CH) [2], Clauser-Horne-Shimony-Holt (CHSH) [3], and information theoretic [4] Bell inequalities, to name but a few. The particular one considered generally depends on the system under consideration. A scheme may violate one Bell inequality but not another. Recently an overview of Bell's inequalities has been given by Peres [5].

A number of experimental tests of Bell's inequality have already been performed [6–12] using entangled *photons*. In this paper we propose an experimental test of Bell's inequality on two macroscopically separated *atoms*. Each atom possesses a two-level system with the states $|0\rangle$ and $|1\rangle$. We describe a scheme which allows us to prepare the atoms in an arbitrary superposition of a maximally entangled state and a product state which is of the form

$$|\varphi\rangle = \frac{\alpha}{\sqrt{2}}(|10\rangle - |01\rangle) + \sqrt{1-|\alpha|^2}|00\rangle \tag{1}$$

in a deterministic way. To do so we make use of the recently proposed idea by Beige *et al.* [13] of how to manipulate the decoherence-free states of N atoms inside a cavity. Together with the control over the prepared state, which can be obtained by following a measurement proposal by Cook [14,15] based on ''electron shelving,'' this allows us to investigate, characterize, and test Bell's inequality with a very high precision and detection efficiency.

The success rate for the preparation of the initial atomic state (1) will be denoted by P_0. If a photon is emitted in the preparation, the scheme fails. If these events are not detected and ignored this leads to a decrease of the observed violation of Bell's inequality. On the other hand, if the scheme succeeds the fidelity of the prepared state is very close to unity. Therefore we estimate that Bell's inequality is violated as long as the preparation probability exceeds 71%, if the scheme is intended to prepare the atoms in the maximally entangled state. In this paper we determine P_0 and show that it can, in principle, be arbitrarily close to unity.

Other tests using atoms or ions have been proposed [16–20]. For instance, an experiment based on the proposal by Cirac and Zoller [16] to entangle two atoms in a cavity has been performed by Hagley *et al.* [21]. Four trapped ions, respectively, have been entangled experimentally in a deterministic fashion by Sackett *et al.* [22] following a proposal by Mølmer and Sørensen [23]. But a test of Bell's inequality using atoms has yet to be realized. The main limiting factor in these experiments is *dissipation* [21,22]. As opposed to this, the scheme proposed here is based on the presence of finite decay rates and should therefore be less demanding experimentally.

The investigation we are examining here is not strictly a strong [24] test of quantum mechanics versus local realism due to the limited spatial separation of the atoms. For a strict test the scheme would require separating the two atoms by a distance larger than the speed of light times the measurement time. However this atom based experiment closes the detection inefficiency loophole while the photon experiments close the causality loopholes [11]. In the scheme we propose, the observable which is expected to violate Bell's inequality is measured in *each* run of the experiment and the state of the two atoms can be determined with almost unity efficiency and a very high precision [15]. Hence this proposed experiment should be seen as complementary to the photon experiments.

The paper is organized as follows. We begin in Sec. II with a description of a simple scheme based on two two-level atoms inside a cavity that can be used to generate the entangled state (1). We describe the single qubit rotation and a way to measure the state of the atoms. The required parameter regime is, however, experimentally demanding. Therefore, in Sec. III a scheme is introduced based on two three-level atoms. This system behaves exactly like in the two-level case described above and the discussion in Sec. II is used to obtain the same results. In Sec. IV we discuss how to

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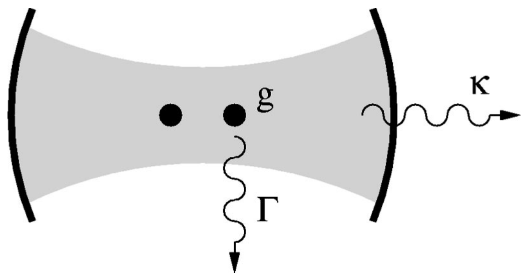

FIG. 1. Experimental setup for the preparation of state (1). The system consists of two two-level atoms placed at fixed positions inside a cavity. Each atom couples to the cavity mode with a constant g and its spontaneous decay rate is given by Γ. The rate κ corresponds to the leakage of photons through the cavity mirrors.

test Bell's inequality and for which parameters a violation of the inequality is expected. A final discussion of the results can be found in Sec. V.

II. A SIMPLE SCHEME USING TWO-LEVEL ATOMS

To prepare two two-level atoms in the entangled state (1) they are placed at fixed positions in a cavity which acts as a resonator for an electromagnetic field. The atoms (or ions) can be stored in the nodes of a standing light field or in a linear trap. In the following $|0\rangle_i$ denotes the ground state and $|1\rangle_i$ the excited state of atom i, respectively, and we assume that the cavity field is in resonance with the atomic transition. We also assume that the coupling constant of each atom with the cavity field is the same and given by g, which can be chosen to be real. The cavity should be nonideal; that is, a photon can leak out with a rate κ as shown in Fig. 1. The spontaneous decay rate of each atom equals Γ. The distance between the atoms inside the cavity should be much larger than an optical wavelength. This allows us to address each atom individually with a laser pulse. The Rabi frequency for atom i will be denoted by $\Omega^{(i)}$ and is in general complex, because we have already chosen g to be real.

