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Twisting invariance of link polynomials derived from ribbon quasi-Hopf algebras

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The construction of link polynomials associated with finite dimensional representations of ribbon quasi-Hopf algebras is discussed in terms of the formulation of an appropriate Markov trace. We then show that this Markov trace is invariant under twisting of the quasi-Hopf structure, which in turn implies twisting invariance of the associated link polynomials. © *2000 American Institute of Physics.* [S0022-2488(00)00407-2]

I. INTRODUCTION

The introduction of quantum algebras by Jimbo[1] and Drinfeld[2] lead to many remarkable developments in diverse areas of mathematical physics. One such was in the field of knot theory whereby a connection between the Yang–Baxter equation and the braid group was quickly established. The quantum algebras, being examples of quasi-triangular Hopf algebras, provide very systematic means to find solutions of the Yang–Baxter equation which in turn gives rise to representations of the braid group. Through a Markov trace formulation defined on each braid group representation, an invariant polynomial can then be computed for the knot or link associated with the closure of the braid.[3–6] Extensions to accomodate the case of quantum superalgebras can be found in Refs. 7 and 8.

Around the same time as the appearance of quantum algebras was Jones's discovery of a new polynomial invariant,[9] an evaluation of which may be undertaken through the simplest quantum algebra $U_q(sl(2))$ in its minimal (two-dimensional) representation. After this breakthrough researchers proceeded to obtain generalizations with the notable examples being the HOMFLY[10] and Kauffman[11] invariant polynomials. What soon became apparent was that the series of link polynomials associated with the fundamental representations of the (nonexceptional) quantum algebras and superalgebras coincided with the two-variable invariants developed in the wake of the discovery of Jones. More precisely, the invariants associated with the fundamental representations of the $U_q(gl(m|n))$ [which includes $U_q(gl(n))$] series belong to the class of HOMFLY invariants while those of the $U_q(osp(m|2n))$ [including both $U_q(o(m))$ and $U_q(sp(2n))$] series are of the Kauffman invariant type.[4,8] It is important to emphasize, however, that by going to higher representations new results are obtainable. In particular, the type I quantum superalgebras consisting of $U_q(gl(m|n))$ and $U_q(osp(2|2n))$ admit one-parameter families of typical representations which give rise to two-variable link invariants in a natural way.[12–14] The work of Reshetikhin and Turaev[15] introduced further the notion of a ribbon Hopf algebra as a particular type of quasi-triangular Hopf algebra. All the quantum algebras fall into the class of ribbon Hopf algebras. The algebraic properties of ribbon Hopf algebras is such that an extension to produce invariants of oriented tangles is permissible. A tangle diagram is analogous to a link diagram with the possibilty of having free ends. An associated invariant takes the form of a tensor operator acting on a product of vector spaces.

As a generalization of Hopf algebras Drinfeld proposed the concept of quasi-Hopf algebras[16] whereby co-associativity of the co-algebra structure is not assumed. Any quasi-Hopf algebra

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generally belongs to an equivalence class where each member is related to the others by twisting.[16] The potential for applications of these structures in the study of integrable systems is vast. They underly elliptic quantum algebras[17–22] which play an important role in obtaining solutions to the dynamical Yang–Baxter equations[23,24] and also twisting lies at the core of the construction of multiparametric quantum spin chains.[25]

In the context of knot theory, Altschuler and Coste[26] have identified the corresponding ribbon quasi-Hopf algebras as the necessary underlying algebraic structure to study tangle invariants (including closed link invariants). Here we wish to make two important observations to this field of study. First, we will show that the class of ribbon quasi-Hopf algebras is closed under twisting; i.e., a twisted ribbon quasi-Hopf algebra is again a ribbon quasi-Hopf algebra. Second, we assert that the link polynomials computed from any finite dimensional representation of a quasi-Hopf algebra are invariant with respect to twisting. Importantly, this implies that link polynomials obtained from twisting the usual quantum algebras give us nothing new. In this respect, one cannot find twist generalizations of the HOMFLY and Kauffman invariants. For a very special class of twists this result has already been noted by Reshetikhin,[27] in which case the twisted quantum algebra is again a ribbon Hopf algebra. Here we will prove the twisting invariance in full generality.

The paper is structured as follows. We begin by presenting the definition of a quasi-Hopf algebra. Next we show how representations of the braid group are obtained from a representation of any quasi-Hopf algebra. The third section deals with defining an appropriate Markov trace on each braid group element which then affords a means to obtain a link invariant. Finally, we demonstrate that the Markov trace is invariant under any twisting.

II. QUASI-HOPF ALGEBRAS

Let us briefly recall the defining relations for quasi-Hopf algebras.[16]

Definition 1: A quasi-Hopf algebra is a unital associative algebra A over a field K which is equipped with algebra homomorphisms ϵ: $A\rightarrow K$ (co-unit), Δ: $A\rightarrow A\otimes A$ (co-product), an invertible element $\Phi\in A\otimes A\otimes A$ (co-associator), an algebra antihomomorphism S: $A\rightarrow A$ (antipode) and canonical elements α, $\beta\in A$, satisfying

$$(I\otimes\Delta)\Delta(a)=\Phi^{-1}(\Delta\otimes I)\Delta(a)\Phi, \quad \forall a\in A, \tag{1}$$

$$(\Delta\otimes I\otimes I)\Phi\cdot(I\otimes I\otimes\Delta)\Phi=(\Phi\otimes 1)\cdot(I\otimes\Delta\otimes I)\Phi\cdot(1\otimes\Phi), \tag{2}$$

$$(\epsilon\otimes I)\Delta=I=(I\otimes\epsilon)\Delta, \tag{3}$$

$$(I\otimes\epsilon\otimes I)\Phi=1, \tag{4}$$

$$m\cdot(1\otimes\alpha)(S\otimes I)\Delta(a)=\epsilon(a)\alpha, \quad \forall a\in A, \tag{5}$$

$$m\cdot(1\otimes\beta)(I\otimes S)\Delta(a)=\epsilon(a)\beta, \quad \forall a\in A, \tag{6}$$

$$m\cdot(m\otimes I)\cdot(1\otimes\beta\otimes\alpha)(I\otimes S\otimes I)\Phi^{-1}=1, \tag{7}$$

$$m\cdot(m\otimes I)\cdot(S\otimes I\otimes I)(1\otimes\alpha\otimes\beta)(I\otimes I\otimes S)\Phi=1. \tag{8}$$

