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Reference Enthalpy Method Developed from Solutions of the Boundary-Layer Equations

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A simple average of local enthalpy over the velocity profile is proposed as the proper definition of reference enthalpy for the purpose of quasi-one-dimensional treatment of compressible boundary layers. Use of Van Driest's nearly exact solutions of the laminar boundary-layer equations shows that this definition produces Eckert's reference enthalpy formulation for the special case of an adiabatic wall. For surfaces other than adiabatic, either Eckert's form should be replaced by that of Young and Janssen, or the coefficient in Eckert's viscous heating term should be slightly modified. A similar analysis was conducted for turbulent flows using Whitfield and High's simplified solutions of the turbulent boundary-layer equations. Dorrance's derivation of reference quantities is also addressed. This work provides a theoretical basis for the empirical reference enthalpy formulas of Eckert and others and supplies practical expressions for the reference enthalpy of both laminar and turbulent compressible boundary layers.

Nomenclature

C	=	Chapman–Rubesin parameter ($=\rho\mu/\rho_e\mu_e$)
C_f	=	skin friction coefficient $=\tau_w/\frac{1}{2}\rho_e u_e^2$
C_h	=	heat transfer coefficient $=q_w/\rho_e u_e c_p(T_{aw}-T_w)$
c_p	=	specific heat at constant pressure
f	=	velocity ratio ($=u/u_e$)
g	=	enthalpy ratio ($=h/h_e$)
h	=	specific enthalpy
M	=	Mach number
Pr	=	Prandtl number
q_w	=	surface heat transfer
Re_x	=	Reynolds number
r	=	adiabatic wall recovery factor
T	=	temperature
u	=	velocity
x, y	=	coordinates
α	=	17.5
β	=	coefficient of viscous heating term in Eckert's reference enthalpy formula
β_1, β_2	=	coefficients in the proposed laminar reference enthalpy formula
γ	=	ratio of specific heats
δ	=	boundary-layer thickness
δ^*	=	boundary-layer displacement thickness
ε	=	$1-P_{rt}$
θ^I, θ^{II}	=	functions of Pr and f in Van Driest's enthalpy profile
η	=	function of f in Van Driest's local shear stress
μ	=	viscosity
ρ	=	density
τ	=	shear stress
ϕ	=	wall enthalpy ratio ($=h_w/h_e$)

Subscripts

aw	=	adiabatic wall
e	=	edge of boundary layer
t	=	turbulent
w	=	wall

Superscript

*	=	reference value

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Introduction

QUASI-ONE-DIMENSIONAL treatments of compressible boundary layers model the real boundary layer, which has continuously variable velocity and gas properties, as a viscous layer with variable velocity but constant or "reference" gas properties. These methods are an attractive alternative to computational solutions of the boundary-layer equations for preliminary design purposes when estimates of C_f and C_h are required. Such treatments have also been used by Mirels[1] to predict boundary-layer effects on test times in shock tubes with reasonable accuracy. An obvious prerequisite for quasi-one-dimensional boundary-layer analyses is the availability of reference gas properties that are adequately representative of the entire boundary-layer cross section. As the temperature can vary significantly across a compressible boundary layer, calculation of reference gas properties requires a reference temperature or reference enthalpy to be established.

The most popular choice for the reference enthalpy of both laminar and turbulent boundary layers is due to Eckert.[2] His equation for the reference enthalpy ratio of a boundary layer is

$$g^* = h^*/h_e = 0.5(1+\phi) + \beta r[(\gamma-1)/2]M_e^2 \tag{1}$$

where $\phi = h_w/h_e$, r is the adiabatic wall recovery factor, and $\beta = 0.22$.