To test Bell's inequality the atoms have to be moved out of the cavity. This can be done by moving the optical lattice or by applying an electric field, respectively, if the atoms are inside a linear ion trap. Another possibility is to let the two atoms fly together through the cavity field during each run of the experiment.

In the experiment we propose, the probability for spontaneous emission of a photon or leakage of a photon through the cavity mirrors will be shown to be small. This immediately suggests that we use the quantum jump approach [25–28]. This method leads to a *conditional* Hamiltonian $H_{\rm cond}$ which gives the time evolution of the system under the condition of no photon emissions. Due to the non-Hermiticity of $H_{\rm cond}$, the norm of the state vector

$$|\psi^0(t)\rangle = e^{-iH_{\rm cond}t/\hbar}|\psi_0\rangle \tag{2}$$

decreases with time and the probability P_0 for *no* photon emission up to time t is given by the squared norm

$$P_0(t) = \| \, |\psi^0(t)\rangle \|^2. \tag{3}$$

If no photon is emitted, the state of the system at time t is the state (2) normalized to unity.

A. The preparation of the entangled state

To prepare the atoms in state (1) we will take advantage of the fact that two-level atoms inside a cavity possess *trapped* states [29–32] which can also be used to obtain an example of a decoherence-free subspace [13,33–35]. If the atoms are in a trapped state they cannot transfer excitation into the resonator field, even if upper levels are populated. Therefore, if the cavity field is empty and spontaneous emission can be neglected no photon can be emitted by the system and the system is in a *decoherence-free* state.

To find the decoherence-free states of the system let us first assume that the two atoms are inside the cavity, but *no* laser field is applied. We choose the interaction picture in a way that the atoms and the cavity mode plus environment are considered as the free system. Then the conditional Hamiltonian equals, as in Ref. [13,29],

$$H_{\rm cond} = i\hbar\, g \sum_{i=1}^{2} (b|1\rangle_{ii}\langle 0| - \text{H.c.}) - i\hbar\,\Gamma \sum_{i=1}^{2} |2\rangle_{ii}\langle 2| - i\hbar\kappa b^\dagger b, \tag{4}$$

where the operator b is the annihilation operator for photons in the cavity mode.

Decoherence-free states arise if no interaction between the system and its environment of free radiation fields takes place. If we neglect spontaneous emissions ($\Gamma=0$) this is exactly the case if the cavity mode is empty [13] and it is $|\psi\rangle = |0\rangle\otimes|\varphi\rangle \equiv |0\varphi\rangle$. In addition, the systems own time evolution due to the interaction between the atoms and the cavity mode should not move the state of the system out of the decoherence-free subspace. Using Eq. (4) this leads as in Ref. [13] to the condition

$$\sum_{i=1}^{2} |0\rangle_{ii}\langle 1|\varphi\rangle = 0, \tag{5}$$

where $|\varphi\rangle$ is the state of the atoms only. From this condition we find that the decoherence-free states are the superpositions of the two atomic states $|g\rangle \equiv |00\rangle$ and

$$|a\rangle \equiv (|10\rangle - |01\rangle)/\sqrt{2} \tag{6}$$

while the cavity mode is empty.

Once prepared in a decoherence-free state the state of the system does not change in time with respect to the chosen interaction picture. The reason for this is $H_{\rm cond}|\psi\rangle = 0$ which can be shown by using Eqs. (4) and (5).

To prepare the atoms in state (1) a weak laser pulse can be used. As in Ref. [13] we assume in the following $\Omega^{(1)} \neq \Omega^{(2)}$ and for all nonvanishing Rabi frequencies

$$\Gamma \ll |\Omega^{(i)}| \ll g \quad \text{and} \quad \kappa \sim g. \tag{7}$$

This corresponds to a strong coupling between the atoms and the cavity mode, while g and κ are of the same order of magnitude. In this parameter regime we can make use of an effect which can easily be understood in terms of the quantum Zeno effect [36–38]. The reason for this is that the entangled state given in Eq. (1) corresponds to a decoherence-free state. We assume now that the system is initially in its ground state which is also decoherence free. If now rapidly repeated measurements are performed on the system of whether the state of the system still belongs to the decoherence-free subspace or not, the laser interaction cannot move the state of the system out of this subspace. Only a time evolution inside the subspace is possible. Hence the laser pulse can introduce entanglement into the system which is not possible in the free atom case. Equivalently we can interpret this inhibition without invoking Zeno effects as a simple consequence of adiabatic elimination using the separation of the frequency scales in Eq. (7) [13].

Let us define ΔT as the time in which a photon leaks out through the cavity mirrors with a probability very close to unity if the system is initially prepared in a state with no overlap with a decoherence-free state. On the other hand, a system in a decoherence-free state will definitely not emit a photon in ΔT. Therefore the observation of the free radiation field over a time interval ΔT can be interpreted as a measurement of whether the system is decoherence free or not [39]. The outcome of the measurement is indicated by an emission or no emission of a photon. This interpretation also holds to a very good approximation in the presence of the laser field because the effect of the laser over a time interval ΔT can be neglected, which is why condition (7) has been chosen. As has been shown in Ref. [39], ΔT is of the order $1/\kappa$ and κ/g^2 and much smaller than $1/|\Omega^{(\pm)}|$,

$$\Omega^{(\pm)} \equiv (\Omega^{(1)} \pm \Omega^{(2)})/\sqrt{2}, \tag{8}$$

the typical time scale for the laser interaction. Here the system continuously interacts with its environment and the system behaves in a very good approximation like a system under continuous observation whose time evolution can easily be predicted with the help of the quantum Zeno effect [36].

