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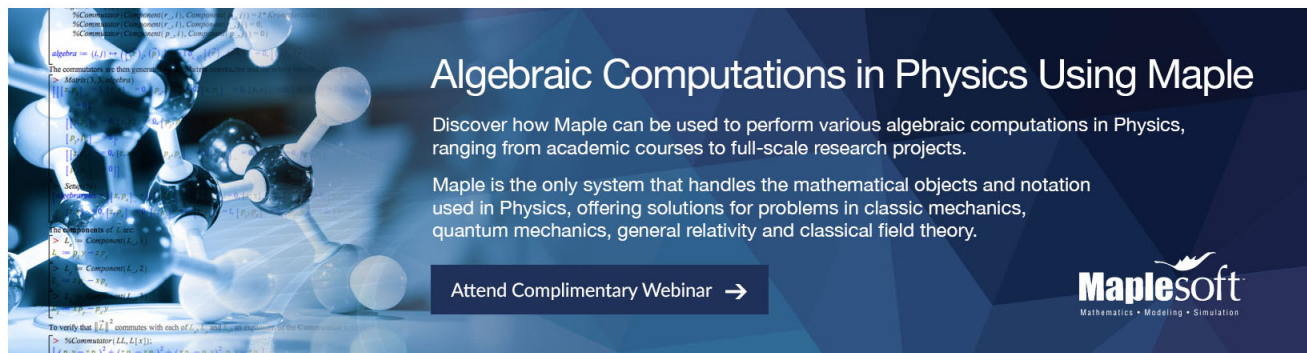
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Casimir invariants from quasi-Hopf (super)algebras

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We show how to construct, starting from a quasi-Hopf (super)algebra, central elements or Casimir invariants. We show that these central elements are invariant under quasi-Hopf twistings. As a consequence, the elliptic quantum (super)groups, which arise from twisting the normal quantum (super)groups, have the same Casimir invariants as the corresponding quantum (super)groups. © 2000 American Institute of Physics. [S0022-2488(99)01312-2]

I. INTRODUCTION

Quasi-Hopf superalgebras are \mathbf{Z}_2 -graded versions of Drinfeld's quasi-Hopf algebras¹ and were introduced in Ref. 2. The potential for applications of these structures, particularly to knot theory and integrable systems, is enormous. They give rise to new (nonstandard) representations of the braid group and corresponding link polynomials.^{3,4} Moreover these remarkable structures underly elliptic quantum (super)groups^{5-9,2} which play an important role in obtaining solutions to the dynamical Yang–Baxter equations.^{10,11}

In applications such as these it is important to have a well defined representation theory. In this paper we investigate several aspects of this theory concerned with the construction and general properties of invariants (invariant bilinear forms, module morphisms, central elements, etc.). In particular, in the quasitriangular case, it is shown how central elements may be systematically constructed utilizing the R -matrix. This construction may be regarded as a natural generalization of that introduced in Refs. 12 and 13, to which it reduces in the case of normal Hopf (super)algebras. However the extension of this paper is by no means straightforward and requires the explicit inclusion of the coassociator into the construction.

We moreover prove the strong result that the Casimir invariants so obtained are invariant under twisting. This implies, in particular, that one will not obtain new Casimir invariants by twisting on quantum (super)groups. As part of our approach we extend the u -operator formalism of Drinfeld–Reshetikhin to the case of quasi-Hopf superalgebras. In particular we prove the surprising result that the u -operator is invariant under twisting. This has some important implications for knot theory which will be investigated elsewhere. It is worth noting that most of our results are new, even in the nongraded case.

II. QUASI-HOPF (SUPER)ALGEBRAS

Let us briefly recall the quasi-Hopf algebras¹ and their super (or \mathbf{Z}_2 graded) versions, quasi-Hopf superalgebras.²

Definition 1: A quasi-Hopf (super)algebra is a (\mathbf{Z}_2 graded) unital associative algebra A over a field K which is equipped with algebra homomorphisms $\epsilon: A \rightarrow K$ (co-unit), $\Delta: A \rightarrow A \otimes A$ (co-product), an invertible homogeneous element $\Phi \in A \otimes A \otimes A$ (co-associator), an (\mathbf{Z}_2 graded) algebra antihomomorphism $S: A \rightarrow A$ (antipode) and homogeneous canonical elements, $\alpha, \beta \in A$, satisfying

$$(1 \otimes \Delta)\Delta(a) = \Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi, \quad \forall a \in A, \quad (2.1)$$

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$$(\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi = (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes \Phi), \tag{2.2}$$

$$(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta, \tag{2.3}$$

$$(1 \otimes \epsilon \otimes 1)\Phi = 1, \tag{2.4}$$

$$m \cdot (1 \otimes \alpha)(S \otimes 1)\Delta(a) = \epsilon(a)\alpha, \quad \forall a \in A, \tag{2.5}$$

$$m \cdot (1 \otimes \beta)(1 \otimes S)\Delta(a) = \epsilon(a)\beta, \quad \forall a \in A, \tag{2.6}$$

$$m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)\Phi^{-1} = 1, \tag{2.7}$$

$$m \cdot (m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha \otimes \beta)(1 \otimes 1 \otimes S)\Phi = 1. \tag{2.8}$$

Here m denotes the usual product map on $A: m \cdot (a \otimes b) = ab, \forall a, b \in A$. Note that since A is associative we have $m \cdot (m \otimes 1) = m \cdot (1 \otimes m)$. For the homogeneous elements $a, b \in A$, the antipode satisfies

$$S(ab) = (-1)^{[a][b]}S(b)S(a), \tag{2.9}$$

which extends to inhomogeneous elements through linearity. (2.2), (2.3), and (2.4) imply that Φ also obeys

$$(\epsilon \otimes 1 \otimes 1)\Phi = 1 = (1 \otimes 1 \otimes \epsilon)\Phi. \tag{2.10}$$

It follows that the co-associator Φ is an even element. Applying ϵ to definitions (2.7) and (2.8) we obtain, in view of (2.4), $\epsilon(\alpha)\epsilon(\beta) = 1$. Thus the canonical elements α, β are both even. By applying ϵ to (2.5), we have $\epsilon(S(a)) = \epsilon(a), \forall a \in A$. Note that the multiplication rule for the tensor products is defined for homogeneous elements $a, b, a', b' \in A$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{[b][a']}(aa' \otimes bb'), \tag{2.11}$$

where $[a] \in \mathbf{Z}_2$ denotes the grading of the element a .

The category of quasi-Hopf (super)algebras is invariant under a kind of gauge transformation. Let $(A, \Delta, \epsilon, \Phi)$ be a quasi-Hopf (super)algebra, with α, β, S satisfying (2.5)–(2.8), and let $F \in A \otimes A$ be an invertible homogeneous element satisfying the co-unit properties

$$(\epsilon \otimes 1)F = 1 = (1 \otimes \epsilon)F. \tag{2.12}$$

It follows that F is even. Throughout we set

$$\Delta_F(a) = F\Delta(a)F^{-1}, \quad \forall a \in A, \tag{2.13}$$

$$\Phi_F = (F \otimes 1)(\Delta \otimes 1)F \cdot \Phi \cdot (1 \otimes \Delta)F^{-1}(1 \otimes F^{-1}). \tag{2.14}$$

Then

Theorem 1: $(A, \Delta_F, \epsilon, \Phi_F)$ defined by (2.13) and (2.14) together with α_F, β_F, S_F given by

$$S_F = S, \quad \alpha_F = m \cdot (1 \otimes \alpha)(S \otimes 1)F^{-1}, \quad \beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F, \tag{2.15}$$

is also a quasi-Hopf (super)algebra. The element F is referred to as a twistor, throughout.

Definition 2: A quasi-Hopf (super)algebra $(A, \Delta, \epsilon, \Phi)$ is called quasitriangular if there exists an invertible homogeneous element $\mathcal{R} \in A \otimes A$ such that

$$\Delta^T(a)\mathcal{R} = \mathcal{R}\Delta(a), \quad \forall a \in A, \tag{2.16}$$

$$(\Delta \otimes 1)\mathcal{R} = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}, \tag{2.17}$$

$$(1 \otimes \Delta)\mathcal{R} = \Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123}. \tag{2.18}$$

\mathcal{R} is referred to as the universal R -matrix.

Throughout, $\Delta^T = T \cdot \Delta$ with T being the graded twist map which is defined, for homogeneous elements $a, b \in A$, by

$$T(a \otimes b) = (-1)^{[a][b]}b \otimes a; \tag{2.19}$$

and Φ_{132} , etc. are derived from $\Phi \equiv \Phi_{123}$ with the help of T ,

$$\Phi_{132} = (1 \otimes T)\Phi_{123},$$

$$\Phi_{312} = (T \otimes 1)\Phi_{132} = (T \otimes 1)(1 \otimes T)\Phi_{123},$$

$$\Phi_{231}^{-1} = (1 \otimes T)\Phi_{213}^{-1} = (1 \otimes T)(T \otimes 1)\Phi_{123}^{-1},$$

and so on.

It is easily shown that the properties (2.16)–(2.18) imply the (graded) Yang–Baxter type equation,

$$\mathcal{R}_{12}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1} = \Phi_{321}^{-1}\mathcal{R}_{23}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}, \tag{2.20}$$

which is referred to as the (graded) quasi-Yang–Baxter equation, and the co-unit properties of \mathcal{R} ,

$$(\epsilon \otimes 1)\mathcal{R} = 1 = (1 \otimes \epsilon)\mathcal{R}. \tag{2.21}$$

Thus the universal R -matrix \mathcal{R} is even. We have

Theorem 2: Denoting by the set $(A, \Delta, \epsilon, \Phi, \mathcal{R})$ a quasitriangular quasi-Hopf (super)algebra, then $(A, \Delta_F, \epsilon, \Phi_F, \mathcal{R}_F)$ is also a quasitriangular quasi-Hopf (super)algebra, with the choice of \mathcal{R}_F given by

$$\mathcal{R}_F = F^T\mathcal{R}F^{-1}, \tag{2.22}$$

where $F^T = T \cdot F \equiv F_{21}$. Here Δ_F and Φ_F are given by (2.13) and (2.14), respectively.