Here m denotes the usual product map on A: $m\cdot(a\otimes b)=ab$, $\forall a,b\in A$. Note that since A is associative we have $m\cdot(m\otimes I)=m\cdot(I\otimes m)$. Above, 1 is used to denote the unit element and I for the identity mapping. For all elements $a,b\in A$, the antipode satisfies

$$S(ab)=S(b)S(a). \tag{9}$$

Equations (2), (3), and (4) imply that Φ also obeys

$$(\epsilon\otimes I\otimes I)\Phi=1=(I\otimes I\otimes\epsilon)\Phi. \tag{10}$$

Applying ϵ to definition (7) and (8) we obtain, in view of (4), $\epsilon(\alpha)\epsilon(\beta)=1$. By applying ϵ to (5), we have $\epsilon(S(a))=\epsilon(a)$, $\forall a\in A$.

A distinguishing feature of quasi-Hopf algebras is that they are in general not co-associative; i.e.,

$$(I\otimes\Delta)\cdot\Delta\neq(\Delta\otimes I)\cdot\Delta.$$

Thus for a given co-product the action extended to the n-fold tensor product space is not uniquely determined. Throughout we will adopt the convention to define a left co-product Δ_L which acts on the tensor algebra $A^{\otimes n}$ according to

$$\Delta_L(a\otimes b\otimes\cdots\otimes c)=\Delta(a)\otimes b\otimes\cdots\otimes c.$$

We then recursively define the action

$$\Delta^{(n)}=\Delta_L.\Delta^{(n-1)} \tag{11}$$

with $\Delta^{(1)}=\Delta$, $\Delta^{(0)}=I$.

The category of quasi-Hopf algebras is invariant under a kind of gauge transformation known as twisting. Let (A,Δ,ϵ,Φ) be a quasi-Hopf algebra, with α,β,S satisfying (5)–(8), and let $F\in A\otimes A$ be an invertible element satisfying the co-unit properties

$$(\epsilon\otimes I)F=1=(I\otimes\epsilon)F. \tag{12}$$

Throughout we set

$$\Delta_F(a)=F\Delta(a)F^{-1},\quad \forall a\in A, \tag{13}$$

$$\Phi_F=F_{12}(\Delta\otimes I)F\cdot\Phi\cdot(I\otimes\Delta)F^{-1}F_{23}^{-1}, \tag{14}$$

where the subscripts above refer to the embedding of the elements in the triple tensor product space. Then

Theorem 1: $(A,\Delta_F,\epsilon,\Phi_F)$ *defined by (13) and (14) together with* α_F,β_F,S_F *given by*

$$S_F=S,\quad \alpha_F=m\cdot(1\otimes\alpha)(S\otimes I)F^{-1},\quad \beta_F=m\cdot(1\otimes\beta)(I\otimes S)F, \tag{15}$$

is also a quasi-Hopf algebra. Throughout, the element F *is referred to as a twistor.*

Definition 2: A quasi-Hopf algebra (A,Δ,ϵ,Φ) *is called quasi-triangular if there exists an invertible element* $\mathcal{R}\in A\otimes A$ *such that*

$$\Delta^T(a)\mathcal{R}=\mathcal{R}\Delta(a),\quad \forall a\in A, \tag{16}$$

$$(\Delta\otimes I)\mathcal{R}=\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}, \tag{17}$$

$$(I\otimes\Delta)\mathcal{R}=\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123}. \tag{18}$$

We refer to $\mathcal{R}$ *as the universal R-matrix.*

Throughout, $\Delta^T=T\cdot\Delta$ with T being the twist map which is defined by

$$T(a\otimes b)=b\otimes a; \tag{19}$$

and Φ_{132}, etc., are derived from $\Phi\equiv\Phi_{123}$ with the help of T

$$\Phi_{132}=(I\otimes T)\Phi_{123},$$

$$\Phi_{312}=(T\otimes I)\Phi_{132}=(T\otimes I)(I\otimes T)\Phi_{123},$$

$$\Phi_{231}^{-1}=(I\otimes T)\Phi_{213}^{-1}=(I\otimes T)(T\otimes I)\Phi_{123}^{-1},$$

and so on. We make special mention that our *definitions* for Φ_{132}, etc., differ from some adopted in the literature; e.g., Ref. 26, but this simply a matter of notation.

It is easily shown that the properties (16)–(18) imply the Yang–Baxter-type equation,

$$\mathcal{R}_{12}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}=\Phi_{321}^{-1}\mathcal{R}_{23}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}, \tag{20}$$

which is referred to as the quasi-Yang–Baxter equation.