Using the measurement interpretation one can easily show that the effect of the laser field on the atomic states can be described by the *effective* Hamiltonian $H_{\rm eff}$ which equals [13]

$$H_{\rm eff} = \mathbb{P}_{\rm DFS} H_{\rm cond} \mathbb{P}_{\rm DFS} \tag{9}$$

and where $\mathbb{P}_{\rm DFS}$ is the projector on the decoherence-free subspace. To obtain the conditional Hamiltonian of the system in the presence of the laser field the Hamiltonian

$$H_{\rm laser\ I} = \frac{\hbar}{2} \sum_{i=1}^{2} (\Omega^{(i)} |1\rangle_{ii}\langle 0| + \text{H.c.}) \tag{10}$$

has to be added to the right-hand side of Eq. (4). If we neglect spontaneous emission $(\Gamma=0)$ this leads to

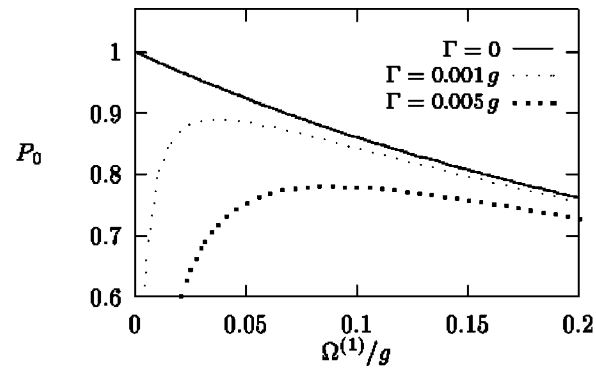

FIG. 2. The probability for no photon emission during the preparation of the maximally entangled state for different Rabi frequencies $\Omega^{(1)}$ and $\Omega^{(2)} = -\Omega^{(1)}$, different spontaneous decay rates Γ and $\kappa = g$.

$$H_{\rm eff} = \frac{\hbar}{2} (\Omega^{(-)} |0a\rangle\langle 0g| + \text{H.c.}). \tag{11}$$

By solving the corresponding time evolution, one finds that a laser pulse of length T prepares the atoms in the state given in Eq. (1) with

$$\alpha = -i \frac{\Omega^{(-)}}{|\Omega^{(-)}|} \sin\left(\frac{|\Omega^{(-)}| T}{2} \right). \tag{12}$$

Varying the length of the laser pulse allowsus to change arbitrarily the value of $|\alpha|$ and the amount of entanglement in the system.

The Hamiltonian in Eq. (11) is Hermitian. Therefore the norm of a vector developing with $H_{\rm eff}$ is not decreasing and in a first approximation, due to Eq. (3), the emission of photons can be neglected. To a very good approximation the cavity mode never does become populated and the success rate of the preparation scheme P_0 equals unity.

Figure 2 shows the probability for no photon emission during the state preparation resulting from a numerical solution of the conditional time evolution of the system using Eqs. (3), (4), and (10). This agrees very well with the approximative results given above. As an example, we assumed

$$T = \pi/|\Omega^{(-)}|, \tag{13}$$

which leads, due to Eq. (12), to the preparation of the maximally entangled state of both atoms. In addition we assumed $\Omega^{(2)} = -\Omega^{(1)}$ [40]. As expected, for $\Gamma=0$ the success rate of the preparation scheme can at least in principle be arbitrarily close to 1. For $\Gamma \neq 0$ the probability P_0 reaches a maximum value for a certain Rabi frequency $\Omega^{(1)}$, but is always smaller than 1. To improve the experiment one can surround the cavity by detectors and repeat it were a decay photon to be registered.

We also determined the state of the atoms at the end of the laser pulse numerically. The fidelity of the prepared state F in case of no photon emission is given by the overlap of the state of Eq. (2) after normalization with the state given in Eq. (1). For the parameters chosen in Fig. 2, F is found to be always higher than 95%.

B. Realization of a single qubit rotation

In this section we describe how the single qubit rotation on atom i, defined by the operator $U_{\rm rot}^{(i)}$,

$$U_{\rm rot}^{(i)}(\xi,\phi)\equiv\cos\xi - i\sin\xi(e^{i\phi}|0\rangle_{ii}\langle 1|+\text{H.c.}), \qquad (14)$$

can be realized, where ξ and ϕ are arbitrary parameters. Thereby the same laser as in Sec. II A can be used. To avoid the situation that the time evolution of the system is restricted to changes inside the decoherence-free subspace, the atom should be moved out of the cavity.

If we neglect again spontaneous emission $(\Gamma=0)$, the laser Hamiltonian which describes the time evolution of atom i is given by

$$H_{\text{laser I}}=\frac{\hbar}{2}(\Omega^{(i)}|1\rangle_{ii}\langle 0|+\text{H.c.}). \qquad (15)$$

Calculating the corresponding time evolution operator for a laser pulse length T leads to Eq. (14) with

$$\xi=\frac{|\Omega^{(i)}|T}{2} \quad \text{and} \quad e^{i\phi}=\frac{\Omega^{(i)}}{|\Omega^{(i)}|}. \qquad (16)$$

To change the phase ϕ, the phase of the Rabi frequency $\Omega^{(i)}$ has to be chosen very carefully, while ξ can easily be varied by varying the length T of the pulse.