Eckert arrived at the factor of $\beta = 0.22$ in the viscous heating term of Eq. (1) by empirically matching the "reference enthalpy corrected" incompressible flat-plate wall shear stress with a database of solutions of the laminar compressible boundary-layer equations by Van Driest.[3] In particular, Eckert corrected the classical expression for laminar incompressible flat-plate skin friction ($C_f = 0.664/\sqrt{Re_x}$) for use in compressible flow by evaluating the gas properties at the reference enthalpy, such that

$$C_f^* = \frac{\tau_w}{1/2\rho^* u_e^2} = \frac{0.664}{\sqrt{Re_x^*}} \tag{2}$$

Rearrangement of Eq. (2) supplies the following expression for τ_w:

$$\tau_w = 0.664\Big/\sqrt{Re_x^*}\,\tfrac{1}{2}\rho^* u_e^2 \tag{3}$$

Substitution of Eq. (3) into the incompressible C_f formula led to the following expression for the skin friction coefficient in compressible flow:

$$C_f\sqrt{Re_x} = 0.664\sqrt{\rho^*\mu^*/\rho_e\mu_e} = 0.664\sqrt{C^*} \tag{4}$$

A value of $\beta = 0.22$ most closely matched Eq. (4) with Van Driest's database.[3] This formula was developed for boundary layers on flat plates; however, it is commonly used for more complicated flows by simply using the local edge conditions.

Although Eckert's reference enthalpy formula is well known throughout the literature, it is interesting to note that Eq. (1) is not unique in satisfying the wall shear stress–based procedure. About equally successful, for example, is the somewhat different form for g^* proposed by Young and Janssen.[4] The purpose of the present paper is to determine how a more fundamentally defined g^* relates to the Eckert and Young and Janssen forms. Limitations will be discussed and improved forms recommended, one for laminar boundary layers and another for turbulent boundary layers. Because the current formulas for g^* are derived directly from solutions of the boundary-layer equations, the results can be said to be derived in some sense from first principles. As far as we know, the only previous such attempt was that of Dorrance,[5] which is discussed.

Proposed Reference Enthalpy

If the ratio of specific enthalpies, $g = h/h_e$, is assumed to depend only on the velocity profile within the boundary layer, $f = u/u_e$, a straightforward definition of a representative reference value, g^*, is

$$g^* = \int_0^1 g(f)\,df \tag{5}$$

Given this definition, our task is to use it to develop a generalized expression for g^* without demanding prior knowledge of $g(f)$; otherwise its utility for quasi-one-dimensional analyses is defeated. This is accomplished through the use of solutions of the boundary-layer equations. For the purposes of the present investigation we assume steady-state flow of a calorically perfect gas ($\gamma = \text{const}$), nearly constant freestream parameters (ρ_e, μ_e, u_e), wall temperature T_w, and relatively constant Prandtl number Pr across the boundary layer compared with variations of other gas properties. As in Ref. 2, generalization can be achieved to some extent by replacing Pr with its reference value Pr^* in the final results. The same may be done for flows with variable γ through the use of γ^*.

Reference Enthalpy Ratio for Laminar Flow

Especially convenient for use in Eq. (5) is the enthalpy profile $g(f)$ developed by Van Driest[3] from the laminar flat-plate boundary-layer equations. With the aforementioned restrictions and the assumption that

$$\tau = \mu\frac{\partial u}{\partial y} = \frac{\eta(f)}{\sqrt{2x}} \tag{6}$$

the boundary-layer equations for the flow of a single-species gas over a flat plate reduce to

$$\eta\eta'' + C\rho_e\mu_e u_e^3 f = 0 \tag{7}$$

$$\eta\left(g'' + Pr\,u_e^2/h_e\right) + (1 - Pr)\eta' g' = 0 \tag{8}$$

where derivatives in Eqs. (7) and (8) are with respect to the single independent variable, f. Based on Crocco's experience[6] that $g(f)$ is practically independent of the viscosity law chosen to specify $C = C(g)$, Van Driest set C to unity in Eq. (7) and used the Blasius velocity distribution for η to solve Eq. (8). His result can be written as

$$g = \phi + (1 - \phi)\theta^I + (\gamma - 1)M_e^2\theta^{II} \tag{9}$$

where θ^I and θ^{II} are functions of Pr and f.