Theorem 2: *Denoting by the set* $(A,\Delta,\epsilon,\Phi,\mathcal{R})$ *a quasi-triangular quasi-Hopf algebra, then* $(A,\Delta_F,\epsilon,\Phi_F,\mathcal{R}_F)$ *is also a quasi-triangular quasi-Hopf algebra, with the choice of* R_F *given by*

$$\mathcal{R}_F=F^T\mathcal{R}F^{-1}, \tag{21}$$

where $F^T=T\cdot F\equiv F_{21}$. *Here* Δ_F *and* Φ_F *are given by (13) and (14), respectively.*

Let us specify some notations, where we adopt a summation convention over all repeated indices. Throughtout the paper,

$$\Phi=X_i\otimes Y_i\otimes Z_i,\quad \Phi^{-1}=\overline{X}_i\otimes\overline{Y}_i\otimes\overline{Z}_i,$$

$$F=f_i\otimes f^i,\quad F^{-1}=\overline{f_i}\otimes\overline{f^i},$$

$$\mathcal{R}=a_\nu\otimes b_\nu,\quad \mathcal{R}^{-1}=c_\nu\otimes d_\nu,$$

$$(I\otimes\Delta)\Delta(a)=\sum a_{(1)}\otimes\Delta(a_{(2)})=\sum a_{(1)}^R\otimes a_{(2)}^R\otimes a_{(3)}^R, \tag{22}$$

$$(\Delta\otimes I)\Delta(a)=\sum \Delta(a_{(1)})\otimes(a_{(2)})=\sum a_{(1)}^L\otimes a_{(2)}^L\otimes a_{(3)}^L,$$

$$(\Phi^{-1}\otimes I)\cdot(\Delta\otimes I\otimes I)\Phi=A_i\otimes B_i\otimes C_i\otimes D_i,$$

$$(\Delta\otimes I\otimes I)\Phi^{-1}\cdot(\Phi\otimes I)=K_i\otimes L_i\otimes M_i\otimes N_i.$$

A important type of twistor is that due to Drinfeld.[16] For any quasi-Hopf algebra A observe that the actions

$$(S\otimes S)\cdot\Delta^T,\quad \Delta^T\cdot S^{-1}$$

both determine algebra antihomomorphisms. It follows that

$$\Delta'\equiv(S\otimes S)\cdot\Delta^T\cdot S^{-1}$$

gives rise to an algebra homomorphism and thus a co-product action on A. Drinfeld showed that the actions Δ and Δ' are related by a twistor; i.e.,

$$\Delta(a)=\mathcal{F}^{-1}((S\otimes S)\Delta^T(S^{-1}(a)))\mathcal{F},\quad \forall a\in A,$$

where

$$\mathcal{F}=(S\otimes S)\Delta^T(X_i)\cdot\gamma\cdot\Delta(Y_i\beta S(Z_i))$$

and

$$\gamma = S(B_i)\alpha C_i \otimes S(A_i)\alpha D_i . \tag{23}$$

It is also useful to define

$$\delta = K_i \beta S(N_i) \otimes L_i \beta S(M_i). \tag{24}$$

Then the following relations can be shown to hold:

$$\Delta(\alpha) = \mathcal{F}^{-1}\gamma, \quad \Delta(\beta) = \delta\mathcal{F}.$$

A quasi-Hopf algebra is said to be of trace type if there exists an invertible element $u \in A$ such that

$$S^2(a) = uau^{-1}, \quad \forall a \in A. \tag{25}$$

In the case A is quasitriangular with R-matrix as in (22) we have the following theorem.[26]

Theorem 3: *The operator defined by*

$$u = S(Y_i \beta S(Z_i)) S(\beta_\nu) \alpha a_\nu X_i \tag{26}$$

satisfies (25). Moreover the inverse is given by

$$u^{-1} = S^{-1}(X_i) S^{-1}(\alpha d_\nu) c_\nu Y_i \beta S(Z_i). \tag{27}$$

An important relation satisfied by u is

$$S(\alpha)u = S(b_\nu)\alpha a_\nu \tag{28}$$

which we will need later.

The significance of trace type quasi-Hopf algebras is that they afford a systematic means to construct Casimir invariants. We have the following result from Ref. 28.

Theorem 4: *Let π be the representation afforded by the finite-dimensional A-module V. Suppose $\eta = \mu_i \otimes \nu_i \otimes \rho_i \in A \otimes End\, V \otimes A$ obeys*

$$(I \otimes \pi \otimes I)(I \otimes \Delta)\Delta(a) \cdot \eta = \eta \cdot (I \otimes \pi \otimes I)(I \otimes \Delta)\Delta(a), \quad \forall a \in A, \tag{29}$$

then

$$\mathrm{tr}(\nu_i \pi(\beta S(\rho_i) S(\alpha) u)) \mu_i \tag{30}$$

is a central element. Similarly if $\overline{\eta} = \overline{\mu_i} \otimes \overline{\nu_i} \otimes \overline{\rho_i} \in A \otimes End\, V \otimes A$ satisfies

$$\overline{\eta} \cdot (I \otimes \pi \otimes I)(\Delta \otimes I)\Delta(a) = (I \otimes \pi \otimes I)(\Delta \otimes I)\Delta(a) \cdot \overline{\eta}, \quad \forall a \in A \tag{31}$$

then

$$\sum \mathrm{tr}(\overline{\nu_i} \pi(u^{-1} S(\beta) S(\overline{\mu_i}) \alpha \overline{\nu_i})) \overline{\rho_i} \tag{32}$$

is a central element.

As a consequence of the above we have

Corollary 1: Suppose $\omega = \Sigma \omega_i \otimes \Omega^i \in A \otimes End\, V$ satisfies

$$(I \otimes \pi)\Delta(a) \cdot \omega = \omega \cdot (I \otimes \pi)\Delta(a), \quad \forall a \in A. \tag{33}$$

Then (29) implies that

$$\tau(\omega)=\mathrm{tr}(\Omega^i\pi(Y_k\beta S(\overline{Z}_jZ_k)S(\alpha)u\overline{Y}_j))\overline{X}_j\omega_iX_k \tag{34}$$

is a central element.