Let us specify some notations. Throughout the paper,

$$\begin{aligned} \Phi &= \sum X_\nu \otimes Y_\nu \otimes Z_\nu, \quad \Phi^{-1} = \sum \bar{X}_\nu \otimes \bar{Y}_\nu \otimes \bar{Z}_\nu, \\ F &= \sum f_i \otimes f^i, \quad F^{-1} = \sum \bar{f}_i \otimes \bar{f}^i, \\ \mathcal{R} &= \sum e_i \otimes e^i, \quad \mathcal{R}^{-1} = \sum \bar{e}_i \otimes \bar{e}^i, \end{aligned} \tag{2.23}$$

$$(1 \otimes \Delta)\Delta(a) = \sum a_{(1)} \otimes \Delta(a_{(2)}) = \sum a_{(1)}^R \otimes a_{(2)}^R \otimes a_{(3)}^R,$$

$$(\Delta \otimes 1)\Delta(a) = \sum \Delta(a_{(1)}) \otimes a_{(2)} = \sum a_{(1)}^L \otimes a_{(2)}^L \otimes a_{(3)}^L.$$

The following lemma is proved in Ref. 14 and will be used frequently in this paper.

Lemma 1: $\forall a \in A$,

$$\begin{aligned}
 \text{(i)} \quad & \sum X_{\nu} a \otimes Y_{\nu} \beta S(Z_{\nu}) (-1)^{[a][X_{\nu}]} = \sum a_{(1)}^L X_{\nu} \otimes a_{(2)}^L Y_{\nu} \beta S(Z_{\nu}) S(a_{(3)}^L) (-1)^{[a_{(2)}^L][X_{\nu}]}, \\
 \text{(ii)} \quad & \sum S(X_{\nu}) \alpha Y_{\nu} \otimes a Z_{\nu} (-1)^{[a][Z_{\nu}]} = \sum S(a_{(1)}^R) S(X_{\nu}) \alpha Y_{\nu} a_{(2)}^R \otimes Z_{\nu} a_{(3)}^R (-1)^{[a_{(2)}^R][Z_{\nu}]}, \\
 \text{(iii)} \quad & \sum a \bar{X}_{\nu} \otimes S(\bar{Y}_{\nu}) \alpha \bar{Z}_{\nu} = \sum \bar{X}_{\nu} a_{(1)}^L \otimes S(a_{(2)}^L) S(\bar{Y}_{\nu}) \alpha \bar{Z}_{\nu} a_{(3)}^L (-1)^{[X_{\nu}][[a_{(1)}^L] + [a_{(2)}^L]}], \\
 \text{(iv)} \quad & \sum \bar{X}_{\nu} \beta S(\bar{Y}_{\nu}) \otimes \bar{Z}_{\nu} a = \sum a_{(1)}^R \bar{X}_{\nu} \beta S(\bar{Y}_{\nu}) S(a_{(2)}^R) \otimes a_{(3)}^R \bar{Z}_{\nu} (-1)^{([a_{(2)}^R] + [a_{(3)}^R])[Z_{\nu}]}.
 \end{aligned}
 \tag{2.24}$$

III. CENTRAL ELEMENTS FROM (ANTI-)ADJOINT ACTIONS

Given an $(\mathbb{Z}_2$ graded) A -module V , we say $v \in V$ an invariant if

$$a \cdot v = \epsilon(a) v, \quad \forall a \in A. \tag{3.1}$$

In particular, A itself constitutes an A -module under the adjoint action defined by

$$\text{Ad } a \cdot b = \sum a_{(1)} b S(a_{(2)}) (-1)^{[b][a_{(2)}]}, \quad \forall a, b \in A. \tag{3.2}$$

It is easily shown that

$$\text{Ad } a \cdot \text{Ad } b = \text{Ad } ab. \tag{3.3}$$

We call $c_1 \in A$ an invariant if it is invariant under the adjoint action, i.e.,

$$\sum a_{(1)} c_1 S(a_{(2)}) (-1)^{[c_1][a_{(2)}]} = \epsilon(a) c_1, \quad \forall a \in A. \tag{3.4}$$

For normal Hopf (super)algebras, the invariants of A are precisely the central elements. This is not true, however, for quasi-Hopf (super)algebras. For instance, the canonical element β is invariant but not generally central. Nevertheless, there is a close connection between central elements and invariants. We have

Proposition 1: Suppose $c_1 \in A$ is even and invariant. Set

$$C_1 = \sum \bar{X}_{\nu} c_1 S(\bar{Y}_{\nu}) \alpha \bar{Z}_{\nu} = m(m \otimes 1) \cdot (1 \otimes c_1 \otimes \alpha)(1 \otimes S \otimes 1) \Phi^{-1}. \tag{3.5}$$

Then (i) $a C_1 = C_1 a, \forall a \in A$, i.e., C_1 is central, and

$$\text{(ii)} \quad c_1 = C_1 \beta = \beta C_1, \tag{3.6}$$

$$\text{(iii)} \quad C_1 = \sum S(X_{\nu}) \alpha Y_{\nu} c_1 S(Z_{\nu}).$$

Proof: Applying $m \cdot (1 \otimes c_1)$ to Lemma 1 (ii) and keeping in mind of (3.4), we obtain (i). We now prove (ii). From (2.2),

$$\begin{aligned}
 C_1 \otimes 1 &= (m(m \otimes 1) \otimes 1) \cdot (1 \otimes c_1 \otimes \alpha \otimes 1)(1 \otimes S \otimes 1 \otimes 1)(\Phi^{-1} \otimes 1) \\
 &= \sum (X_{\nu} \bar{X}_{\sigma} \bar{X}_{\rho}^{(1)} c_1 S(\bar{X}_{\rho}^{(2)}) S(\bar{Y}_{\sigma}) S(X_{\mu}) S(Y_{\nu}^{(1)}) \alpha Y_{\nu}^{(2)} Y_{\mu} \bar{Z}_{\sigma}^{(1)} \bar{Y}_{\rho} \otimes Z_{\nu} Z_{\mu} \bar{Z}_{\sigma}^{(2)} \bar{Z}_{\rho}) (-1)^x,
 \end{aligned}
 \tag{3.7}$$

where

$$\begin{aligned}
 x = & [\bar{Z}_\sigma][X_\nu] + [X_\mu][Y_\nu] + [Z_\mu][Z_\nu] + ([\bar{Z}_\sigma^{(1)}] + [\bar{Y}_\rho])([Z_\mu] \\
 & + [Z_\nu]) + [\bar{Y}_\rho][\bar{Z}_\sigma^{(2)}] + [\bar{X}_\rho]([X_\nu] + [\bar{X}_\sigma]).
 \end{aligned}
 \tag{3.8}$$

By (3.4) and (2.5),

$$\begin{aligned}
 C_1 \otimes 1 = & \sum (X_\nu \bar{X}_\sigma \epsilon(\bar{X}_\rho) c_1 S(\bar{Y}_\sigma) S(X_\mu) \epsilon(Y_\nu) \alpha Y_\mu \bar{Z}_\sigma^{(1)} \bar{Y}_\rho \otimes Z_\nu Z_\mu \bar{Z}_\sigma^{(2)} \bar{Z}_\rho) (-1)^x \\
 = & \sum (X_\nu \epsilon(Y_\nu) \otimes Z_\nu) (\bar{X}_\sigma c_1 S(\bar{Y}_\sigma) S(X_\mu) \alpha Y_\mu \bar{Z}_\sigma^{(1)} \otimes Z_\mu \bar{Z}_\sigma^{(2)}) (\epsilon(\bar{X}_\rho) \bar{Y}_\rho \otimes \bar{Z}_\rho) (-1)^{[Z_\mu][Z_\sigma^{(1)}]} \\
 = & \sum (\bar{X}_\sigma c_1 S(\bar{Y}_\sigma) S(X_\mu) \alpha Y_\mu \bar{Z}_\sigma^{(1)} \otimes Z_\mu \bar{Z}_\sigma^{(2)}) (-1)^{[Z_\mu][\bar{Z}_\sigma^{(1)}]}
 \end{aligned}
 \tag{3.9}$$

by (2.4) and (2.10).

Applying $m \cdot (1 \otimes \beta)(1 \otimes S)$ gives rise to

$$\begin{aligned}
 C_1 \beta = & \sum \bar{X}_\sigma c_1 S(\bar{Y}_\sigma) S(X_\mu) \alpha Y_\mu \bar{Z}_\sigma^{(1)} \beta S(\bar{Z}_\sigma^{(2)}) S(Z_\mu) (-1)^{[Z_\mu][\bar{Z}_\sigma]} \\
 = & \sum \bar{X}_\sigma c_1 S(\bar{Y}_\sigma) S(X_\mu) \alpha Y_\mu \epsilon(\bar{Z}_\sigma) \beta S(Z_\mu)
 \end{aligned}$$

by (2.6)

$$= \sum \bar{X}_\sigma c_1 S(\bar{Y}_\sigma) \epsilon(\bar{Z}_\sigma) = c_1,
 \tag{3.10}$$

by (2.8) and (2.10), thus proving (ii). (iii) is the direct consequence of (i) and (ii).

The above gives a very clear picture of the connection between invariants and central elements. In particular, we have

Corollary 1: Suppose $c \in A$ is even. Then c is an invariant if and only if there exists a central element C such that

$$c = C\beta = \beta C.
 \tag{3.11}$$

A also admits an antiadjoint action defined by

$$\overline{\text{Ad}} a \cdot b = \sum S(a_{(1)}) b a_{(2)} (-1)^{[b][a_{(1)}]}, \quad \forall a, b \in A.
 \tag{3.12}$$

We have

$$\overline{\text{Ad}} a \cdot \overline{\text{Ad}} b = \overline{\text{Ad}}(ba).
 \tag{3.13}$$

We call $c_2 \in A$ a pseudoinvariant if it is invariant under the antiadjoint action, i.e.,

$$\sum S(a_{(1)}) c_2 a_{(2)} (-1)^{[c_2][a_{(1)}]} = \epsilon(a) c_2, \quad \forall a \in A.
 \tag{3.14}$$

Proposition 2: Suppose $c_2 \in A$ i.e., even and pseudoinvariant. Set

$$C_2 = \sum S(X_\nu) c_2 Y_\nu \beta S(Z_\nu) = m(m \otimes 1) \cdot (1 \otimes c_2 \otimes \beta)(S \otimes 1 \otimes S) \Phi.
 \tag{3.15}$$

Then (i) $aC_2 = C_2a, \forall a \in A$, i.e., C_2 is central, and

$$(ii) \quad c_2 = C_2\alpha = \alpha C_2,$$

$$(iii) \quad C_2 = \sum \bar{X}_\nu \beta S(\bar{Y}_\nu) c_2 \bar{Z}_\nu. \tag{3.16}$$

Proof: Similar to the proof of Proposition 1. Applying $m \cdot (1 \otimes c_2)(S \otimes 1)$ to Lemma 1(i), we obtain (i). Applying $m \cdot (1 \otimes \alpha)$ to

$$C_2 \otimes 1 = (m(m \otimes 1) \otimes 1) \cdot (1 \otimes c_2 \otimes \beta)(S \otimes 1 \otimes S)(\Phi \otimes 1), \tag{3.17}$$

leads to (ii). Finally, (iii) is a direct consequence of (i) and (ii).

