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Free field realization of current superalgebra $gl(m|n)_k$

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We construct the free field representation of the affine currents, energy-momentum tensor, and screening currents of the first kind of the current superalgebra $gl(m|n)_k$ uniformly for $m=n$ and $m\neq n$. The energy-momentum tensor is given by a linear combination of two Sugawara tensors associated with the two independent quadratic Casimir elements of $gl(m|n)$. *© 2007 American Institute of Physics.*
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I. INTRODUCTION

Current superalgebras or affine superalgebras have emerged in a wide range of physical areas ranging from high energy physics to condensed matter physics. In high energy theoretical physics, sigma models with supermanifold target spaces naturally appear in the quantization of superstring theory in the AdS-type backgrounds. It was argued in Ref. 1 that even without a Wess-Zumino-Novikov-Witten (WZNW) term the sigma model on $PSL(n|n)$ supergroup is already conformally invariant. A WZNW term with any integer coefficient may then be added whenever necessary, without violating the conformal invariance. In Ref. 2, the $PSU(1,1|2)$ sigma model was used to quantize superstring theory on the $\mathrm{AdS}_3\times S^3$ background with Ramond-Ramond (RR) flux. In condensed matter physics, the supersymmetric treatment of quenched disorders leads to current superalgebras of zero superdimension. It is believed that critical behaviors of certain disordered systems such as the integer quantum Hall transition are described by sigma models or their WZNW generalizations based on supergroups of zero superdimension.[3–7]

As can be seen from the work in Ref. 8, in most above-mentioned applications one expects to work with sigma models on some kind of coset supermanifolds[9,10] with a WZNW term, and thus models of interest are more complicated than WZNW models on noncoset supergroups. However, even for such (noncoset) supergroup WZNW models, little has been known in general,[11] largely due to the technical difficulties in dealing with atypical and indecomposable representations which are common features for most superalgebras.

The Wakimoto free field realization[12,13] has been proved to be a powerful method in the study of Conformal Field Theories (CFTs) such as WZNW models. Motivated by the above-mentioned applications, in this paper we construct free field representations of the current superalgebra $gl(m|n)_k$ associated with the $GL(m|n)$ WZNW model for $m=n$ and $m\neq n$ in a unified way.

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Free field realization of the $gl(m|n)$ currents, *in principle*, can be obtained by a general method outlined in Refs. 14–18 where differential realizations of the corresponding finite dimensional Lie (super) algebras play a key role. However, *in practice*, such constructions for an *explicit* expression of the currents become very complicated for higher-rank algebras.[16–20] In this paper, we find a way to overcome the complication. In our approach, the construction of the differential operator realization becomes much simpler (cf. Refs. 18 and 19). We demonstrated this by working out the differential realization of $gl(4|4)$ in Ref. 21. Here we provide the complete results of $gl(m|n)$ for any m and n.

This paper is organized as follows. In Sec. II, we briefly review the definitions of finite-dimensional superalgebra $gl(m|n)$ and the associated current algebra, which also serves as an introduction our notation and some basic ingredients. In Sec. III, we construct explicitly the differential operator realization of $gl(m|n)$ in the standard basis. In Secs. IV and V, we construct the free field representation of the affine currents associated with $gl(m|n)$ at a generic level k, and the corresponding energy-momentum tensor. We, moreover, construct, in Sec. VI, the screening currents of the first kind. Section VII is for conclusions.

II. NOTATION AND PRELIMINARIES

Let us first fix our notation for the underlying nonaffine superalgebra $gl(m|n)$ which is nonsemisimple for both $m=n$ and $m\neq n$.[22,23]

$gl(m|n)$ is $\mathbb{Z}_2$ graded and is generated by the elements $\{E_{i,j}|i,j=1,\ldots,m+n\}$ which satisfy the following (anti)commutation relations:

$$[E_{i,j},E_{k,l}] = \delta_{jk}E_{i,l} - (-1)^{([i]+[j])([k]+[l])}\delta_{il}E_{k,j}. \tag{2.1}$$

Here and throughout, we adopt the convention $[a,b]=ab-(-1)^{[a][b]}ba$. The $\mathbb{Z}_2$ grading of the generators is $[E_{i,j}]=[i]+[j]$ with $[1]=\cdots=[m]=0$, $[m+1]=\cdots=[m+n]=1$. $E_{i,j}$, $1\leqslant i\neq j\leqslant m+n$, are raising/lowering generators. For a unified treatment of the $m=n$ and $m\neq n$ cases, we have chosen $E_{i,i}, i=1,\ldots,m+n$, to be the elements of the Cartan subalgebra (CSA) of $gl(m|n)$.

