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Exact steady-state Wigner function for a nondegenerate parametric oscillator

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We consider the model of a nondegenerate parametric oscillator, in the adiabatic limit of strongly damped pump mode, and find an *exact analytic* solution for the joint two-mode Wigner function in the steady-state regime. The one-mode Wigner functions for the individual signal and idler modes are derived as well. This gives an explicit and complete phase-space representation of this two-mode nonequilibrium quantum system. Simple examples are given illustrating the phase-space images of optical bistability and the phase diffusion effect.

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The concept of quantum-mechanical phase space and associated quasiprobability distributions has proven to be extremely useful and appealing in many fundamental applications of quantum mechanics. The oldest and most famous phase-space quasiprobability is the Wigner function [1]. Recently, there have been major experimental successes in reconstructing the Wigner function for a number of quantum-mechanical systems. The original proposal—referred to as quantum state tomography—was suggested by Vogel and Risken [2] for measuring the quantum state of light in quantum optics. The technique was then successfully realized in a series of experiments [3]. This in turn has greatly increased the interest in employing and further developing the Wigner function and phase-space methods (see, e.g., [4]). The problems under investigation have now been extended to molecular vibrational states [5], motional states of atoms in traps and atomic beams [6], and Bose-Einstein condensates [7].

Despite these achievements, there is a known disadvantage—from the methodological and theoretical points of view—of the Wigner function method, which continues to serve as a challenging problem. The disadvantage is in the difficulty of solving realistic problems directly in terms of the Wigner function, which is due to the complexity of the corresponding evolution equation. Even for the simplest models involving nonlinear interactions (such as those due to quadratic or cubic nonlinearities in quantum optics) and dissipation, the Wigner function evolution equation contains third- or higher-order derivative terms. While numerical techniques can routinely be employed here (see, e.g., [8]), no general *analytic* methods are known for solving this type of partial differential equation to find the Wigner function.

This is in contrast to employing generalized P representations [9,10] of the density matrix. The P-representation method often produces a Fokker-Planck evolution equation (having no higher than the second-order derivatives) which can be treated using well-known methods [10,11]. Of especial importance here are the models that possess exact analytic solutions in either the positive or complex P representation [9,12,13]. The disadvantage of the P representations, however, is that the positive P method requires a doubled number of phase-space variables, while the complex P representation is defined on specific contours in complex planes of the phase space. This is not the case of the Wigner function; hence (together with the historical reasons) its wider usage and popularity.

In this paper we employ an indirect method to find an *exact analytic* solution for the Wigner function for one of the basic models in nonlinear and quantum optics—the nondegenerate parametric oscillator (NDPO) [13–16]. In the limit of adiabatically eliminated pump mode, and provided that the damping rates and cavity detunings for the signal and idler modes are equal, the reduced model has an exact steady-state solution [13,16] to the corresponding Fokker-Planck equation in the complex P representation.

We use this solution to derive an exact and rather simple analytic result for the joint two-mode Wigner function describing the steady state of this nonequilibrium dissipative quantum system. Single-mode Wigner functions for the individual signal and idler modes are derived as well. To our best knowledge, this is the first example of an exact analytic Wigner function solution for a *two-mode* nonlinear dissipative problem. Earlier solutions of this type were only found for single-mode models of a degenerate parametric oscillator [17,18] and a driven anharmonic oscillator [19]. The phase-space representation of the individual signal and idler modes is itself a result for the model of a NDPO that has not been discussed before to our knowledge.

The model of NDPO that we consider describes three quantized modes of the radiation field—the pump, signal, and idler. The modes interact via a quadratic nonlinearity in a triply resonant cavity. The pump mode is driven externally by a coherent driving field, and all three modes are assumed to be coupled to zero-temperature reservoirs, resulting in the usual single-photon losses. This system can be modeled by the following Hamiltonian, in the rotating-wave approximation [13,15]:

$$H=\sum_{i=0}^{2}\hbar\omega_i a_i^\dagger a_i+i\hbar\kappa(a_1^\dagger a_2^\dagger a_0-a_1a_2a_0^\dagger)$$

$$+i\hbar(Ee^{-i\omega_L t}a_0^\dagger-E^*e^{i\omega_L t}a_0)+\sum_{i=0}^{2}(a_i\Gamma_i^\dagger+a_i^\dagger\Gamma_i). \qquad (1)$$

Here $a_i^\dagger$ and a_i are boson creation and annihilation operators for the cavity modes at frequencies ω_i, where $i=0$, 1, and 2 refer to the pump, signal, and idler, respectively. The parametric coupling κ is due to the second-order susceptibility ($\chi^{(2)}$) of the nonlinear medium, and we assume a phasematching condition so that $\omega_0 \simeq \omega_1 + \omega_2$. In addition, E is the amplitude of the coherent driving field with the carrier frequency $\omega_L \simeq \omega_0$, and $\Gamma_i^\dagger$, Γ_i are the reservoir operators which give rise to the cavity losses with decay rates γ_i.

