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Consistent Mass and Exact Displacement Shape Function for a Tapered Curved Frame Element

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Abstract: The principle of virtual force is employed to derive the exact shape function for a tapered-curved frame element in space and these shape functions can be employed to calculate the exact consistent mass and geometric stiffness matrixes for curved-tapered frame elements. The lack of any displacement shape function with exact fulfilment of equilibrium equations by an accurate force interpolation is the salient feature of this approach. The formulation adopts the linear elastic behaviour of the material and the strain compatibility is satisfied based on the Nevier-Bernoulli hypothesis. Shear deformations are considered and the Saint-Venat hypothesis for torsion is adopted. The efficiency and accuracy of the formulation are verified using some numerical examples.

Keywords: force interpolation, Navier-Bernoulli hypothesis, Saint-Venant torsion, tapered element.

1 Introduction

The finite element method can be formulated in the framework of stiffness, flexibility or mixed (hybrid) methods [1]. Generally, stiffness and flexibility formulations have the same degree of approximations due to coupling between compatibility and equilibrium equations. However, the stiffness method or mixed methods rooted from stiffness-based techniques are more popular than flexibility formulation, because the stiffness formulation is straightforward while the compatibility fulfilment in a flexibility method is somewhat more complex. However, this is not the case always and in specific types of structures, such as for a frame, by adopting some simplifying assumptions the equilibrium equations become uncoupled with compatibility and they can be satisfied exactly [2]. In such a case the formulation accuracy is improved without increasing the number of degrees of freedom per element.

Tapered-curved frame elements are used in various types of structures subjected to dynamic and static loadings. Analysing these types of systems in an efficient way is one of the structural engineers' concerns. Mostly the present finite element formulations use the direct stiffness method to establish the stiffness, mass and geometric stiffness matrix of the frame elements [3]. The concept of stiffness-based elements with more degrees of accuracy has been utilized for limited specific cases to derive the mass and stability matrices more accurately [4-6]. Moreover in free vibration analysis the accuracy of results can be improved by taking higher order terms in the eigenvalue expansion and application of Eulerian stress in the moveable coordinate system is another approach to improve the efficiency of the stability analysis [7, 8]. The flexibility method for frame element formulation has extensively been used by some researchers for static, cyclic and dynamic analysis of reinforced concrete frames including material nonlinearities [9-11]. However, less attention has been paid to the capability of method for deriving consistent mass and geometric stiffness matrices of element [12].

In this paper, the flexibility approach is adopted for deriving the exact stiffness matrix of the element [12]. The exact elastic curves of the element (exact in the sense of the adopted assumptions) are obtained by using the principle of virtual force for different degrees of freedom with respect to the exact element stiffness sub-matrices. These elastic curves are collected in a matrix of element shape functions and are used for calculating the consistent mass. The possible application of the method for deriving the stability matrix is discussed briefly and the accuracy of the formulation is verified by some simple numerical examples for free vibration and bucking analysis.

2 Flexibility and stiffness matrices of element

The flexibility and stiffness matrix of elements are derived by direct fulfilment of the equilibrium, compatibility and the constitutive law of the material which is taken to be linear elastic in this study.

2.1 Equilibrium equations at element level

The axis of the curved element in global coordinate system, XYZ, can be represented by, $\Gamma(s) = [X(s) \quad Y(s) \quad Z(s)]$, where s denotes the arc length in the curvilinear coordinate system (Figure 1a). The section local coordinate system is denoted by xyz where the $x-$axis is perpendicular to section plane (Figure 1b).

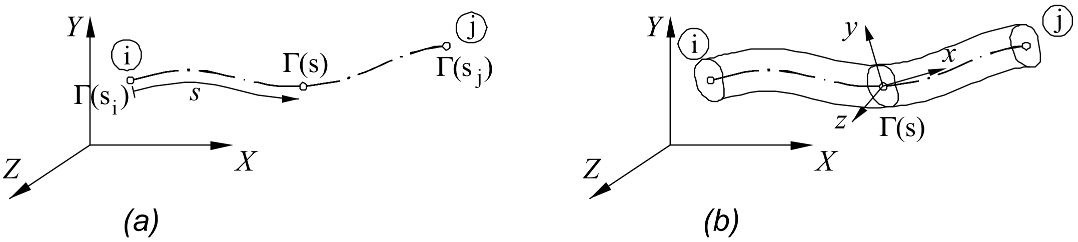

Figure 1 (a) element axis in global coordinate, (b) section local coordinate system.