Let us remark that other bases of the CSA widely used by most physicists do not seem suitable for the unified treatment because the CSA elements for $m=n$ and $m\neq n$ in those bases are different. This is seen as follows. Let

$$I = \sum_{i=1}^{m+n} E_{ii}, \quad J = \sum_{i=1}^{m+n} (-1)^{[i]}E_{ii}.$$

In the fundamental representation of $gl(m|n)$, I is the $(m+n)\times(m+n)$ identity matrix and J is the diagonal matrix $J=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$. For $m\neq n$ the usual choice of the $gl(m|n)$ CSA elements is $\{I, H_i=(-1)^{[i]}E_{i,i}-(-1)^{[i+1]}E_{i+1,i+1}, i=1,\ldots,m+n-1\}$. However, this choice is inappropriate for $m=n$ because in this case I and H_i, $i=1,\ldots,2n-1$ become dependent,

$$H_n = \frac{1}{n}\left\{ I - \sum_{i=1}^{n-1}[iH_i + (n-i)H_{n+i}]\right\}.$$

That is, for $m=n$ one cannot simultaneously choose, say, H_n and I, as part of the $gl(n|n)$ CSA elements, in contrast to the $m\neq n$ case. One popular choice of $gl(n|n)$ CSA elements is $\{I,J,H_i, 1\leqslant i\neq n\leqslant 2n-1\}$. On the other hand, for $m\neq n$, J is a linear combination of I and H_i, $1\leqslant i\leqslant m+n-1$. Note also that for $m=n$, $I\in sl(n|n)$ and both $sl(n|n)$ and $gl(n|n)$ are nonsemisimple; for $m\neq n$, I is in $gl(m|n)$ but not in $sl(m|n)$.

With our above choice of generators, it is easy to check that the usual quadratic Casimir element of $gl(m|n)$ is

$$C_1 = \sum_{i,j=1}^{m+n} (-1)^{[j]} E_{i,j} E_{j,i}. \tag{2.2}$$

Since $gl(m|n)$ is nonsemisimple, there exists another independent quadratic Casimir element

$$C_2 = \sum_{i,j=1}^{m+n} E_{i,i} E_{j,j} = \left(\sum_{i=1}^{m+n} E_{i,i} \right)^2. \tag{2.3}$$

These two Casimir elements are useful in the following for the construction of the correct energy-momentum tensor.

For any m and n, $gl(m|n)$ has a nondegenerate and invariant metric or bilinear form given by[22,23]

$$(E_{i,j}, E_{k,l}) = \mathrm{str}(e_{i,j} e_{k,l}). \tag{2.4}$$

Here $e_{i,j}$, which is the $(m+n)\times(m+n)$ matrix with entry 1 at the ith row and jth column and zero elsewhere, is the fundamental or defining representation of $E_{i,j}$; str denotes the supertrace, i.e., str $(a) = \Sigma_i (-1)^{[i]} a_{ii}$.

The $gl(m|n)$ current algebra is generated by the currents $E_{i,j}(z)$ associated with the generators $E_{i,j}$ of $gl(m|n)$. The current algebra at a general level k obeys the following Operator Product Expansions (OPEs):[13]

$$E_{i,j}(z) E_{l,m}(w) = k \frac{(E_{i,j}, E_{l,m})}{(z-w)^2} + \frac{1}{(z-w)} (\delta_{jl} E_{i,m}(w) - (-1)^{([i]+[j])([l]+[m])} \delta_{im} E_{l,j}(w)). \tag{2.5}$$

III. DIFFERENTIAL OPERATOR REALIZATION OF *gl*(*m*|*n*)

As mentioned in the Introduction, practically it would be very involved (if not impossible) to obtain the *explicit* free field realization of higher-rank algebras such as $gl(m|n)$ for a larger value of $m+n$ by the general method outlined in Refs. 14–18. We have found a way to overcome the complication. In our approach, the construction of the differential operator realization becomes much simpler.

Let us introduce $\frac{1}{2}((m(m-1)+n(n-1))$ bosonic coordinates $\{x_{i,j},\ x_{m+k,m+l} | 1 \leqslant i < j \leqslant m,\ 1 \leqslant k < l \leqslant n\}$ with the $\mathbb{Z}_2$ grading: $[x_{i,j}]=0$, and $m \times n$ fermionic coordinates $\{\theta_{i,m+j} | 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ with the $\mathbb{Z}_2$ grading: $[\theta_{i,m+j}]=1$. These coordinates satisfy the following (anti)commutation relations:

$$[x_{i,j}, x_{k,l}] = 0, \quad [\partial_{x_{i,j}}, x_{k,l}] = \delta_{ik}\delta_{jl},$$

$$[\theta_{i,m+j}, \theta_{k,m+l}] = 0, \quad [\partial_{\theta_{i,m+j}}, \theta_{k,m+l}] = \delta_{ik}\delta_{jl},$$

and the other (anti)commutation relations are vanishing. Let $\langle\Lambda|$ be the lowest weight vector in the associated representation of $gl(m|n)$, satisfying the following conditions:

$$\langle\Lambda| E_{j+1,j} = 0, \quad 1 \leqslant j \leqslant m+n-1, \tag{3.1}$$

$$\langle\Lambda| E_{i,i} = \lambda_i \langle\Lambda|, \quad 1 \leqslant i \leqslant m+n. \tag{3.2}$$

An arbitrary vector in this representation space is parametrized by $\langle\Lambda|$ and the coordinates (x and θ) as

$$\langle\Lambda, x, \theta| = \langle\Lambda| G_+(x,\theta), \tag{3.3}$$

where $G_+(x,\theta)$ is given by (cf. Ref. 18)

$$G_+(x,\theta) = (G_{m+n-1,m+n}) \cdots (G_{j,m+n} \cdots G_{j,j+1}) \cdots (G_{1,m+n} \cdots G_{1,2}). \tag{3.4}$$

