

## Non destructive inspection of plates using frequency measurements

Kannappan, L., Shankar, K.

*School of Aerospace, Civil and Mechanical Engineering, Australian Defence Force Academy,  
University of New South Wales, Canberra*

**Abstract:** This paper examines the use of natural frequency measurements and energy formulation of vibration theory to detect damage present in a structure. The extent of change in natural frequencies of the structure due to damage, which is highly dependent on the location and size of damage, is used to detect and characterise damage. An analytical solution to detect and characterise damage in plates from just change in natural frequencies is presented here for the first time. Damage is modelled as a spring with finite stiffness which is less than that of the undamaged structure. Using the strain energies stored in the plate and that of spring, calculated from discrete values of deflection mode shapes, the location and size of the damage is determined. The applicability of this approach is demonstrated through a numerical study, where the location and size of through thickness centre cracks in simply supported plates is deduced with greater than 95 percent accuracy.

**Keywords:** crack detection, centre crack, plates, structural health monitoring, vibration.

### 1 Introduction

In the last two decades several vibration based techniques of non destructive inspection have been proposed. Based on the principle that damage in a structure changes its dynamic properties like natural frequencies, damping and mode shapes, damage can be detected and characterised using those parameters.

One class of the vibration based damage detection techniques relies on monitoring structures' mode shapes and their derivatives. The presence of a local defect produces only an insignificant variation in the overall mode shape, but a sharp discrepancy in the second derivative (curvature) due to its direct relation to the stiffness. A major disadvantage of using mode shape based technique is that obtaining accurate mode shapes involves arduous and meticulous measurement of displacement or acceleration over a large number of points on the structure before and after damage. The accuracy in measurement of mode shapes is highly dependent on the number and distribution of sensors employed. On the other hand, measurement of natural frequencies is more reliable, repeatable and more accurate [1]. Also, natural frequency measurement can be accomplished with a single sensor.

By comparing variations in measured frequencies, the inverse problem of identifying the damage location and severity can be solved by accurate numerical or analytical models. Previous studies using the analytical procedure have used Euler Bernoulli beam vibration theory [2, 3] which cannot be extended to complex structures. The damage detection method using natural frequencies and strain energies stored in the structure as well as the crack was envisaged by Hu and Liang [4]. They calculated the energies using undamaged structure's mode shape deflections as they implemented this technique for damage detection in a simply supported beam. Patil and Maiti [5, 6] also used this energy approach to detect damage in cantilever beams.

The main disadvantage of the above analytical approach is that mode shapes of the beam used to calculate strain energies are obtained from theory, restricting the implementation of this approach to simple structures for which exact solutions are available. Kannappan *et al.* [7, 8] demonstrated that natural frequencies of a structure before and after damage combined with numerical data of the undamaged structure's mode shapes obtained from numerical modelling or experiments to calculate the strain energies can be employed detect and characterise damage. They validated this new hybrid technique by applying it to detect through thickness cracks in cantilever beams.

The inverse problem of characterising damage in plate structures, using the change in natural frequencies, employing an analytical model has never been solved. For the first time, in this paper, a mathematical model using energy approach is developed for detection and assessment of cracks in plates from just the change in natural frequencies. The theory for the inverse problem is presented and also validated using numerical simulation.

The first three modal frequencies of an undamaged and damaged plate and mode shapes of the plate in pristine state are obtained by numerical simulation using commercial Finite Element Analysis (FEA) software, ANSYS10. The strain energy of the plate in each mode is computed from the discrete values of deflection shapes while the change in energy due to the presence of damage is computed in terms of the energy stored in the spring. Applying the change in frequencies and the energies calculated to the analytical model, cracks are located and assessed.

## 2 Analytical Modelling

Cracks can be modelled as a local change in bending stiffness (inverse of flexibility) which introduces a finite increase in rotation at the location of the crack. This increase in rotation is represented by a massless, rotational spring. A cracked plate vibration theory is developed by representing the crack as a rotational spring with stiffness,  $k_r$ , proportional to the size of the crack. If the crack is large, the change in flexibility is more and hence a spring of less stiffness is used.

