Bell Inequalities for Continuous-Variable Correlations

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We derive a new class of correlation Bell-type inequalities. The inequalities are valid for any number of outcomes of two observables per each of n parties, including continuous and unbounded observables. We show that there are no first-moment correlation Bell inequalities for that scenario, but such inequalities can be found if one considers at least second moments. The derivation stems from a simple variance inequality by setting local commutators to zero. We show that above a constant detector efficiency threshold, the continuous-variable Bell violation can survive even in the macroscopic limit of large n. This method can be used to derive other well-known Bell inequalities, shedding new light on the importance of noncommutativity for violations of local realism.

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Einstein, Podolsky, and Rosen (EPR), in their famous 1935 paper [1], demonstrated the incompatibility between the premises of *local realism* and the completeness of quantum mechanics. The original EPR paper used continuous position and momentum variables, and relied on their commutation relations, via the corresponding uncertainty principle. Bohm [2] introduced, in 1951, his version of the EPR paradox with spin observables. This was the version that was used by Bell [3] to prove his famous theorem showing that quantum mechanics predicts results that can rule out the whole class of local hidden-variable (LHV) theories. It is hard to overemphasize the importance of this result, which has even been called "the most profound discovery of science" [4]. However, the original Bell inequality, and all of its generalizations, are directly applicable only to the case of discrete observables. The main purpose of this Letter is to close the circle and derive a class of Bell-type inequalities applicable to continuous variables correlations, together with multipartite generalizations.

We derive a class of inequalities for local realism that directly use correlations of measurements, with no restriction to spin measurements or discrete binning. The new inequalities are remarkably simple. They place no restriction on the number of possible outcomes, and the contrast between the classical and quantum bounds involves commutation relations in a central way. They must be satisfied by any observations in an LHV theory, whether having discrete, continuous, or unbounded outcomes. We can immediately rederive previously known Bell-type inequalities, obtaining at the same time their quantum-mechanical bounds by considering the noncommutativity of the observables involved. We also display quantum states that directly violate the new inequalities for continuous, unbounded measurements, even in the macroscopic, large n limit [5–8]. We show that the new Bell violations survive the effects of finite generation and detection efficiency. This is very surprising, in view of the many examples in which decoherence rapidly destroys macroscopic superpositions [9].

Apart from this intrinsic interest, these inequalities are relevant to an important scientific problem. No experiment has yet produced a Bell inequality violation without introducing either locality or detection loopholes. One path towards this goal is to use continuous variables (CV) and efficient homodyne detection, which allows much higher detection efficiency than is feasible with discrete spin or photodetection measurements. A number of loop-hole free proposals exist in the literature, but they all use Bell [10–14] or Hardy [15] inequalities with a dichotomic binning of the results (which usually lead to small violations), or else a parity or pseudospin approach [16–18] which cannot be realized with efficient homodyne detection. Are there Bell inequalities that can be derived *without* the assumption of a finite number of outcomes and therefore are directly applicable to CV—with no need to bin the results?

For n parties, m measurements per party, and o outcomes, it is well known that the set of correlations allowed by LHV theories can be represented as a *convex polytope,* a multidimensional geometrical structure formed by all convex combinations (linear combinations where the coefficients are probabilities; i.e., they are non-negative and sum to one) of a finite number of vertices. The vertices of this polytope are the classical pure states—the states with well-defined values for all variables [7,19,20]. The tight Bell inequalities are associated with the linear facets of the polytope. It is a computationally hard problem to list all Bell inequalities for given (n, m, o), and full numerical characterizations have been accomplished only for small values of those parameters.

However, no class of Bell inequalities has previously been derived without *any* reference to the number of outcomes or to their bound. Any real experiment will always yield a finite number of outcomes; but are there constraints imposed by LHV theories that are independent of any *particular* discretization, and can be explicitly written even in the limit $o \to \infty$? Our answer is yes; and the derivation is much more straightforward than in the case of the usual Bell-type inequalities, which are restricted to a particular set of outputs.