Here, for $i<j$, $G_{i,j}$ are given by

$$G_{i,j} = \begin{cases} e^{x_{i,j}E_{i,j}} & \text{if } [E_{i,j}] = 0 \\ e^{\theta_{i,j}E_{i,j}} & \text{if } [E_{i,j}] = 1. \end{cases} \tag{3.5}$$

One can define a differential operator realization $\rho^{(d)}$ of the generators of $gl(m|n)$ by

$$\rho^{(d)}(g)\langle\Lambda,x,\theta| \equiv \langle\Lambda,x,\theta|g, \qquad \forall\, g \in gl(m|n). \tag{3.6}$$

Here $\rho^{(d)}(g)$ is a differential operator of the bosonic and ferminic coordinates associated with the generator g, which can be obtained from the defining relation (3.6). Moreover, the defining relation also assures that the differential operator realization is actually a representation of $gl(m|n)$. Therefore it is sufficient to obtain the associated differential operators which are related to the simple roots, and the others can be constructed through the simple ones by (anti)commutation relations (3.16)–(3.17) (see below). By using the relation (3.6) and the Baker-Campbell-Hausdorff formula, after some algebraic manipulations, we obtain the following differential operator representation of the simple generators in the standard (distinguished) basis:[23]

$$\rho^{(d)}(E_{j,j+1}) = \sum_{k\leqslant j-1} x_{k,j}\partial_{x_{k,j+1}} + \partial_{x_{j,j+1}}, \quad 1 \leqslant j \leqslant m-1, \tag{3.7}$$

$$\rho^{(d)}(E_{m,m+1}) = \sum_{k\leqslant m-1} x_{k,m}\partial_{\theta_{k,m+1}} + \partial_{\theta_{m,m+1}}, \tag{3.8}$$

$$\rho^{(d)}(E_{m+j,m+1+j}) = \sum_{k\leqslant m} \theta_{k,m+j}\partial_{\theta_{k,m+1+j}} + \sum_{k\leqslant j-1} x_{m+k,m+j}\partial_{x_{m+k,m+1+j}} + \partial_{x_{m+j,m+1+j}}, \quad 1 \leqslant j \leqslant n-1, \tag{3.9}$$

$$\rho^{(d)}(E_{j,j}) = \sum_{k\leqslant j-1} x_{k,j}\partial_{x_{k,j}} - \sum_{j+1\leqslant k\leqslant m} x_{j,k}\partial_{x_{j,k}} - \sum_{k\leqslant n} \theta_{j,m+k}\partial_{\theta_{j,m+k}} + \lambda_j, \quad 1 \leqslant j \leqslant m-1, \tag{3.10}$$

$$\rho^{(d)}(E_{m,m}) = \sum_{k\leqslant m-1} x_{k,m}\partial_{x_{k,m}} - \sum_{k\leqslant n} \theta_{m,m+k}\partial_{\theta_{m,m+k}} + \lambda_m, \tag{3.11}$$

$$\begin{aligned}\rho^{(d)}(E_{m+j,m+j}) = {}& \sum_{k\leqslant m} \theta_{k,m+j}\partial_{\theta_{k,m+j}} + \sum_{k\leqslant j-1} x_{m+k,m+j}\partial_{x_{m+k,m+j}} - \sum_{j+1\leqslant k\leqslant n} x_{m+j,m+k}\partial_{x_{m+j,m+k}} + \lambda_{m+j}, \\ & 1 \leqslant j \leqslant n,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\rho^{(d)}(E_{j+1,j}) = {}& \sum_{k\leqslant j-1} x_{k,j+1}\partial_{x_{k,j}} - \sum_{j+2\leqslant k\leqslant m} x_{j,k}\partial_{x_{j+1,k}} - \sum_{k\leqslant n} \theta_{j,m+k}\partial_{\theta_{j+1,m+k}} - x_{j,j+1}\Bigg(\sum_{j+1\leqslant k\leqslant m} x_{j,k}\partial_{x_{j,k}} \\ & + \sum_{k\leqslant n} \theta_{j,m+k}\partial_{\theta_{j,m+k}}\Bigg) + x_{j,j+1}\Bigg(\sum_{j+2\leqslant k\leqslant m} x_{j+1,k}\partial_{x_{j+1,k}} + \sum_{k\leqslant n} \theta_{j+1,m+k}\partial_{\theta_{j+1,m+k}}\Bigg) \\ & + x_{j,j+1}(\lambda_j - \lambda_{j+1}), \quad 1 \leqslant j \leqslant m-1,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\rho^{(d)}(E_{m+1,m}) = {}& \sum_{k\leqslant m-1} \theta_{k,m+1}\partial_{x_{k,m}} + \sum_{2\leqslant k\leqslant n} \theta_{m,m+k}\partial_{x_{m+1,m+k}} \\ & - \theta_{m,m+1}\Bigg(\sum_{2\leqslant k\leqslant n} (\theta_{m,m+k}\partial_{\theta_{m,m+k}} + x_{m+1,m+k}\partial_{x_{m+1,m+k}})\Bigg) + \theta_{m,m+1}(\lambda_m + \lambda_{m+1}),\end{aligned} \tag{3.14}$$