Using perturbation theory, Gudmundson [9] derived the relation between the eigen frequency changes and the strain energy in uncracked and cracked structure as

$$\frac{\omega_n'^2}{\omega_n^2} = 1 - \frac{\Delta U_n}{U_n}$$

where,  $\omega_n'$  is  $n^{\text{th}}$  mode natural frequency of the cracked structure,  
 $\omega_n$  is  $n^{\text{th}}$  mode natural frequency of the uncracked structure ,  
 $\Delta U_n$  is increase  $n^{\text{th}}$  mode strain energy due to the finite bending at the location crack, equal to the strain energy stored in the spring, and,  
 $U_n$  is the strain energy of the undamaged structure in  $n^{\text{th}}$  mode.

A first order approximation of the above equation yields

$$\frac{\Delta \omega_n}{\omega_n} = \frac{1}{2} \frac{\Delta U_n}{U_n} \quad (1)$$

where,  $\Delta \omega_n = \omega_n - \omega_n'$ .

For an applied bending moment  $M_x$  perpendicular to x-axis and  $M_y$  perpendicular to y-axis, if  $\theta$  is the additional bending due to the presence of a crack of length  $2b$ , oriented at an angle  $\Phi$  wrt x-axis, then the energy stored in the crack is given by,

$$\begin{aligned} \Delta U &= \frac{1}{2} M \theta \\ \text{Since } \theta &= \frac{M}{k_r} \text{ and } M = M_n * 2b \\ \Rightarrow \Delta U &= \frac{2b^2 M_n^2}{k_r} \end{aligned} \quad (2)$$

where,  $k_r$  is the stiffness of the rotational spring, and

$$\begin{aligned} M_n &= M_x \cos \phi + M_y \sin \phi \\ M_x &= -D(\kappa_x + \nu \kappa_y), \quad M_y = -D(\kappa_y + \nu \kappa_x) \end{aligned} \quad (3)$$

Here,  $D$  is the flexural rigidity of the plate,  $\nu$  is poisson's ratio and  $\kappa$  is the curvature of the lateral deflection of the plate ( $\psi$ ) defined as

$$\kappa_x = \frac{\partial^2 \psi}{\partial x^2}, \quad \kappa_y = \frac{\partial^2 \psi}{\partial y^2}$$

Substituting Eq. (3) in Eq. (2)

$$\Delta U = \frac{2b^2 D^2}{k_r} \left[ (\kappa_x + \nu \kappa_y) \cos \phi + (\kappa_y + \nu \kappa_x) \sin \phi \right]^2 \quad (4)$$

where,  $k_r$  is the spring stiffness and  $b$  is the semi-crack length

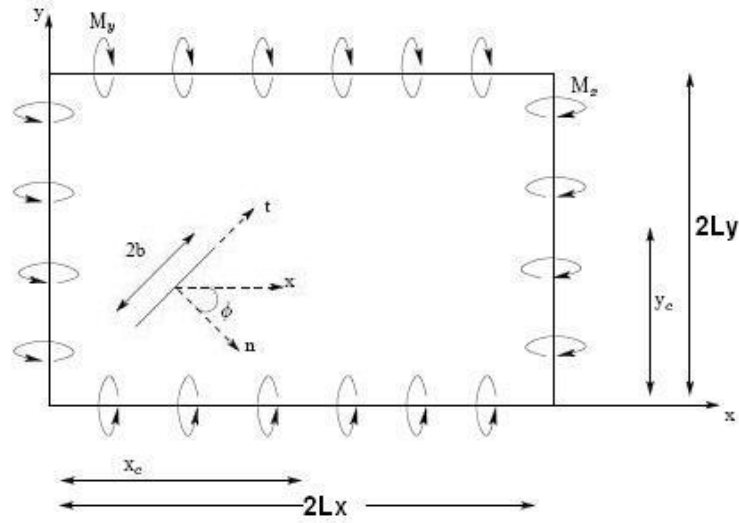


Figure 1: Plate containing crack oriented at an angle  $\Phi$  wrt x-axis

The total energy stored in the plate can be calculated from *Kirchoff's classical plate theory* as

$$U = \frac{1}{2} D \int_0^{2Ly} \int_0^{2Lx} \left\{ \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \right\} dx dy \quad (5)$$

