5th Australasian Congress on Applied Mechanics, ACAM 2007
10-12 December 2007, Brisbane, Australia

A half-car model for dynamic analysis of vehicles with random parameters

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Abstract: A half-car model is used to investigate the dynamic response of cars with uncertainty under random road input excitations in this paper. The mass of the vehicle body, mass moment of inertia of the vehicle body, masses of the front/rear wheels, damping coefficients and spring stiffness of front/rear suspensions, distances of the front/rear suspension locations to the centre of gravity of the vehicle body and the stiffness of front/rear tires are considered as random variables. The road irregularity is considered a Gaussian random process and modeled by means of a simple exponential power spectral density. The mean value and standard deviation of the vehicle's natural frequencies and mean square value of vehicle's random response are obtained by using the Monte-Carlo simulation method. The influences of the randomness of the vehicle's parameters on the vehicle's dynamic characteristic and response are investigated in detail using a practical example.

Keywords: stochastic half car model, vehicle, random variables, random response.

1 Introduction

The vibration of an on-road vehicle is predominantly excited by the unevenness of the road surface on which the vehicle travels. Vehicle dynamic analysis has been a hot research topic for many years due to its important role in ride comfort, vehicle safety and overall vehicle performance. Numerous papers about the theoretical and experimental investigation on the dynamic behaviour of passively and actively suspended road vehicles have been published [1-3]. The quarter-car model [4], half-car model [5] and full-vehicle model [6] have been developed with researches related to the dynamic behaviour of vehicle and its vibration control.

Although mathematical modelling tools for analysis/computation have experienced a tremendous growth, most research in vehicle dynamics was based on the assumption that all parameters of vehicle systems are deterministic. Actually, the spring stiffness and damping rate may vary with respect to the nominal value due to production tolerances and/or wear, ageing... etc. The vehicle body mass and the tyre radial stiffness can have stochastic variations due to the variety of possible vehicle loading conditions and to the uncertainty of the inflating pressure of poorly maintained tyres [7]. In cars and buses, weight and placements of passengers often exhibit significant variability. In addition, even same brand and type vehicles leaving the production line may have uncertainties in size, mass and performance and so on. Hence, the problem of vehicle vibration subject to uncertain parameters is of great significance in realistic engineering applications.

In this paper, a four-degree-of-freedom half car model is used to investigate the vibration response of cars with uncertainty under random road input excitations. The vehicle's parameters are considered as random variables and the road unevenness is considered a Gaussian random process and modelled by means of a simple exponential power spectral density (PSD), the so-called "one slope PSD". The first two statistical moments of the dynamic characteristic and response are obtained by using conventional Monte-Carlo simulation method. A practical example is used to investigate the influences of the uncertainty of the vehicle's parameters on the vehicle's dynamic behaviour.

2 Vehicle model and dynamic analysis

Consider the model of a passenger car subjected to irregular excitation from a road surface as shown in Figure 1. The equations of motion for the vehicle body and the front/rear wheels are given by

$$m_s\ddot{x}_s + c_{s1}(\dot{x}_{s1} - \dot{x}_{u1}) + c_{s2}(\dot{x}_{s2} - \dot{x}_{u2}) + k_{s1}(x_{s1} - x_{u1}) + k_{s2}(x_{s2} - x_{u2}) = 0 \qquad (1)$$

$$I_s\ddot{q}_s + l_1(c_{s1}(\dot{x}_{s1} - \dot{x}_{u1}) + k_{s1}(x_{s1} - x_{u1})) - l_2(c_{s2}(\dot{x}_{s2} - \dot{x}_{u2}) + k_{s2}(x_{s2} - x_{u2})) = 0 \qquad (2)$$

$$m_{u1}\ddot{x}_{u1} - c_{s1}(\dot{x}_{s1} - \dot{x}_{u1}) - k_{s1}(x_{s1} - x_{u1}) + k_{t1}(x_{u1} - x_{r1}) = 0 \qquad (3)$$

$$m_{u2}\ddot{x}_{u2} - c_{s2}(\dot{x}_{s2} - \dot{x}_{u2}) - k_{s2}(x_{s2} - x_{u2}) + k_{t2}(x_{u2} - x_{r2}) = 0 \tag{4}$$

and the constraints are given by [8]