Using this g and the form of $C(g)$ appropriate to Sutherland's viscosity law, he then returned to Eq. (7) to get the final $\eta(f)$. For our current purposes, however, his expression for enthalpy ratio is of greatest interest, as substitution of Eq. (9) into our definition of g^* gives the Young and Janssen form of the reference enthalpy:

$$g^* = \beta_1 + (1 - \beta_1)\phi + \beta_2 r[(\gamma - 1)/2]M_e^2 \tag{10}$$

where $r = \sqrt{Pr}$ and the constants β_1 and β_2 are given by

$$\beta_1 = \int_0^1 \theta^I\,df, \qquad \beta_2 = \frac{2}{r}\int_0^1 \theta^{II}\,df \tag{11}$$

Integrations of Van Driest's tabulated functions (θ^I and θ^{II}) to obtain β_1 and β_2 are shown in Table 1 for selected Prandtl numbers. These can be compared with Young and Janssen's values of $\beta_1 = 0.42$ and $\beta_2 = 0.19$ for air, which were determined empirically from the surface shear stress. Note that the results for $Pr = 1$ can be derived from Crocco's relation; that is, $\theta^I = f$ and $\theta^{II} = 0.5f(1 - f)$.

Table 1 Coefficients in Van Driest's enthalpy formulation

Pr	β_1	β_2
0.50	0.4134	0.1520
0.725	0.4595	0.1609
1.00	0.5000	0.1667

Table 2 Coefficient β for conditions where the current formulation reduces to the Eckert form

Pr	$\beta\,(Pr = 1)$	$\beta\,(\phi = 1)$	β (adiabatic wall)
0.50	——	0.152	0.239
0.725	——	0.161	0.201
1.00	0.167	0.167	0.167

For the purpose of comparing how the expression in Eq. (10) relates to Eckert's form for g^* in Eq. (1), Eq. (10) can be rewritten as

$$g^* = 0.5(1 + \phi) + r\left(\frac{\gamma - 1}{2}\right)M_e^2\left[\beta_2 + \frac{(\phi - 1)(0.5 - \beta_1)}{r[(\gamma - 1)/2]M_e^2}\right] \tag{12}$$

leading to an expression corresponding to Eckert's β:

$$\beta = \beta_2 + \frac{(\phi - 1)(0.5 - \beta_1)}{r[(\gamma - 1)/2]M_e^2} \tag{13}$$

Examination of Eq. (13) indicates that Eqs. (1) and (10) are of similar form in all of the following situations: 1) $Pr = 1$, so that $\beta_1 = 0.5$ and Eq. (13) reduces to $\beta = \beta_2$; 2) $\phi = 1.0$, corresponding to the surface cooled to the freestream temperature, where $\beta = \beta_2$ once again; and 3) an adiabatic wall, where Eq. (13) reduces to $\beta = \beta_2 + 0.5 - \beta_1$.

The values of β that correspond to situations 1–3 are listed in Table 2 for several Prandtl numbers. These values show that only for an airlike gas flowing over an adiabatic surface are the results close to the $\beta = 0.22$ used by Eckert. This is perhaps not surprising because almost half the solutions Eckert used to determine his g^* involved an adiabatic wall, and the rest were for a wall with $\phi > 1$. The key result here is, however, that use of the proposed definition of g^* and a generalized enthalpy profile developed from solutions of the boundary-layer equations leads to a formula for g^* that reduces to Eckert's reference enthalpy formula for the special case of an adiabatic wall.

Recommended g^* for Laminar Boundary Layers

The definition of g^* in Eq. (5) and the presented analysis suggest a generalization to the two-constant representation for g^* given in Eq. (10). This expression is expected to be more appropriate than that of Eckert, as it is developed from solutions of the boundary-layer equations with no assumptions involving the value of the wall enthalpy ratio. For Prandtl numbers roughly in the range of 0.67 for inert gases and 0.72 for air, the entries in Table 1 suggest the engineering approximation

$$g^* = 0.45 + 0.55\phi + 0.16r[(\gamma - 1)/2]M_e^2 \tag{14}$$

with $r = \sqrt{Pr^*}$ and γ based on reference conditions.