Substituting Eq. (4) and Eq. (5) in Eq. (1)

$$\frac{\Delta \omega_n}{\omega_n} = \frac{2b^2 D}{K_r} \frac{\left[ (\kappa_x + \nu \kappa_y) \cos \phi + (\kappa_y + \nu \kappa_x) \sin \phi \right]^2}{\int_0^{2Ly} \int_0^{2Lx} \left\{ \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \right\} dx dy} \quad (6)$$

Let,  $K = k_r / 2b^2 D$

$$\text{Eq. (6)} \rightarrow \frac{\Delta \omega_n}{\omega_n} = \frac{1}{K} \frac{\left[ (\kappa_x + \nu \kappa_y) \cos \phi + (\kappa_y + \nu \kappa_x) \sin \phi \right]^2}{\int_0^{2Ly} \int_0^{2Lx} \left\{ \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \right\} dx dy}$$

$$\text{or } K = \frac{1}{\left( \frac{\Delta \omega_n}{\omega_n} \right)} \frac{\left[ (\kappa_x + \nu \kappa_y) \cos \phi + (\kappa_y + \nu \kappa_x) \sin \phi \right]^2}{\int_0^{2Ly} \int_0^{2Lx} \left\{ \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \right\} dx dy} \quad (7)$$

Let,  $X = (\kappa_x + \nu \kappa_y) \cos \phi$ ,  $Y = (\kappa_y + \nu \kappa_x) \sin \phi$

$$C = \left( \frac{\Delta \omega_n}{\omega_n} \right) \int_0^{2Ly} \int_0^{2Lx} \left\{ \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \right\} dx dy$$

$$\text{Eq. (7)} \rightarrow K = \frac{[X \cos \phi + Y \sin \phi]^2}{C} = \frac{k_r}{2b^2 D} \quad (8)$$

For solving the inverse problem of detecting and characterising damage in plates, Eq. (8) is the basis. Using the first three natural frequencies of the plate before and after damage and using Eq. (8), the spring stiffness (K) is calculated for every point on the plate using finite difference approximation. For three modal frequencies, three surfaces of K varying with respect to location along the plate are obtained. As explained before, since the stiffness of the spring modelled is independent of the mode of

vibration, the point of intersection of the surfaces will give the location of damage. This process can be followed for more number of modes for accurate determination of location and extent of damage.

## 2.1 Point of intersection

To find the point where the surfaces intersect, the distance between the surfaces is calculated for every point on the plate using Eq. (9). The location where the root of sum of square of the distances between the curves becomes minimum is considered to be the point of intersection. From this location obtained, the spring stiffness is determined from the surfaces used to calculate the minimum distance (K Vs x Vs y surfaces). The title of the paper should be flush left of page using sentence case. The title should be brief and reflecting what the paper is about.

$$d_{xy} = \sqrt{\sum_{i=1}^{nm-1} \sum_{j=i}^{nm} (K_i - K_j)^2} \quad (9)$$

where, nm is the number of modes used in the damage detection algorithm and  $d_{xy}$  represents the distance between the surfaces for each point  $(x_1, y_1), \dots, (x_{max}, y_{max})$  along the plate.

## 2.2 Plate containing crack perpendicular to x- axis

For a crack which is perpendicular to the x-axis, as shown in Fig. 2, moment  $M_y$  has no effect on the crack and the additional deflection in the plate due to the presence of crack is completely due to the moment  $M_x$  since  $\Phi = 0$ . This is evident by substituting  $\Phi=0$  in Eq. (3).