$$x_s = (l_2 x_{s1} + l_1 x_{s2})/l \tag{5}$$

$$\theta_s = (x_{s1} - x_{s2})/l \tag{6}$$

where m_s is the mass for the vehicle body, I_s is the mass moment of inertia for the vehicle body, m_{u1} and m_{u2} are the masses of the front/rear wheels respectively, c_{s1} and c_{s2} are the damping coefficients of front/rear suspensions respectively, k_{s1} and k_{s2} are the spring stiffness of front/rear suspensions respectively, k_{t1} and k_{t2} are the stiffness of front/rear tires respectively. x_s is the vertical displacement of the vehicle body at the centre of gravity, θ_s is the rotary angle of the vehicle body at the centre of gravity, x_{u1} and x_{u2} are the vertical displacements of the front/rear wheels, x_{r1} and x_{r2} are the irregular excitations from the road surface, x_{s1} and x_{s2} are the vertical displacements of the vehicle body at the front/rear suspension locations, l_1 and l_2 are the distances of the front/rear suspension locations, with reference to the centre of gravity of the vehicle body, and $l_1 + l_2 = l$.

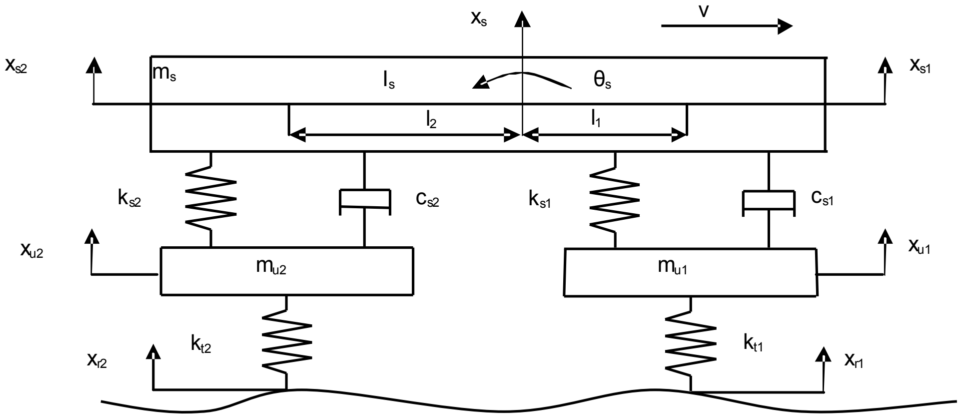

Figure 1. The half-car model of the vehicle.

Equations (1) to (4) can be rewritten as

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{P\} \tag{7}$$

where

$$[M] = \begin{bmatrix} l_2 m_s/l & 0 & l_1 m_s/l & 0 \\ I_s/l & 0 & -I_s/l & 0 \\ 0 & m_{u1} & 0 & 0 \\ 0 & 0 & 0 & m_{u2} \end{bmatrix}, \quad [C] = \begin{bmatrix} c_{s1} & -c_{s1} & c_{s2} & -c_{s2} \\ l_1 c_{s1} & -l_1 c_{s1} & -l_2 c_{s2} & l_2 c_{s2} \\ -c_{s1} & c_{s1} & 0 & 0 \\ 0 & 0 & -c_{s2} & c_{s2} \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_{s1} & -k_{s1} & k_{s2} & -k_{s2} \\ l_1 k_{s1} & -l_1 k_{s1} & -l_2 k_{s2} & l_2 k_{s2} \\ -k_{s1} & k_{s1} + k_{t1} & 0 & 0 \\ 0 & 0 & -k_{s2} & k_{s2} + k_{t2} \end{bmatrix}, \quad \{P\} = \begin{Bmatrix} 0 \\ 0 \\ k_{t1} x_{r1} \\ k_{t2} x_{r2} \end{Bmatrix}, \quad \{X\} = \begin{Bmatrix} x_{s1} \\ x_{u1} \\ x_{s2} \\ x_{u2} \end{Bmatrix} \tag{8}$$

The displacement x_{r1} and x_{r2} may be represented by a random variable defined by a stationary and ergodic stochastic process with zero mean value. The power spectral density of the process may be determined on the basis of experimental measurements and in the literature there are many different formulations for it. In this paper for sake of simplicity, the following spectrum [7] is considered

$$S_{x_{r1}}(\omega) = S_{x_{r2}}(\omega) = (A_b v)/\omega^2 \tag{9}$$

From equations (8) and (9), the power spectral density matrix $[S_P(\omega)]$ of $\{P\}$ can be obtained

$$[S_P(\omega)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{t1}^2 \dfrac{A_b v}{\omega^2} & 0 \\ 0 & 0 & 0 & k_{t2}^2 \dfrac{A_b v}{\omega^2} \end{bmatrix} \tag{10}$$