As an example we construct the so-called quadratic invariants. Suppose $\omega = \sum \omega_i \otimes \omega^i \in A \otimes A$ is even and satisfies

$$\Delta(a)\omega = \omega\Delta(a), \quad \forall a \in A. \tag{3.18}$$

Applying $m \cdot (1 \otimes \beta)(1 \otimes S)$ gives

$$\sum a_{(1)} \omega_i \beta S(\omega^i) S(a_{(2)}) = \epsilon(a) \sum \omega_i \beta S(\omega^i), \quad \forall a \in A, \tag{3.19}$$

which implies that

$$c_1 \equiv \sum \omega_i \beta S(\omega^i) \tag{3.20}$$

is an invariant. Similarly, applying $m \cdot (1 \otimes \alpha)(S \otimes 1)$, one can show that

$$c_2 \equiv \sum S(\omega_i) \alpha \omega^i \tag{3.21}$$

is a pseudoinvariant. It follows from propositions 1 and 2 that

$$\begin{aligned} C_1 &= \sum \bar{X}_\nu \omega_i \beta S(\omega^i) S(\bar{Y}_\nu) \alpha \bar{Z}_\nu = \sum S(X_\nu) \alpha Y_\nu \omega_i \beta S(\omega^i) Z_\nu, \\ C_2 &= \sum S(X_\nu) S(\omega_i) \alpha \omega^i Y_\nu \beta S(Z_\nu) = \sum \bar{X}_\nu \beta S(\bar{Y}_\nu) S(\omega_i) \alpha \omega^i \bar{Z}_\nu \end{aligned} \tag{3.22}$$

are central elements. The invariants (3.22) are usually referred to as quadratic invariants.

IV. TWISTING INVARIANCE OF CENTRAL ELEMENTS C_1 AND C_2

Lemma 2: Let $c_1 \in A$ be even and invariant, and $c_2 \in A$ be even and pseudoinvariant. For any $\eta \in A \otimes A, \xi \in A \otimes A \otimes A$, we have, $\forall a, b \in A$,

$$\begin{aligned} (i) \quad & m \cdot (1 \otimes c_1)(1 \otimes S)(\eta \Delta(a)) = \epsilon(a) m \cdot (1 \otimes c_1)(1 \otimes S) \eta, \\ (ii) \quad & m \cdot (1 \otimes c_2)(S \otimes 1)(\Delta(a) \eta) = \epsilon(a) m \cdot (1 \otimes c_2)(S \otimes 1) \eta, \\ (iii) \quad & m(m \otimes c_1 \otimes c_2)(1 \otimes S \otimes 1)[(1 \otimes \Delta(a)) \cdot \xi \cdot (\Delta(b) \otimes 1)] \\ & = \epsilon(a) \epsilon(b) m(m \otimes 1) \cdot (1 \otimes c_1 \otimes c_2)(1 \otimes S \otimes 1) \xi, \end{aligned} \tag{4.1}$$

$$\begin{aligned} (i v) \quad & m(m \otimes c_2 \otimes c_1)(S \otimes 1 \otimes S)[(\Delta(a) \otimes 1) \cdot \xi \cdot (1 \otimes \Delta(b))] \\ & = \epsilon(a)\epsilon(b)m(m \otimes 1) \cdot (1 \otimes c_2 \otimes c_1)(S \otimes 1 \otimes S)\xi. \end{aligned}$$

The proof of this lemma is a straightforward computation, which we omit.

Lemma 3:

$$\begin{aligned} c_1^F &= m \cdot (1 \otimes c_1)(1 \otimes S)F = \sum f_i c_1 S(f^i), \\ c_2^F &= m \cdot (1 \otimes c_2)(S \otimes 1)F^{-1} = \sum S(\bar{f}_i) c_2 \bar{f}^i, \end{aligned} \tag{4.2}$$

are invariant and pseudoinvariant, respectively, under the twisted structure $\Delta_F(a) = F\Delta(a)F^{-1}$, $\forall a \in A$.

Proof: Write

$$\Delta_F(a) = \sum a_{(1)}^F \otimes a_{(2)}^F = F \sum a_{(1)} \otimes a_{(2)} F^{-1}. \tag{4.3}$$

Then, $\forall a \in A$,

$$\begin{aligned} \sum S(a_{(1)}^F) c_2^F a_{(2)}^F &= \sum S(f_i a_{(1)} \bar{f}_j) S(\bar{f}_k) c_2 \bar{f}^k f^i a_{(2)} \bar{f}^j (-1)^{[f^i][a_{(1)}] + [\bar{f}_j] + [a_{(2)}][\bar{f}_j]} \\ &= \sum S(\bar{f}_j) S(a_{(1)}) S(\bar{f}_k f_i) c_2 \bar{f}^k \bar{f}^i a_{(2)} \bar{f}^j (-1)^{[f_i][\bar{f}_k] + [a][\bar{f}_j]} \\ &= \sum S(\bar{f}_j) S(a_{(1)}) \epsilon(F^{-1}F) c_2 a_{(2)} \bar{f}^j (-1)^{[a][\bar{f}_j]} \end{aligned}$$

by (3.14)

$$= \epsilon(a) \sum S(\bar{f}_j) c_2 \bar{f}^j = \epsilon(a) c_2^F. \tag{4.4}$$

Similarly, one can prove $\sum a_{(1)}^F c_1^F S(a_{(2)}^F) = \epsilon(a) c_1^F$, $\forall a \in A$.

We thus arrive at the following central elements induced by twisting with F :

$$\begin{aligned} C_1^F &= \sum S(X_v^F) \alpha_F Y_v^F c_1^F S(Z_v^F) = m(m \otimes 1) \cdot (1 \otimes \alpha_F \otimes c_1^F)(S \otimes 1 \otimes S) \Phi_F, \\ C_2^F &= \sum S(X_v^F) c_2^F Y_v^F \beta_F S(Z_v^F) = m(m \otimes 1) \cdot (1 \otimes c_2^F \otimes \beta_F)(S \otimes 1 \otimes S) \Phi_F, \end{aligned} \tag{4.5}$$

which corresponds to (3.6) and (3.15), respectively. Here α_F and β_F are the twisted canonical elements given in (2.15).

Theorem 3: *The central elements (4.5) induced by twisting with F coincide precisely with the central elements C_1 , C_2 defined by (3.6) and (3.15), respectively. In other words, the central elements C_1 and C_2 are invariant under twisting.*

To prove this theorem, we first notice

Lemma 4: For any elements $\eta \in A \otimes A$ and $\xi \in A \otimes A \otimes A$,

- (i) $m \cdot (1 \otimes c_1^F)(1 \otimes S) \eta = m \cdot (1 \otimes c_1)(1 \otimes S)(\eta F)$,
- (ii) $m \cdot (1 \otimes c_2^F)(S \otimes 1) \eta = m \cdot (1 \otimes c_2)(S \otimes 1)(F^{-1} \eta)$,

$$\begin{aligned}
 & (iii) \quad m \cdot (m \otimes 1) \cdot (1 \otimes c_1^F \otimes c_2^F)(1 \otimes S \otimes 1)\xi \\
 & \qquad \qquad = m \cdot (m \otimes 1) \cdot (1 \otimes c_1 \otimes c_2)(1 \otimes S \otimes 1)[(1 \otimes F^{-1}) \cdot \xi \cdot (F \otimes 1)], \\
 & (iv) \quad m \cdot (m \otimes 1) \cdot (1 \otimes c_2^F \otimes c_1^F)(S \otimes 1 \otimes S)\xi \\
 & \qquad \qquad = m \cdot (m \otimes 1) \cdot (1 \otimes c_2 \otimes c_1)(S \otimes 1 \otimes S)[(F^{-1} \otimes 1) \cdot \xi \cdot (1 \otimes F)].
 \end{aligned}
 \tag{4.6}$$

This lemma is proven by direct computation. Now with the help of (2.14), lemmas 3 and 4 and using the obvious fact that β, α are invariant and pseudoinvariant of A , respectively, one can easily show that indeed $C_1^F = C_1$ and $C_2^F = C_2$.

For the quadratic-type invariants (3.20) and (3.21), we have the central elements [cf. (3.22)],

$$\begin{aligned}
 C_1 &= \sum S(X_\nu) \alpha Y_\nu \omega_i \beta S(\omega^i) S(Z_\nu) = m(m \otimes 1) \cdot (S \otimes \alpha \otimes \beta S)[\Phi(1 \otimes \omega)], \\
 C_2 &= \sum S(X_\nu) S(\omega_i) \alpha \omega^i Y_\nu \beta S(Z_\nu) = m(m \otimes 1) \cdot (S \otimes \alpha \otimes \beta S)[(\omega \otimes 1)\Phi].
 \end{aligned}
 \tag{4.7}$$

By (4.2) and Lemma 4(i)(ii), one has

$$\begin{aligned}
 c_1^F &= \sum f_j c_1 S(f^j) = m \cdot (1 \otimes \beta S)(F \omega) = m \cdot (1 \otimes \beta_F S) \omega_F, \\
 c_2^F &= \sum S(\bar{f}_j c_2 \bar{f}^j) = m \cdot (S \otimes \alpha)(\omega F^{-1}) = m \cdot (S \otimes \alpha_F) \omega_F,
 \end{aligned}
 \tag{4.8}$$

where

$$\omega_F = F \omega F^{-1}, \tag{4.9}$$

obviously commutes with the action of the twisted coproduct $\Delta_F : \Delta_F(a) \omega_F = \omega_F \Delta_F(a), \forall a \in A$. In this notation, we have central elements

$$\begin{aligned}
 C_1^F &= m(m \otimes 1) \cdot (1 \otimes \alpha_F \otimes \beta_F)(S \otimes 1 \otimes S)[\Phi_F(1 \otimes \omega_F)], \\
 C_2^F &= m(m \otimes 1) \cdot (1 \otimes \alpha_F \otimes \beta_F)(S \otimes 1 \otimes S)[(\omega_F \otimes 1)\Phi_F],
 \end{aligned}
 \tag{4.10}$$

which, as a corollary of Theorem 3, reduce to C_1 and C_2 defined in (4.7), respectively, independent of the twist applied.

In the case that A is quasitriangular with the universal R -matrix \mathcal{R} , where $\Delta(a) \mathcal{R}^T \mathcal{R} = \mathcal{R}^T \mathcal{R} \Delta(a), \forall a \in A$, so we can take $\omega = (\mathcal{R}^T \mathcal{R})^m, m \in \mathbf{Z}$. Then we obtain families of Casimir invariants C_1^m and $C_2^m, m \in \mathbf{Z}$, which are invariant under twisting.