$$\rho^{(d)}(E_{m+1+j,m+j}) = \sum_{k\leq m}\theta_{k,m+1+j}\partial_{\theta_{k,m+j}} + \sum_{k\leq j-1}x_{m+k,m+1+j}\partial_{x_{m+k,m+j}} - \sum_{j+2\leq k\leq n}x_{m+j,m+k}\partial_{x_{m+1+j,m+k}}$$
$$- x_{m+j,m+1+j}\sum_{j+1\leq k\leq n}x_{m+j,m+k}\partial_{x_{m+j,m+k}} + x_{m+j,m+1+j}\sum_{j+2\leq k\leq n}x_{m+1+j,m+k}\partial_{x_{m+1+j,m+k}}$$
$$+ x_{m+j,m+1+j}(\lambda_{m+j} - \lambda_{m+1+j}), \quad 1\leq j\leq n-1. \tag{3.15}$$

The generators associated with the nonsimple roots can be constructed through the simple ones by the (anti)commutation relations

$$\rho^{(d)}(E_{i,j}) = [\rho^{(d)}(E_{i,k}),\rho^{(d)}(E_{k,j})], \quad 1\leq i<k<j\leq m+n \text{ and } 2\leq j-i, \tag{3.16}$$

$$\rho^{(d)}(E_{j,i}) = [\rho^{(d)}(E_{j,k}),\rho^{(d)}(E_{k,i})], \quad 1\leq i<k<j\leq m+n \text{ and } 2\leq j-i. \tag{3.17}$$

A direct computation shows that the differential realization [Eqs. (3.7)–(3.17)] of $gl(m|n)$ satisfies the commutation relation (2.1). Alternatively, one may check that Eqs. (3.7)–(3.15) satisfy the (anti)commutation relations corresponding to the simple roots together with the Serre relations.[23]

IV. FREE FIELD REALIZATION OF $gl(m|n)_k$

With the help of the differential realization obtained in the last section we can construct the free field representation of the $gl(m|n)$ current algebra in terms of $\frac{1}{2}(m(m-1)+n(n-1))$ bosonic $\beta-\gamma$ pairs $\{(\beta_{i,j},\gamma_{i,j}),\ 1\leq i<j\leq m;\ (\bar{\beta}_{i,j}\bar{\gamma}_{i,j}),\ 1\leq i<j\leq n\}$, $m\times n$ fermionic $b-c$ pairs $\{(\psi^{\dagger}_{i,j},\psi_{i,j}),\ 1\leq i\leq m,\ 1\leq j\leq n\}$, and $m+n$ free scalar fields ϕ_i, $i=1,\ldots,m+n$. These free fields obey the following OPEs:

$$\beta_{i,j}(z)\gamma_{k,l}(w) = -\gamma_{k,l}(z)\beta_{i,j}(w) = \frac{\delta_{ik}\delta_{jl}}{(z-w)}, \quad 1\leq i<j\leq m, \quad 1\leq k<l\leq m, \tag{4.1}$$

$$\bar{\beta}_{i,j}(z)\bar{\gamma}_{k,l}(w) = -\bar{\gamma}_{k,l}(z)\bar{\beta}_{i,j}(w) = \frac{\delta_{ik}\delta_{jl}}{(z-w)}, \quad 1\leq i<j\leq n, \quad 1\leq k<l\leq n, \tag{4.2}$$

$$\psi_{i,j}(z)\psi^{\dagger}_{k,l}(w) = \psi^{\dagger}_{k,l}(z)\psi_{i,j}(w) = \frac{\delta_{ik}\delta_{jl}}{(z-w)}, \quad 1\leq i,k\leq m, \quad 1\leq j,l\leq n, \tag{4.3}$$

$$\phi_i(z)\phi_j(w) = (E_{i,i},E_{j,j})\ln(z-w) = (-1)^{[i]}\delta_{ij}\ln(z-w), \quad 1\leq i,j\leq m+n, \tag{4.4}$$

and the other OPEs are trivial.

The free field realization of the $gl(m|n)$ current algebra (2.5) is obtained by the substitution in the differential realization [Eqs. (3.7)–(3.15)] of $gl(m|n)$,

$$x_{i,j}\to\gamma_{i,j}(z), \quad \partial_{x_{i,j}}\to\beta_{i,j}(z), \quad 1\leq i<j\leq m,$$

$$x_{m+i,m+j}\to\bar{\gamma}_{i,j}(z), \quad \partial_{x_{m+i,m+j}}\to\bar{\beta}_{i,j}(z), \quad 1\leq i<j\leq n,$$

$$\theta_{i,m+j}\to\psi^{\dagger}_{i,j}(z), \quad \partial_{\theta_{i,m+j}}\to\psi_{i,j}(z), \quad 1\leq i\leq m \text{ and } 1\leq j\leq n,$$

$$\lambda_j\to\sqrt{k+m-n}\,\partial\phi_j(z) - \frac{(-1)^{[j]}(1+\alpha)}{2\sqrt{k+m-n}}\sum_{l=1}^{m+n}\phi_l(z), \quad 1\leq j\leq m+n,$$

with $\alpha=1+2k/(m-n)-2\sqrt{k(k+m-n)}/(m-n)$, followed by the addition of anomalous terms linear in $\partial\psi^{\dagger}(z)$, $\partial\gamma(z)$, and $\partial\bar{\gamma}(z)$ in the expressions of the currents. It is remarked that for $m=n$, α is