We focus on the correlation functions of observables for n sites or observers, each equipped with m possible apparatus settings to make their causally separated measurements. We consider any real, complex, or vector function $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots)$ of local observations X_i, Y_i, Z_i at each site i, which in an LHV theory are all functions of hidden variables λ. In a real experiment the different terms in F may not all be measurable at once, because they may involve different choices of incompatible observables. The assumption of *locality* enters the reasoning by requiring that the local choice of observable does not affect the correlations between variables at different sites, and therefore that the averages are taken over the same hidden-variable ensemble $P(\lambda)$ for all terms. We introduce averages over the LHV ensemble (there is no loss of generality in considering deterministic LHVs [21]),

$$\langle F\rangle = \int P(\lambda)F(\mathbf{X}(\lambda), \mathbf{Y}(\lambda), \mathbf{Z}(\lambda), \ldots)d\lambda. \tag{1}$$

Our LHV inequality uses the simple result that any function of random variables has a non-negative variance,

$$|\langle F\rangle|^2 \leq \langle |F|^2\rangle. \tag{2}$$

We can also give a bound $\langle |F|^2\rangle \leq \langle |F|^2\rangle_{\text{sup}}$, where the subscript denotes the supremum (least upper bound), in which products of incompatible observables are replaced by their maximum achievable values. This is necessary since, if we are not able to measure both X_i and Y_i simultaneously, a general LHV model could predict any achievable correlation [22].

The same variance inequality applies to the corresponding Hermitian operator $\hat{F}$ in quantum mechanics. While the observables at different sites commute—they can be simultaneously measured—those at the same site do not, so operator ordering must be included. This enables us to see how quantum theory can violate the variance bound for an LHV.

As an example, we apply this variance inequality to a well-known case. Consider two dichotomic observables X_i, Y_i per site i, the outcomes of which are ±1. We define $F_1 \equiv X_1$, $F_1' \equiv Y_1$, and then inductively construct [23]

$$F_n \equiv \tfrac{1}{2}(F_{n-1} + F'_{n-1})X_n + \tfrac{1}{2}(F_{n-1} - F'_{n-1})Y_n, \tag{3}$$

where F'_n can be obtained from F_n by the exchange $X_i \leftrightarrow Y_i$. In calculating F_n^2 we will keep track of the local commutators just to make the contrast with quantum mechanics clearer. For real variables X, Y, the commutator is defined in the same way as for the corresponding operators, i.e., $[X, Y] \equiv XY - YX$. The anticommutator is defined by $[X, Y]_+ \equiv XY + YX$. Then

$$\begin{aligned} F_n^2 = \tfrac{1}{4}\{&(F_{n-1}^2 + F'^2_{n-1})(X_n^2 + Y_n^2) + [F_{n-1}, F'_{n-1}]_+(X_n^2 - Y_n^2) \\ &+ (F_{n-1}^2 - F'^2_{n-1})[X_n, Y_n]_+ - [F_{n-1}, F'_{n-1}][X_n, Y_n]\}. \end{aligned} \tag{4}$$

Since $\hat{X}_n^2 = \hat{Y}_n^2 = 1$, we can show that $F_n^2 = F'^2_n$ and

$$F_n^2 = F_{n-1}^2 - \tfrac{1}{4}[F_{n-1}, F'_{n-1}][X_n, Y_n]. \tag{5}$$

In a LHV theory, the term that involves commutators will be zero since $[X(\lambda), Y(\lambda)] = X(\lambda)Y(\lambda) - Y(\lambda)X(\lambda) = 0$. Hence by induction $F_n^2 = F_1^2 = 1$ and the variance inequality (2) becomes $-1 \leq \langle F_n\rangle \leq 1$. This is the Mermin-Ardehali-Belinskii-Klyshko (MABK) [5,24,25] Bell inequality, which reduces to the well-known Bell Clauser-Horne-Shimony-Holt (CHSH) [26] inequality for $n = 2$.

We can now calculate the quantum-mechanical bound by writing the variance inequality (2) and substituting the functions in (5) by their corresponding operators

$$\begin{aligned} \langle \hat{F}_n\rangle_Q^2 &\leq \langle \hat{F}_n^2\rangle_Q = \langle \hat{F}_{n-1}^2 - \tfrac{1}{4}[\hat{F}_{n-1}, \hat{F}'_{n-1}][\hat{X}_n, \hat{Y}_n]\rangle_Q \\ &\leq \|\hat{F}_{n-1}^2\| + \tfrac{1}{4}\|[\hat{F}_{n-1}, \hat{F}'_{n-1}]\|\|[\hat{X}_n, \hat{Y}_n]\|, \end{aligned} \tag{6}$$