For an $(n+1)$-fold tensor product space and $\omega=\Sigma\,\omega_i\otimes\Omega^i\in A^{\otimes n}\otimes\mathrm{End}\,V$ we define

$$\tau_n(\omega)=\mathrm{tr}(\Omega^i\pi(Y_k\beta S(\overline{Z}_jZ_k)S(\alpha)u\overline{Y}_j))\Delta^{(n-1)}(\overline{X}_j)\omega_i\Delta^{(n-1)}(X_k). \tag{35}$$

III. REPRESENTATIONS OF THE BRAID GROUP

Given any representation π of a quasi-Hopf algebra A we set

$$\check{R}=P\cdot(\pi\otimes\pi)\mathcal{R} \tag{36}$$

and

$$\Phi_i=(\Delta^{(i-2)}\otimes I\otimes I)\Phi.$$

In terms of $\check{R}$ the quasi-Yang–Baxter equation may be written

$$\Phi\check{R}_{23}\Phi^{-1}\check{R}_{12}\Phi\check{R}_{23}\Phi^{-1}=\check{R}_{12}\Phi\check{R}_{23}\Phi^{-1}\check{R}_{12}, \tag{37}$$

where throughout we use the same symbols Φ and Φ_i for both the algebraic objects and their matrix representatives.

Theorem 5: *Define n operators on the* $(n+1)$*-fold tensor product space by*

$$\sigma_i=\Phi_i\check{R}_{i(i+1)}\Phi_i^{-1},\quad i=1,2,\ldots,n. \tag{38}$$

These give rise to a representation of the braid group B_n *by satisfying the defining relations*

$$\sigma_i\sigma_j=\sigma_j\sigma_i,\quad j\neq i\pm 1, \tag{39}$$

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{40}$$

The above result was given in Ref. 26. Here we want to present a detailed proof.

First we establish that the braid generators (38) are invariant with respect to the co-product action $\Delta^{(n)}$ of A; i.e.,

$$[\sigma_i,\Delta^{(n)}(a)]=0,\quad \forall a\in A. \tag{41}$$

It is clear from definition (36) that

$$[\check{R},\Delta(a)]=0,\quad \forall a\in A$$

which immediately implies that

$$[\sigma_1,\Delta^{(n)}(a)]=0,\quad \forall a\in A.$$

Next consider

$$\begin{aligned}\sigma_2\Delta^{(j)}(a)&=\Phi\check{R}_{23}\Phi^{-1}(\Delta\otimes I^{\otimes(j-1)})\Delta^{(j-1)}(a)\\&=\Phi\check{R}_{23}(I\otimes\Delta\otimes I^{(j-2)})\Delta^{(j-1)}(a)\Phi^{-1}\\&=\Phi(I\otimes\Delta\otimes I^{(j-2)})\Delta^{(j-1)}(a)\check{R}_{23}\Phi^{-1}\\&=(\Delta\otimes I^{\otimes(j-1)})\Delta^{(j-1)}(a)\Phi\check{R}_{23}\Phi^{-1}\end{aligned}$$

$$=\Delta^{(j)}(a)\sigma_2. \tag{42}$$

Observing that the action (11) enjoys the property

$$\Delta^{(i)}\cdot\Delta^{(j)}=\Delta^{(i+j)}$$

and applying $\Delta^{(k)}\otimes I^{\otimes j}$ to (42) now yields (41) by choosing $k=i-2$, $j=n-i-2$.

Since $\check{R}$ commutes with the co-product action we immediately deduce for $i>1$

$$\begin{aligned}
\sigma_1\sigma_i&=\check{R}_{12}\Phi_i\check{R}_{i(i+1)}\Phi_i^{-1}\\
&=\Phi_i\check{R}_{12}\check{R}_{i(i+1)}\Phi_i^{-1}\\
&=\Phi_i\check{R}_{i(i+1)}\check{R}_{12}\Phi_i^{-1}\\
&=\Phi_i\check{R}_{i(i+1)}\Phi_i^{-1}\check{R}_{12}\\
&=\sigma_i\sigma_1.
\end{aligned}$$

Consider now for $l>3$

$$\begin{aligned}
\sigma_2\sigma_l&=\sigma_2(\Delta^{(l-2)}\otimes I\otimes I)\Phi\cdot\check{R}_{l(l+1)}(\Delta^{(l-2)}\otimes I\otimes I)\Phi^{-1}\\
&=(\Delta^{(l-2)}\otimes I\otimes I)\Phi\cdot\sigma_2\check{R}_{l(l+1)}(\Delta^{(l-2)}\Phi^{-1}\otimes I\otimes I)\\
&=(\Delta^{(l-2)}\otimes I\otimes I)\Phi\cdot\Phi_{123}\check{R}_{23}\Phi_{123}^{-1}\check{R}_{l(l+1)}(\Delta^{(l-2)}\otimes I\otimes I)\Phi^{-1}\\
&=(\Delta^{(l-2)}\otimes I\otimes I)\Phi\cdot\check{R}_{l(l+1)}\Phi_{123}\check{R}_{23}\Phi_{123}^{-1}(\Delta^{(l-2)}\otimes I\otimes I)\Phi^{-1}\\
&=(\Delta^{(l-2)}\otimes I\otimes I)\Phi\cdot\check{R}_{l(l+1)}\sigma_2(\Delta^{(l-2)}\otimes I\otimes I)\Phi^{-1}\\
&=(\Delta^{(l-2)}\otimes I\otimes I)\Phi\cdot\check{R}_{l(l+1)}(\Delta^{(l-2)}\otimes I\otimes I)\Phi^{-1}\cdot\sigma_2\\
&=\sigma_2\sigma_l.
\end{aligned}\tag{43}$$

Applying $\Delta^{(k)}\otimes I\otimes I$ to (43) yields (39) for $i\geqslant 2$ by choosing $k=i-2$, $l=j-i+2$.