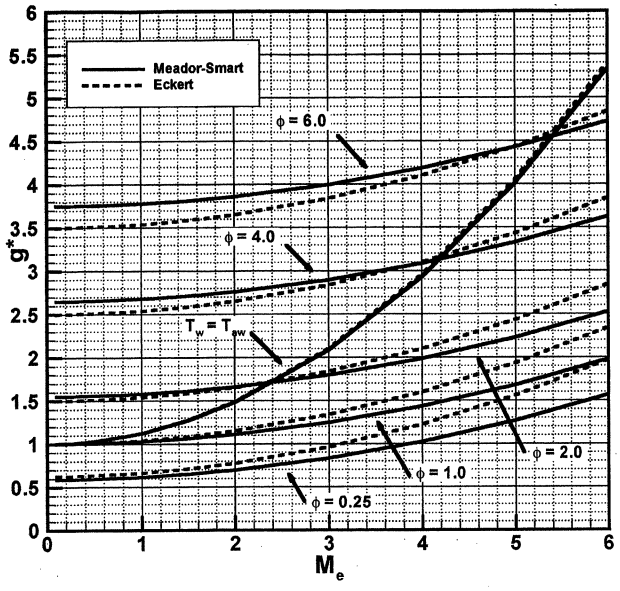

Fig. 1 Eckert and Meador–Smart reference enthalpies.

Figure 1 shows a plot of Eq. (14) for a large range of wall temperature conditions up to Mach 6 (labeled Meador–Smart). The Eckert reference enthalpy is also included for comparison purposes. The first thing to note from Fig. 1 is that g^* increases with wall enthalpy ratio (ϕ) and Mach number for both methods. This is to be expected, because a meaningful reference enthalpy must account for heat input or removal at the wall, as well as viscous heating. Another feature, already noted in the preceding section, is that both formulations produce an almost identical g^* for adiabatic wall conditions. For cold wall conditions, defined as when $T_w < T_{aw}$ (i.e., the flow is being cooled), Fig. 1 shows that the current reference enthalpy formulation supplies a lower g^* than Eckert. This difference increases with Mach number and the amount by which T_w is less than T_{aw}. For $\phi = 0.25$ and $M_e = 6.0$, the current formulation is 20.5% lower than Eckert. For hot wall conditions, where $T_w > T_{aw}$ (i.e., the flow is being heated), the current formulation supplies a higher g^* than Eckert. For $\phi = 6.0$ and $M_e = 0.0$, this difference is 7.1%. The usefulness of both formulations for prediction of laminar flat-plate boundary layer properties is quantified in a later section by comparison with numerical solutions of the boundary-layer equations.

Comparison with Dorrance's Reference Enthalpy

A possible concern with the proposed definition of g^* is that it does not generally agree with a similar expression deduced by Dorrance.[5] With $C = \rho\mu/\rho_e\mu_e$ and the requirement that $(\tau_w)^* = \tau_w$, Dorrance correctly used a formal solution of the momentum boundary layer equation to write the reciprocal of C^* as the ratio of two integrals. However, his next step, near the bottom of p. 137 in Ref. 5, where the integral ratio is replaced with an approximation, is enigmatic. Our best guess, based on use of his previous equations to evaluate the denominator integral and to simplify the numerator integral, is that an essential intermediate step is

$$\frac{1}{C^*} = \int_0^1 \left(\frac{C}{C^*}\right)\frac{1}{C}\,\mathrm{d}f \approx \frac{C_{\text{avg}}}{C^*}\int_0^1 \frac{1}{C}\,\mathrm{d}f \approx \int_0^1 \frac{1}{C}\,\mathrm{d}f \qquad (15)$$

whereas the equality clearly states that nothing informative is determined (it is simply $1 = 1$). The approximate equalities used subsequently by Dorrance demonstrate the risk of replacing part of an integrand with its average and taking it outside the integral. Accordingly, we do not accept his result

$$(g^*)^{1-n} = \int_0^1 [g(f)]^{1-n}\,\mathrm{d}f \qquad (16)$$

for viscosity proportional to T^n, nor do we regard the agreement with Eq. (5) for $n = 0$ as anything but fortuitous. The reader should observe that the notation here is different from that of Dorrance, including our f being his f'.