where the norm $\|A\|$ denotes the modulus of the maximum value of $\langle \hat{A}\rangle_Q$ over all quantum states. The norm of the second commutator has the bound $\|[\hat{X}_n, \hat{Y}_n]\| \leq 2$. It is easy to show that $[\hat{F}_n, \hat{F}'_n] = \hat{F}_{n-1}^2[\hat{X}_n, \hat{Y}_n] + [\hat{F}_{n-1}, \hat{F}'_{n-1}]$ and therefore $\|[\hat{F}_n, \hat{F}'_n]\| \leq 2\|\hat{F}_{n-1}^2\| + \|[\hat{F}_{n-1}, \hat{F}'_{n-1}]\|$. Solving the recursion relation by noting that $\|\hat{F}_1^2\| = \frac{1}{2}\|[\hat{X}_1, \hat{Y}_1]\| = 1$ we finally arrive at the bound $\langle \hat{F}_n\rangle_Q^2 \leq 2^{n-1}$. This can be attained with the generalized Greenberger-Horne-Zeilinger (GHZ) states [23], which therefore violate (2).

Inspired by those results, we now demonstrate an LHV inequality that is directly applicable to unbounded continuous variables, in particular, field quadrature operators. The choice of the function F_n in (3) is not optimal though, since the variance in general involves incompatible operator products that have no upper bound.

To overcome this problem, consider a complex function C_n of the local real observables $\{X_k, Y_k\}$ defined as

$$C_n = \tilde{X}_n + i\tilde{Y}_n = \prod_{k=1}^{n}(X_k + iY_k), \tag{7}$$

so that the modulus square involves only compatible operator products, i.e., $|C_n|^2 = \prod_{k=1}^{n}(X_k^2 + Y_k^2)$. Applying the variance inequality to both $\tilde{X}_n$ and $\tilde{Y}_n$, we find that

$$\langle \tilde{X}_n\rangle^2 + \langle \tilde{Y}_n\rangle^2 \leq \left\langle \prod_{k=1}^{n}(X_k^2 + Y_k^2)\right\rangle. \tag{8}$$

This is our main result. Given the assumption of local hidden variables, this inequality must be satisfied for any set of observables X_k, Y_k, regardless of their spectrum.

The fact that we have neglected the commutators in deriving (8) hints that quantum mechanics might predict a violation. We define quadrature operators

$$\begin{aligned} \hat{X}_k &= \hat{a}_k e^{-i\theta_k} + \hat{a}_k^\dagger e^{i\theta_k}, \\ \hat{Y}_k &= \hat{a}_k e^{-i(\theta_k + s_k\pi/2)} + \hat{a}_k^\dagger e^{i(\theta_k + s_k\pi/2)}, \end{aligned} \tag{9}$$

where $\hat{a}_k$, $\hat{a}_k^\dagger$ are the boson annihilation and creation operators at site k and $s_k \in \{-1, 1\}$.

We now define the operator $\hat{Z}_k \equiv \hat{X}_k + i\hat{Y}_k$ and note that it follows that $\hat{C}_n = \prod_{k=1}^n \hat{Z}_k$. The definition of $\hat{Y}_k$ allows for the choice of the relative phase with respect to $\hat{X}_k$ to be $\pm\pi/2$. Depending on s_k, for each k either $\hat{Z}_k = 2\hat{a}_k e^{-i\theta_k}$ or $\hat{Z}_k = 2\hat{a}_k^\dagger e^{i\theta_k}$. Denoting $\hat{A}_k(1) = \hat{a}_k$ and $\hat{A}_k(-1) = \hat{a}_k^\dagger$, the term in the left-hand side (LHS) of (8) in quantum mechanics is then $|\langle\prod_k \hat{Z}_k\rangle_Q|^2 = |2^n e^{i\sum_k s_k\theta_k}\langle\prod_k \hat{A}_k(s_k)\rangle_Q|^2$. The right-hand side (RHS) becomes $\langle\prod_{k=1}^n(4\hat{a}_k^\dagger\hat{a}_k + 2)\rangle_Q$ regardless of the phase choices. To violate (8) we must therefore find a state that satisfies

$$\left|\left\langle\prod_k \hat{A}_k(s_k)\right\rangle_Q\right|^2 > \left\langle\prod_k\left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right)\right\rangle_Q, \tag{10}$$

which is surprisingly insensitive to relative phases between the quadrature measurements at different sites.