Satisfying the equilibrium equations for a differential element of the curved member without element loads and then integrating, the resulting equations yields [12],

$$\mathbf{R}(s) = \mathbf{N}(s,s_j)\,\mathbf{R}_j \tag{1}$$

$$\mathbf{N}(s,s_j) = \left[\begin{array}{ccc|c} & \mathbf{I}_{3\times3} & & \mathbf{0}_{3\times3} \\ \hline 0 & (Z-Z_j) & -(Y-Y_j) & \\ -(Z-Z_j) & 0 & (X-X_j) & \mathbf{I}_{3\times3} \\ (Y-Y_j) & -(X-X_j) & 0 & \end{array}\right] \tag{2}$$

where, $\mathbf{R}(s) = [F_x \quad F_y \quad F_z \quad M_x \quad M_y \quad M_z]^T$ and $\mathbf{R}_j = [F_{Xj} \quad F_{Yj} \quad F_{Zj} \quad M_{Xj} \quad M_{Yj} \quad M_{Zj}]^T$ are the generalized internal force vector at section s and end node j, respectively, $\mathbf{N}(s,s_j)$ is the exact force interpolation function, $\mathbf{I}_{3\times3}$ and $\mathbf{0}_{3\times3}$ denote the 3 by 3 identity and zero matrices, respectively.

2.2 Compatibility equations

The Navier-Bresse strain-displacement relationships for a spatially curved element (see Figure 1a), clamped at end i, leads to the following compatibility relationship

$$\boldsymbol{\Delta}_j = \int_i^j \mathbf{N}^T(s,s_j)\,\boldsymbol{\varepsilon}_{XY}(s)\,ds \tag{3}$$

where, $\varepsilon_{XY}(s) = [\varepsilon_X \quad \varepsilon_Y \quad \varepsilon_Z \quad \phi_X \quad \phi_Y \quad \phi_Z]^T$ is the generalized strain vector at section s and $\boldsymbol{\Delta}_j = [u_{Xj} \quad u_{Yj} \quad u_{Zj} \quad \theta_{Xj} \quad \theta_{Yj} \quad \theta_{Zj}]^T$ denotes vector of generalized displacement at end j.

2.3 Main assumptions and section stiffness matrix

The section stiffness matrix is derived by adopting the following assumptions: (i) the material is assumed to be linear elastic; (ii) the mechanical properties of the material are constant over each section; (iii) Saint-Venant hypothesis for free warping of sections subjected to torsion is adopted; and (iv) Navier hypothesis for compatibility of axial strains at section level is accepted. With regards to assumption (iv) the tangential and normal strains (stresses) are uncoupled.

Equilibrium between stresses acting over the section and generalized internal forces (i.e. axial force, shear forces and bending moments) in a section free of initial strains (stresses) leads to

$$\boldsymbol{\varepsilon}_{xy}(s) = \mathbf{f}_s\,\mathbf{r}(s) \tag{4}$$

$$\mathbf{f}_s = \begin{bmatrix} \frac{1}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{GA_{sy}} + \frac{z_c^2}{GJ} & -\frac{y_c z_c}{GJ} & \frac{z_c}{GJ} & 0 & 0 \\ 0 & -\frac{y_c z_c}{GJ} & \frac{1}{GA_{sz}} + \frac{y_c^2}{GJ} & -\frac{y_c}{GJ} & 0 & 0 \\ 0 & \frac{z_c}{GJ} & -\frac{y_c}{GJ} & \frac{1}{GJ} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I_z}{E(I_y I_z - I_{yz}^2)} & \frac{I_{yz}}{E(I_y I_z - I_{yz}^2)} \\ 0 & 0 & 0 & 0 & \frac{I_{yz}}{E(I_y I_z - I_{yz}^2)} & \frac{I_y}{E(I_y I_z - I_{yz}^2)} \end{bmatrix} \tag{5}$$

where, $\mathbf{r}(s)$ is the section generalized internal force vector, $\mathbf{f}_s(s)$ is the section flexibility matrix, $\boldsymbol{\varepsilon}_{xy}(s)$ is the section generalized strain vector, (y_c, z_c) is the position of the section shear centre in the local system, $I_{y\,(z)}$ and J are the second moment of inertia and warping constant of the section, respectively. If $\mathbf{T}_s$ and $\mathbf{F}_{jj}$ denote the local to global coordinate transformation matrix at s and the flexibility sub-matrix of end j in the configuration ij clamped at end i, then virtual force principle yields

$$\mathbf{F}_{jj} = \int_i^j \mathbf{N}^T(s, s_j)\, \mathbf{T}_s\, \mathbf{f}_s\, \mathbf{T}_s^T\, \mathbf{N}(s, s_j)\, ds \tag{6}$$

