5th Australasian Congress on Applied Mechanics, ACAM 2007
10-12 December 2007, Brisbane, Australia

Application of the direct integration method to analysis of elasticity and thermoelasticity problems for inhomogeneous solids

Yuriy V. Tokovyy[1,2] and Chien-Ching Ma[1]

[1]*Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.*
[2]*Pidstryhach Institute for Applied Problems of Mechanics and Mathematics NASU, Lviv, Ukraine*

Abstract: The application of the direct integration method for finding the solutions of 2D elasticity and thermoelasticity problems for the radially inhomogeneous ring and for the strip inhomogeneous with respect to width is presented. The main feature of this approach is the integration of the equilibrium equations, which do not depend on material properties. This gives the possibility to express all the stresses in terms of a governing one, as well as to deduce the integral equilibrium conditions for all of the stress tensor components. In such way, the original problem can be reduced to finding the governing stress from compatibility equation. The governing equation is reduced to the Volterra type integral equation and it can be solved by simple iterations.

Keywords: inhomogeneous ring and strip, plane problems, Volterra type integral equations.

1 Introduction

The development of new approaches, mathematical models, and methodologies taking many characteristics of various natures into account are necessary toward the solution of the current problems of the solid mechanics. Particularly, construction of analytical and semi-analytical solutions to the elasticity and thermoelasticity problems for inhomogeneous materials attracts considerable attention in recent years. The functionally graded materials can be regarded as the special and very important example of materials which are inhomogeneous in one of spatial directions [1]. Some existing analytical methods are valid for special cases of material properties. Hence there is strong need to develop methods which are applicable for rather general material properties as well as for different form of domains investigated.

This paper presents the application of direct integration method [2,3] to solving the plane elasticity and thermoelasticity problems in terms of stresses for the inhomogeneous strip and radially inhomogeneous rind. Because the essence of the method is the integration of the equilibrium equations, which are independent on the material properties, this method is applicable for different kinds of domains (simply- and multiply-connected, finite and infinite, etc.) in different coordinate systems (Cartesian, cylindrical, etc.). Employment of the Volterra type integral equations theory gives the possibility to avoid restrictions for the material properties of the solids.

2 Problems Formulation

It is well-known that the 2D problems of elasticity and thermoelasticity for the strip correspond to 3D problems for the thin long plate under generalized plane stress condition as well as for the thick plate under plane strain condition. Analogously, the 2D problems for the ring correspond to 3D problems for both hollow cylinder in the event of plane strain and thin annular disk in the plane stress case. For simplicity, the absence of axial (perpendicular to the plane of consideration) strain and displacement is assumed below for the plane strain cases [4]. This, however, do not restrain the universality of the method.

2.1 Statement of the problems for the inhomogeneous strip

Let us consider plane elasticity and thermoelasticity problems for the strip $\Omega_1 = \{(x, y) \in (-\infty, \infty) \times [-1, 1]\}$ which is inhomogeneous with respect to width. In absence of body forces, these problems are governed by the equilibrium equations

$$\partial\sigma_x / \partial x + \partial\sigma_{xy} / \partial y = 0, \qquad \partial\sigma_{xy} / \partial x + \partial\sigma_y / \partial y = 0, \tag{1}$$

the compatibility equation in terms of strains

$$\partial^2 e_x / \partial y^2 + \partial^2 e_y / \partial x^2 = \partial^2 e_{xy} / (\partial x \partial y), \tag{2}$$

and the strain-stress relations

$$\bar{E}e_x = \sigma_x - \bar{\nu}\sigma_y + \bar{\alpha}\bar{E}T, \quad \bar{E}e_y = \sigma_y - \bar{\nu}\sigma_x + \bar{\alpha}\bar{E}T, \quad Ge_{xy} = \sigma_{xy}. \tag{3}$$

Here $\sigma_j, \sigma_{xy}; e_j, e_{xy}$ $(j = x, y)$ are the stress tensor and strain tensor components; x, y are the dimensionless Cartesian coordinates normalized by the half of strip's width;

$$\bar{E} = \begin{cases} E/(1-\nu^2), \text{ plane strain}, \\ E, \text{ plane stress}; \end{cases} \quad \bar{\nu} = \begin{cases} \nu/(1-\nu), \text{ plane strain}, \\ \nu, \text{ plane stress}; \end{cases} \quad \bar{\alpha} = \begin{cases} \alpha(1+\nu), \text{ plane strain}, \\ \alpha, \text{ plane stress}; \end{cases} \tag{4}$$