Again, for $\Gamma\neq 0$ a photon may be emitted spontaneously during the single qubit rotation which leads to a failure of the experiment and therefore to a further decrease of the success rate of the scheme to test Bell's inequality proposed here.

C. State measurement on a single atom

Whether an atom i is in state $|0\rangle_i$ or $|1\rangle_i$ can be measured with a very high precision following a proposal by Cook [14]. To do this, we make use of a short strong laser pulse and an auxiliary level 2. The probe pulse couples one of the states, for instance the state $|0\rangle_i$ to state $|2\rangle_i$, and has the Rabi frequency Ω_2. The spontaneous decay rate of the auxiliary level is Γ_2. If the length of the laser pulse T fulfills a minimum length,

$$T\gg\max\{1/\Gamma_2,\Gamma_2/\Omega_2^2\}, \qquad (17)$$

the absence or occurrence of photons from the 0-2 transition indicates whether the atom is found in state $|0\rangle_i$ or $|1\rangle_i$, respectively. If the system is initially prepared in level 0 photons are emitted until the end of the pulse. If the atom is in $|1\rangle_i$ the laser has no effect on the atomic state and no photon emissions will occur. For an arbitrary state of the atom

$$|\varphi\rangle=\alpha_0|0\rangle_i+\alpha_1|1\rangle_i \qquad (18)$$

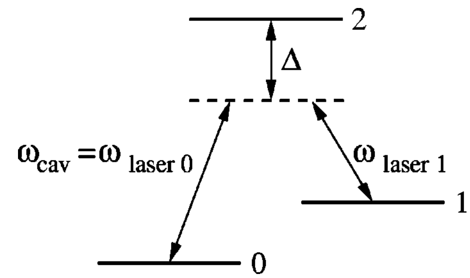

FIG. 3. Atomic three-level scheme. The cavity mode and laser 0 couple to the 0-2 transition of the atom with the same detuning Δ laser 1 has with respect to the 1-2 transition.

it has been shown by Beige and Hegerfeldt [15] that photons are emitted with probability $|\alpha_0|^2$ as predicted for an ideal measurement. The proposition for this scheme to work is that the laser pulse is long enough that an atom initially in state $|0\rangle_i$ emits definitively a photon which leads to condition (18). As discussed in Ref. [15] the precision of this measurement can be very high, even if the efficiency of the detectors measuring the photons from the 0-2 transition is very low. The population difference between the two levels is given by

$$\langle\sigma_z^{(i)}\rangle=1-2|\alpha_0|^2 \qquad (19)$$

averaged over many runs.

III. AN IMPROVED SCHEME USING THREE-LEVEL ATOMS

To observe a violation of Bell's inequality the preparation of the maximally entangled state $|a\rangle$ should succeed with a probability above 71% in each run of the experiment. For this, as can be seen in Fig. 2, the coupling constant g has to be at least 100 times larger than the spontaneous decay rate Γ. This is difficult to achieve experimentally using optical frequencies, and has only been realized in microcavities with circular Rydberg atoms coupled to a microwave cavity [41].

In the following we describe how this problem can be circumvented easily by making use of an additional atomic level. It is known that certain three-level atoms behave in a good approximation like the two-level atoms described in Sec. II C. We will show that they possess the same decoherence-free states and again a weak laser pulse can be used to create entanglement between the atoms. We describe how to perform a single qubit rotation and how to measure the state of an atom.

A. The preparation of the entangled state

We consider now two three-level atoms with a Λ configuration as shown in Fig. 3. The states $|0\rangle_i$ and $|1\rangle_i$ are the ground states of atom i and couple to an excited state denoted by $|2\rangle_i$. To prepare the atoms in state (1) they have to be moved into a cavity as described in Sec. II. In the following $\hbar\omega_i$ denotes the energy of level i. The frequency $\omega_{\rm cav}$ of the single cavity mode to which the atoms are coupled equals

$$\omega_{\rm cav}=\omega_2-\omega_0-\Delta, \qquad (20)$$

where Δ denotes a detuning. A laser field with the same detuning and frequency

$$\omega_{\text{laser }1} = \omega_2 - \omega_1 - \Delta \tag{21}$$

excites the 1-2 transition of each atom with a Rabi frequency Ω_1. In addition, at time $t=0$ a laser pulse with the frequency $\omega_{\text{laser }0} = \omega_{\text{cav}}$ is applied which couples to the 0-2 transition in atom i with a Rabi frequency $\Omega_0^{(i)}$.

In the following we assume again that the coupling constant g of each atom to the cavity mode is the same for both atoms. The Rabi frequencies Ω_1 and $\Omega_0^{(i)}$ are chosen to be much smaller than g, while g and the spontaneous decay rate of the excited state Γ are much smaller than the detuning Δ, such that

$$|\Omega_0^{(i)}| = |\Omega_1| \ll g \ll \Delta \quad \text{and} \quad \Gamma \ll \Delta. \tag{22}$$

We will see later that the decay rate κ of photons inside the cavity should fulfill the condition

$$\kappa \sim g \cdot \Omega_1 / \Delta \tag{23}$$

and is therefore now much smaller than g.