The flexibility sub-matrix of end i, is obtained by a similar procedure for the configuration clamped at end j, then by mathematical transformation the element stiffness matrix $\mathbf{K}^e$ is obtained as

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{N}(s_i, s_j)[\mathbf{F}_{jj}]^{-1}\mathbf{N}^T(s_i, s_j) & -\mathbf{N}(s_i, s_j)[\mathbf{F}_{jj}]^{-1} \\ -[\mathbf{F}_{jj}]^{-1}\mathbf{N}^T(s_i, s_j) & [\mathbf{F}_{jj}]^{-1} \end{bmatrix} \tag{7}$$

This stiffness matrix is the simplified form of the stiffness matrix obtained by Molins et al. [12] and it is the exact stiffness matrix of the element for the adopted assumptions.

3 Consistent mass matrix of element

In the displacement-based formulation the mass matrix of the element $\mathbf{M}^e$ is calculated as follows

$$\mathbf{M}^e = \begin{bmatrix} \mathbf{m}_{ii} & \mathbf{m}_{ij} \\ \mathbf{m}_{ij}^T & \mathbf{m}_{jj} \end{bmatrix} = \begin{bmatrix} \int_i^j \boldsymbol{\Psi}^T m(s) \boldsymbol{\Psi}\, ds & \int_i^j \boldsymbol{\Psi}^T m(s) \boldsymbol{\Lambda}\, ds \\ \int_i^j \boldsymbol{\Lambda}^T m(s) \boldsymbol{\Psi}\, ds & \int_i^j \boldsymbol{\Lambda}^T m(s) \boldsymbol{\Lambda}\, ds \end{bmatrix} \tag{8}$$

where, $m(s)$ represents the mass distribution with respect to the arc length s, $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ are shape functions matrices corresponding to degrees of freedom at node i and j, respectively, and are:

$$\boldsymbol{\Psi} = [\psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6] \tag{9}$$

$$\boldsymbol{\Lambda} = [\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \quad \lambda_6] \tag{10}$$

In (9) and (10) matrices, $\boldsymbol{\psi}_k = [\mathrm{y}_x \quad \mathrm{y}_y \quad \mathrm{y}_z \quad \mathrm{y}_{xx} \quad \mathrm{y}_{yy} \quad \mathrm{y}_{zz}]^T$ denotes the exact shape function corresponding to k^{th} degree of freedom at end node i and $\boldsymbol{\lambda}_k = [\mathrm{l}_x \quad \mathrm{l}_y \quad \mathrm{l}_z \quad \mathrm{l}_{xx} \quad \mathrm{l}_{yy} \quad \mathrm{l}_{zz}]^T$ is the exact shape function corresponding to k^{th} degree of freedom at end node j. These exact shape functions can be obtained by superimposing the elastic curves of the cantilever configurations subjected to nodal unit forces at different degrees of freedom at the free end (Figure 2).

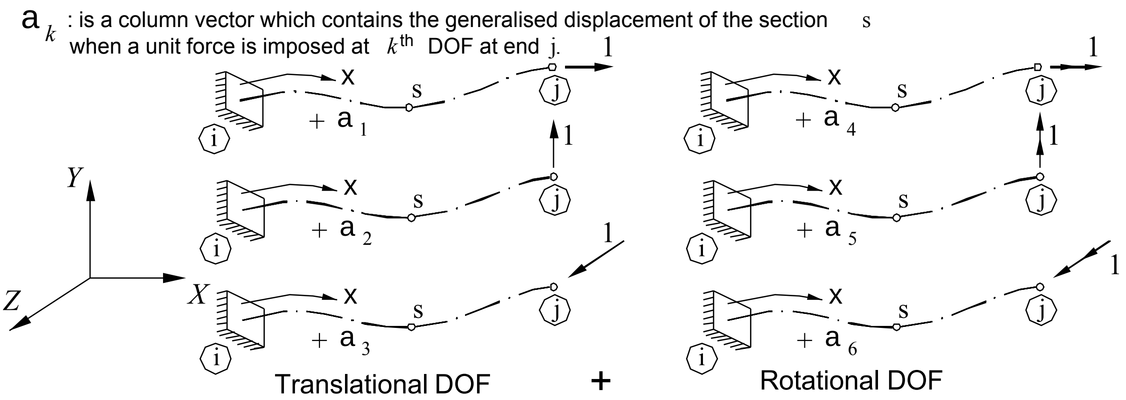

Figure 2 Outline of the clamped configuration subjected to unit virtual forces at end i.