In the limit of strongly damped pump mode, $\gamma_0 \gg \gamma_{1,2}$, the pump mode can be adiabatically eliminated [13]. The reduced model can then be described by the following interaction picture master equation for the density operator ρ of the signal and idler modes:

$$\frac{\partial \rho}{\partial t} = i \sum_{i=1,2} \Delta_i [a_i^\dagger a_i, \rho] + \frac{1}{i\hbar}[H_{eff}, \rho] + \sum_{i=1,2} \gamma_i (2 a_i \rho a_i^\dagger - a_i^\dagger a_i \rho - \rho a_i^\dagger a_i) + g(2 a_1 a_2 \rho a_1^\dagger a_2^\dagger - a_1^\dagger a_2^\dagger a_1 a_2 \rho - \rho a_1^\dagger a_2^\dagger a_1 a_2). \quad (2)$$

Here the effective interaction Hamiltonian is given by

$$H_{eff} = i\hbar (\mu a_1^\dagger a_2^\dagger - \mu^* a_1 a_2) + \hbar \chi a_1^\dagger a_2^\dagger a_1 a_2. \quad (3)$$

In addition, $\Delta_0 = \omega_0 - \omega_L$ is the cavity detuning for the pump mode and $\Delta_{1,2} = \omega_{1,2} - (\omega_L/2 \pm \epsilon)$ are the cavity detunings for the signal and the idler modes, with $\omega_L/2 \pm \epsilon$ being their characteristic rotating-frame frequencies. We have also defined

$$\mu = \frac{\kappa E}{\overline{\gamma_0}}, \quad \chi = -\frac{\kappa^2 \Delta_0}{|\overline{\gamma_0}|^2}, \quad g = \frac{\kappa^2 \gamma_0}{|\overline{\gamma_0}|^2}, \quad (4)$$

where $\overline{\gamma_0} = \gamma_0 + i\Delta_0$.

The above form of the master equation and the effective Hamiltonian (3) simply implies that, after adiabatic elimination of the pump mode, we are dealing with a two-mode model that describes two cavity modes (signal and idler) that are driven parametrically with a driving strength μ, and are subject to the usual one-photon (linear) losses at rates γ_i *and* to a nondegenerate two-photon (nonlinear) loss mechanism with a rate g. In addition, the two modes are coupled by a Kerr type interaction due to an effective cubic ''nonlinearity'' χ, causing cross-phase modulation. This χ term would be absent in the case of exact resonance, $\Delta_0 = 0$. We note that the cross-phase modulation and nondegenerate two-photon absorption terms could explicitly be included into the original Hamiltonian (1), representing the possibility of combining different nonlinear optical processes in a single cavity. The corresponding nonlinear constants would then be simply added to the above ''effective'' couplings χ and g, thus providing additional degrees of freedom for the control over the system parameters.

Using the complex P representation [9] of the density operator, the above master equation is transformed into the Fokker-Planck equation, which can then be solved exactly for a steady state [13], using the method of potential equations. The solution is available for the case of symmetric decay rates $\gamma_1 = \gamma_2 \equiv \gamma$ and cavity detunings $\Delta_1 = \Delta_2 \equiv \Delta$. The resulting steady-state P function has the following form [13,16]:

$$P(\beta_1, \beta_2, \beta_1^\dagger, \beta_2^\dagger) = \mathcal{N} (\beta_1 \beta_2 - \varepsilon)^{\nu - 1} (\beta_1^\dagger \beta_2^\dagger - \varepsilon^*)^{\nu^* - 1} \times \exp[2(\beta_1 \beta_1^\dagger + \beta_2 \beta_2^\dagger)]. \quad (5)$$

Here β_1, β_2, $\beta_1^\dagger$, and $\beta_2^\dagger$ are independent complex variables, corresponding to the operators a_1, a_2, $a_1^\dagger$, and $a_2^\dagger$, respectively, $\mathcal{N}$ is the normalization constant, and we have introduced the following dimensionless parameters:

$$\varepsilon \equiv \frac{\mu}{g + i\chi} = \frac{E}{\kappa}, \quad \nu \equiv \frac{2\overline{\gamma}^*}{g - i\chi} = \frac{2\overline{\gamma}^* \overline{\gamma}_0^*}{\kappa^2}, \quad (6)$$

where $\overline{\gamma} = \gamma + i\Delta$.