V. INVARIANT BILINEAR FORMS AND INVARIANT FORMS

Let V, W be two (graded) A -modules, and $\mathcal{L}(V, W)$ the space of vector space maps (i.e., linear maps) from V to W . We make $\mathcal{L}(V, W)$ into a (graded) A -module with the definition

$$(a \cdot f)(v) = \sum a_{(1)} f(S(a_{(2)})v) (-1)^{[f][a_{(2)}]}, \quad \forall a \in A, v \in V, f \in \mathcal{L}(V, W). \tag{5.1}$$

We call f invariant if $a \cdot f = \epsilon(a)f, \forall a \in A$. Or equivalently, $\forall a \in A, v \in V,$

$$\sum a_{(1)} f(S(a_{(2)} v)) (-1)^{[f][a_{(2)}]} = \epsilon(a) f(v). \tag{5.2}$$

In the case of normal Hopf (super)algebras, such invariants correspond precisely to A -module homomorphisms, provided they are even. This is not the case for quasi-Hopf (super)algebras. Nevertheless, there is a close connection between such invariants and A -module homomorphisms.

Proposition 3: Suppose $f \in \mathcal{L}(V, W)$ is even and invariant. Set

$$\tilde{f}(v) = \sum S(X_\nu) \alpha Y_\nu f(S(Z_\nu) v), \quad \forall v \in V. \tag{5.3}$$

Then (i) $\tilde{f} \in \mathcal{L}(V, W)$ is an A -module homomorphism, and

$$(ii) \quad \beta \tilde{f}(v) = f(v), \quad \forall v \in V, \tag{5.4}$$

$$(iii) \quad \tilde{f}(v) = \sum \bar{X}_\nu f(S(\bar{Y}_\nu) \alpha \bar{Z}_\nu v), \quad \forall v \in V.$$

Proof: Applying $m \cdot (1 \otimes S)$ to Lemma 1(ii) and using (5.2), one derives,

$$\tilde{f}(S(a) v) = S(a) \tilde{f}(v), \quad \forall a \in A, \quad v \in V. \tag{5.5}$$

Thus \tilde{f} is an A -module homomorphism. This proves (i). As for (ii), note

$$\tilde{f}(v) = m(m \otimes 1) \cdot (S \otimes \alpha \otimes f S) \cdot \Phi \cdot (1 \otimes 1 \otimes v). \tag{5.6}$$

Then by (2.2),

$$\begin{aligned} 1 \otimes \tilde{f}(v) &= \sum \bar{X}_\nu \bar{X}_\mu X_\sigma^{(1)} X_\rho \otimes S(Y_\rho) S(X_\sigma^{(2)}) S(\bar{Y}_\mu) S(\bar{Y}_\nu^{(1)}) \alpha \bar{Y}_\nu^{(2)} \bar{Z}_\mu Y_\sigma Z_\rho^{(1)} \\ &\quad \cdot f(S(Z_\rho^{(2)}) S(Z_\sigma) S(\bar{Z}_\nu) v) (-1)^y, \end{aligned} \tag{5.7}$$

where

$$y = ([X_\sigma] + [Z_\rho])([\bar{X}_\nu] + [\bar{X}_\mu]) + [X_\sigma^{(2)}][Z_\rho] + [\bar{Z}_\nu][X_\sigma] + [\bar{X}_\mu][\bar{Y}_\nu + [Z_\rho][X_\sigma]]. \tag{5.8}$$

By (2.5) and (5.2),

$$\begin{aligned} 1 \otimes \tilde{f}(v) &= \sum \bar{X}_\nu \bar{X}_\mu X_\sigma^{(1)} X_\rho \otimes S(Y_\rho) S(X_\sigma^{(2)}) S(\bar{Y}_\mu) \epsilon(\bar{Y}_\nu) \alpha \bar{Z}_\mu Y_\sigma \epsilon(Z_\rho) f(S(Z_\sigma) S(\bar{Z}_\nu) v) (-1)^y, \\ &= (1 \otimes m) \cdot (1 \otimes S \otimes 1) \sum (\bar{X}_\nu \bar{X}_\mu X_\sigma^{(1)} \otimes \epsilon(\bar{Y}_\nu) \bar{Y}_\mu X_\sigma^{(2)} \otimes \alpha \bar{Z}_\mu Y_\sigma f(S(Z_\sigma) S(\bar{Z}_\nu) v)) (X_\rho \otimes Y_\rho \\ &\quad \otimes \epsilon(Z_\rho)) (-1)^{[X_\sigma]([\bar{Z}_\nu] + [\bar{X}_\mu]) + [\bar{Z}_\nu][X_\sigma] + [\bar{Y}_\mu][X_\sigma^{(2)}]} \\ &= \sum \bar{X}_\nu \bar{X}_\mu X_\sigma^{(1)} \otimes \epsilon(\bar{Y}_\nu) S(X_\sigma^{(2)}) S(\bar{Y}_\mu) \alpha \bar{Z}_\mu Y_\sigma f(S(\bar{Z}_\nu Z_\sigma) v) (-1)^{[\bar{Z}_\nu][Z_\sigma] + [X_\sigma][\bar{X}_\mu]} \end{aligned}$$

by (2.10)

$$\begin{aligned}
 &= (1 \otimes m) \cdot (1 \otimes 1 \otimes fS) \sum (\bar{X}_\nu \otimes \epsilon(\bar{Y}_\nu) \otimes \bar{Z}_\nu) (\bar{X}_\mu X_\sigma^{(1)} \otimes S(X_\sigma^{(2)}) S(\bar{Y}_\mu) \alpha Z_\mu Y_\sigma \otimes Z_\sigma \mathbf{v}) (-1)^{[X_\sigma][\bar{X}_\mu]} \\
 &= \sum \bar{X}_\mu X_\sigma^{(1)} \otimes S(X_\sigma^{(2)}) S(\bar{Y}_\mu) \alpha Z_\mu Y_\sigma f(S(Z_\sigma) \mathbf{v}) (-1)^{[X_\sigma][\bar{X}_\mu]} \tag{5.9}
 \end{aligned}$$

by (2.4).

Applying $m \cdot (1 \otimes \beta)$ gives

$$\begin{aligned}
 \beta \tilde{f}(\mathbf{v}) &= \sum \bar{X}_\mu X_\sigma^{(1)} \beta S(X_\sigma^{(2)}) S(\bar{Y}_\mu) \alpha Z_\mu Y_\sigma f(S(Z_\sigma) \mathbf{v}) (-1)^{[X_\sigma][\bar{X}_\mu]} \\
 &= \sum \bar{X}_\mu \epsilon(X_\sigma) \beta S(\bar{Y}_\mu) \alpha Z_\mu Y_\sigma f(S(Z_\sigma) \mathbf{v}) (-1)^{[X_\sigma][\bar{X}_\mu]}
 \end{aligned}$$

by (2.6)

$$= \sum \epsilon(X_\sigma) Y_\sigma f(S(Z_\sigma) \mathbf{v}) = f(\mathbf{v}), \tag{5.10}$$

by (2.7) and (2.10), which proves (ii). (iii) is a direct consequence of (ii) and (i).

In the special case where $W = \mathbf{C}$ is one-dimensional, we obtain the dual space $V^* = l(V, \mathbf{C})$ which thus becomes a graded A -module with the definition,

$$a \cdot f(\mathbf{v}) = \sum \epsilon(a_{(1)}) f(S(a_{(2)}) \mathbf{v}) (-1)^{[f][a_{(2)}]} = (-1)^{[f][a]} f(S(a) \mathbf{v}), \quad \forall a \in A, \mathbf{v} \in V, f \in V^*. \tag{5.11}$$

We note that $f \in V^*$ is an A -invariant if and only if

$$\epsilon(a) f(\mathbf{v}) = a \cdot f(\mathbf{v}) = (-1)^{[f][a]} f(S(a) \mathbf{v}), \quad \forall a \in A, \tag{5.12}$$

or equivalently, since $\epsilon(a) = 0$ if $[a] = 1$ and $\epsilon(S^{-1}(a)) = \epsilon(a)$,

$$\epsilon(a) f(\mathbf{v}) = (S^{-1} \cdot f)(\mathbf{v}) = f(a \mathbf{v}), \quad \forall a \in A. \tag{5.13}$$

A bilinear form $(\ , \)$ on V and W is equivalent to an element $\xi \in (V \otimes W)^*$ defined by

$$\xi(\mathbf{v} \otimes \mathbf{w}) = (\mathbf{v}, \mathbf{w}), \quad \forall \mathbf{v} \in V, \mathbf{w} \in W. \tag{5.14}$$

We say the form is invariant if ξ is invariant. From (5.13) this is equivalent to

$$\epsilon(a) \xi(\mathbf{v} \otimes \mathbf{w}) = \xi(\Delta(a)(\mathbf{v} \otimes \mathbf{w})) = \sum \xi(a_{(1)} \mathbf{v} \otimes a_{(2)} \mathbf{w}) (-1)^{[v][a_{(2)}]}, \quad \forall a \in A. \tag{5.15}$$

Thus a bilinear form is invariant iff

$$\sum (a_{(1)} \mathbf{v}, a_{(2)} \mathbf{w}) (-1)^{[v][a_{(2)}]} = \epsilon(a) (\mathbf{v}, \mathbf{w}), \quad \forall a \in A, \mathbf{v} \in V, \mathbf{w} \in W. \tag{5.16}$$

In particular, a bilinear form $(\ , \)$ on A itself is called invariant iff

$$\sum (\text{Ad } a_{(1)} \cdot b, \text{Ad } a_{(2)} \cdot c) (-1)^{[b][a_{(2)}]} = \epsilon(a) (b, c), \quad \forall a, b, c \in A. \tag{5.17}$$

Of particular interest are linear forms on A which correspond to elements ξ of A^* . Such a linear form ξ is called invariant if it is an invariant element of A^* , i.e., $a \cdot \xi = \epsilon(a)\xi$, $\forall a \in A$. Equivalently, $\xi \in A^*$ is called an invariant linear form on A^* if

$$\xi(\text{Ad } a \cdot b) = (S^{-1}(a) \cdot \xi)(b) = \epsilon(a)\xi(b), \quad \forall a, b \in A. \tag{5.18}$$

A linear form $\xi \in A^*$ is called pseudoinvariant if

$$\xi(\overline{\text{Ad}} a \cdot b) = \epsilon(a)\xi(b), \quad \forall a, b \in A. \tag{5.19}$$

Summarizing, $\xi \in A^*$ is called a pseudoinvariant linear form on A if

$$\sum \xi(S(a_{(1)})ba_{(2)})(-1)^{[b][a_{(1)}]} = \epsilon(a)\xi(b), \quad \forall a, b \in A, \tag{5.20}$$

and an invariant linear form on A if

$$\sum \xi(a_{(1)}bS(a_{(2)}))(-1)^{[b][a_{(2)}]} = \epsilon(a)\xi(b), \quad \forall a, b \in A. \tag{5.21}$$

It is easily seen that given any (graded) A -module V , the even invariants of $V^* = l(V, \mathbb{C})$ correspond precisely with the A -module homomorphisms $f \in V^*$. Thus the even invariant forms on A correspond to A -module homomorphisms $\xi \in A^*$, regarding A as a module under the adjoint actions.