It is assumed that $\boldsymbol{\beta} = [\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\beta}_3 \quad \boldsymbol{\beta}_4 \quad \boldsymbol{\beta}_5 \quad \boldsymbol{\beta}_6]$ and $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \boldsymbol{\alpha}_3 \quad \boldsymbol{\alpha}_4 \quad \boldsymbol{\alpha}_5 \quad \boldsymbol{\alpha}_6]$ are matrices of elastic curves corresponding to DOFs at end i and j, respectively; where $\boldsymbol{\beta}_k$ is a column vector containing the generalized displacement of the section at s, for a cantilever clamped at end j and subjected to unit force at the k^{th} DOF at end i, and $\boldsymbol{\alpha}_k$ is a column vector containing the generalized displacement of the section at s, for a cantilever clamped at end i and subjected to unit force at k^{th} DOF at end j (Figure 2). Application of the displacement compatibility within superposition of the elastic curves leads to the following relationships between elastic curves and shape function matrices:

$$\boldsymbol{\alpha}\,\mathbf{k}_{jj} = \boldsymbol{\Lambda} \tag{11}$$

$$\boldsymbol{\beta}\,\mathbf{k}_{ii} = \boldsymbol{\Psi} \tag{12}$$

Rewriting (3) for the section at s of the element clamped at end i (see Figure 2) leads to

$$\boldsymbol{\Delta}_s = \int_i^s \mathbf{N}^T(\mathrm{x}, s)\; \boldsymbol{\varepsilon}_{XY}(\mathrm{x})\; d\mathrm{x} \tag{13}$$

Using local to global coordinate transformation along with (1) and (4) in the recent relationship yields

$$\boldsymbol{\Delta}_s = \boldsymbol{\Phi}\; \mathbf{R}_j \tag{14}$$

$$\boldsymbol{\Phi} = \int_i^s \mathbf{N}^T(\mathrm{x}, s)\,\mathbf{T}_\mathrm{x}\; \mathbf{f}_\mathrm{x}\; \mathbf{T}_\mathrm{x}^T\; \mathbf{N}(\mathrm{x}, s_j)\, d\mathrm{x} \tag{15}$$

Matrix $\boldsymbol{\Phi}$ relates the vector of generalized displacements, $\boldsymbol{\Delta}_s$, to the vector of nodal forces imposed at end j. Thus, regarding the definition of matrix $\boldsymbol{\alpha}$, it is concluded that $\boldsymbol{\alpha} = \boldsymbol{\Phi}$ and

$$\boldsymbol{\alpha} = \int_i^s \mathbf{N}^T(\mathrm{x}, s)\,\mathbf{T}_\mathrm{x}\; \mathbf{f}_\mathrm{x}\; \mathbf{T}_\mathrm{x}^T\; \mathbf{N}(\mathrm{x}, s_j)\, d\mathrm{x} \tag{16}$$

Matrix $\boldsymbol{\beta}$ can be obtained by following a similar procedure for the member clamped at end j and subjected to nodal forces at end i, as

$$\boldsymbol{\beta} = \int_j^s \mathbf{N}^T(\mathrm{x}, s)\,\mathbf{T}_\mathrm{x}\,\mathbf{f}_\mathrm{x}\,\mathbf{T}_\mathrm{x}^T\,\mathbf{N}(\mathrm{x}, s_i)\,d\mathrm{x} \tag{17}$$

where $\mathbf{T}_\mathrm{x}$ denotes the transformation matrix from local curvilinear coordinate, x, to global coordinate system at any arbitrary section along the element. Having the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ determined, the mass matrix of the element is obtained by substituting (11) and (12) into (8), giving

$$\mathbf{M}^e = \left[\begin{array}{c|c} \mathbf{k}_{ii} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{k}_{jj} \end{array}\right] \left\{ \int_i^j \left[\begin{array}{c|c} \boldsymbol{\alpha}^T\boldsymbol{\alpha} & \boldsymbol{\alpha}^T\boldsymbol{\beta} \\ \hline \boldsymbol{\beta}^T\boldsymbol{\alpha} & \boldsymbol{\beta}^T\boldsymbol{\beta} \end{array}\right] m(s)\,ds \right\} \left[\begin{array}{c|c} \mathbf{k}_{ii} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{k}_{jj} \end{array}\right] \tag{18}$$