In order to show that (40) is satisfied we see from (37) that

$$\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2$$

is certainly true. Now through (37), the invariance of $\check{R}$ and repeated use of the pentagonal relation (2) we find

$$\begin{aligned}
\sigma_2\sigma_3\sigma_2&=\Phi_2\check{R}_{23}\Phi_2^{-1}\Phi_3\check{R}_{34}\Phi_3^{-1}\Phi_2\check{R}_{23}\Phi_2^{-1}\\
&=\Phi_2\check{R}_{23}\Phi_2^{-1}\Phi_3(I\otimes I\otimes\Delta)\Phi\cdot\check{R}_{34}(I\otimes I\otimes\Delta)\Phi^{-1}\cdot\Phi_3^{-1}\Phi_2\check{R}_{23}\Phi_2^{-1}\\
&=\Phi_2\check{R}_{23}(I\otimes\Delta\otimes I)\Phi\cdot(I\otimes\Phi)\check{R}_{34}(I\otimes\Phi^{-1})(I\otimes\Delta\otimes I)\Phi^{-1}\cdot\check{R}_{23}\Phi_2^{-1}\\
&=\Phi_2(I\otimes\Delta\otimes I)\Phi\cdot[\check{R}_{23}(I\otimes\Phi)\check{R}_{34}(I\otimes\Phi^{-1})\check{R}_{23}](I\otimes\Delta\otimes I)\Phi^{-1}\cdot\Phi_2^{-1}\\
&=\Phi_2(I\otimes\Delta\otimes I)\Phi\cdot[(I\otimes\Phi)\check{R}_{34}(I\otimes\Phi^{-1})\check{R}_{23}(I\otimes\Phi)\check{R}_{34}(I\otimes\Phi^{-1})](I\otimes\Delta\otimes I)\Phi^{-1}\cdot\Phi_2^{-1}\\
&=\Phi_3(I\otimes I\otimes\Delta)\Phi\cdot\check{R}_{34}(I\otimes\Phi^{-1})\check{R}_{23}(I\otimes\Phi)\check{R}_{34}(I\otimes I\otimes\Delta)\Phi^{-1}\cdot\Phi_3^{-1}\\
&=\Phi_3\check{R}_{34}(I\otimes I\otimes\Delta)\Phi\cdot(I\otimes\Phi^{-1})\check{R}_{23}(I\otimes\Phi)(I\otimes I\otimes\Delta)\Phi^{-1}\cdot\check{R}_{34}\Phi_3^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \Phi_3 \check{R}_{34} \Phi_3^{-1} \Phi_2 (I \otimes \Delta \otimes I) \Phi \cdot \check{R}_{23} (I \otimes \Delta \otimes I) \Phi^{-1} \cdot \Phi_2^{-1} \Phi_3 \check{R}_{23} \Phi_3^{-1} \\
&= \Phi_3 \check{R}_{34} \Phi_3^{-1} \Phi_2 \check{R}_{23} \Phi_2^{-1} \Phi_3 \check{R}_{34} \Phi_3^{-1} \\
&= \sigma_3 \sigma_2 \sigma_3 . \qquad (44)
\end{aligned}
$$

Finally, acting $\Delta^{(i-2)} \otimes I^{\otimes 3}$ on (44) above yields

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} .$$

IV. LINK POLYNOMIALS FROM RIBBON QUASI-HOPF ALGEBRAS

In Ref. 26 the following definition was proposed for the ribbon quasi-Hopf algebras.

Definition 3: Let A be a quasi-triangular quasi-Hopf algebra. We say that A is a ribbon quasi-Hopf algebra if there exists a central element $v \in A$ such that

(1) $v^2 = uS(u)$,
(2) $S(v) = v$,
(3) $\epsilon(v) = 1$,
(4) $\Delta(uv^{-1}) = \mathcal{F}^{-1}(S \otimes S)\mathcal{F}_{21} \cdot uv^{-1} \otimes uv^{-1}$,

where $\mathcal{F}$ is the Drinfeld twist discussed earlier.

Given a ribbon quasi-Hopf algebra, a prescription was also provided in Ref. 26 to define a Markov trace on the braid group representation which in turn may be used to compute link polynomials in the usual way. From here on we will omit the symbol π denoting the representation for ease of notation.

Theorem 6: Let Ψ be a word in the braid generators (38) for a fixed finite dimensional irreducible representation of a ribbon Hopf-algebra A. *Then a Markov trace θ_n on the $(n+1)$-fold tensor product space may be defined by*

$$\theta_n(\Psi) = \mathrm{tr}(\Psi \Delta^{(n)}(\beta S(\alpha) u v^{-1}))$$

which satisfies the Markov properties

(1) $\theta_n(\Psi_1 \Psi_2) = \theta_n(\Psi_2 \Psi_1), \quad \forall \Psi_1, \Psi_2 \in B_n$,
(2) $\theta_n(\Psi \sigma^{\pm 1}) = z^{\pm} \theta_{n-1}(\Psi), \quad \forall \Psi \in B_{n-1} \subset B_n$,

where $z^{\pm}$ are the eigenvalues of the central operators $v^{\mp 1}$ in the representation π.

The importance of the Markov trace is that from it one can define a link polynomial $L(\hat{\Psi})$ through

$$L(\hat{\Psi}) = (z^+ z^-)^{n/2} \left(\frac{z^-}{z^+} \right)^{e(\Psi)/2} \theta(\Psi), \quad \Psi \in B_n , \qquad (45)$$

where $e(\Psi)$ is the sum of the exponents of the σ_i's appearing in Ψ. The functional $L(\hat{\Psi})$ enjoys the following properties:

(1) $L(\widehat{\Psi \eta}) = L(\widehat{\eta \Psi}), \quad \forall \Psi, \eta \in B_M$,
(2) $L(\widehat{\Psi \sigma_{n-1}^{\pm 1}}) = L(\hat{\Psi}), \quad \forall \Psi \in B_{n-1} \subset B_n$,

and is an invariant of ambient isotopy.

The first Markov property follows easily from the invariance of the braid generators $\sigma^{\pm 1}$ and the cyclic rule of traces. To establish the second Markov property requires some work and was stated in Ref. 26 without proof. Here we provide the details.