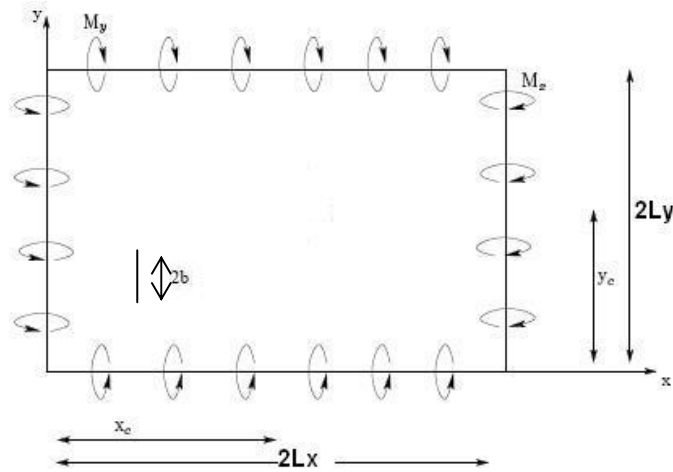


Figure 2: Plate containing perpendicular crack

$$M_n = M_x \cos \phi + M_y \sin \phi = M_x$$

$$\text{Eq. (2)} \rightarrow \Delta U = \frac{2b^2 M_n^2}{k_r} \quad (10)$$

Substituting Eq. (10) and Eq. (5) in Eq. (1)

$$\frac{\Delta \omega_n}{\omega_n} = \frac{1}{K} \frac{(\kappa_x + \nu \kappa_y)^2}{\int_0^{2Ly} \int_0^{2Lx} \left\{ \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \right\} dx dy} \quad (11)$$

## 2.3 Determination of crack length

Kannappan et al. [10] used the stress intensity factor equations derived by Boduroglu and Erdogan [11] to relate spring stiffness and crack length in case of beams containing through thickness centre cracks. Here, using a similar procedure as [10], the relation between the spring stiffness and crack length for plates with a width to thickness ratio of 10 is derived as

$$K = \frac{1}{D} \left[ \frac{Eh^3}{72Ly^2\gamma^2g(\gamma)} \right] \quad (12)$$

where, E, h and Ly are Young's modulus, thickness and semi width of the plate,  $\gamma = b/Ly$  and

$$g(\gamma)_{wh=10} = 1057.1\gamma^{10} - 4181.2\gamma^9 + 7295.8\gamma^8 - 7422.3\gamma^7 + 4894.9\gamma^6 - 2228.9\gamma^5 + 751.8\gamma^4 - 202.8\gamma^3 + 43.8\gamma^2 - 6.5\gamma + 1$$

### 3 Results - Numerical Simulation

The derived analytical model was validated by applying frequencies of simply supported plate before and after damage obtained from numerical simulation. In this study, plates containing through thickness cracks (shown in Fig. 2), perpendicular to the x-axis was only considered. The frequency values and mode shapes were obtained from commercial FEA software, ANSYS 10. For this numerical study, a plate 150mm long, 100mm wide and 5mm thick was considered and was assumed to be made of steel with Young's modulus 192.3GPa and density  $7.81 \times 10^{-6} \text{kg/mm}^3$ . Cracks of different sizes were modelled at various locations on the plate. The first 3 natural frequencies of vibration of the plate before and after the damage was introduced, shown in Table. 1, were input to the damage detection algorithm. All computations including calculation of strain energies were carried out using MATLAB.