$\alpha=\lim_{m\to n}(1+2k/(m-n)-2\sqrt{k(k+m-n)}/(m-n))=0$. Here we present the realization of the currents associated with the simple roots,

$$E_{j,j+1}(z)=\sum_{l\leq j-1}\gamma_{l,j}(z)\beta_{l,j+1}(z)+\beta_{j,j+1}(z),\quad 1\leq j\leq m-1, \tag{4.5}$$

$$E_{m,m+1}(z)=\sum_{l\leq m-1}\gamma_{l,m}(z)\psi_{l,1}(z)+\psi_{m,1}(z), \tag{4.6}$$

$$E_{m+j,m+j+1}(z)=\sum_{l\leq m}\psi^{\dagger}_{l,j}(z)\psi_{l,j+1}(z)+\sum_{l\leq j-1}\bar{\gamma}_{l,j}(z)\bar{\beta}_{l,j+1}(z)+\bar{\beta}_{j,j+1}(z),\quad 1\leq j\leq n-1, \tag{4.7}$$

$$\begin{aligned}E_{j,j}(z)=&\sum_{l\leq j-1}\gamma_{l,j}(z)\beta_{l,j}(z)-\sum_{j+1\leq l\leq m}\gamma_{j,l}(z)\beta_{j,l}(z)-\sum_{l\leq n}\psi^{\dagger}_{j,l}(z)\psi_{j,l}(z)+\sqrt{k+m-n}\,\partial\phi_j(z)\\&-\frac{1+\alpha}{2\sqrt{k+m-n}}\sum_{l=1}^{m+n}\partial\phi_l(z),\quad 1\leq j\leq m,\end{aligned} \tag{4.8}$$

$$\begin{aligned}E_{m+j,m+j}(z)=&\sum_{l\leq m}\psi^{\dagger}_{l,j}(z)\psi_{l,j}(z)+\sum_{l\leq j-1}\bar{\gamma}_{l,j}(z)\bar{\beta}_{l,j}(z)-\sum_{j+1\leq l\leq n}\bar{\gamma}_{j,l}(z)\bar{\beta}_{j,l}(z)+\sqrt{k+m-n}\,\partial\phi_{m+j}(z)\\&+\frac{1+\alpha}{2\sqrt{k+m-n}}\sum_{l=1}^{m+n}\partial\phi_l(z),\quad 1\leq j\leq n,\end{aligned} \tag{4.9}$$

$$\begin{aligned}E_{j+1,j}(z)=&\sum_{l\leq j-1}\gamma_{l,j+1}(z)\beta_{l,j}(z)-\sum_{j+2\leq l\leq m}\gamma_{j,l}(z)\beta_{j+1,l}(z)-\sum_{l\leq n}\psi^{\dagger}_{j,l}(z)\psi_{j+1,l}(z)-\gamma_{j,j+1}(z)\\&\times\left(\sum_{j+1\leq l\leq m}\gamma_{j,l}(z)\beta_{j,l}(z)+\sum_{l\leq n}\psi^{\dagger}_{j,l}(z)\psi_{j,l}(z)\right)+\gamma_{j,j+1}(z)\left(\sum_{j+2\leq l\leq m}\gamma_{j+1,l}(z)\beta_{j+1,l}(z)\right.\\&\left.+\sum_{l\leq n}\psi^{\dagger}_{j+1,l}(z)\psi_{j+1,l}(z)\right)+\sqrt{k+m-n}\,\gamma_{j,j+1}(z)(\partial\phi_j(z)-\partial\phi_{j+1}(z))\\&+(k+j-1)\partial\gamma_{j,j+1}(z),\quad 1\leq j\leq m-1,\end{aligned} \tag{4.10}$$

$$\begin{aligned}E_{m+1,m}(z)=&\sum_{l\leq m-1}\psi^{\dagger}_{l,1}(z)\beta_{l,m}(z)+\sum_{2\leq l\leq n}\psi^{\dagger}_{m,l}(z)\bar{\beta}_{1,l}(z)-\psi^{\dagger}_{m,1}(z)\left(\sum_{2\leq l\leq n}(\psi^{\dagger}_{m,l}(z)\psi_{m,l}(z)\right.\\&\left.+\bar{\gamma}_{1,l}(z)\bar{\beta}_{1,l}(z))\right)+\sqrt{k+m-n}\,\psi^{\dagger}_{m,1}(z)(\partial\phi_m(z)+\partial\phi_{m+1}(z))+(k+m-1)\partial\psi^{\dagger}_{m,1}(z),\end{aligned} \tag{4.11}$$

$$\begin{aligned}E_{m+j+1,m+j}(z)=&\sum_{l\leq m}\psi^{\dagger}_{l,j+1}(z)\psi_{l,j}(z)+\sum_{l\leq j-1}\bar{\gamma}_{l,j+1}(z)\bar{\beta}_{l,j}(z)-\sum_{j+2\leq l\leq n}\bar{\gamma}_{j,l}(z)\bar{\beta}_{j+1,l}(z)-\bar{\gamma}_{j,j+1}(z)\\&\times\left(\sum_{j+1\leq l\leq n}\bar{\gamma}_{j,l}(z)\bar{\beta}_{j,l}(z)-\sum_{j+2\leq l\leq n}\bar{\gamma}_{j+1,l}(z)\bar{\beta}_{j+1,l}(z)\right)+\sqrt{k+m-n}\,\bar{\gamma}_{j,j+1}(\partial\phi_{m+j}(z)\\&-\partial\phi_{m+j+1}(z))-(k+m+1-j)\partial\bar{\gamma}_{j,j+1}(z),\quad 1\leq j\leq n-1.\end{aligned} \tag{4.12}$$