Before proceeding, we need to determine the co-product action of the element $S(\alpha)uv^{-1}$. Using the Drinfeld twistor we find

$$\begin{aligned}\Delta(S(\alpha)) &= \mathcal{F}^{-1}((S\otimes S)\Delta^{T}(\alpha))\mathcal{F}\\ &= \mathcal{F}^{-1}((S\otimes S)(\mathcal{F}_{21}^{-1}\gamma_{21}))\mathcal{F}\\ &= \mathcal{F}^{-1}(S\otimes S)\gamma_{21}\cdot(S\otimes S)\mathcal{F}_{21}^{-1}\cdot\mathcal{F}.\end{aligned}$$

Now through using (23) and definition (3) we find that

$$\Delta(S(\alpha)uv^{-1}) = \mathcal{F}^{-1}(S(D_i)S(\alpha)uv^{-1}A_i\otimes S(C_i)S(\alpha)uv^{-1}B_i). \tag{46}$$

We will also need the following result.

Lemma 1: Let $\mathcal{C}\in End(V\otimes V)$ *be any invariant operator; i.e.,*

$$[\mathcal{C},\Delta(a)] = 0, \quad \forall a\in A.$$

Then

$$(A_i\otimes B_i)\mathcal{C}(K_j\beta S(D_iN_j)\otimes L_j\beta S(C_iM_j)) = (\overline{X_j}\otimes\overline{Y_j})\mathcal{C}(X_i\beta\otimes Y_i\beta S(\overline{Z_j}Z_i)).$$

The above result follows directly from the definitions (22). We may now see that

$$\begin{aligned}\theta_n(\Psi) &= \mathrm{tr}(\Psi\Delta^{(n)}(\beta S(\alpha)uv^{-1}))\\ &= \mathrm{tr}(\Psi\cdot(\Delta^{(n-1)}\otimes I)\delta\cdot\Delta^{(n-1)}(S(D_i)S(\alpha)uv^{-1}A_i)\otimes S(C_i)S(\alpha)uv^{-1}B_i)\\ &= \mathrm{tr}(\Psi\cdot\Delta^{(n-1)}(K_j\beta S(N_j)S(D_i)S(\alpha)uv^{-1}A_i)\otimes L_j\beta S(M_j)S(C_i)S(\alpha)uv^{-1}B_i)\\ &= \mathrm{tr}(\Psi\cdot\Delta^{(n-1)}(X_i\beta S(\alpha)uv^{-1}\overline{X_j})\otimes Y_i\beta S(Z_i)S(\overline{Z_j})S(\alpha)uv^{-1}\overline{Y_j})\\ &= \mathrm{tr}(\tau_n(\Psi)\Delta^{(n-1)}(\beta S(\alpha)uv^{-1}))\\ &= \theta_{n-1}(\tau_n(\Psi)),\end{aligned} \tag{47}$$

where the element u in the definition (35) has now been replaced by uv^{-1}. It is apparent also from (35) that for $\Psi\in B_{n-1}$ then

$$\tau_n(\Psi\sigma_n^{\pm 1}) = \Psi\tau_n(\sigma_n^{\pm 1}).$$

To evaluate $\tau_n(\sigma_n^{\pm 1})$ we can appeal to the pentagonal relation (2) to find that

$$\begin{aligned}\Phi_{n+1}^{-1}\sigma_n^{\pm 1}\Phi_{n+1} &= \Phi_{n+1}^{-1}\Phi_n\check{R}_{n(n+1)}^{\pm 1}\Phi_n^{-1}\Phi_{n+1}\\ &= I^{(n-2)}\otimes((I\otimes I\otimes\Delta)\Phi\cdot(I\otimes\Phi^{-1})(I\otimes\Delta\otimes I)\Phi^{-1}\\ &\quad\cdot(I\otimes\check{R}^{\pm 1}\otimes I)\cdot(I\otimes\Delta\otimes I)\Phi\cdot(I\otimes\Phi)(I\otimes I\otimes\Delta)\Phi^{-1})\\ &= I^{(n-2)}\otimes((I\otimes I\otimes\Delta)\Phi\cdot(I\otimes\Phi^{-1})(I\otimes\check{R}^{\pm 1}\otimes I)(I\otimes\Phi)(I\otimes I\otimes\Delta)\Phi^{-1})\end{aligned}$$

which, upon using (5), (6), and (10), leads us to conclude that

$$\tau_n(\sigma_n^{\pm 1}) = I^{\otimes(n-1)}\otimes\tau(\check{R}^{\pm 1}).$$

An algebraic exercise shows that

$$\tau(\check{R}) = \overline{X_j}b_\nu Y_l\beta S(Z_l)S(\overline{Z_j})S(\alpha)uv^{-1}\overline{Y_j}a_\nu X_l,$$

$$\tau(\check{R}^{-1}) = X_j c_\nu Y_k \beta S(Z_k) S(\overline{Z}_j) S(\alpha) u v^{-1} \overline{Y}_j d_\nu X_k$$

and hence we can conclude that $z^{\pm}$ are given by the eigenvalues of the central operators $v^{\mp 1}$ if we can show that the following relations hold.

Lemma 2:

$$\overline{X}_j b_\nu Y_l \beta S(Z_l) S(\overline{Z}_j) S(\alpha) u \overline{Y}_j a_\nu X_l = 1,$$

$$\overline{X}_j c_\nu Y_k \beta S(Z_k) S(\overline{Z}_j) S(\alpha) u \overline{Y}_j d_\nu X_k = v^2.$$

Through use of (7) and (28) we obtain

$$\begin{aligned} 1 &= \overline{X}_i \beta S(\overline{Y}_i) \alpha \overline{Z}_i \\ &= \overline{X}_j \beta S(\overline{Y}_i) S(d_\nu) S(b_\mu) \alpha a_\mu c_\nu \overline{Z}_i \\ &= \overline{X}_j \beta S(\overline{Y}_i) S(d_\nu) S(\alpha) u c_\nu \overline{Z}_i \\ &= \overline{X}_i \beta S(\overline{Y}_i) S(d_\nu) S(\alpha) S^2(c_\nu) S^2(\overline{Z}_i) u. \end{aligned} \tag{48}$$