VI. CASIMIR INVARIANTS FROM INVARIANT FORMS

We now investigate the construction of central elements utilizing invariant and pseudoinvariant linear forms on A . In the case A is quasitriangular, we shall see how such central elements may be constructed, corresponding to any finite dimensional A -module, utilizing the universal R -matrix.

Proposition 4: Suppose $\theta = \sum a_i \otimes b_i \otimes c_i \in A^{\otimes 3}$ obeys

$$(1 \otimes \Delta)\Delta(a) \cdot \theta = \theta \cdot (1 \otimes \Delta)\Delta(a), \quad \forall a \in A. \tag{6.1}$$

If $\xi \in A^*$ is an even invariant form, then

$$C = (1 \otimes \xi)(1 \otimes m)(1 \otimes 1 \otimes \beta S)\theta = \sum a_i \xi(b_i \beta S(c_i)) \tag{6.2}$$

is a central element. Similarly if $\bar{\theta} = \sum \bar{a}_i \otimes \bar{b}_i \otimes \bar{c}_i \in A^{\otimes 3}$ satisfies

$$\bar{\theta} \cdot (\Delta \otimes 1)\Delta(a) = (\Delta \otimes 1)\Delta(a) \cdot \bar{\theta}, \quad \forall a \in A, \tag{6.3}$$

and $\bar{\xi} \in A^*$ is an even pseudoinvariant form then

$$\bar{C} = (\bar{\xi} \otimes 1)(m \otimes 1)(S \otimes \alpha \otimes 1)\bar{\theta} = \sum \bar{\xi}(S(\bar{a}_i) \alpha \bar{b}_i) \bar{c}_i \tag{6.4}$$

is a central element.

Proof: Applying $(1 \otimes m)(1 \otimes 1 \otimes \beta S)$ to (6.1), one has

$$\text{l.h.s.} = \sum a_{(1)} a_i \otimes \text{Ad } a_{(2)} \cdot (b_i \beta S(c_i)) (-1)^{[a_i][a_{(2)}]},$$

$$\begin{aligned} \text{r.h.s.} &= \sum a_i a_{(1)}^R \otimes b_i a_{(2)}^R \beta S(a_{(3)}^R) S(c_i) (-1)^{[c_i]([a_{(2)}^R] + [a_{(3)}^R]) + [a_{(1)}^R]([b_i] + [c_i])} \\ &= \sum a_i a_{(1)} \epsilon(a_{(2)}) \otimes b_i \beta S(c_i) (-1)^{[a]([b_i] + [c_i])} \end{aligned}$$

by (2.6)

$$= \sum a_i a \otimes b_i \beta S(c_i) (-1)^{[a]([b_i] + [c_i])}. \tag{6.5}$$

by (2.3).

Applying $(1 \otimes \xi)$ gives

$$\begin{aligned} \text{l.h.s.} &= \sum a_{(1)} a_i \otimes \xi(\text{Ad } a_{(2)} \cdot (b_i \beta S(c_i))) (-1)^{[a_i][a_{(2)}]} \\ &= \sum (a_{(1)} a_i \xi(\text{Ad } a_{(2)} \cdot (b_i \beta S(c_i))) \otimes 1 (-1)^{[a_i][a_{(2)}]} \\ &= \sum a_{(1)} a_i \epsilon(a_{(2)}) \xi(b_i \beta S(c_i)) \otimes 1 = a C \otimes 1, \end{aligned}$$

by (5.18) and (2.3),

$$\begin{aligned} \text{r.h.s.} &= \sum a_i a \otimes \xi(b_i \beta S(c_i)) (-1)^{[a]([b_i] + [c_i])} \\ &= \sum a_i a \xi(b_i \beta S(c_i)) \otimes 1 = C a \otimes 1, \end{aligned} \tag{6.6}$$

where in the second last equality we have used the fact that ξ is even, i.e., $\xi(a) = 0$ if $[a] = 1$. This proves the first part of the proposition. The second part can be proved in a similar way.

It is easily shown that

$$\theta = \Phi^{-1}(\omega \otimes 1)\Phi, \quad \bar{\theta} = \Phi(1 \otimes \omega)\Phi^{-1} \tag{6.7}$$

satisfy (6.1) and (6.3), respectively. Thus as a corollary of Proposition 4 we have the central elements,

$$\begin{aligned} C &= \sum \xi(\bar{Y}_\nu \omega^i Y_\mu \beta S(\bar{Z}_\nu Z_\mu)) \bar{X}_\nu \omega_i X_\mu (-1)^{[X_\mu][\bar{Y}_\nu] + [Y_\mu][\bar{Z}_\nu] + [\omega_i]([X_\mu] + [\bar{Y}_\nu])} \\ &= (1 \otimes \xi)(1 \otimes m)(1 \otimes 1 \otimes \beta S)[\Phi^{-1}(\omega \otimes 1)\Phi], \\ \bar{C} &= \sum \bar{\xi}(S(X_\nu \bar{X}_\mu) \alpha Y_\nu \omega_i \bar{Y}_\mu) Z_\nu \omega^i \bar{Z}_\mu (-1)^{[Z_\nu][\bar{Z}_\mu] + [Y_\nu][\bar{X}_\mu] + [\omega_i]([Z_\nu] + [\bar{Y}_\mu])} \\ &= (\bar{\xi} \otimes 1)(m \otimes 1)(S \otimes \alpha \otimes 1)[\Phi(1 \otimes \omega)\Phi^{-1}]. \end{aligned} \tag{6.8}$$

A quasi-Hopf (super)algebra is said to be of trace type if there exists an invertible even element $u \in A$ such that

$$S^2(a) = u a u^{-1}, \quad \forall a \in A. \tag{6.9}$$

In the case A is quasitriangular with the R -matrix as in (2.23) we have

Proposition 5: The operator defined by

$$u = \sum S(Y_\nu \beta S(Z_\nu)) S(e^i) \alpha e_i X_\nu (-1)^{[e_i] + [X_\nu]} \tag{6.10}$$

satisfies (6.9). Moreover the inverse is given by

$$u^{-1} = S^2(u^{-1}) = \sum S^{-1}(X_\nu) S^{-1}(\alpha e^{-i}) \bar{e}_i Y_\nu \beta S(Z_\nu) (-1)^{[\bar{e}_i]}. \tag{6.11}$$

Proof: The nonsuper case was proven in Ref. 3. We here prove the super case. First observe

$$\begin{aligned} S^2(a)u &= \sum S^2(a_{(3)}^L) S(Y_\nu \beta S(Z_\nu)) S(e^i) S(a_{(1)}^L) \alpha a_{(2)}^L e_i X_\nu \\ &\quad \times (-1)^{([a_{(1)}^L] + [a_{(2)}^L])([e_i] + [a_{(3)}^L] + [X_\nu]) + [e_i] + [X_\nu]} \end{aligned}$$

by (2.5) and (2.3)

$$\begin{aligned} &= \sum S^2(a_{(3)}^L) S(Y_\nu \beta S(Z_\nu)) m \cdot [(S \otimes \alpha)(a_{(1)}^L \otimes a_{(2)}^L)(e^i \otimes e_i)] \\ &\quad \times X_\nu (-1)^{([a_{(1)}^L] + [a_{(2)}^L])([a_{(3)}^L] + [X_\nu]) + [e_i] + [X_\nu]} \\ &= \sum S^2(a_{(3)}^L) S(Y_\nu \beta S(Z_\nu)) S(a_{(2)}^L) S(e^i) \alpha e_i a_{(1)}^L \\ &\quad \times X_\nu (-1)^{([a_{(1)}^L] + [a_{(2)}^L])([a_{(3)}^L] + [X_\nu]) + [e_i] + [X_\nu] + [a_{(1)}^L] + [a_{(2)}^L]} \end{aligned}$$

by (2.16)

$$\begin{aligned} &= \sum S(a_{(2)}^L) Y_\nu \beta S(Z_\nu) S(a_{(3)}^L) S(e^i) \alpha e_i a_{(1)}^L X_\nu \\ &\quad \times (-1)^{[a_{(1)}^L]([a_{(2)}^L] + [a_{(3)}^L]) + ([a_{(1)}^L] + [a_{(3)}^L])[X_\nu] + [X_\nu] + [e_i]}. \tag{6.12} \end{aligned}$$

Applying $m \cdot T \cdot [\sum S(e^i) \alpha e_i (-1)^{[e_i]} \otimes S]$ to Lemma 1(i), one has

$$\begin{aligned} ua &= \sum S(a_{(2)}^L) Y_\nu \beta S(Z_\nu) S(a_{(3)}^L) S(e^i) \alpha e_i a_{(1)}^L \\ &\quad \times X_\nu (-1)^{[a_{(1)}^L]([a_{(2)}^L] + [a_{(3)}^L]) + ([a_{(1)}^L] + [a_{(3)}^L])[X_\nu] + [X_\nu] + [e_i]} \\ &= S^2(a)u \tag{6.13} \end{aligned}$$

by (6.12).