Here and throughout normal ordering of free fields is implied whenever necessary. The free field realization of currents associated with the nonsimple roots can be obtained from the OPEs of the simple ones, similar to Eqs. (3.16) and (3.17). It is straightforward to check that the above free field realizations of the currents satisfy the OPEs of the $gl(m|n)$ current algebra. Moreover, for the case $n=0$ our results reduce to those in, Ref. 15, giving the free field realization of the $gl(m)$ current algebra.

Some remarks are in order. We have obtained the free field realization of $gl(m|n)$ current algebra uniformly for any m and n for the CSA basis we have chosen. It is easy to make simple basis transforms of the CSA to get expressions for the more familar CSA bases. This is seen as follows. Introduce new free scalar fields through linear combinations of the original free scalar fields $\phi_i(z)$,

$$\phi_I(z) = \sum_{i=1}^{m+n} \phi_i(z), \quad \phi_J(z) = \sum_{i=1}^{m+n} (-1)^{[i]}\phi_i(z),$$

$$\phi_{H_i}(z) = (-1)^{[i]}\phi_i(z) - (-1)^{[i+1]}\phi_{i+1}(z). \tag{4.13}$$

In terms of the new scalar fields, the currents associated with I,J, and H_i take the form, as can easily be seen from Eqs. (4.8) and (4.9),

$$I(z) = \sum_{i=1}^{m+n} E_{i,i}(z) = \sqrt{k}\partial\phi_I(z),$$

$$J(z) = \sum_{i=1}^{m+n} (-1)^{[i]}E_{i,i}(z) = \sqrt{k+m-n}\partial\phi_J(z) - \frac{(m+n)(1+\alpha)}{2\sqrt{k+m-n}}\partial\phi_I(z) - \sum_{j=1}^{n}\sum_{l\leqslant n} \psi^{\dagger}_{jl}(z)\psi_{jl}(z)$$
$$- \sum_{j=1}^{n}\sum_{l\leqslant m} \psi^{\dagger}_{lj}(z)\psi_{lj}(z),$$

$$H_i(z) = (-1)^{[i]}E_{i,i}(z) - (-1)^{[i+1]}E_{i+1,i+1}(z) = \tilde{H}_i(z) + \sqrt{k+m-n}\partial\phi_{H_i}(z), \quad 1 \leqslant i \leqslant m+n-1, \tag{4.14}$$

where $\tilde{H}_i(z)$ are functions of the $\beta-\gamma$ and $b-c$ pairs only. Now for $m=n$, replacing the $2n$ original free scalar fields $\phi_i(z)$ by $\{\phi_{H_i}(z),\ 1\leqslant i\neq n\leqslant 2n-1,\ \phi_I(z),\ \phi_J(z)\}$ and moreover using the relation

$$\phi_{H_n}(z) = \frac{1}{n}\left[\phi_I(z) - \sum_{i=1}^{n-1}(i\phi_{H_i}(z) + (n-i)\phi_{H_{n+i}}(z))\right]$$

to eliminate $\phi_{H_n}(z)$, then we obtain the $gl(n|n)$ currents $\{E_{i,j}(z),\ 1\leqslant i\neq j\leqslant 2n-1;\ I(z),\ J(z),\ H_l(z),\ 1\leqslant l\neq n\leqslant 2n-1\}$ in the new basis in terms of the new free scalar fields defined above together with the original $\beta-\gamma$ and $b-c$ pairs. Similarly for $m\neq n$, we replace the $m+n$ original free scalar fields $\phi_i(z)$ by $\{\phi_{H_i}(z),\ 1\leqslant i\leqslant m+n-1,\ \phi_I(z)\}$ to obtain the $gl(m|n)$ currents $\{E_{i,j}(z),\ 1\leqslant i\neq j \leqslant 2n-1;\ I(z),\ H_l(z),\ 1\leqslant l\leqslant m+n-1\}$ in the new basis.

Note that for $m=n$, $\phi_J(z)$ only appears in $J(z)$. Thus the free field realization of $sl(n|n)$ current algebra may be obtained from that of the $gl(n|n)$ current algebra by simply dropping $J(z)$. The free field realization of $psl(n|n)=sl(n|n)/I$ current algebra is obtained by setting $\phi_I(z)=0$ and thus $I(z)=0$ in the realization of the $sl(n|n)$ current algebra. For $m\neq n$, since $\phi_I(z)$ only appears in $I(z)$ in the new basis, one may obtain the free field realization of the $sl(m|n)$ current algebra by simply dropping $I(z)$ in the realization of the $gl(m|n)$ current algebra.