From Eq. (18) we see that

$$\mathcal{R}_{13}^{-1} \Phi_{312}^{-1} (I \otimes \Delta) \mathcal{R} = \Phi_{213}^{-1} \mathcal{R}_{12} \Phi_{123}$$

which expressed in terms of the tensor components reads

$$c_\nu \overline{Z}_j a_l \otimes \overline{X}_j b_l^{(1)} \otimes d_\nu \overline{Y}_j b_l^{(2)} = \overline{Y}_j a_\nu X_l \otimes \overline{X}_j b_\nu Y_l \otimes \overline{Z}_j Z_l .$$

We can now write

$$\begin{aligned} \overline{X}_j b_\nu Y_l \beta S(Z_l) S(\overline{Z}_j) S(\alpha) u \overline{Y}_j a_\nu X_l &= \overline{X}_j b_l^{(1)} \beta S(b_l^{(2)}) S(\overline{Y}_j) S(d_\nu) S(\alpha) u c_\nu \overline{Z}_j a_l \\ &= \epsilon(b_l) \overline{X}_j \beta S(\overline{Y}_j) S(d_\nu) S(\alpha) u c_\nu \overline{Z}_j a_l \\ &= \overline{X}_i \beta S(\overline{Y}_i) S(d_\nu) S(\alpha) S^2(c_\nu) S^2(\overline{Z}_i) u \\ &= 1. \end{aligned} \tag{49}$$

Next we see that

$$\begin{aligned} u &= S(\overline{X}_i \beta S(\overline{Y}_i) \alpha \overline{Z}_i) u \\ &= S(\overline{Z}_i) S(\alpha) S^2(\overline{Y}_i) S(\beta) S(\overline{X}_i) u \\ &= S(\overline{Z}_i) S(\alpha) u \overline{Y}_i S^{-1}(\beta) S^{-1}(\overline{X}_i) \\ &= S(\overline{Z}_i) S(b_\nu) \alpha a_\nu \overline{Y}_i S^{-1}(\beta) S^{-1}(\overline{X}_i), \end{aligned} \tag{50}$$

where in the last step we have used Eq. (28). Consequently,

$$S(u) = \overline{X}_i \beta S(\overline{Y}_i) S(a_\nu) S(\alpha) S^2(b_\nu) S^2(\overline{Z}_i).$$

From Eq. (17) we have

$$\mathcal{R}_{23} \Phi_{123}^{-1} (\Delta \otimes I) \mathcal{R}^{-1} = \Phi_{132}^{-1} \mathcal{R}_{13}^{-1} \Phi_{231}$$

which we may express as

$$\overline{X}_j c_\nu^{(1)} \otimes a_\mu \overline{Y}_j c_\nu^{(2)} \otimes b_\mu \overline{Z}_j d_\nu = \overline{X}_j c_\nu Y_k \otimes \overline{Z}_j Z_k \otimes \overline{Y}_j d_\nu X_k .$$

This relation leads us to deduce that

$$\begin{aligned}
\overline{X}_j c_\nu Y_k \beta S(Z_k) S(\overline{Z}_j) S(\alpha) u \overline{Y}_j d_\nu X_k &= \overline{X}_j c_\nu^{(1)} \beta S(c_\nu^{(2)}) S(\overline{Y}_j) S(a_\mu) S(\alpha) u b_\mu \overline{Z}_j d_\nu \\
&= \overline{X}_j \beta S(\overline{Y}_j) S(\alpha_\mu) S(\alpha) S^2(b_\mu) S^2(\overline{Z}_j) u \\
&= S(u) u \\
&= v^2 \qquad (51)
\end{aligned}$$

which proves lemma 2 and completes the proof of theorem 6.

V. TWISTING INVARIANCE OF THE MARKOV TRACE

Now we are in a position to show twisting invariance of the link polynomials. Let us begin with the following result.

Proposition 1: Every twisted ribbon quasi-Hopf algebra is again a ribbon quasi-Hopf algebra.

Recall from definition (3) that the first three conditions of a ribbon quasi-Hopf algebra are properties of the algebra structure rather than the co-algebra. Thus, to this end we need only show that if

$$\Delta(uv^{-1}) = \mathcal{F}^{-1}(S \otimes S)\mathcal{F}_{21} . uv^{-1} \otimes uv^{-1}$$

then

$$\Delta_F(uv^{-1}) = \mathcal{F}_F^{-1}(S \otimes S)(\mathcal{F}_F)_{21} . uv^{-1} \otimes uv^{-1},$$

where $\mathcal{F}_F$ denotes the Drinfeld twistor for the twisted quasi-Hopf algebra. Recalling that the Drinfeld twist $\mathcal{F}$ is determined by

$$\mathcal{F}\Delta(a)\mathcal{F}^{-1} = (S \otimes S)(\Delta^T(S^{-1}(a))), \quad \forall a \in A$$

shows that

$$\begin{aligned}
F\Delta(a)F^{-1} &= \Delta_F(a) \\
&= \mathcal{F}_F^{-1}((S \otimes S)\Delta_F^T(S^{-1}(a)))\mathcal{F}_F \\
&= \mathcal{F}_F^{-1}((S \otimes S)(F_{21}\Delta^T(S^{-1}(a))F_{21}^{-1}))\mathcal{F}_F \\
&= \mathcal{F}_F^{-1}((S \otimes S)F_{21}^{-1} \cdot (S \otimes S)\Delta^T(S^{-1}(a)).(S \otimes S)F_{21})\mathcal{F}_F \\
&= \mathcal{F}_F^{-1}(S \otimes S)F_{21}^{-1} \cdot \mathcal{F}\Delta(a)\mathcal{F}^{-1} \cdot (S \otimes S)F_{21}\mathcal{F}_F
\end{aligned}$$