It remains to show that u is invertible. First, we have

Lemma 5:

$$S(\alpha)u = \sum S(e^i) \alpha e_i (-1)^{[e_i]} = m \cdot (S \otimes \alpha) \mathcal{R}^T. \tag{6.14}$$

Proof: Note

$$\begin{aligned}
 u \otimes 1 &= \left[m \cdot \left(S \otimes \sum S(e^i) \alpha e_i (-1)^{[e_i]} \right) (m \otimes 1) \otimes 1 \right] \\
 &\quad \cdot (1 \otimes \beta S \otimes 1 \otimes 1) (1 \otimes T \otimes 1) (T \otimes 1 \otimes 1) (\Phi \otimes 1) \\
 &= \sum S(X_\nu^{(2)} Y_\mu \bar{X}_\sigma \bar{Y}_\rho^{(1)}) \beta S(\bar{Y}_\sigma^{(2)}) S(Y_\nu Z_\mu^{(1)} \bar{Y}_\sigma) S(e^i) \alpha e_i X_\nu^{(1)} X_\mu \bar{X}_\rho \otimes Z_\nu Z_\mu^{(2)} \bar{Z}_\sigma \bar{Z}_\rho (-1)^z
 \end{aligned} \tag{6.15}$$

by (2.2), where,

$$\begin{aligned}
 z &= [e_i] + [X_\nu^{(1)}][X_\nu^{(2)}] + ([X_\mu] + [X_\nu^{(1)}])([Y_\mu] + [Y_\nu] + [\bar{Z}_\sigma] + [\bar{Y}_\rho] + [Z_\mu^{(1)}]) + [\bar{X}_\sigma]([Z_\mu] + [Z_\nu] \\
 &\quad + [\bar{Z}_\sigma] + [\bar{Y}_\rho] + [Z_\mu^{(1)}]) + [\bar{X}_\sigma]([Z_\mu] + [X_\nu]) + [\bar{Y}_\sigma]([Z_\nu] + [Z_\mu^{(2)}]) + [Z_\mu][X_\nu] + [Z_\nu][Z_\mu^{(1)}] \\
 &\quad + [\bar{Y}_\rho]([Z_\mu] + [X_\nu] + [\bar{X}_\sigma]).
 \end{aligned} \tag{6.16}$$

By (2.6), one has

$$u \otimes 1 = \sum [S(X_\nu^{(2)} Y_\mu \bar{X}_\sigma \beta S(Y_\nu Z_\mu^{(1)} \bar{Y}_\sigma)) S(e^i) \alpha e_i X_\nu^{(1)} X_\mu \otimes Z_\nu Z_\mu^{(2)} \bar{Z}_\sigma] (\bar{X}_\rho \epsilon(\bar{Y}_\rho) \otimes \bar{Z}_\rho) (-1)^{\bar{z}}, \tag{6.17}$$

where

$$\begin{aligned}
 \bar{z} &= [e_i] + [X_\nu^{(1)}][X_\nu^{(2)}] + ([X_\mu] + [X_\nu^{(1)}])([Y_\mu] + [Y_\nu] + [\bar{Z}_\sigma] + [Z_\mu^{(1)}]) \\
 &\quad + [\bar{X}_\sigma]([Z_\mu] + [X_\nu]) + [\bar{Y}_\sigma]([Z_\nu] + Z_\mu^{(2)}) + [Z_\mu][X_\nu] + [Z_\nu][Z_\mu^{(1)}].
 \end{aligned} \tag{6.18}$$

By (2.4), one gets

$$\begin{aligned}
 u \otimes 1 &= \sum S(X_\nu^{(2)} Y_\mu \bar{X}_\sigma \beta S(Y_\nu Z_\mu^{(1)} \bar{Y}_\sigma)) S(e^i) \alpha e_i X_\nu^{(1)} X_\mu \otimes Z_\nu Z_\mu^{(2)} \bar{Z}_\sigma (-1)^{\bar{z}} \\
 &= \sum S(Y_\mu \bar{X}_\sigma \beta S(Y_\nu Z_\mu^{(1)} \bar{Y}_\sigma)) S(e^i) S(X_\nu^{(1)}) \alpha X_\nu^{(2)} e_i X_\mu \otimes Z_\nu Z_\mu^{(2)} \bar{Z}_\sigma \\
 &\quad \times (-1)^{\bar{z} + [X_\nu^{(2)}]([Y_\mu] + [Y_\nu] + [\bar{Z}_\sigma] + [Z_\mu^{(1)}]) + [e_i] + [e_i][X_\nu]}
 \end{aligned}$$

by (2.16)

$$\begin{aligned}
 &= \sum (\epsilon(X_\nu) S^2(Y_\nu) \otimes X_\nu) (S(Y_\mu \bar{X}_\sigma \beta S(Z_\mu^{(1)} \bar{Y}_\sigma)) S(e^i) \alpha e_i X_\mu \otimes Z_\mu^{(2)} \bar{Z}_\sigma) \\
 &\quad \times (-1)^{[e_i] + [X_\mu]([Y_\mu] + [\bar{Z}_\sigma] + [Z_\mu^{(1)}]) + [\bar{X}_\sigma][Z_\mu] + [\bar{Y}_\sigma][Z_\mu^{(2)}]}
 \end{aligned}$$

by (2.5)

$$\begin{aligned}
 &= \sum S(Y_\mu \bar{X}_\sigma \beta S(\bar{Y}_\sigma) S(Z_\mu^{(1)})) S(e^i) \alpha e_i X_\mu \otimes Z_\mu^{(2)} \bar{Z}_\sigma \\
 &\quad \times (-1)^{[e_i] + [X_\mu]([Y_\mu] + [\bar{Z}_\sigma] + [Z_\mu^{(1)}]) + [\bar{X}_\sigma][Z_\mu] + [\bar{Y}_\sigma][Z_\mu]}
 \end{aligned} \tag{6.19}$$

by (2.10).

Applying $m \cdot (1 \otimes S(\alpha)) \cdot T \cdot (1 \otimes S)$ gives

$$\begin{aligned}
 S(\alpha)u &= \sum S(\bar{Z}_\sigma)S(S(Z_\mu^{(1)})\alpha Z_\mu^{(2)})S(Y_\mu\bar{X}_\sigma\beta S(\bar{Y}_\sigma))S(e^i)\alpha e_i X_\mu \\
 &\quad \times (-1)^{[Z_\mu]+[\bar{Z}_\sigma]+[e_i]+[X_\mu]([Y_\mu]+[\bar{Z}_\sigma])} \\
 &= \sum S(\bar{Z}_\sigma)\epsilon(Z_\mu)S(\alpha)S(Y_\mu\bar{X}_\sigma\beta S(\bar{Y}_\sigma))S(e^i)\alpha e_i X_\mu (-1)^{[\bar{Z}_\sigma]+[e_i]+[X_\mu]([Y_\mu]+[\bar{Z}_\sigma])}
 \end{aligned}$$

by (2.5)

$$\begin{aligned}
 &= \sum S(\bar{Z}_\sigma)S(\alpha)S(\bar{X}_\sigma\beta S(\bar{Y}_\sigma))(-1)^{[\bar{Z}_\sigma]}m \cdot T \cdot \left(\sum S(e^i)\alpha e_i (-1)^{[e_i]} \otimes S \right) (X_\mu \otimes Y_\mu \epsilon(Z_\mu)) \\
 &= \sum S(\bar{Z}_\sigma)S(\alpha)S(\bar{X}_\sigma\beta S(\bar{Y}_\sigma))S(e^i)\alpha e_i (-1)^{[e_i]+[\bar{Z}_\sigma]}
 \end{aligned}$$

by (2.10)

$$= \sum S(\bar{X}_\sigma\beta S(\bar{Y}_\sigma)\alpha\bar{Z}_\sigma)S(e^i)\alpha e_i (-1)^{[e_i]} = \sum S(e^i)\alpha e_i (-1)^{[e_i]} \tag{6.20}$$

by (2.7),
thus proving lemma 5.

Lemma 6:

$$u \sum S^{-1}(\alpha e^{-i})\bar{e}_i (-1)^{[e_i]} = \alpha. \tag{6.21}$$

This lemma is easily proved with the help of (6.13) and Lemma 5.

Now we are in a position to prove Proposition 5,

$$1 = \sum S(X_\nu)\alpha Y_\nu\beta S(Z_\nu)$$

by (2.8)

$$= \sum S(X_\nu)uS^{-1}(\alpha e^{-i})\bar{e}_i Y_\nu\beta S(Z_\nu)(-1)^{[\bar{e}_i]}$$

by Lemma 6

$$= u \sum S^{-1}(X_\nu)S^{-1}(\alpha e^{-i})\bar{e}_i Y_\nu\beta S(Z_\nu)(-1)^{[\bar{e}_i]}$$

by (6.13)

$$= S^2 \left(\sum S^{-1}(X_\nu)S^{-1}(\alpha e^{-i})\bar{e}_i Y_\nu\beta S(Z_\nu)(-1)^{[\bar{e}_i]} \right) u \tag{6.22}$$

by (6.13).

It follows that u is invertible, with u^{-1} given by (6.11). This completes our proof for Proposition 5.

Corollary 2: If A is a quasitriangular quasi-Hopf (super) algebra, then A is of trace-type. In particular $S^2(u) = u$, and $uS(u) = S(u)u$ is a central element.

Below we assume that A is a quasi-Hopf (super)algebra of trace type. Let V be finite-dimensional (graded) A -module. Then,

Proposition 6: $\xi \in A^*$ defined by

$$\xi(a) = \text{Str}_V(uS^{-1}(\alpha)a), \quad \forall a \in A \tag{6.23}$$

determines an invariant linear form, and $\bar{\xi} \in A^*$ defined by

$$\bar{\xi}(a) = \text{Str}_V(u^{-1}S(\beta)a), \quad \forall a \in A \tag{6.24}$$

determines a pseudoinvariant linear form.

Proof: By means of (6.9) and the (super)trace property,

$$\text{Str}_V(ab) = (-1)^{[a][b]} \text{Str}_V(ba), \quad \forall a, b \in A \tag{6.25}$$

one has, $\forall a, b \in A$,

$$\begin{aligned} \xi(\text{Ad } a \cdot b) &= \sum \text{Str}_V(uS^{-1}(\alpha)a_{(1)}bS(a_{(2)}))(-1)^{[b][a_{(2)}]} \\ &= \sum \text{Str}_V(S(a_{(2)})uS^{-1}(\alpha)a_{(1)}b)(-1)^{[a_{(1)}][a_{(2)}]} \end{aligned}$$

by (6.25)

$$= \sum \text{Str}_V(uS^{-1}(a_{(2)})S^{-1}(\alpha)a_{(1)}b)(-1)^{[a_{(1)}][a_{(2)}]}$$

by (6.9)

$$\begin{aligned} &= \sum \text{Str}_V(uS^{-1}(S(a_{(1)})\alpha a_{(2)})b) \\ &= \epsilon(a) \cdot \text{Str}_V(uS^{-1}(\alpha)b) \end{aligned}$$

by (2.5)

$$= \epsilon(a)\xi(b). \tag{6.26}$$

Thus we have proved the first part of the proposition. The second part of the proposition is proved in a similar fashion.