4 Stability matrix of the element

In a displacement-based formulation, the geometric stiffness (stability) matrix of the element $\mathbf{K}_G^e$ is calculated as follows:

$$\mathbf{K}_G^e = \begin{bmatrix} \int_i^j \boldsymbol{\Psi}'^T N\,\boldsymbol{\Psi}'\,ds & \int_i^j \boldsymbol{\Psi}'^T N\,\boldsymbol{\Lambda}'\,ds \\ \int_i^j \boldsymbol{\Lambda}'^T N\,\boldsymbol{\Psi}'\,ds & \int_i^j \boldsymbol{\Lambda}'^T N\,\boldsymbol{\Lambda}'\,ds \end{bmatrix} \tag{19}$$

where, N is the axial force function along the element and $\boldsymbol{\Psi}'$ and $\boldsymbol{\Lambda}'$ denote the first derivative of the shape function matrices with respect to the coordinate system. Since dealing with an analytical form of the curves in a general FE code is cumbersome, the present method can be used with a super-parametric FE formulation. In this case, the complex spatial geometry of the element axis is approximated by a shape function (polynomial) and the integrals in the formulation are analytically or numerically estimated within this shape function.

5 Numerical examples

5.1 Buckling load of a simply supported tapered column

Figure 3 shows a tapered simply supported column. The second moment of inertia varies parabolically along the length with $I(x) = I_0\,(1 + x/L)^2$. The axial and shear deformations are neglected.

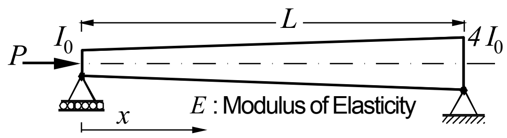

Figure 3 Geometry of the tapered simply supported column.

The buckling load of the column obtained from the method presented in this paper with a single element is $P_{cr} = 20.25EI_0/L^2$; where as the analytical solution by Timoshenko [13] is $P_{cr} \cong 20.75EI_0/L^2$, and where a single element based on a stiffness method with a hermitian displacement shape function gives $P_{cr} = 23.52EI_0/L^2$. Thus the method based on the flexibility formulation developed in this study provides an improved accuracy, and efficiency, compared to the stiffness method. It is also worthy of mention that the flexibility method in buckling analysis converges to the exact critical load from the lower side, which is conservative from a designers point of view.

5.2 Free vibration of a circular arch

Consider a 90° cantilever (clamped-end) circular arch with radius of $r = 200$ mm and square cross section $b = h = 10$ mm that can vibrate freely in the XY plane. The material properties are,

$E = 10^5$ MPa (elasticity modulus), $G = 5 \times 10^4$ MPa (shear modulus) and $r = 50\ \text{Ton/m}^3$ (mass density). By the method presented in this paper, the circular arch is approximated by a super-parametric function which includes 8 equal distance points along the arch. The resulting first and second natural frequencies of the in-plane vibration modes are given in Table 1 which clearly shows the efficiency of the formulation.

Table 1 The two first natural frequency of the in-plane vibration modes in (Hz).

Formulation type	Present formulation	Hermitian displacement-based formulation		
	(1 super-parametric element)	5 element	10 element	20 element
1st natural frequency	0.7568	0.7509	0.7535	0.7542
2nd natural frequency	0.7675	0.7632	0.7677	0.7688

6 Conclusions and recommendations

The exact displacement shape function of the curved-tapered frame element is derived with respect to the exact stiffness and implemented in a concise matrix form suitable for finite element programming. These shape functions can be used independently or in combination with ordinary displacement-based codes to calculate the consistent mass and geometric stiffness (stability) matrices within a static or dynamic linear elastic analysis. The procedure is not limited, however, to just linear elastic cases and can be used for nonlinear dynamic as well as stability analysis of frames using an appropriate iterative procedure. The computer implementation of the present method can be facilitated by recasting it in the super-parametric FE formulation framework without significant loss of accuracy.

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