This violation of a continuous-variable Bell inequality can be realized within quantum mechanics. Consider an even number of sites, choosing $s_k = 1$ for the first half of them and $s_k = -1$ for the remaining. To maximize the LHS, we need a superposition of terms that are coupled by that product of annihilation/creation operators. One choice is a state of type

$$|\Psi_S\rangle = c_0|0,\ldots,0,1,\ldots,1\rangle + c_1|1,\ldots,1,0,\ldots,0\rangle, \tag{11}$$

where in the first term the first $n/2$ modes are occupied by zero photons and the remaining by 1; conversely for the second term. With that choice of state the LHS of (10) becomes $|c_0|^2|c_1|^2$, which is maximized by $|c_0|^2 = |c_1|^2 = \frac{1}{2}$. The RHS is $(\frac{3}{2})^{n/2}(\frac{1}{2})^{n/2}$ independently of the amplitudes c_0, c_1. Dividing the LHS by the RHS, inequality (10) becomes $\frac{1}{4}(\frac{4}{3})^{n/2} \le 1$, which is violated for $n \ge 10$, and the violation grows exponentially with the number of sites.

While setting up the homodyne detectors necessary for this observation is challenging, the complexity of this task scales linearly with the number of modes. A more stringent constraint is most likely in the state preparation, but we can relate state (11) to a class of states of great experimental interest. They can be achieved from a generalized GHZ state of $n/2$ photons, $\frac{1}{\sqrt{2}}(|H\rangle^{\otimes n/2} + |V\rangle^{\otimes n/2})$—where $|H\rangle$ and $|V\rangle$, respectively, represent single-particle states of horizontal and vertical polarization—by splitting each mode with a polarizing beam splitter. Therefore violation of (8) can be observed in the ideal case with a 5-qubit photon polarization GHZ state and homodyne detection.

An interesting question is the effect of decoherence, both from state preparation error [27] and detector inefficiency. The usual Bell CHSH violations have an efficiency threshold [28] of 83%. This has not yet been achieved for single-photon counting. Homodyne detection is remarkably efficient by comparison, with up to 99% efficiencies being reported. However, the effect of detector efficiency is easily included by assuming that each detected photon mode is preceded by a beam splitter with intensity transmission $\eta < 1$. This changes both the LHS and RHS, so that the inequality becomes $\frac{4\eta^2}{2\eta+1} \le 4^{2/n}$, giving a threshold efficiency requirement of $\eta > \eta_{\min}$, where $\eta_{\min} = (1 + \sqrt{1 + 4^{1-2/n}})/4^{1-2/n}$.

This *reduces* at large n to an asymptotic value of $\eta_\infty = 0.809\,02$. Unexpectedly, the Bell violation (which signifies a quantum superposition) is less sensitive to detector inefficiency in the macroscopic, large n limit. The minimum detector efficiency η_n at finite n is plotted in Fig. 1, together with the minimum state preparation fidelity $\epsilon_{\min}$ in the case of ideal detectors, where we model the density matrix as $\hat{\rho} = \epsilon|\Psi_S\rangle\langle\Psi_S| + (1-\epsilon)\hat{I}$.

We finally prove that there are no LHV inequalities possible if one considers only the first-moment correlations between continuous variables in different sites. We show this explicitly for the simplest case and indicate how to generalize to arbitrary numbers of parties and settings. Consider first $n = 2$ parties, Alice and Bob, each of whom can choose between $m = 2$ observables: X_a, Y_a for Alice and X_b, Y_b for Bob. Each measurement yields an outcome in the real numbers. The first-moment correlation functions for each of the 4 possible configurations are just the averages $\langle X_a X_b\rangle$, $\langle X_a Y_b\rangle$, $\langle Y_a X_b\rangle$, $\langle Y_a Y_b\rangle$. Given those 4 experimental outcomes, can we find a local hidden-variable model that reproduces them?