It immediately follows from Propositions 4 and 6 that one has

Proposition 7: Let π be the representation afforded by the finite-dimensional (graded) A -module V . Suppose $\theta = \sum a_i \otimes B_i \otimes c_i \in A \otimes \text{End } V \otimes A$ obeys

$$(1 \otimes \pi \otimes 1)(1 \otimes \Delta)\Delta(a) \cdot \theta = \theta \cdot (1 \otimes \pi \otimes 1)(1 \otimes \Delta)\Delta(a), \quad \forall a \in A, \tag{6.27}$$

then

$$C = \sum \text{Str}_V(uS^{-1}(\alpha)B_i\beta S(c_i))a_i \tag{6.28}$$

is a central element. Similarly if $\bar{\theta} = \sum \bar{a}_i \otimes \bar{B}_i \otimes \bar{c}_i \in A \otimes \text{End } V \otimes A$ satisfies

$$\bar{\theta} \cdot (1 \otimes \pi \otimes 1)(\Delta \otimes 1)\Delta(a) = (1 \otimes \pi \otimes 1)(\Delta \otimes 1)\Delta(a) \cdot \bar{\theta}, \quad \forall a \in A. \tag{6.29}$$

Then,

$$\bar{C} = \sum \text{Str}_V(u^{-1}S(\beta)S(\bar{a}_i)\alpha\bar{B}_i)\bar{c}_i \tag{6.30}$$

is a central element.

Corollary 3: Suppose $\omega = \sum \omega_i \otimes \Omega^i \in A \otimes \text{End } V$ satisfies

$$(1 \otimes \pi)\Delta(a) \cdot \omega = \omega \cdot (1 \otimes \pi)\Delta(a), \quad \forall a \in A. \tag{6.31}$$

Then the first equation of (6.8) implies that

$$C = \sum \text{Str}_V(uS^{-1}(\alpha)\bar{Y}_\nu\Omega^i Y_\mu\beta S(\bar{Z}_\nu Z_\mu))\bar{X}_\nu\omega_i X_\mu (-1)^{[X_\mu][\bar{X}_\nu]+[Y_\mu][\bar{Y}_\nu]+[\omega_i]([X_\mu]+\bar{Y}_\nu)} \tag{6.32}$$

is a central element. Similarly if $\bar{\omega} = \sum \bar{\Omega}_i \otimes \bar{\omega}^i \in \text{End } V \otimes A$ satisfies

$$(\pi \otimes 1)\Delta(a) \cdot \bar{\omega} = \bar{\omega} \cdot (\pi \otimes 1)\Delta(a), \quad \forall a \in A. \tag{6.33}$$

Then the second equation of (6.8) means that

$$\bar{C} = \sum \text{Str}_V(u^{-1}S(\beta)S(X_\nu\bar{X}_\mu)\alpha Y_\nu\bar{\Omega}_i\bar{Y}_\mu)Z_\nu\bar{\omega}^i\bar{Z}_\mu (-1)^{[Z_\nu][\bar{Z}_\mu]+[Y_\nu][\bar{X}_\mu]+[\omega^i]([Z_\nu]+\bar{Y}_\mu)} \tag{6.34}$$

is a central element.

Corollary 4: In the case that A is quasitriangular, one takes $\omega = (\mathcal{R}^T\mathcal{R})^m, m \in \mathbb{Z}$. Then we obtain the following families of Casimir invariants associated with $\mathcal{R}^T\mathcal{R}$ and its powers:

$$\begin{aligned} C_m &= (1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \cdot \Phi^{-1}((\mathcal{R}^T\mathcal{R})^m \otimes 1)\Phi, \\ \bar{C}_m &= (\text{Str}_V \otimes 1)(m \otimes 1)(u^{-1}S(\beta)S \otimes \alpha 1) \cdot \Phi(1 \otimes (\mathcal{R}^T\mathcal{R})^m)\Phi^{-1}. \end{aligned} \tag{6.35}$$

Remark: The above invariants are natural generalizations of those obtained in Refs. 12 and 13 to which they reduce in the case of normal Hopf (super)algebras (for which $\Phi = 1 \otimes 1 \otimes 1$).

VII. TWISTING INVARIANCE OF CENTRAL ELEMENTS C AND \bar{C}

In this section we show that the trace-type central elements C and \bar{C} are invariant under twisting. Associated with F , we have the twisted coassociator Φ_F and in the quasitriangular case, the twisted R -matrix \mathcal{R}_F . We write,

$$\begin{aligned} \Phi_F &= \sum X_\nu^F \otimes Y_\nu^F \otimes Z_\nu^F, \\ \Phi_F^{-1} &= \sum \bar{X}_\nu^F \otimes \bar{Y}_\nu^F \otimes \bar{Z}_\nu^F, \\ \mathcal{R}_F &= \sum e_i^F \otimes e_i^F. \end{aligned} \tag{7.1}$$

Lemma 7:

$$\beta = \sum \bar{f}_i \beta_F S(\bar{f}^i), \quad \alpha = \sum S(f_i) \alpha_F f^i. \tag{7.2}$$

This lemma is proven by direct computation by means of (2.15).

Associated with a twist F on a quasitriangular quasi-Hopf (super)algebra, we have the u -operator in terms of the twisted structure, denoted u_F ,

$$u_F = \sum S(Y_\nu^F \beta_F S(Z_\nu^F)) S(e_F^t) \alpha_F e_i^F X_\nu^F (-1)^{[e_i^F] + [X_\nu^F]}. \tag{7.3}$$

Theorem 4: *The u -operator, given explicitly in Proposition 5, is invariant under twisting.*
Proof: We compute u_F . By (2.14), one has

$$u_F = \sum S(f^i f_j^{(2)} Y_{w\bar{j}(1)} \bar{f}_\lambda \beta_F S(\bar{f}^l) S(\bar{f}_k^{(2)} S(f^j Z_\nu)) S(e_F^t) \alpha_F e_i^F f_i f_j^{(1)} X_{w\bar{k}} (-1)^{[e_i^F] + r}, \tag{7.4}$$

where

$$\begin{aligned} r = & [f_i] + ([f_j^{(1)}] + X_\nu) + [\bar{f}_k] ([f^i] + [f_j^{(2)}] + [f^j] + [\bar{f}^k] + [X_\nu]) \\ & + [f^j] ([\bar{f}_k] + [Z_\nu]) + [f^k] ([f^j] + [Z_\nu]) + [f_j^{(2)}] ([\bar{f}_k] + [X_\nu]) + [\bar{f}_k] [X_\nu]. \end{aligned} \tag{7.5}$$

By Lemma 7,

$$\begin{aligned} u_F = & \sum S\left(f^i f_j^{(2)} Y_{w\bar{j}(1)} \bar{f}_\lambda \beta S(\bar{f}_k^{(2)} S(f^j Z_\nu)) S(e_F^t) \alpha_F e_i^F f_i f_j^{(1)} X_{w\bar{k}} (-1)^{[e_i^F] + r}, \right. \\ = & \sum S(f^i f_j^{(2)} Y_\nu \epsilon(\bar{f}^k) \beta S(f^j Z_\nu)) S(e_F^t) \alpha_F e_i^F f_i f_j^{(1)} X_{w\bar{k}} \\ & \times (-1)^{[e_i^F] + [f_i] + [X_\nu] + [f^j] [Y_\nu] + [X_\nu] [f^i] + [f_j^{(1)}] ([f_j^{(2)}] + [f^i] + [f^j] + [X_\nu])} \end{aligned}$$

by (2.6)

$$\begin{aligned} = & \sum S(f_j^{(2)} Y_\nu \beta S(f^j Z_\nu)) S(e_F^t) \alpha_F e_i^F f_i f_j^{(1)} X_\nu \\ & \times (-1)^{[e_i^F] + [f_i] + [X_\nu] + [f^j] [e_i^F] + [f^j] [Y_\nu] + [f_j^{(1)}] ([f_j^{(2)}] + [f^j] + [X_\nu])} \end{aligned}$$

by (2.12)

$$\begin{aligned} = & \sum S(f_j^{(2)} Y_\nu \beta S(f^j Z_\nu)) S(e^t) S(f^i) \alpha_F f^i e_i f_j^{(1)} X_\nu \\ & \times (-1)^{[e_i] + [X_\nu] + [f^j] [Y_\nu] + [f_j^{(1)}] ([f_j^{(2)}] + [f^j] + [X_\nu])} \end{aligned}$$

by (2.16)

$$\begin{aligned} = & \sum S(f_j^{(2)} Y_\nu \beta S(f^j Z_\nu)) S(e^t) \alpha e_i f_j^{(1)} X_\nu \\ & \times (-1)^{[e_i] + [X_\nu] + [f^j] [Y_\nu] + [f_j^{(1)}] ([f_j^{(2)}] + [f^j] + [X_\nu])} \end{aligned}$$

by Lemma 7

$$\begin{aligned}
 &= \sum S(Y_\nu \beta S(f^j Z_\nu)) S(e^t f_j^{(2)}) \alpha e_i f_j^{(1)} X_\nu \\
 &\quad \times (-1)^{[e_i] + [X_\nu] + [f_j^{(2)}][e_i] + [f_j^{(1)}][f_j^{(2)}] + [f_j] + [f_j][Z_\nu]} \\
 &= \sum S(Y_\nu \beta S(f^j Z_\nu)) S(e^t) S(f_j^{(1)}) \alpha f_j^{(2)} e_i X_\nu (-1)^{[e_i] + [X_\nu] + [f_j][e_i] + [f_j] + [f_j][Z_\nu]}
 \end{aligned}$$

by (2.16)

$$= \sum S(Y_\nu \beta S(f^j Z_\nu)) S(e^t) \alpha e_i \epsilon(f_j) X_\nu (-1)^{[e_i] + [X_\nu]}$$

by (2.5)

$$= \sum S(Y_\nu \beta S(Z_\nu)) S(e^t) \alpha e_i X_\nu (-1)^{[e_i] + [X_\nu]}$$

by (2.12)

$$= u. \tag{7.6}$$

Thus we end up with the same u -operator, independently of the twist applied.

Corollary 5:

$$S(u)S(\beta) = \sum e_i \beta S(e^i) = m \cdot (\beta \otimes S) \mathcal{R}. \tag{7.7}$$

Proof: We apply Theorem 4 and Lemma 5 to the special case where F is the Drinfeld twist¹ F_D . In Ref. 14, we proved

$$S(\beta) = \alpha_{F_D}, \quad (S \otimes S) \mathcal{R} = \mathcal{R}_{F_D}. \tag{7.8}$$

Then from Lemma 5 and Theorem 4,

$$S(\alpha_{F_D})u = m \cdot (S \otimes \alpha_{F_D}) \mathcal{R}_{F_D}^T, \tag{7.9}$$

which gives rise to, on using (7.8),

$$S^2(\beta)u = \sum S^2(e^i)S(\beta)S(e_i)(-1)^{[e_i]} = S \sum e_i \beta S(e^i). \tag{7.10}$$

Namely,

$$\sum e_i \beta S(e^i) = S^{-1}(u)S(\beta) = S(u)S(\beta), \tag{7.11}$$

where we have used $S^2(u) = u \cdot u \cdot u^{-1} = u$.