Comparison with Numerical Solutions of the Boundary-Layer Equations

Reference enthalpy predictions of skin friction coefficient and other boundary-layer properties can be tested for accuracy by comparison with numerical solutions of the boundary-layer equations. In the current work the compressible boundary-layer code listed in the text by Cebeci and Bradshaw[7] has been used for this purpose. This code solves the compressible boundary-layer equation set with the assumption that both γ and Pr are constant through the layer. Figure 2 shows a comparison of C_f vs Mach number plots at different values of ϕ calculated using the proposed reference enthalpy formula, Eckert's reference enthalpy formula, and numerical solutions for laminar flow over flat plates. Under adiabatic wall conditions both reference enthalpy formulations predict C_f very accurately, with the maximum error reaching 1.2% and 1.4% for the proposed and Eckert formulas, respectively, at $M_e = 6.0$. For cold wall conditions it has already been noted that the proposed formulation produces a lower g^* than Eckert, so it is not surprising that the proposed formulation predicts a higher C_f than Eckert. It is also interesting to note that the numerical solutions of cold wall flows fall between the two reference enthalpy predictions, except for very cold walls at low Mach number, where both reference enthalpy formulas overpredict C_f. For hot wall conditions both methods predict C_f with an error of less than 1.2%.

In general, both formulations predict laminar C_f very well but become slightly less accurate as Mach number increases. Table 3 shows the percentage differences between the C_f calculated by numerical solutions and the reference enthalpy formulas at $M_e = 6.0$. The proposed formula remains within 1.2% of the numerical solutions, whereas Eckert's formula differs by a maximum of 1.9% at $\phi = 0.25$.

The Reynolds analogy relates values of C_f and C_h in boundary-layer flows through

$$C_f/2C_h = Pr^{\frac{2}{3}} \qquad (17)$$

Table 3 Percentage error in laminar flat plate C_f at Mach 6 of both reference enthalpy formulations

ϕ	Δ (Meador–Smart)	Δ (Eckert)
0.25	+1.12	−1.86
1.00	+0.85	−1.65
2.00	+0.55	−1.39
4.00	−0.13	−1.20
$T_w = T_{aw}$	−1.22	−1.45

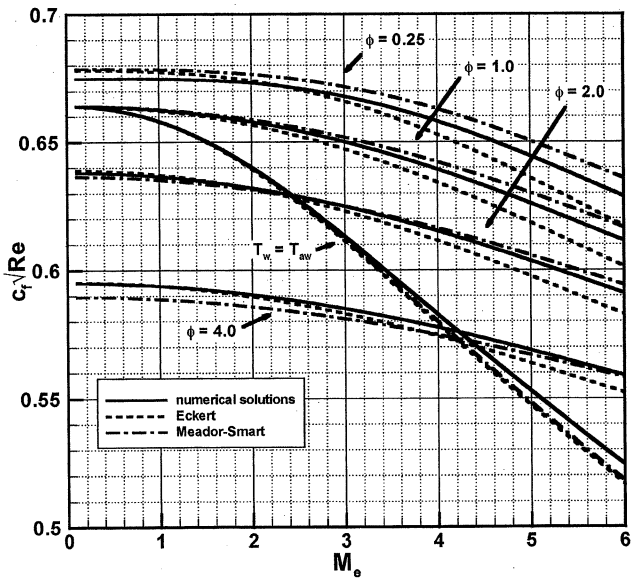

Fig. 2 Laminar C_f prediction comparison with numerical solutions.