To describe the time evolution of the system under the condition of no photon emission we make again use of the quantum jump approach [25]. We chose the interaction picture with respect to the sum of the atomic Hamiltonian

$$H_0 = \hbar \sum_{i=1}^{2} \sum_{j=0}^{2} (\omega_j |j\rangle_{ii}\langle j| - \Delta |2\rangle_{ii}\langle 2|) \tag{24}$$

and the Hamiltonian describing the energy of the cavity mode and the free radiation fields forming the environment of the system. Then the conditional Hamiltonian is time independent and given by

$$\begin{aligned} H_{\text{cond}} = {} & i\hbar g \sum_{i=1}^{2} (b|2\rangle_{ii}\langle 0| - \text{H.c.}) \\ & + \frac{\hbar}{2} \sum_{i=1}^{2} (\Omega_0^{(i)} |2\rangle_{ii}\langle 0| + \Omega_1 |2\rangle_{ii}\langle 1| + \text{H.c.}) \\ & - i\hbar(\Gamma + i\Delta) \sum_{i=1}^{2} |2\rangle_{ii}\langle 2| - i\hbar \kappa b^\dagger b. \end{aligned} \tag{25}$$

The (unnormalized) state of the system $|\psi^0\rangle$ defined in Eq. (2) will be written in the following as

$$|\psi^0\rangle = \sum_{n=0}^{\infty} \sum_{j_1, j_2 = 0}^{2} c_{n\, j_1 j_2} |n j_1 j_2\rangle. \tag{26}$$

Due to the parameter choice (22) there are very different time scales in the (conditional) time evolution of the system. We first investigate the coefficients $c_{0 j_1 j_2}$ that change on the very short time scale proportional to $1/\Delta$. If we assume that only states with $n=0$ are populated, we find from Eq. (25)

$$\begin{aligned} \dot{c}_{002} &= -\frac{i}{2}(\Omega_0^{(1)*} c_{022} + \Omega_0^{(2)} c_{000} + \Omega_1 c_{001}) - (\Gamma + i\Delta) c_{002}, \\ \dot{c}_{020} &= -\frac{i}{2}(\Omega_0^{(1)} c_{000} + \Omega_0^{(2)*} c_{022} + \Omega_1 c_{010}) - (\Gamma + i\Delta) c_{020}, \\ \dot{c}_{012} &= -\frac{i}{2}(\Omega_0^{(2)} c_{010} + \Omega_1 c_{011} + \Omega_1^* c_{022}) - (\Gamma + i\Delta) c_{012}, \\ \dot{c}_{021} &= -\frac{i}{2}(\Omega_0^{(1)} c_{001} + \Omega_1 c_{011} + \Omega_1^* c_{022}) - (\Gamma + i\Delta) c_{021}, \\ \dot{c}_{022} &= -\frac{i}{2}(\Omega_0^{(1)} c_{002} + \Omega_0^{(2)} c_{020} + \Omega_1 c_{012} + \Omega_1 c_{021}) \\ & \quad - 2(\Gamma + i\Delta) c_{022}. \end{aligned} \tag{27}$$

Because we are only interested in the time evolution of the system on the a time scale much longer than $1/\Delta$ we can adiabatically eliminate level 2 by eliminating the fast varying coefficients. They adapt essentially immediately to the state of the other levels and we can set their derivatives in Eq. (27) to zero to obtain

$$\begin{aligned} c_{002} &= -\frac{1}{2\Delta}(\Omega_0^{(2)} c_{000} + \Omega_1 c_{001}), \\ c_{020} &= -\frac{1}{2\Delta}(\Omega_0^{(1)} c_{000} + \Omega_1 c_{010}), \\ c_{012} &= -\frac{1}{2\Delta}(\Omega_0^{(2)} c_{010} + \Omega_1 c_{011}), \\ c_{021} &= -\frac{1}{2\Delta}(\Omega_0^{(1)} c_{001} + \Omega_1 c_{011}), \\ c_{022} &= 0, \end{aligned} \tag{28}$$

where all terms that are due to Eq. (22) much smaller than $|\Omega_1|/\Delta$ have been neglected.

To determine the decoherence-free states of the system we assume now that the weak laser field is not applied and set $\Omega_0^{(1)} = \Omega_0^{(2)} = 0$. For the same reasons as in the Sec. II the system is only decoherence-free if there are no photons inside the cavity. In addition, the cavity mode should never become populated [13]. The derivatives of all coefficients with $n=1$ have to vanish, if initially only states with $n=0$ are populated. In this case we have, from Eqs. (25) and (28),

$$\begin{aligned} \dot{c}_{100} &= g \frac{\Omega_1}{2\Delta}(c_{001} + c_{010}), \\ \dot{c}_{101} &= \dot{c}_{110} = g \frac{\Omega_1}{2\Delta} c_{011}, \\ \dot{c}_{111} &= 0. \end{aligned} \tag{29}$$

The system is therefore in a decoherence-free state if and only if

$$c_{001}+c_{010}=c_{011}\equiv 0 \tag{30}$$

and the decoherence-free states are the same as in Sec. II—the superpositions of the two states $|0g\rangle$ and $|0a\rangle$.