V. ENERGY-MOMENTUM TENSOR

In this setion we construct the free field realization of the Sugawara energy-momentum tensor associated with the $gl(m|n)$ current algebra. After a tedious calculation, we find that the Sugawara tensor corresponding to the quadratic Casimir C_1 is given by

$$
\begin{aligned}
T_1(z) &= \frac{1}{2(k+m-n)}\sum_{i,j=1}^{m+n}(-1)^{[j]}{:}E_{i,j}(z)E_{j,i}(z)\\
&{:}= \frac{1}{2}\sum_{l=1}^{m+n}(-1)^{[l]}\partial\phi_l(z)\partial\phi_l(z) - \frac{1}{2\sqrt{k+m-n}}\partial^2\left(\sum_{i=1}^{m}(m-n-2i+1)\phi_i(z)\right.\\
&\quad\left.-\sum_{j=1}^{n}(m+n-2j+1)\phi_{m+j}(z)\right) + \sum_{i<j}^{m}\partial\gamma_{i,j}(z)\beta_{i,j}(z) + \sum_{i<j}^{n}\partial\bar{\gamma}_{i,j}(z)\bar{\beta}_{i,j}(z)\\
&\quad+\sum_{i=1}^{m}\sum_{j=1}^{n}\partial\psi_{i,j}^{\dagger}(z)\psi_{i,j}(z) - \frac{1}{2(k+m-n)}\partial\phi_I(z)\partial\phi_I(z).
\end{aligned}
\tag{5.1}
$$

On the other hand, the Sugawara tensor corresponding to the quadratic Casimir C_2 is

$$
T_2(z) = \frac{1}{2(k+m-n)}\sum_{i,j=1}^{m+n}{:}E_{i,i}(z)E_{j,j}(z){:}= \frac{k}{2(k+m-n)}\partial\phi_I(z)\partial\phi_I(z).
\tag{5.2}
$$

In order that all currents $E_{i,j}(z)$ are primary fields with conformal dimensional one, we define the energy-momentum tensor $T(z)$ as follows:

$$
\begin{aligned}
T(z) &= T_1(z) + \frac{1}{k}T_2(z) = \frac{1}{2}\sum_{l=1}^{m+n}(-1)^{[l]}\partial\phi_l(z)\partial\phi_l(z) - \frac{1}{2\sqrt{k+m-n}}\partial^2\left(\sum_{i=1}^{m}(m-n-2i+1)\phi_i(z)\right.\\
&\quad\left.-\sum_{j=1}^{n}(m+n-2j+1)\phi_{m+j}(z)\right) + \sum_{i<j}^{m}\partial\gamma_{i,j}(z)\beta_{i,j}(z) + \sum_{i<j}^{n}\partial\bar{\gamma}_{i,j}(z)\bar{\beta}_{i,j}(z) + \sum_{i=1}^{m}\sum_{j=1}^{n}\partial\psi_{i,j}^{\dagger}(z)\psi_{i,j}(z).
\end{aligned}
\tag{5.3}
$$

It is straightforward to check that $T(z)$ satisfy the following OPE:

$$
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},
\tag{5.4}
$$

where the central charge $c=0$ for $m=n$, and

$$
c = \frac{k((m-n)^2-1)}{k+m-n} + 1
\tag{5.5}
$$

for $m \neq n$. Moreover, we find that with regard to the energy-momentum tensor $T(z)$ defined by Eq. (5.3) all currents $E_{i,j}(z)$ are indeed primary fields with conformal dimensional one, namely,

$$
T(z)E_{i,j}(w) = \frac{E_{i,j}(w)}{(z-w)^2} + \frac{\partial E_{i,j}(w)}{(z-w)}, \quad 1 \leq i,j \leq m+n.
\tag{5.6}
$$

Therefore, $T(z)$ is the very energy-momentum tensor of the $gl(m|n)$ current algebra.

VI. SCREENING CURRENTS

Important objects in applying the free field realization to the computation of correlation functions of the associated CFT are screening currents. A screening current is a primary field with conformal dimension one and has the property that the singular part of its OPE with the affine currents is a total derivative. These properties ensure that integrated screening currents (screening charges) may be inserted into correlators while the conformal or affine Ward identities remain intact. This in turn makes them very useful in the computation of correlation functions.[24,25]

Free field realization of the $gl(m|n)$ screening currents of the first kind may be constructed from certain differential operators $\{s_{i,j}|1\leqslant i<j\leqslant n+m\}$,[15,18] which we define as given from the relation

$$s_{i,j}\langle\Lambda,x,\theta|\equiv\langle\Lambda|E_{i,j}G_+(x,\theta)\quad\text{for } 1\leqslant i,j\leqslant m+n. \tag{6.1}$$