which leads us to

$$\mathcal{F}_F = (S \otimes S)F_{21}^{-1} \cdot \mathcal{F}F^{-1}.$$

Now we observe that

$$\begin{aligned}
&\mathcal{F}_F^{-1}(S \otimes S)(\mathcal{F}_F)_{21} \cdot uv^{-1} \otimes uv^{-1} \\
&\quad = F\mathcal{F}^{-1}(S \otimes S)F_{21} \cdot (S \otimes S)((S \otimes S)F^{-1} \cdot (\mathcal{F}F^{-1})_{21})uv^{-1} \otimes uv^{-1} \\
&\quad = F\mathcal{F}^{-1}(S \otimes S)\mathcal{F}_{21} \cdot (S^2 \otimes S^2)F^{-1} \cdot uv^{-1} \otimes uv^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= F\mathcal{F}^{-1}(S\otimes S)\mathcal{F}_{21}\cdot uv^{-1}\otimes uv^{-1}\cdot F^{-1}\\
&= F\Delta(uv^{-1})F^{-1}\\
&= \Delta_F(uv^{-1})
\end{aligned}
$$

thus establishing that the twisted ribbon quasi-Hopf algebra is also of ribbon type.

By induction the co-product action on the $(n+1)$-fold space assumes the form

$$\Delta_F^{(n)}(a)=\chi_n\Delta^{(n)}(a)\chi_n^{-1},$$

where

$$\chi_n=F_{12}\cdot(\Delta\otimes I)F_{12}\cdot(\Delta^2\otimes I)F_{12}\cdots(\Delta^{(n-1)}\otimes I)F.$$

Consider next

$$
\begin{aligned}
\chi_n\sigma_i\chi_n^{-1}&=F_{12}(\Delta\otimes I)F_{12}.(\Delta^2\otimes I)F_{12}\cdots(\Delta^{(n-1)}\otimes I)F\sigma_i\times(\Delta^{(n-1)}\otimes I)F^{-1}\cdots(\Delta\otimes I)F_{12}^{-1}\cdot F_{12}^{1}\\
&=F_{12}(\Delta\otimes I)F_{12}.(\Delta^2\otimes I)F_{12}\cdots(\Delta^{(i-1)}\otimes I)F\sigma_i\times(\Delta^{(i-1)}\otimes I)F^{-1}\cdots(\Delta\otimes I)F_{12}^{-1}\cdot F_{12}^{1}\\
&=\chi_i\sigma_i\chi_i^{-1}.
\end{aligned}
$$

We now determine the representations of the braid generators under twisting; i.e.,

$$
\begin{aligned}
\sigma_i^F&=(\Delta_F^{(i-2)}\otimes I\otimes I)\Phi_F\cdot\check{R}_{i(i+1)}^F\cdot(\Delta_F^{(i-2)}\otimes I\otimes I)\Phi_F^{-1}\\
&=\chi_{i-2}(\Delta^{(i-2)}\otimes I\otimes I)(F_{12}(\Delta\otimes I)F\cdot\Phi\cdot(I\otimes\Delta)F^{-1}\cdot F_{23}^{-1})\chi_{i-2}^{-1}F_{i(i+1)}\check{R}_{i(i+1)}F_{i.i+1}^{-1}\\
&\quad\times\chi_{i-2}\Delta^{(i-2)}(F_{23}(I\otimes\Delta)F\cdot\Phi^{-1}(\Delta\otimes I)F^{-1}\cdot F_{12}^{-1})\chi_{i-2}^{-1}\\
&=\chi_i\Delta\Phi\cdot(\Delta^{(i-2)}\otimes\Delta)F^{-1}\cdot F_{i(i+1)}^{-1}\chi_{i-2}^{-1}F_{i(i+1)}\check{R}_{i(i+1)}F_{i(i+1)}^{-1}\chi_{i-2}F_{i(i+1)}\\
&\quad\times(\Delta^{(i-2)}\otimes\Delta)F\cdot\Delta^{(i-2)}\Phi^{-1}\cdot\chi_i^{-1}\\
&=\chi_i\Delta^{(i-2)}\Phi\cdot\check{R}_{i(i+1)}\Delta^{(i-2)}\Phi^{-1}\cdot\chi_i^{-1}\\
&=\chi_i\sigma_i\chi_i^{-1}\\
&=\chi_n\sigma_i\chi_n^{-1}
\end{aligned}
$$

which shows that the representation of the braid generators under twisting are related to those of the untwisted case by a basis transformation. Thus for any word in the generators of the braid group we can write

$$\Psi^F=\chi_n\Psi\chi_n^{-1}$$

in an obvious notation.

Using the relations (15) we may write

$$\alpha_F=S(\overline{f_i})\alpha\overline{f^i},\quad \beta_F=f_i\beta S(f^i)$$

and proceed to calculate

$$
\begin{aligned}
\theta_n^F(\Psi)&=\mathrm{tr}(\Psi^F\Delta_F^{(n)}(\beta_F S(\alpha_F)uv^{-1}))\\
&=\mathrm{tr}(\chi_n\Psi\Delta^{(n)}(\beta_F S(\alpha_F)uv^{-1})\chi_n^{-1})\\
&=\mathrm{tr}(\Psi\Delta^{(n)}(f_i\beta S(f^i)S(\overline{f^j})S(\alpha)S^2(\overline{f_j})uv^{-1}))
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{tr}(\Psi\Delta^{(n)}(f_i\beta S(\overline{f^j}f^i)S(\alpha)uv^{-1}\overline{f_j}))\\
&= \mathrm{tr}(\Delta^{(n)}(\overline{f_j})\Psi\Delta^{(n)}(f_i\beta S(\overline{f^j}f^i)S(\alpha)uv^{-1}))\\
&= \mathrm{tr}(\Psi\Delta^{(n)}(\overline{f_j}f_i\beta S(\overline{f^j}f^i)S(\alpha)uv^{-1}))\\
&= \mathrm{tr}(\Psi\Delta^{(n)}(\beta S(\alpha)uv^{-1}))\\
&= \theta_n(\Psi)
\end{aligned}
$$

which proves twisting invariance of the Markov trace and consequently the associated link polynomials.

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