The two-mode Wigner function that we are interested in is defined as follows:

$$W(\alpha_1, \alpha_2) = \left(\frac{1}{\pi^2}\right)^2 \int \int d^2\delta_1 d^2\delta_2 \times \mathrm{Tr}(\rho e^{\delta_1 a_1^\dagger - \delta_1^* a_1 + \delta_2 a_2^\dagger - \delta_2^* a_2}) \times e^{\delta_1^* \alpha_1 - \delta_1 \alpha_1^* + \delta_2^* \alpha_2 - \delta_2 \alpha_2^*}. \quad (7)$$

This can be expressed in terms of the complex P representation as

$$W(\alpha_1, \alpha_2) = \left(\frac{2}{\pi}\right)^2 e^{-2(|\alpha_1|^2 + |\alpha_2|^2)} \times \int_{C_{\beta_1}} \int_{C_{\beta_2}} \int_{C_{\beta_1^\dagger}} \int_{C_{\beta_2^\dagger}} d\beta_1 d\beta_2 d\beta_1^\dagger d\beta_2^\dagger \times P(\beta_1, \beta_1^\dagger, \beta_2, \beta_2^\dagger) \times e^{-2(\beta_1 \beta_1^\dagger + \beta_2 \beta_2^\dagger) + 2(\alpha_1^* \beta_1 + \alpha_1 \beta_1^\dagger) + 2(\alpha_2^* \beta_2 + \alpha_2 \beta_2^\dagger)}, \quad (8)$$

where the integrals are contour integrals in the individual complex planes for the independent variables β_1, β_2, $\beta_1^\dagger$, and $\beta_2^\dagger$, according to the way the complex P representation is defined [9].

Substituting the steady-state complex P function (5) into Eq. (8), and assuming that the integrations over β_j and $\beta_j^\dagger$ variables are along the same contours in the respective complex planes (i.e., the contour C_{β_1} is the same as $C_{\beta_1^\dagger}$, and the contour C_{β_2} is the same as $C_{\beta_2^\dagger}$), one can express the Wigner function $W(\alpha_1, \alpha_2)$ as follows:

$$W(\alpha_1, \alpha_2) = \mathcal{N} \left(\frac{2}{\pi}\right)^2 e^{-2(|\alpha_1|^2 + |\alpha_2|^2)} |I|^2, \quad (9)$$

where we have defined

$$I \equiv I(\alpha_1^*, \alpha_2^*; \varepsilon, \nu^*) = \int_{C_{\beta 1}} \int_{C_{\beta_2}} d\beta_1 d\beta_2 (\beta_1 \beta_2 - \varepsilon)^{\nu^* - 1} e^{2(\alpha_1^* \beta_1 + \alpha_2^* \beta_2)}, \tag{10}$$

so that $I^* = [I(\alpha_1^*, \alpha_2^*; \varepsilon, \nu^*)]^* = I(\alpha_1, \alpha_2; \varepsilon^*, \nu)$.

To evaluate the integral I we transform to new variables $t = 2\varepsilon \alpha_2^*/\beta_1$ and $\tau = 2\alpha_2^*(\beta_1\beta_2 - \varepsilon)/\beta_1$:

$$I = -(\varepsilon)^{\nu^*} \int_{C_\tau} d\tau \tau^{\nu^* - 1} e^{\tau} \int_{C_t} dt t^{-\nu^* - 1} e^{t + 4\varepsilon \alpha_1^* \alpha_2^*/t}. \tag{11}$$