**Table 1: Natural frequencies of undamaged and damaged plate (in Hz.)**

| Damage Case | Non-dimensional x location of crack ( $\alpha=x/Lx$ ) | Non-dimensional y location of crack ( $\beta =y/Ly$ ) | Crack length ( $\gamma=b/Ly$ ) | Mode1  | Mode2  | Mode3  |
|-------------|---|---|--------------------------------|--------|--------|--------|
|             | Undamaged plate                                       |   |                                | 1721.5 | 3310.6 | 5296.9 |
| 1           | 0.2   | 0.2   | 0.2                            | 1719.4 | 3290.2 | 5286.7 |
| 2           | 0.3   | 0.3   | 0.1                            | 1719.7 | 3301.0 | 5292.8 |
| 3           | 0.3   | 0.3   | 0.3                            | 1705.2 | 3229.2 | 5262.7 |
| 4           | 0.4   | 0.2   | 0.2                            | 1716.3 | 3302.2 | 5274.8 |
| 5           | 0.4   | 0.4   | 0.4                            | 1671.7 | 3207.8 | 5252.2 |

The first three mode shapes were obtained from 120 points on the plate using FEA. The strain energies were calculated from these mode shapes using **Finite Difference Method (FDM)** approach. The accuracy of prediction of damage location and size is highly dependent on the mesh size used in the FDM. In this study, the element length for FDM was selected as 0.5mm for better accuracy after comparing the results obtained from different element lengths. Since mode shape of simply supported plate is symmetrical, only a quarter of the plate was considered as it reduced the computational time significantly (by four times).

Substituting the change in natural frequencies and the calculated energies in Eq.(11), the variation of K at each point for each mode was obtained. The distances between the surfaces was calculated using Eq. (9), from which the point of intersection of the surfaces and thus the location of crack was obtained. From the point of intersection of surfaces, the spring stiffness (K) was obtained for each surface. Though the K values must ideally be same for each surface at that crack location obtained, due to the inaccuracy in obtaining the point of intersection, average of the three K values was taken to be the ultimate spring stiffness. With this spring stiffness implemented in Eq. (12), the crack length was also determined. The actual location and size of crack was compared with those predicted from the damage detection algorithm (Table. 2). Here, error was calculated as the difference between the predicted and actual value expressed as a percentage of the beam length or beam width when calculating crack location error or crack size error respectively. This helps eliminate the spurious dependency of the error magnitude when comparing locations and sizes of the crack.

**Table 2: Comparison of actual and predicted crack location and size**

| Damage Case | Actual crack characteristics |         |          | Predicted crack characteristics |         |          | %Error in prediction |         |          |
|-------------|------------------------------|---------|----------|---------------------------------|---------|----------|----------------------|---------|----------|
|             | $\alpha$                     | $\beta$ | $\gamma$ | $\alpha$                        | $\beta$ | $\gamma$ | $\alpha$             | $\beta$ | $\gamma$ |
| 1           | 0.2                          | 0.2     | 0.2      | 0.16                            | 0.19    | 0.21     | 4.0                  | 1.0     | 1.3      |
| 2           | 0.3                          | 0.3     | 0.1      | 0.32                            | 0.32    | 0.09     | 2.3                  | 1.7     | 1.2      |
| 3           | 0.3                          | 0.3     | 0.3      | 0.31                            | 0.28    | 0.26     | 0.5                  | 1.9     | 4.5      |
| 4           | 0.4                          | 0.2     | 0.2      | 0.40                            | 0.22    | 0.16     | 0.3                  | 1.8     | 4.4      |
| 5           | 0.4                          | 0.4     | 0.4      | 0.38                            | 0.37    | 0.37     | 2.2                  | 2.6     | 3.2      |

## 4 Conclusion

The development of a hybrid methodology for damage detection in plates using combination of frequency measurements and mode shapes has been presented for the first time. Some work has been done using combination of natural frequencies and mode shapes, but applied only to beams. In this study, the theory for damage detection in plates has been derived and implemented. The theory to deduce the size of through thickness crack has also been extended for plate like structures. The accuracy of the method in locating and characterising the damage is demonstrated with frequencies obtained from FEA. The maximum error in prediction of location and crack length is less than 5%.

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