To prepare the atoms in the entangled state (1) the same idea as in Sec. II can be used, because the three-level atoms considered here behave to a very good approximation like the two-level atoms discussed in Sec. II if conditions (22) and (23) are fulfilled. Despite the values of the precise frequencies the differential equations (29) are exactly the same as one obtains from Eq. (4) by neglecting spontaneous emission by the two-level atoms. A comparison of both sets of differential equations gives the value of the constant $g_{\rm eff}$,

$$g_{\rm eff}=-g\Omega_1/(2\Delta), \tag{31}$$

that describes the effective coupling strength of the three-level atoms to the cavity mode. To determine the effective Rabi frequencies $\Omega_{\rm eff}^{(i)}$ of the weak laser pulse we calculate the derivatives of the coefficients c_{000}, c_{001}, c_{010}, and c_{011}. If the cavity mode is not populated Eqs. (22), (25), and (28) lead to

$$\dot{c}_{000}=\frac{i}{4\Delta}(\Omega_0^{(1)*}\Omega_1 c_{010}+\Omega_0^{(2)*}\Omega_1 c_{001}+2|\Omega_1|^2 c_{000}),$$

$$\dot{c}_{001}=\frac{i}{4\Delta}(\Omega_0^{(1)*}\Omega_1 c_{011}+\Omega_0^{(2)}\Omega_1^* c_{000}+2|\Omega_1|^2 c_{001}),$$

$$\dot{c}_{010}=\frac{i}{4\Delta}(\Omega_0^{(1)}\Omega_1^* c_{000}+\Omega_0^{(2)*}\Omega_1 c_{011}+2|\Omega_1|^2 c_{010}),$$

$$\dot{c}_{011}=\frac{i}{4\Delta}(\Omega_0^{(1)}\Omega_1^* c_{001}+\Omega_0^{(2)*}\Omega_1^* c_{010}+2|\Omega_1|^2 c_{011}). \tag{32}$$

Except for the last term in each equation, the differential equations (32) are exactly the same as one obtains from Eq. (4) neglecting spontaneous emission. A comparison with Eq. (32) gives the effective Rabi frequency $\Omega_{\rm eff}^{(i)}$,

$$\Omega_{\rm eff}^{(i)}=-\Omega_0^{(i)}\Omega_1^*/(2\Delta). \tag{33}$$

Here we can neglect the terms proportional $|\Omega_1|^2$ in Eq. (32) because they correspond to a level shift of the states $|0\rangle_i$ and $|1\rangle_i$, which is the same for all states and causes therefore only an overall phase factor of the prepared state. This factor does not affect the outcome of the Bell measurement described in Sec. IV. The effective spontaneous decay rate $\Gamma_{\rm eff}$ of the atoms inside the cavity can be bounded from above using Eq. (28) and we find

$$\Gamma_{\rm eff}<\Gamma|\Omega_1|^2/(2\Delta^2), \tag{34}$$

which is much smaller than $g_{\rm eff}$ even if Γ and g are of the same order of magnitude. The rate κ does not change due to the presence of level 2 and it is $\kappa_{\rm eff}=\kappa$.

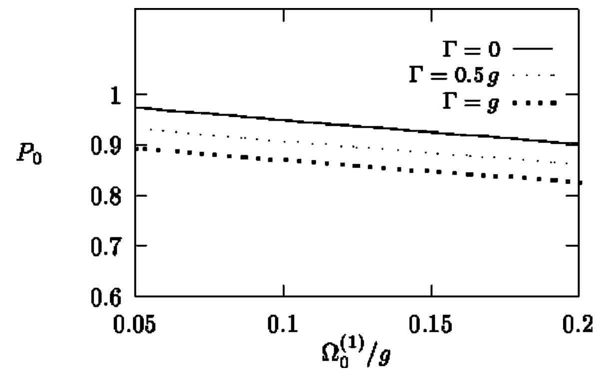

FIG. 4. The probability for no photon emission during the preparation of the maximally entangled state for different Rabi frequencies $\Omega_0^{(1)}$ and $\Omega_1=-\Omega_0^{(2)}=\Omega_0^{(1)}$, $\Delta=50g$, different spontaneous decay rates Γ, and $\kappa=g_{\rm eff}$.

By analogy to Sec. II, Eq. (7), we assume now

$$\Gamma_{\rm eff}\ll|\Omega_{\rm eff}^{(i)}|\ll g_{\rm eff} \quad \text{and} \quad \kappa_{\rm eff}\sim g_{\rm eff}, \tag{35}$$

which leads to the conditions (22) and (23). If this condition is fulfilled we expect that the weak laser pulse with the Rabi frequencies $\Omega_0^{(i)}$ does not move the system out of the decoherence-free subspace, if the system is initially in the ground state $|000\rangle$ and its effect can again be described by the effective Hamiltonian $H_{\rm eff}$ given in Eq. (11). One only has to replace in Eq. (33) the Rabi frequencies $\Omega_0^{(i)}$ by $\Omega_{\rm eff}^{(i)}$ to obtain $\Omega^{(-)}$.

Figure 4 shows the probability for no photon emission for a laser pulse of the length

$$T=2\sqrt{2}\pi\Delta/|\Omega_1(\Omega_0^{(1)}-\Omega_0^{(2)})| \tag{36}$$

obtained from a numerical integration of Eq. (2). Due to Eqs. (13) and (33), the laser field prepares the atoms in the maximally entangled state $|a\rangle$. The fidelity is found to be always higher than 87% for the parameters chosen in Fig. 4. As expected, the success rate of the scheme can now be very close to unity even if the spontaneous decay rate Γ is of the same order of magnitude as the coupling constant g. The results are the better the larger the detuning Δ becomes compared to Γ and g.