We construct an explicit example. Consider a hidden-variable state S where the hidden variables are the measured values $\mathbf{X}$, $\mathbf{Y}$, in an equal mixture of four classical pure states $S_k = (X_a, Y_a, X_b, Y_b)_k$ defined by

$$\begin{aligned} S_1 &= 2(1, 0, \langle X_a X_b\rangle, 0), & S_2 &= 2(1, 0, 0, \langle X_a Y_b\rangle), \\ S_3 &= 2(0, 1, \langle Y_a X_b\rangle, 0), & S_4 &= 2(0, 1, 0, \langle Y_a Y_b\rangle). \end{aligned} \tag{12}$$

Each of the states S_k assigns a nonzero value to only one of the 4 correlation functions. Since the probability of each

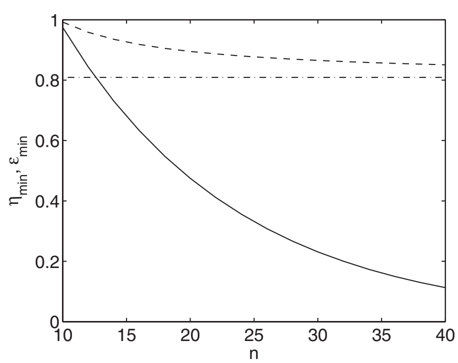

FIG. 1. Minimum state preparation fidelity $\epsilon_{\min}$ for ideal detectors (solid line), and minimum detection efficiency $\eta_{\min}$ for ideal state preparation (dashed line) required for violation of (8) as a function of the number of modes. The asymptotic value of $\eta_{\min}$ is indicated by the dash-dotted line.

of the states in the equal mixture is $1/4$, we have, for example, $\langle X_a X_b\rangle_S = \frac{1}{4}\sum_i \langle X_a X_b\rangle_{S_i} = \langle X_a X_b\rangle$.

Satisfying the two-site correlations using the state S defined by (12) leaves us with uncontrolled values for the single-site correlations, for instance, $\langle X_b\rangle_S = \frac{1}{2}(\langle X_a X_b\rangle + \langle Y_a X_b\rangle)$. One might object to the fact that this is not equal to $\langle X_b\rangle$ in general. However, we may correct these lower order correlations by adding four more states (S_5 to S_8) and changing the prefactors multiplying S_1 to S_4 to compensate for their reduced weight in the equal mixture. Crucially, adding these extra states to S in this manner does not modify the values of correlations such as $\langle X_a X_b\rangle$. As an example, we exhibit the state $S_5 = 8(0, 0, \langle X_b\rangle - (\langle X_a X_b\rangle + \langle Y_a X_b\rangle)/\sqrt{8}, 0)$, which corrects the single expectation value $\langle X_b\rangle_S$ to $\langle X_b\rangle$.

The proof generalizes easily to arbitrary n and m. In that case, there are m^n possible combinations of measurements that yield n-site correlations. Denoting the jth observable at site i by X_i^j, each combination is specified by a sequence of indices $(j_1, j_2, \ldots, j_n)$. For each combination of measurements, we define a hidden-variable state that assigns nonzero values only to the variables that appear in the associated correlation function $\langle\prod_{i=1}^n X_i^{j_i}\rangle$. In analogy to the example above, we can always choose the values of the hidden variables associated with $X_i^{j_i}$ such that their product is equal to $m^n\langle\prod_{i=1}^n X_i^{j_i}\rangle$. Since all other $m^n - 1$ states defined in this way will give a value of zero to this particular correlation function, and given that the probability associated with each of those states is $1/m^n$, we reproduce all correlations as desired. As indicated in the example, additional first-moment correlations involving *less* than n sites can be included in the LHV model by adding additional states to S in a way that does not affect the n-site correlations. Thus, any possible observation of first-moment correlations may be explained using a LHV model, and hence these correlations alone cannot violate any Bell inequality. In other words, the minimum requirement for a correlation Bell inequality with continuous, unbounded variables, is to use not just the first but also the second moments at each site.

In conclusion, we have derived a new class of Bell-type inequalities valid for continuous and unbounded experimental outcomes. We have shown that the same procedure allows one to derive the MABK class of Bell inequalities and their corresponding quantum bounds. That derivation makes it explicit that nonzero commutators—associated with the incompatibility of the local observables—are the essential ingredient responsible for the discrepancy between quantum mechanics and local hidden-variable theories. The new Bell-type inequality derived here can be directly applied to continuous variables without the need for a specific binning of the measurement outcomes. Surprisingly, quantum mechanics predicts exponentially increasing violations of the inequality for macroscopically large numbers of sites, even including realistic decoherence effects like inefficient state preparation, and a detector loss at *every* site.

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