The integrals over τ and t can now be identified, respectively, with contour integral representations of the gamma function $\Gamma(\nu^*)$ and the Bessel function $J_{\nu^*}(\sqrt{-16\varepsilon \alpha_1^* \alpha_2^*})$ [20], where both the integration contours C_τ and C_t start at infinity on the negative $\tau(t)$ axis, encircle the origin counterclockwise, and return to $-\infty$. As a result, we obtain the result that $I \propto J_{\nu^*}(\sqrt{-16\varepsilon \alpha_1^* \alpha_2^*})/(-16\varepsilon \alpha_1^* \alpha_2^*)^{\nu^*/2}$. The steady-state Wigner function $W(\alpha_1, \alpha_2)$ is then given by

$$W(\alpha_1, \alpha_2) = N e^{-2(|\alpha_1|^2 + |\alpha_2|^2)} \left| \frac{J_\nu(\sqrt{-16\varepsilon^* \alpha_1 \alpha_2})}{(-16\varepsilon^* \alpha_1 \alpha_2)^{\nu/2}} \right|^2, \tag{12}$$

where N is the normalization constant to be found from $\int\int d^2\alpha_1 d^2\alpha_2 W(\alpha_1, \alpha_2) = 1$. Using the expansion of the Bessel function

$$J_\nu(\sqrt{t}) = \frac{t^{\nu/2}}{2^\nu \Gamma(\nu)} \sum_{k=0}^{\infty} \frac{(-t/4)^k \Gamma(\nu+1)}{k! \Gamma(k+\nu+1)}, \tag{13}$$

one can rewrite the above Wigner function as

$$W(\alpha_1, \alpha_2) = N' e^{-2(|\alpha_1|^2 + |\alpha_2|^2)} \left| \sum_{k=0}^{\infty} \frac{(4\varepsilon^* \alpha_1 \alpha_2)^k \ \Gamma(\nu+1)}{k! \Gamma(k+\nu+1)} \right|^2. \tag{14}$$

The constant $N' = N|2^\nu \Gamma(\nu)|^{-2}$ can now be easily found from the normalization condition by carrying out the integrations in polar coordinates $[\alpha_j = r_j \exp(i\varphi_j)]$ and using $\int_0^{2\pi} d\varphi \exp[i(k-m)\varphi] = 2\pi \delta_{km}$:

$$N' = \left(\frac{2}{\pi}\right)^2 \left[\sum_{k=0}^{\infty} \frac{(4|\varepsilon|^2)^k |\Gamma(\nu+1)|^2}{|\Gamma(k+\nu+1)|^2} \right]^{-1}. \tag{15}$$

Equations (12) and (14) are the key results of the present paper. We point out that the Wigner function $W(\alpha_1, \alpha_2)$ is positive everywhere. Therefore it can act as a local hidden variable theory for quadrature phase measurements and thus cannot violate a Bell inequality, in contrast to the results that can be obtained with a nondissipative model of NDPO (see, e.g., [21–23]).

We next use the two-mode Wigner function $W(\alpha_1, \alpha_2)$ to calculate the Wigner function for each of the individual (signal or idler) modes. This is obtained by integrating $W(\alpha_1, \alpha_2)$ with respect to one of the phase-space variables, α_2 or α_1. Due to the symmetry of $W(\alpha_1, \alpha_2)$ with respect to α_1 and α_2, the one-mode Wigner functions for the signal and the idler modes are equal to each other and are given, from Eq. (14), by

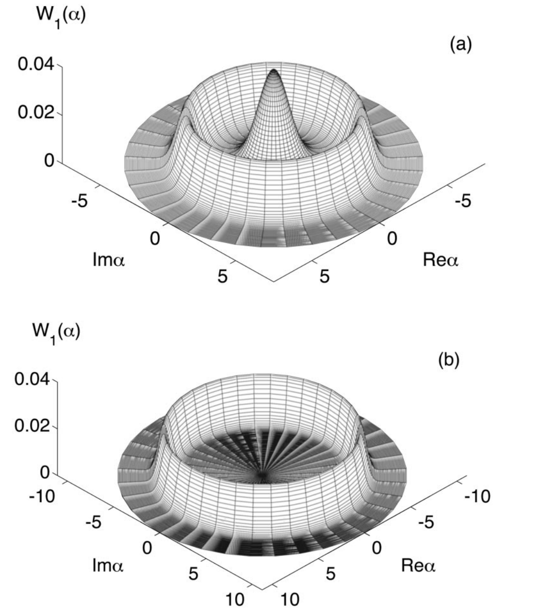

FIG. 1. One-mode Wigner function for $\mathrm{Re}\,\nu = -30$, $\mathrm{Im}\,\nu = -30$, and two values of the driving field parameter: $|\varepsilon| = 16.5$ (a), and $|\varepsilon| = 45$ (b).