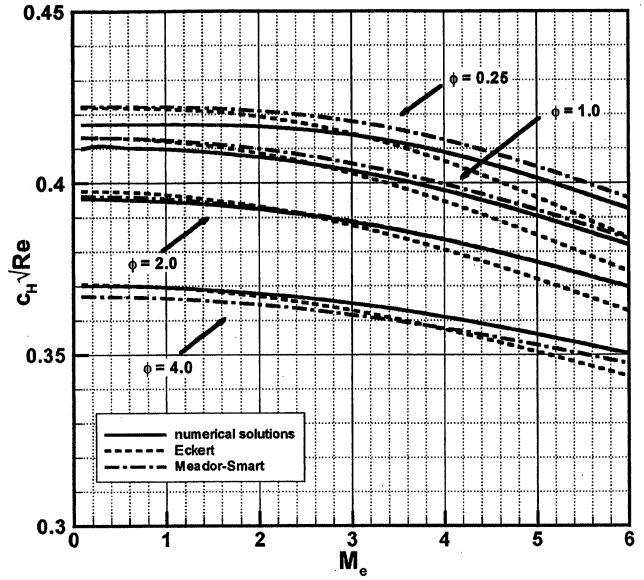

Fig. 3 Laminar C_h prediction comparison with numerical solutions.

Use of Eq. (17) in conjunction with the C_f values calculated with the reference enthalpy formulas supplies values for C_h that are compared with numerical solutions in Fig. 3. The same trends indicated in Fig. 2 for C_f are repeated for C_h, with the proposed reference enthalpy formulation being closest to the numerical solutions over a wide range of wall temperature conditions and Mach numbers. The results presented in Figs. 2 and 3 not only indicate that the proposed reference enthalpy formula supplies a slightly more accurate prediction of laminar C_f and C_h than Eckert's formula, but confirm that useful reference enthalpy formulas can be developed directly from the boundary-layer equations with no empirical dependence on numerical solutions or experimental data.

Turbulent Boundary Layers

In a manner similar to our use of Van Driest's enthalpy ratio to recommend a g^* for laminar boundary layers, we use Whitfield and High's first-order theory[8] to get a g^* for turbulent boundary layers. Based on their approximation

$$\tau(f)/\tau_w \approx (\rho/\rho_w)\exp(-4f^\alpha) \tag{18}$$

deduced from the assumption that the Reynolds stress is proportional to the local turbulent kinetic energy and that the mean velocity obeys a $\frac{1}{7}$ power law ($f = \{y/\delta\}^{1/7}$), these authors solved a turbulent version of Eq. (8) obtained by replacing the molecular Prandtl number with a turbulent Prandtl number (Pr_t) containing molecular and eddy contributions. Their resulting ordinary differential equation is

$$g'' + (\gamma-1)M_e^2 - \varepsilon\left[(\gamma-1)M_e^2 + (4)(\alpha)f^{\alpha-1}g' + (g')^2/g\right] = 0.0 \tag{19}$$

where differentiation is once again with respect to the single independent variable f.

With $\varepsilon = 1 - Pr_t$ taken to be a small number, Whitfield and High[8] produced zero- and first-order perturbation solutions of Eq. (19) for both an adiabatic wall and $\phi = 1$. Use of the hypersonic flow approximation $M_e^2 \gg 1$ enables simplification of these solutions, such that for an adiabatic wall we get

$$g_{aw} \approx 1 + \left(\frac{\gamma-1}{2}\right) \times M_e^2\left\{1 - f^2 + \varepsilon\left[1 - f^2 + \frac{8\alpha}{(\alpha+1)(\alpha+2)}(1 - f^{\alpha+2}) + 2(1+f)\ln(1+f) + 2(1-f)\ln(1-f) - 4\ln 2\right]\right\} \tag{20}$$

Application of the wall boundary condition $g_{aw}(0) = 1 + r[(\gamma-1)/2]M_e^2$ and substitution of $r = \sqrt[3]{Pr}$ and $\alpha = 17.5$ (from Ref. 8) yields

$$\varepsilon = 1 - Pr_t = (1-r)\left[4\ln 2 - 1 - \frac{8\alpha}{(\alpha+1)(\alpha+2)}\right]^{-1} = 0.074914 \tag{21}$$

giving $Pr_t = 0.9251$. Integration of Eq. (20) according to the definition of g^* in Eq. (5) leads to

$$g^* = 1 + \left(\frac{\gamma-1}{2}\right)M_e^2\left\{1 - 2\varepsilon\left[1 - \frac{6\alpha}{(\alpha+1)(\alpha+3)}\right]\right\} \tag{22}$$