B. Realization of a single qubit rotation

To perform the single qubit rotation on atom i we propose to make use of an adiabatic population transfer [42,43]. In order to do this, the atom has to be moved out of the cavity. Then the same two lasers as in Sec. III A with the Rabi frequencies $\Omega_0^{(i)}$ and Ω_1, respectively, are applied simultaneously on atom i. With respect to the interaction picture defined in Eq. (24) the conditional Hamiltonian is now given by

$$H_{\rm cond}=\frac{\hbar}{2}(\Omega_0^{(i)}|2\rangle_{ii}\langle 0|+\Omega_1|2\rangle_{ii}\langle 1|+\text{H.c.}) -i\hbar(\Gamma+i\Delta)|2\rangle_{ii}\langle 2|. \tag{37}$$

Equation (22) allows us again to eliminate adiabatically level 2. Neglecting all terms of higher order in $1/\Delta$ we find that the system can effectively be described by the Hamiltonian [43]

$$H=-\frac{\hbar}{4}\left[\left(\frac{\Omega_0^{(i)*}\Omega_1}{\Delta}|0\rangle_{ii}\langle 1|+\text{H.c.}\right)+\frac{|\Omega_1|^2}{\Delta}(|0\rangle_{ii}\langle 0|+|1\rangle_{ii}\langle 1|)\right], \tag{38}$$

where Eq. (22) has been used. This Hamiltonian does not depend on Γ and spontaneous emission can again be neglected to a very good approximation. The time evolution operator corresponds up to a total phase factor with the operator given in Eq. (14) and we have

$$U(T,0)=\exp\left(i\frac{|\Omega_1|^2T}{4\Delta}\right)U_{\text{rot}}^{(i)}(\xi,\phi) \tag{39}$$

with

$$\xi=\frac{|\Omega_1|^2T}{4\Delta} \quad \text{and} \quad e^{i\phi}=-\frac{\Omega_0^{(i)*}\Omega_1}{|\Omega_1|^2}. \tag{40}$$

We will see later that the additional phase factor does not affect the outcome of the Bell measurement described in the next section. We can therefore ignore this factor and use the Hamiltonian (37) to realize the single qubit rotation.

C. State measurement on a single atom

To measure whether atom i is in state $|0\rangle_i$ or $|1\rangle_i$, respectively, the same scheme as described in Sec. II C can be used.

IV. A TEST OF THE BELL INEQUALITY

Given that the state (1) can be generated, the next interesting question is whether such a state will violate one of Bell's inequalities? For certain parameters it must but what physical measurements are necessary to characterize this disagreement with local realism?

A. The Bell inequality

The *spin* (or correlation function) Bell inequality [1,3] may be written formally as

$$B_S=|E(\theta_1,\theta_2)-E(\theta_1,\theta_2')+E(\theta_1',\theta_2)+E(\theta_1',\theta_2')|\leqslant 2, \tag{41}$$

where the correlation function $E(\theta_1,\theta_2)$ is given by

$$E(\theta_1,\theta_2)=\langle\sigma_{\theta_1}^{(1)}\sigma_{\theta_2}^{(2)}\rangle. \tag{42}$$

Here θ_1 and θ_2 are real parameters. In the following the operator $\sigma_a^{(i)}$ with $a=x$, y, or z is the a Pauli spin operators for the two-level system of atom i and the operator $\sigma_{\theta_i}^{(i)}$ is defined as

$$\sigma_{\theta_i}^{(i)}=\cos\theta_i\sigma_x^{(i)}+\sin\theta_i\sigma_y^{(i)}. \tag{43}$$

We describe now how the inequality (41) could be tested experimentally.

B. Description of the experimental test

To test Bell's inequality the atoms have to be prepared first in a state for which a violation of Bell's inequality (41) is expected. This can be done with the help of the scheme discussed in Sec. II A by preparing the atoms in state (1). The parameter α can be varied by changing the length T of the laser pulse.

For certain initial states and in certain cases (including here) the correlation function depends only on the difference between the angles θ_1 and θ_2 and we have

$$E(\theta_1,\theta_2)=E(\theta_1-\theta_2,0). \tag{44}$$

This can be proven easily and holds because the state $|11\rangle$ is not populated. Populating $|11\rangle$ by the preparation schemes proposed here is not possible, because the time evolution of the system is restricted to decoherence-free states [44]. As an example to test Bell's inequality we choose $\vartheta=\theta_1-\theta_2=\theta_2-\theta_1'=\theta_1'-\theta_2'$. This leads to $\theta_1-\theta_2'=3\vartheta$. Using Eq. (44) the inequality (41) simplifies for this parameter choice to

$$B_S=|3E(\vartheta,0)-E(3\vartheta,0)|\leqslant 2. \tag{45}$$

A violation of this inequality corresponds to $|B_S|>2$.