$$W_1(\alpha) = \int d^2\alpha_2 W(\alpha, \alpha_2) = \frac{\pi N'}{2} e^{-2|\alpha|^2} \sum_{k=0}^{\infty} \frac{(8|\varepsilon|^2 |\alpha|^2)^k |\Gamma(\nu+1)|^2}{k! |\Gamma(k+\nu+1)|^2}. \tag{16}$$

This expression can be rewritten in a compact form, in terms of the hypergeometric function ${}_0F_2$ [20]:

$$W_1(\alpha) = \frac{\pi N'}{2} e^{-2r^2} {}_0F_2(\nu+1, \nu^*+1, 8|\varepsilon|^2 r^2), \tag{17}$$

where $r = |\alpha|$.

Examples of $W_1(\alpha)$ are plotted in Fig. 1. The Wigner function $W_1(\alpha)$ depends only on the radial coordinate $r = |\alpha|$, and is uniformly distributed with respect to the phase φ. The phase distribution function, defined as an integral over the radial coordinate $\Phi_1(\varphi) = \int_0^\infty r\, dr\, W_1(\alpha = re^{i\varphi})$, can be found directly using the normalization condition, giving the result that $\Phi_1(\varphi) = 1/(2\pi)$. This radial symmetry of $W_1(\alpha)$ reflects the known phenomenon of phase diffusion occurring in NDPOs [15], according to which the phases of the signal and idler modes do not have well-defined values.

In Fig. 1(a), the coexistence of the central peak together with the outer ring represents the phenomenon of bistability known to occur in the NDPO for $\Delta\Delta_0 > \gamma\gamma_0$ and experimentally observed in [24]. In terms of the semiclassical steady-state solutions [14–16] for the intensities of the signal and idler modes, $n_1^{(0)}=n_2^{(0)}\equiv n^{(0)}$, the bistability is referred to the overlap of the stability regions for the below-threshold solution $n^{(0)}=0$ and the above-threshold solution with $n^{(0)}\neq 0$. Using the parameters ν and ε, the above-threshold solution can be written as

$$n^{(0)}=-\mathrm{Re}(\nu/2)\pm\sqrt{|\varepsilon|^2-[\mathrm{Im}(\nu/2)]^2}. \qquad (18)$$

The bistability in the NDPO occurs if $\mathrm{Re}(\nu/2)<0$, and the corresponding region—as a function of the scaled driving field intensity $|\varepsilon|^2$—is determined by $|\varepsilon^{(-)}|^2<|\varepsilon|^2<|\varepsilon_{th}|^2$. Here $|\varepsilon_{th}|^2=|\nu|^2/4$ gives the threshold value of $|\varepsilon|^2$, while $|\varepsilon^{(-)}|^2=[\mathrm{Im}(\nu/2)]^2$ is the turning point where the system returns from the above-threshold regime of oscillation to the below-threshold regime. In Fig. 1(a) the value of $|\varepsilon|$ is chosen to be within the bistability region, so that the central peak corresponds to the below-threshold solution, while the outer ring represents the above-threshold solution. For small values of $|\varepsilon|$ the Wigner function only has the central peak, while large values of $|\varepsilon|$ lead to disappearance of the central peak, and we observe only the above-threshold outer ring, as in Fig. 1(b).

Thus, the Wigner function solutions found here provide us with an explicit phase-space representation of the steady state of this two-mode nonequilibrium quantum system, for arbitrary values of relevant parameters. These are the key objects for quantum tomographic applications [25], and the results might be especially useful for analyzing Bell type inequalities where the knowledge of phase-space probabilities to a great degree of accuracy can often be of crucial importance [23,26,27]. We note that the results presented here are valid in the threshold region and in the extreme quantum regime of operation characteristic of high nonlinearities and low damping rates. We also point out a remarkable spin-off (from the methodological point of view) of the solution found here. This follows from the comparison of the *two-mode* Wigner function with the earlier known solutions to *single-mode* models of the degenerate parametric oscillator and anharmonic oscillator: Quite surprisingly, the structure of the Wigner functions in all these three quite distinct exactly soluble models appears similar, once the solutions are expressed [18,19] in terms of Bessel functions. This suggests an intriguing possibility of finding a *direct* method of solving at least a class of Wigner function evolution equations with third-order derivative terms, having the form generic to these models. Hopefully this would provide previously unknown solutions to other problems.

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