Equation (7) presents a set of coupled differential equations. If the vehicle is initially considered at rest, then its solution can be obtained in terms of the decoupling transform and Duhamel integral [9]

$$\{u(t)\} = \int_0^t [\varphi][h(t-\tau)][\varphi]^T \{P(\tau)\} d\tau \tag{11}$$

where $[\varphi]$ is the normal modal matrix of the vehicle. $[h(t)]$ is the impulse response function matrix of the vehicle, and can be expressed as

$$[h(t)] = diag\{h_j(t)\}, \quad h_j(t) = \begin{cases} \dfrac{1}{\omega_{jd}} \exp(-\zeta_j \omega_j t) \sin \omega_{jd} t & t \ge 0 \\ 0 & t < 0 \end{cases}, \qquad j = 1,2,3,4 \tag{12}$$

where ω_j and ζ_j are respectively the j^{th} natural frequency and modal damping of the vehicle, and $\omega_{jd} = \omega_j (1-\zeta_j^2)^{1/2}$.

Using Rayleigh's quotient, the j^{th} natural frequency can be expressed as

$$\omega_j^2 = (\{\varphi_j\}^T [K] \{\varphi_j\}) / (\{\varphi_j\}^T [M] \{\varphi_j\}) \tag{13}$$

ζ_j can be obtained from the following equation [10]

$$\zeta_j = \{\varphi_j\}^T [C] \{\varphi_j\} / (2\omega_j \{\varphi_j\}^T [M] \{\varphi_j\}) \tag{14}$$

From equation (11), the correlation function matrix of the displacement response of the vehicle $[R_u(\varepsilon)]$ can be obtained

$$[R_u(\varepsilon)] = E(\{u(t)\}\{u(t+\varepsilon)\}^T) = \int_0^t \int_0^t [\varphi][h(\tau)][\varphi]^T [R_P(\tau - \tau_1 + \varepsilon)][\varphi][h(\tau_1)]^T [\varphi]^T d\tau \, d\tau_1 \tag{15}$$

where $[R_P(\tau - \tau_1 + \varepsilon)]$ is the correlation function matrix of the $\{P(t)\}$. By performing a $[R_u(\varepsilon)]$ Fourier transformation, the power spectral density matrix of the displacement response $[S_u(\omega)]$ is

$$[S_u(\omega)] = [\varphi][H(\omega)][\varphi]^T [S_P(\omega)][\varphi][H^*(\omega)][\varphi]^T \tag{16}$$

where $[H^*(\omega)]$ is the conjugate matrix of $[H(\omega)]$, $[H(\omega)]$ is the frequency response function matrix of the vehicle and can be expressed as

$$[H(\omega)] = diag[H_j(\omega)], \quad H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + i \cdot 2\xi_j \omega_j \omega}, \qquad i = \sqrt{-1}, \; j = 1,2,3,4 \tag{17}$$

Integrating $[S_u(\omega)]$ within the frequency domain, the mean square value matrix of the vehicle's displacement response, that is, $[\psi_u^2]$ can be obtained

$$[\psi_u^2] = \int_{-\infty}^{\infty} [S_u(\omega)] d\omega = \int_{-\infty}^{\infty} [\varphi][H(\omega)][\varphi]^T [S_P(\omega)][\varphi][H^*(\omega)][\varphi]^T d\omega \tag{18}$$

$[\psi_u^2]$ is a 4×4 matrix and can be expressed as

$$[\psi_u^2]=\begin{bmatrix}\psi_{u11}^2 & \psi_{u12}^2 & \psi_{u13}^2 & \psi_{u14}^2\\ \psi_{u21}^2 & \psi_{u22}^2 & \psi_{u23}^2 & \psi_{u24}^2\\ \psi_{u31}^2 & \psi_{u32}^2 & \psi_{u33}^2 & \psi_{u34}^2\\ \psi_{u41}^2 & \psi_{u42}^2 & \psi_{u43}^2 & \psi_{u44}^2\end{bmatrix} \tag{19}$$

From equations (5), (6) and (18), the mean square values of m_s and θ_s can be respectively obtained as follows

$$\psi_{m_s}^2=(l_2^2\psi_{u11}^2+l_2l_1\psi_{u13}^2+l_1l_2\psi_{u31}^2+l_1^2\psi_{u33}^2)/l^2 \tag{20}$$

$$\psi_{\theta_s}^2=(\psi_{u11}^2-\psi_{u13}^2-\psi_{u31}^2+\psi_{u33}^2)/l^2 \tag{21}$$