The following result follows as a special case of Proposition 6 applied to the twisted quasi-Hopf (super) algebra structure.

Lemma 8: $\xi_F \in A^*$ defined by

$$\xi_F(a) = \text{Str}_V(uS^{-1}(\alpha_F)a), \quad \forall a \in A \tag{7.12}$$

determines a linear form invariant under the twisted quasi-Hopf (super) algebra structure. Similarly $\bar{\xi} \in A^*$ defined by

$$\bar{\xi}_F(a) = \text{Str}_V(u^{-1}S(\beta_F)a), \quad \forall a \in A \tag{7.13}$$

determines a pseudoinvariant linear form under the twisted structure.

Following Proposition 4, if $\theta \in A^{\otimes 3}$ satisfies (6.1) and $\bar{\theta} \in A^{\otimes 3}$ satisfies (6.3), then we have trace type invariants

$$\begin{aligned} C &= (1 \otimes \text{Str})(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S)\theta, \\ \bar{C} &= (\text{Str} \otimes 1)(m \otimes 1)(u^{-1}S(\beta)S \otimes \alpha \otimes 1)\bar{\theta}. \end{aligned} \tag{7.14}$$

Lemma 9: Suppose $\theta \in A^{\otimes 3}$ satisfies (6.1). Then,

$$\theta_F \equiv (1 \otimes F)(1 \otimes \Delta)F \cdot \theta \cdot (1 \otimes \Delta)F^{-1}(1 \otimes F^{-1}) \tag{7.15}$$

also satisfies (6.1) for the twisted structure; viz.,

$$(1 \otimes \Delta_F)\Delta_F(a) \cdot \theta_F = \theta_F \cdot (1 \otimes \Delta_F)\Delta_F(a), \quad \forall a \in A. \tag{7.16}$$

Similarly if $\bar{\theta} \in A^{\otimes 3}$ satisfies (6.3). Then,

$$\bar{\theta}_F \equiv (F \otimes 1)(\Delta \otimes 1)F \cdot \bar{\theta} \cdot (\Delta \otimes 1)F^{-1}(F^{-1} \otimes 1) \tag{7.17}$$

also satisfies (6.3) for the twisted structure; viz.,

$$(\Delta_F \otimes 1)\Delta_F(a) \cdot \bar{\theta}_F = \bar{\theta}_F \cdot (\Delta_F \otimes 1)\Delta_F(a), \quad \forall a \in A. \tag{7.18}$$

Proof: Applying $(1 \otimes F)(1 \otimes \Delta)F$ to the left and $(1 \otimes \Delta)F^{-1}(1 \otimes F^{-1})$ to the right of (6.1) gives (7.16). Similarly, applying $(F \otimes 1)(\Delta \otimes 1)F$ to the left and $(\Delta \otimes 1)F^{-1}(F^{-1} \otimes 1)$ to the right of (6.3), one gets (7.18).

We thus arrive at the following central elements obtained by twisting those of (7.14) with F :

$$\begin{aligned} C_F &= (1 \otimes \text{Str})(1 \otimes m)(1 \otimes uS^{-1}(\alpha_F) \otimes \beta_F S)\theta_F, \\ \bar{C}_F &= (\text{Str} \otimes 1)(m \otimes 1)(u^{-1}S(\beta_F)S \otimes \alpha_F \otimes 1)\bar{\theta}_F. \end{aligned} \tag{7.19}$$

We shall show that these invariants coincide precisely with those of (7.14). Namely,

Theorem 5: The trace type central elements (7.14) are invariant under twisting.

To prove this theorem, we first state

Lemma 10: $\forall a \in A, \xi \in A^{\otimes 3}$, we have

$$\begin{aligned} (i) \quad & (1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \cdot \xi(1 \otimes \Delta(a)) \\ &= (1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \cdot (1 \otimes \Delta(a))\xi \\ &= \epsilon(a)(1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \cdot \xi, \\ (ii) \quad & (\text{Str}_V \otimes 1)(m \otimes 1)u^{-1}S(\beta)S \otimes \alpha \otimes 1 \cdot (\Delta(a) \otimes 1)\xi \\ &= (\text{Str}_V \otimes 1)(m \otimes 1)(u^{-1}S(\beta)S \otimes \alpha \otimes 1) \cdot \xi(\Delta(a) \otimes 1) \\ &= \epsilon(a)(\text{Str}_V \otimes 1)(m \otimes 1)(u^{-1}S(\beta)S \otimes \alpha \otimes 1) \cdot \xi, \\ (iii) \quad & (1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha_F) \otimes \beta_F S) \cdot \xi \\ &= (1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S)[(1 \otimes F^{-1}) \cdot \xi \cdot (1 \otimes F)], \end{aligned}$$

$$(iv) \quad (\text{Str}_V \otimes 1)(m \otimes 1)(u^{-1}S(\beta_F)S \otimes \alpha_F \otimes 1) \cdot \xi(\text{Str}_V \otimes 1) \\ \times (m \otimes 1)(u^{-1}S(\beta)S \otimes \alpha \otimes 1)[(F^{-1} \otimes 1) \cdot \xi \cdot (F \otimes 1)]. \quad (7.20)$$

Proof: This lemma is proven by direct computations using (6.25), (6.9), (2.5), and (2.6). For demonstration, we show the details for proving some of the relations. Write $\xi = \sum x_i \otimes y_i \otimes z_i \in A^{\otimes 3}$. Then,

$$(1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \cdot (1 \otimes \Delta(a)) \xi \\ = \sum (1 \otimes \text{Str}_V)(x_i \otimes uS^{-1}(\alpha)a_{(1)}y_i\beta S(z_i)S(a_{(2)}))(-1)^{[x_i][a]+[a_{(2)}]} \\ = \sum (1 \otimes \text{Str}_V)(x_i \otimes S(a_{(2)})uS^{-1}(\alpha)a_{(1)}y_i\beta S(z_i))(-1)^{[x_i][a]+[a_{(1)}][a_{(2)}]}$$

by (6.25)

$$= \sum (1 \otimes \text{Str}_V)(x_i \otimes uS^{-1}(S(a_{(1)})\alpha a_{(2)})y_i\beta S(z_i))(-1)^{[z_i][a]}$$

by (6.9)

$$= \epsilon(a)(1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \cdot \xi \quad (7.21)$$

by (2.5).

Other relations in (i) and (ii) are proven similarly. We now prove (iii),

$$(1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha_F) \otimes \beta_F S) \cdot \xi \\ = \sum (1 \otimes \text{Str}_V)(x_i \otimes uS^{-1}(\bar{f}^j)S^{-1}(\alpha)\bar{f}_j y_i f_k \beta S(z_i f^k))(-1)^{[\bar{f}_j]+[z_i][f_k]}$$

by (2.15)

$$= \sum (1 \otimes \text{Str}_V)(x_i \otimes S(\bar{f}^j)uS^{-1}(\alpha)\bar{f}_j y_i f_k \beta S(z_i f^k))(-1)^{[\bar{f}_j]+[z_i][f_k]}$$

by (6.9)

$$= \sum (1 \otimes \text{Str}_V)(x_i \otimes uS^{-1}(\alpha)\bar{f}_j y_i f_k \beta S(\bar{f}^j z_i f^k))(-1)^{[\bar{f}_j]([y_i]+[f_k])+[z_i][f_k]}$$

by (6.25)

$$= (1 \otimes \text{Str}_V)(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S)[1 \otimes F^{-1}] \cdot \xi \cdot (1 \otimes F). \quad (7.22)$$

(iv) Can be proved in a similar fashion.

We are now in a position to prove Theorem 5. From (7.19), one has, by Lemma 10(iii) and (7.15),

$$C_F = (1 \otimes \text{Str})(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S)[(1 \otimes \Delta)F \cdot \theta \cdot (1 \otimes \Delta)F^{-1}] \\ = (1 \otimes \text{Str})(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S)[(1 \otimes \epsilon)F \cdot \theta \cdot (1 \otimes \epsilon)F^{-1}]$$

by Lemma 10(i)

$$= (1 \otimes \text{Str})(1 \otimes m)(1 \otimes uS^{-1}(\alpha) \otimes \beta S) \theta$$

by (2.12)

$$= C. \tag{7.23}$$

Similarly, one can show $\bar{C}_F = \bar{C}$. This completes the proof of Theorem 5.

In the quasitriangular case it is worth noting that when $\theta, \bar{\theta}$ have the special form of (6.7) with $\omega = (\mathcal{R}^T \mathcal{R})^m, \epsilon m \in \mathbf{Z}$, then their twisted analogues are given by

$$\theta_F = \Phi_F^{-1}(\omega_F \otimes 1)\Phi_F, \quad \bar{\theta}_F = \Phi_F(1 \otimes \omega_F)\Phi_F^{-1}, \quad \omega_F = (\mathcal{R}_F^T \mathcal{R}_F)^m, \tag{7.24}$$

which agree precisely with the prescription of Lemma 9. It follows, as a special case of Theorem 5, that the central elements of (6.35) are invariant under twisting.

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¹V. G. Drinfeld, Leningrad Math. J. **1**, 1419 (1990).

²Y.-Z. Zhang and M. D. Gould, J. Math. Phys. **40**, 5264 (1999).

³D. Altschuler and A. Coste, Commun. Math. Phys. **150**, 83 (1992).

⁴G. Mack and V. Schomerus, Phys. Lett. B **267**, 213 (1991).

⁵O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, and H. Yan, Lett. Math. Phys. **32**, 259 (1994).

⁶G. Felder, "Elliptic quantum groups," Proceedings of the ICMP, Paris, 1994, p. 211, 1995 (unpublished).

⁷C. Fronsdal, Lett. Math. Phys. **40**, 134 (1997).

⁸M. Jimbo, H. Konno, S. Odake, and J. Shiraishi, "Quasi-Hopf twistors for elliptic quantum groups," e-print q-alg/9712029.

⁹B. Enriquez and G. Felder, "Elliptic quantum group $E_{\tau, \eta}(sl_2)$ and quasi-Hopf algebras," e-print q-alg/9703018.

¹⁰O. Babelon, D. Bernard, and E. Billey, Phys. Lett. B **375**, 89 (1996).

¹¹D. Arnaudon, E. Buffenoir, E. Ragoucy, and Ph. Roche, "Universal solutions of quantum dynamical Yang-Baxter equations," e-print q-alg/9712037.

¹²J. R. Links and M. D. Gould, Rep. Math. Phys. **31**, 91 (1992).

¹³R. B. Zhang and M. D. Gould, J. Math. Phys. **32**, 3261 (1991).

¹⁴M. D. Gould, Y.-Z. Zhang, and P. S. Isaac, "On quasi-Hopf superalgebras," e-print math.QA/9811062.