The above-defined operators $s_{i,j}$ give a differential operator realization of a subalgebra of $gl(m|n)$. Again it is sufficient to construct $s_{j,j+1}$ related to the simple generators $E_{j,j+1}$, $1\leqslant j\leqslant m+n-1$ of $gl(m|n)$. Let us denote those differential operators by s_j. Using Eq. (6.1) and the Baker-Campbell-Hausdorff formula, after some algebraic manipulations, we obtain the following explicit expressions of s_j:

$$s_j=\sum_{j+2\leqslant l\leqslant m}x_{j+1,l}\partial_{x_{j,l}}+\sum_{l\leqslant n}\theta_{j+1,m+l}\partial_{\theta_{j,m+l}}+\partial_{x_{j,j+1}},\quad 1\leqslant j\leqslant m-1, \tag{6.2}$$

$$s_m=\sum_{2\leqslant l\leqslant n}x_{m+1,m+l}\partial_{\theta_{m,m+l}}+\partial_{\theta_{m,m+1}}, \tag{6.3}$$

$$s_{m+j}=\sum_{j+2\leqslant l\leqslant n}x_{m+j+1,m+l}\partial_{x_{m+j,m+l}}+\partial_{x_{m+j,m+j+1}},\quad 1\leqslant j\leqslant n-1. \tag{6.4}$$

From (anti)communication relations similar to Eq. (3.16), one may obtain the differential operators $s_{i,j}$ associated with the nonsimple generators of $gl(m|n)$. Following the procedure similar to Refs. 15 and 18, we find the free field realization of the screening currents S_j corresponding to the differential operators s_j,

$$S_j(z)=\left(\sum_{j+2\leqslant l\leqslant m}\gamma_{j+1,l}(z)\beta_{j,l}(z)+\sum_{l=1}^{n}\psi^\dagger_{j+1,l}(z)\psi_{j,l}(z)+\beta_{j,j+1}(z)\right)\tilde{s}_j(z),\quad 1\leqslant j\leqslant m-1, \tag{6.5}$$

where

$$S_m(z)=\left(\sum_{2\leqslant l\leqslant n}\bar{\gamma}_{1,l}(z)\psi_{m,l}(z)+\psi_{m,1}(z)\right)\tilde{s}_m(z), \tag{6.6}$$

$$S_{m+j}(z)=\left(\sum_{j+2\leqslant l\leqslant n}\bar{\gamma}_{j+1,l}(z)\bar{\beta}_{j,l}(z)+\bar{\beta}_{j,j+1}(z)\right)\tilde{s}_{m+j}(z),\quad 1\leqslant j\leqslant n-1, \tag{6.7}$$

where

$$\tilde{s}_j(z)=e^{-1/\sqrt{k+m-n}(\phi_j(z)-\phi_{j+1}(z))},\quad 1\leqslant j\leqslant m-1, \tag{6.8}$$

$$\tilde{s}_m(z)=e^{-1/\sqrt{k+m-n}(\phi_m(z)+\phi_{m+1}(z))}, \tag{6.9}$$

$$\tilde{s}_{m+j}(z)=e^{1/\sqrt{k+m-n}(\phi_{m+j}(z)-\phi_{m+j+1}(z))},\quad 1\leqslant j\leqslant n-1. \tag{6.10}$$

The OPEs of the screening currents with the energy-momentum tensor and the $gl(m|n)$ currents (4.5)–(4.12) are

$$T(z)S_j(w)=\frac{S_j(w)}{(z-w)^2}+\frac{\partial S_j(w)}{(z-w)}=\partial_w\left\{\frac{S_j(w)}{(z-w)}\right\},\quad 1\leqslant j\leqslant m+n-1, \tag{6.11}$$

$$E_{i+1,i}(z)S_j(w) = (-1)^{[i]+[i+1]}\delta_{ij}\partial_w\left\{\frac{k\tilde{s}_j(w)}{(z-w)}\right\}, \quad 1 \leqslant i,j \leqslant m+n-1, \tag{6.12}$$

$$E_{i,i+1}(z)S_j(w) = 0, \quad 1 \leqslant i,j \leqslant m+n-1, \tag{6.13}$$

$$E_{i,i}(z)S_j(w) = 0, \quad 1 \leqslant i \leqslant m+n, \quad 1 \leqslant j \leqslant m+n-1. \tag{6.14}$$

The screening currents obtained this way are screening currents of the first kind.[26] Moreover, $S_m(z)$ is fermionic and the others are bosonic.

VII. DISCUSSIONS

We have studied the $gl(m|n)$ current algebra at general level k. We have constructed its Wakimoto free field realization [Eqs. (4.5)–(4.12)] for $m=n$ and $m\neq n$ in a unified way, and the corresponding energy-momentum tensor (5.3) which is a linear combination of two Sugawara tensors associated with two quadratic Casimir elements of $gl(m|n)$. We have also found $m+n-1$ screening currents, Eqs. (6.5)–(6.10), of the first kind. Our results reduce to those in Ref. 15 for $n=0$ (i.e., in the bosonic case), and recover those in Ref. 21 for $m=n=4$, thus providing a complete proof of the results in that paper.

To fully take the advantage of the free field approach in applications mentioned in the Introduction, explicit construction of primary fields in terms of free fields is needed. It is well known that there exist two types of representations for the underlying finite dimensional superalgebra $gl(m|n)$: typical and atypical representations. Atypical representations, which are often indecomposable, have no counterparts in the bosonic algebra setting and the understanding of such representations is still very much incomplete. Although the construction of the primary fields associated with typical representations are similar to the bosonic algebra cases, it is a highly nontrivial task to construct the primary fields associated with atypical representations.[20]

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