To find a way to measure the correlation functions $E(\vartheta,0)$ we make use of the relation

$$U_{\text{rot}}^{(i)\dagger}(\xi,\phi)\sigma_z^{(i)}U_{\text{rot}}^{(i)}(\xi,\phi)=\cos 2\xi\sigma_z^{(i)}-\sin 2\xi(\cos\phi\sigma_y^{(i)}+\sin\phi\sigma_x^{(i)}). \tag{46}$$

This allows us to rewrite $\sigma_{\theta_i}^{(i)}$ in terms of $\sigma_z^{(i)}$. By choosing $\xi=\pi/4$ and by making use of some trigonometric relations one obtains from Eq. (43)

$$\sigma_{\theta_i}^{(i)}=U_{\text{rot}}^{(i)\dagger}\left(\frac{\pi}{4},\frac{3\pi}{2}-\theta_i\right)\sigma_z^{(i)}U_{\text{rot}}^{(i)}\left(\frac{\pi}{4},\frac{3\pi}{2}-\theta_i\right), \tag{47}$$

where $U_{\text{rot}}^{(i)}$ is the single qubit rotation defined in Eq. (14). Using this and Eqs. (42) and (44) one can show that

$$E(\vartheta,0)=\left\langle U_{\text{rot}}^{(1)\dagger}\left(\frac{\pi}{4},\frac{3\pi}{2}-\vartheta\right)\sigma_z^{(1)}U_{\text{rot}}^{(1)}\left(\frac{\pi}{4},\frac{3\pi}{2}-\vartheta\right)\times U_{\text{rot}}^{(2)\dagger}\left(\frac{\pi}{4},\frac{3\pi}{2}\right)\sigma_z^{(2)}U_{\text{rot}}^{(2)}\left(\frac{\pi}{4},\frac{3\pi}{2}\right)\right\rangle. \tag{48}$$

This expectation value can be measured in the following way. First, the single qubit rotation described in Sec. II B has to be applied on both atoms with $\xi=\pi/4$ and $\phi=3\pi/2-\vartheta$ for atom 1 and $\xi=\pi/4$ and $\phi=3\pi/2$ for atom 2. Afterwards the observables $\sigma_z^{(1)}$ and $\sigma_z^{(2)}$ have to be measured. This can

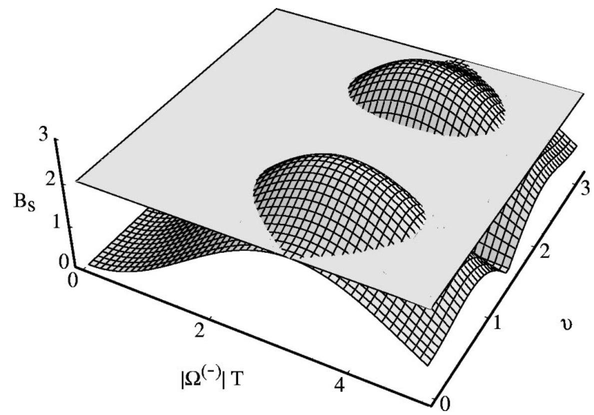

FIG. 5. Plot of $|B_S|$ versus $|\Omega^{(-)}|T$ and ϑ. A violation of the *spin* Bell's inequality occurs for $|B_S|>2$ and are displayed as *islands* in the $|\Omega^{(-)}|T$-ϑ plane. The angles have been chosen so as to maximise the violation utilizing the maximally entangled state.

be done by measuring whether the atoms are in their ground state or not as described in Sec. II C or III C, respectively. In an analogous way $E(3\vartheta,0)$ can be determined experimentally.

It is important to point out that the correlation function represents an ensemble average obtained by performing the measurements over many runs, each time repreparing the initial state.

C. Expected violation of Bill's inequality

It is straightforward to show that the correlation function for the initial state (1) is given by

$$E(\vartheta,0) = -|\alpha|^2 \cos\vartheta \tag{49}$$

and hence Eq. (45) can assume a maximum of $|B_S| = 2\sqrt{2}|\alpha|^2$ where we have chosen $\vartheta=\pi/4$. Therefore, a violation of the *spin* Bell inequality is possible for $|\alpha|^2 > 1/\sqrt{2}$. The quantity $|\alpha|^2$ can be expressed in terms of the fundamental system parameter $|\Omega^{(-)}|T$ only with the help of Eq. (12). In Fig. 5 we plot $|B_S|$ versus $|\Omega^{(-)}|T$ and ϑ.

A significant region of violation is observed with the maximum of $|B_S| = 2\sqrt{2}$ occurring at $|\Omega^{(-)}|T=\pi$. The state of the atoms at such a time is a maximally entangled state. This test on Bell's inequality should be feasible with current technology.

V. DISCUSSION

In this paper we have made use of a recently proposed scheme [13] to prepare in a controlled way with a very high success rate two atoms in an arbitrary superposition of a maximally entangled state and a product state. We show how the spin Bell inequality [1,3] can be characterized, tested, and violated closing the detection loophole. To do so we use the highly efficient measurement proposal by Cook [14] based on ''electron shelving.'' The system discussed here has the appeal that the atoms are massive particles compared with photons and hence our proposal tests quantum mechanics in an all new macroscopic regime. In addition, while the photon experiments close the casualty loophole, the proposed atom experiment would close the detection efficiency loophole. Therefore, the experiment we discuss is complementary to the current photon experiments being performed.

To summarize, entanglement is a necessary quantum resource used in quantum information. While entangled photons have to date been the engine of much recent work, their ''flying'' nature renders them inappropriate for the storage of information. We have discussed a means in which trapped ions or atoms become entangled in a controled way using dissipation, and the degree to which the resulting entanglement can be measured through Bell correlations.

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