3 random response analysis of vehicle with uncertain parameters

The vehicle's parameters corresponding to m_s, I_s, m_{u1}, m_{u2}, c_{s1}, c_{s2}, k_{s1}, k_{s2}, k_{t1}, k_{t2}, l_1 and l_2 are simultaneously considered as random variables. The randomness of vehicle's parameters will result in randomness of the matrices $[M]$ and $[K]$ and $[C]$, and consequently the natural frequencies ω_j, mode matrix $[\phi]$ and modal damping ζ_j. The random variables are each given a mean value (μ) and standard deviation (σ), for example, $m_s=\mu_{m_s}\pm\sigma_{m_s}$. A further parameter used in this paper is the variation coefficient ν, defined by the ratio of the standard deviation to the mean value, that is $\nu=\sigma/\mu$.

In the MCSM, N samples of the random variables are generated in given ranges. The implementation of the method consists in the numerical simulation of these samples associated to the random quantities of the physical problem, the procedure used for a deterministic analysis is repeated for each sample of the simulation process, obtaining then N responses that are computed to get the first two statistical moments of the response. For the four-degree-freedom system, the computational effort is acceptable for analysis of the mean value and standard deviation of vehicle's dynamic characteristics and random response.

4 Numerical examples

The mean values of vehicle's parameters for this study are given in Table 1 [11]. In the following simulations, $A_b=1.4e-5(m)$ and $v=50(m/s)$ are taken into consideration. In order to investigate the effect of the uncertainty of random variables m_s, I_s, m_{u1}, m_{u2}, c_{s1}, c_{s2}, k_{s1}, k_{s2}, k_{t1}, k_{t2}, l_1 and l_2 on the vehicle's dynamic characteristics and responses, the values of their variation coefficients ν_{m_s}, ν_{I_s}, $\nu_{m_{u1}}$, $\nu_{m_{u2}}$, $\nu_{c_{s1}}$, $\nu_{c_{s2}}$, $\nu_{k_{s1}}$, $\nu_{k_{s2}}$, $\nu_{k_{t1}}$, $\nu_{k_{t2}}$, ν_{l_1} and ν_{l_2} are respectively taken as different groups. The computational results of natural frequencies and mean square responses are respectively given in Tables 2 and 3, in which 10000 simulations are used. In these tables, symbol ν denotes $\nu=\nu_{m_s}=\nu_{I_s}=\nu_{m_{u1}}=\nu_{m_{u2}}=\nu_{c_{s1}}=\nu_{c_{s2}}=\nu_{k_{s1}}=\nu_{k_{s2}}=\nu_{k_{t1}}=\nu_{k_{t2}}=\nu_{l_1}=\nu_{l_2}$.

Table 1. The mean values of vehicle system parameters for the half-car model

Parameters	Mean values	Parameters	Mean values
m_s	1794.4 kg	k_{s1}	66824.4 N/m
I_s	3443.05 kgm^2	k_{s2}	18615.0 N/m
m_{u1}	87.15 kg	c_{s1}	1190 Ns/m
m_{u2}	140.4 kg	c_{s2}	1000 Ns/m
l_1	1.271 m	k_{t1}	101115.0 N/m
l_2	1.716 m	k_{t2}	101115.0 N/m

Table 2. The computational results of natural frequencies (unit: rad/s)

Model	m_{w_1}	s_{w_1}	m_{w_2}	s_{w_2}	m_{w_3}	s_{w_3}	m_{w_4}	s_{w_4}
$n=0$	4.6806	0	6.3951	0	29.2741	0	44.2143	0
$n_{m_s}=0.1$	4.4124	1.5346	6.4156	0.1570	29.2744	0.0030	44.2159	0.0181
$n_{I_s}=0.1$	4.5001	1.2705	6.4191	0.1757	29.2746	0.0046	44.2158	0.0153
$n_{m_{u1}}=0.1$	4.6806	1.7761e-5	6.3951	0.0046	29.2741	1.4463e6	44.3664	2.2328
$n_{m_{u2}}=0.1$	4.6806	0.0011	6.3951	3.0496e-5	29.3760	1.4738	44.2143	4.3299e-7
$n_{k_{s1}}=0.1$	4.6804	0.0015	6.3863	0.1854	29.2741	1.9989e-7	44.2132	0.9091
$n_{k_{s2}}=0.1$	4.6745	0.1929	6.3953	0.0027	29.2753	0.2373	44.2143	2.2220e-6
$n_{k_{t1}}=0.1$	4.6805	0.0010	6.3851	0.1336	29.2741	1.9962e-6	44.1889	1.2904
$n_{k_{t2}}=0.1$	4.6768	0.0394	6.3951	5.5457e-4	29.2400	1.2322	44.2143	1.4359e-7
$n_{l_1}=0.1$	4.3879	1.5790	6.4118	0.3002	29.2741	1.3219e-5	44.2162	0.0294
$n_{l_2}=0.1$	4.3393	1.7091	6.4089	0.0377	29.2746	0.0091	44.2143	3.9161e-5
$n=0.1$	4.0998	2.3417	6.4495	0.4498	29.3639	2.0153	44.3473	2.7629