This expression for g^* can be shown to have the same form as the Eckert reference enthalpy formula of Eq. (1):

$$g^* = 0.5(1+\phi) + \beta r[(\gamma-1)/2]M_e^2$$

but with

$$\beta = \frac{2}{3r}\left\{1 - 2\varepsilon\left[1 - \frac{6\alpha}{(\alpha+1)(\alpha+3)}\right]\right\} - 0.5 = 0.1632 \tag{23}$$

Likewise, for $\phi = 1$, the first-order solution of Eq. (19) with the assumption of hypersonic flow can be written as

$$g(\phi=1) \approx 1 + \left(\frac{\gamma-1}{2}\right)M_e^2\left\{f(1-f) + \varepsilon\left[\frac{8\alpha f}{(\alpha+1)(\alpha+2)}(1 - f^{\alpha+1}) - \frac{4f}{\alpha+1}(1-f^\alpha) + f(1-f) + f\ln f - (1-f)\ln(1-f)\right]\right\} \tag{24}$$

Integration according to the proposed definition of g^* also leads to Eq. (1), but with

$$\beta = \frac{1}{6r}\left\{1 - 2\varepsilon\left[1 - \frac{6\alpha(\alpha-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}\right]\right\} = 0.1646 \tag{25}$$

Contrary to the different β (adiabatic wall) and $\beta(\phi=1)$ in Table 2 for an airlike laminar boundary layer on a flat plate, here we have near equality of the two values of β. At least partly responsible is the somewhat closer to unity $Pr_t = 0.9251$. These results suggest that a reasonable reference enthalpy formulation for turbulent boundary layers is

$$g^* = 0.5(1+\phi) + 0.16r[(\gamma-1)/2]M_e^2 \tag{26}$$

The reference enthalpy formulation of Eq. (26) may now be used to estimate the properties of turbulent compressible boundary layers. Several relations for incompressible turbulent flat-plate skin friction are offered in the literature. We have chosen the formula

$$C_f = \frac{0.02296}{Re_x^{0.139}} \tag{27}$$

listed by Schlichting,[9] which is based on the data of Nikuradse[10] at Reynolds numbers between 1.7×10^6 and 18×10^6. Following the procedure used to develop Eq. (4) for laminar boundary layers, we arrive at the following relation for the skin friction coefficient in turbulent compressible boundary layers, where the reference quantities are calculated at the reference enthalpy indicated by Eq. (26):

$$C_f = \frac{0.02296}{Re_x^{0.139}}\left(\frac{\rho^*}{\rho_e}\right)^{0.861}\left(\frac{\mu^*}{\mu_e}\right)^{0.139} \tag{28}$$

The accuracy of the proposed reference enthalpy formulations for the prediction of turbulent skin friction can be gauged by comparing the results of Eq. (28) with values calculated using Van Driest II.[11]

Table 4 Percentage difference relative to Van Driest II in turbulent flat plate C_f at Mach 6 of both reference enthalpy formulations

ϕ	Δ (Meador–Smart)	Δ (Eckert)
0.25	+9.2	−6.5
1.00	+0.7	−12.0
2.00	−6.2	−15.5
4.00	−16.1	−23.2
$T_w = T_{aw}$	−26.6	−30.9

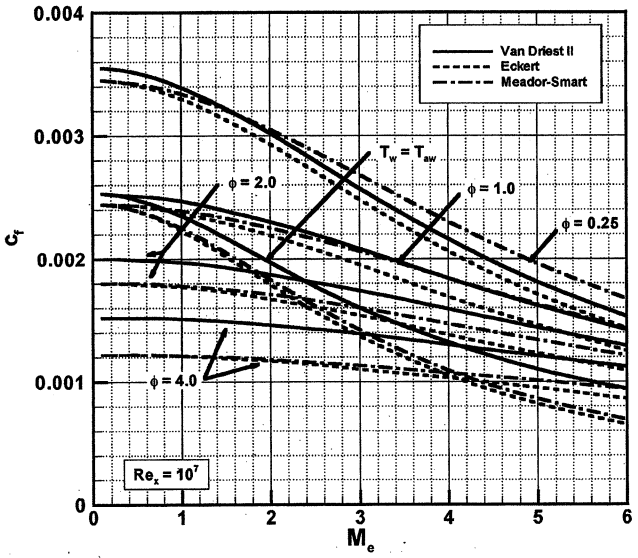

Fig. 4 Predictions of turbulent skin friction.