Table 3. The computational results of random responses

Model	$m_{y^2_{m_s}}$ $\times 10^2$ m^2	$s_{y^2_{m_s}}$ m^2	$m_{y^2_{q_S}}$	$s_{y^2_{q_S}}$	$m_{y^2_{m_{u1}}}$ $\times 10^3$ m^2	$s_{y^2_{m_{u1}}}$ m^2	$m_{y^2_{m_{u2}}}$ $\times 10^2$ m^2	$s_{y^2_{m_{u2}}}$ m^2
$n=0$	2.7531	0	61.2658	0	1.0414	0	8.7759	0
$n_{m_s}=0.1$	2.7873	73.6973	61.7905	4.9045	1.0245	133.1999	8.7487	112.4368
$n_{I_s}=0.1$	2.7309	41.9330	61.8825	2.8877	1.0242	126.1182	8.7191	55.0714
$n_{m_{u1}}=0.1$	2.7508	7.4168	61.2414	2.2477	1.0405	16.5686	8.7733	5.3070
$n_{m_{u2}}=0.1$	2.7454	18.4273	61.2085	4.9764	1.0371	46.5648	8.7515	29.8518
$n_{c_{s1}}=0.1$	2.7531	0.1774	61.2657	0.0353	1.0414	1.4590	8.7758	1.5383
$n_{c_{s2}}=0.1$	2.7542	4.4651	61.2821	0.9915	1.0430	34.9298	8.7874	28.9713
$n_{k_{s1}}=0.1$	2.7830	21.5635	60.3418	6.6719	1.0294	98.9414	8.7456	16.8911
$n_{k_{s2}}=0.1$	2.7690	10.0367	61.5951	1.4093	1.0403	19.9863	8.7749	26.9630
$n_{k_{t1}}=0.1$	2.7814	10.8305	61.8612	1.0039	1.0606	146.0841	8.8048	78.4056
$n_{k_{t2}}=0.1$	2.7536	19.7375	61.2539	5.6703	1.0405	86.6988	8.8639	89.0602
$n_{l_1}=0.1$	2.6483	23.2473	61.6481	11.0713	1.0239	160.8900	8.6768	62.6106
$n_{l_2}=0.1$	2.9052	84.4541	62.7997	10.5128	1.0301	105.0050	8.8633	42.9070
$n=0.1$	3.1133	237.3606	55.4829	36.7227	1.0395	940.1963	8.8847	516.9753

From Table 2 and 3, it can be obtained that the uncertainty of the vehicle's natural frequencies is dependent on the uncertainty of vehicle's parameters. The randomness of the distances of the rear/front suspension locations to the centre of gravity of the vehicle body, that is geometric parameters l_2 and l_1, respectively produce the greatest effect on the vehicle's first and second natural frequency. However, the change of masses of the rear/front wheels, that is m_{u2} and m_{u1}, respectively produce the greatest effect on the vehicle's third and fourth natural frequency. The uncertainty geometric parameter l_2 and vehicle body's mass m_s produce greatest effect on the mean square displacement of vehicle body and real wheel, respectively. The randomness of geometric parameter l_1 produces notable effect on vehicle's random responses, especially for rotary angle of the vehicle body and front wheel. Comparing with the case that only one of the uncertainty of vehicle's parameters is taken into account, the change of the vehicle's dynamic characteristics and response are greater when their uncertainty are considered simultaneously.

It should be noted that when the randomness of all vehicle's parameters are considered, the standard deviations of vehicle's random response are too big as given in Table 3. Therefore, the probabilistic method seems not applicable and interval analytic methods are more suitable to find the change range (lower and upper bounds) of vehicle's responses.

4 Conclusions

In this paper, a stochastic half-car model is used to investigate the dynamic response of cars with uncertainty. The effect of uncertainty in the vehicle's parameters on the randomness of the natural frequencies and vehicle's random responses are presented by using the MCSM. The dynamic characteristics and random response of stochastic vehicles are obtained expediently. This method will also be applied to the dynamic analysis of random vehicles by using stochastic full-car models.

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