Figure 4 shows plots of C_f vs Mach number for each of these methods at different values of ϕ at $Re_x = 10^7$. Values calculated from Eq. (28) using the Eckert formulation are added for reference. The first thing to note from Fig. 4 is that both reference enthalpy formulations predict a C_f that follows the same trends as in Van Driest II for values of ϕ between 0.25 and 4.0. However, the typical 20% difference between the reference enthalpy-generated C_f and Van Driest II is considerably greater than the error associated with reference enthalpy predictions of laminar flow skin friction (Fig. 2). Under adiabatic wall conditions, both reference enthalpy formulations underpredict C_f at all Mach numbers, with maximum errors of 27 and 31% for the proposed and Eckert formulas, respectively, occurring at $M_e = 6.0$. These differences are due in part to the offset between the incompressible correlation of Eq. (27) and Van Driest II at $M_e = 0.0$.

Both reference enthalpy formulations predict C_f values closest to Van Driest II for cold wall conditions. Errors become larger with increased ϕ. This is partially due to the discrepancy between both reference enthalpy formulas and Van Driest II at $M_e = 0$ and large ϕ(i.e., hot walls), which could be alleviated through the use of an incompressible correlation that accounted for the effect of heat transfer. Table 4 shows the percentage difference between Van Driest II and reference enthalpy formulas at $M_e = 6.0$. The proposed method predicts C_f values within 16% of Van Driest II for all cases except an adiabatic wall, whereas Eckert's method predicts C_f values within 23% of Van Driest II for the same range of wall conditions. It is also interesting to note that Eckert's method is slightly closer to Van Driest II for $\phi = 0.25$ and $M_e = 6.0$. Given the relative simplicity of applying reference enthalpy formulas, this level of accuracy is considered adequate for preliminary design calculations of viscous drag and heat transfer in high-speed flows.

Conclusions

A simple average of the local enthalpy over the velocity profile is proposed as the proper definition of reference enthalpy for the purpose of quasi-one-dimensional treatment of compressible boundary layers. Substitution of generalized enthalpy profiles obtained from solutions of the boundary-layer equations into this definition leads to simple expressions for the reference enthalpy. In particular, for airlike gases with Prandtl number not deviating too much from 0.7, expressions for the reference enthalpy ratio applicable to laminar and turbulent flow over flat plates are

$$g^*(\text{lam}) = 0.45 + 0.55\phi + 0.16r[(\gamma - 1)/2]M_e^2$$

$$g^*(\text{turb}) = 0.5(1 + \phi) + 0.16r[(\gamma - 1)/2]M_e^2$$

Reference enthalpy–based predictions of flat plate laminar skin friction and heat transfer were compared with exact numerical solutions of the boundary-layer equations. The proposed and Eckert formulas predicted skin friction coefficients within 1.3 and 1.9%, respectively, of the numerical solutions for both hot and cold walls up to Mach 6. Similar accuracy was obtained for the heat transfer coefficient through use of the Reynolds analogy. Overall, the proposed laminar formula was slightly more accurate than Eckert for the prediction of both C_f and C_h.

Predictions of the flat-plate turbulent skin friction coefficient were compared with Van Driest II, using an incompressible correlation based on high Reynolds number turbulent data. Once again, the proposed formula predicted C_f slightly more accurately than the Eckert formula over a wide range of wall temperature conditions up to Mach 6. However, the accuracy of both the proposed and Eckert reference enthalpy formulas was less impressive than for laminar flow, with typical discrepancies of 20% relative to Van Driest II.

The development of these reference enthalpy formulas required no empirical dependence on numerical solutions or experimental data but followed directly from solutions of the compressible boundary-layer equations. This work therefore provides a theoretical basis for the empirical reference enthalpy formulas of Eckert and others.

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