E and ν are the Young modulus and the Poisson ratio; $G = E/(2+2\nu)$ and α denote the shear modulus and the linear temperature expansion coefficient; $T(x, y)$ is the temperature which is given or determined from corresponding heat-transfer problem [5]. The material properties (4) are arbitrary functions of y. We do not impose any restrictions on them aside from the existence of derivatives of necessary order. The stresses in the strip Ω_1 are caused by the above-mentioned temperature and by the external force loadings

$$\sigma_y\big|_{y=1} = -p_1(x), \qquad \sigma_y\big|_{y=-1} = -p_2(x), \qquad \sigma_{xy}\big|_{y=1} = q_1(x), \qquad \sigma_{xy}\big|_{y=-1} = q_2(x). \tag{5}$$

Using (1) and (3), we represent the compatibility equation (2) in terms of stresses

$$\Delta_{xy}\left(\frac{\sigma}{\bar{E}} + \bar{\alpha}T\right) = \frac{\sigma_y}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{1}{G}\right), \qquad \Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{6}$$

where σ is the total stress, $\sigma = \sigma_x + \sigma_y$. We are aimed to determine the stress tensor components from equations (1) and (6) under given boundary conditions (5) and temperature distribution, $T(x, y)$.

2.2 Statement of the problems for the radially inhomogeneous ring

Analogously to the former section, the plane non-axisymmetric problems of elasticity and thermoelasticity for the ring region $\Omega_2 = \{(\rho, \varphi) \in [k, 1] \times [0, 2\pi]\}$ are governed by the equilibrium equations

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2\sigma_r\right) + \frac{\partial\sigma_{r\varphi}}{\partial\varphi} = \sigma, \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2\sigma_{r\varphi}\right) + \frac{\partial\sigma_\varphi}{\partial\varphi} = 0, \tag{7}$$

the compatibility equation in terms of strains

$$\frac{\partial^2}{\partial\rho\partial\varphi}\left(\rho e_{r\varphi}\right) = \frac{\partial^2 e_r}{\partial\varphi^2} + \frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial e_\varphi}{\partial\rho}\right) - \rho\frac{\partial e_r}{\partial\rho}, \tag{8}$$

the physical relations (3) in which r instead of x and φ instead of y should be substituted. Here $\{\rho, k\} = \{r, R_1\}/R_2$; r and φ are the dimensional radial and angular coordinates of the cylindrical polar coordinate system; R_1 and R_2 are the inner and outer radii of the ring Ω_2; $\sigma_j, \sigma_{r\varphi}$ and $e_j, e_{r\varphi}$ $(j = r, \varphi)$ denote the in-plane components of stress tensor and strain tensor; $\sigma = \sigma_r + \sigma_\varphi$. The material properties (4) are assumed to be functions of ρ. The ring Ω_2 is stresses by external force loadings

$$\sigma_r\big|_{\rho=k} = -p_3(\varphi), \; \sigma_r\big|_{\rho=1} = -p_4(\varphi), \; \sigma_{r\varphi}\big|_{\rho=k} = q_3(\varphi), \; \sigma_{r\varphi}\big|_{\rho=1} = q_4(\varphi), \tag{9}$$

as well as by the temperature $T(\rho, \varphi)$. Moreover, because the ring Ω_2 is a multiply-connected domain, we need to impose the additional conditions [4,5]. Such conditions can be represented in the form

$$\int_0^{2\pi}\left[ke_r(k,\varphi) + e_r(1,\varphi) - k^2\frac{\partial e_\varphi}{\partial\rho}\bigg|_{\rho=k} - \frac{\partial e_\varphi}{\partial\rho}\bigg|_{\rho=1}\right]\begin{Bmatrix}\sin\varphi \\ \cos\varphi\end{Bmatrix}\mathrm{d}\varphi = \int_0^{2\pi}\left[ke_{r\varphi}(k,\varphi) + e_{r\varphi}(1,\varphi)\right]\begin{Bmatrix}\cos\varphi \\ -\sin\varphi\end{Bmatrix}\mathrm{d}\varphi \tag{10}$$

by integration of Cauchy strain-displacement relations [4]. Analogously to (6), the equation (8) can be represented in terms of stresses

$$\Delta_{\rho\varphi}\left(\frac{1}{\overline{E}}\sigma+\overline{\alpha}T\right)=\frac{\sigma_r}{2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(\frac{1}{G}\right)+\frac{\sigma_\varphi}{2\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right),\quad \Delta_{\rho\varphi}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}. \tag{11}$$

3 Construction of solutions

3.1 Analysis of strip inhomogeneous with respect to width

It is simply to deduce the equation

$$\Delta_{xy}\sigma_y=\partial^2\sigma/\partial x^2, \tag{12}$$

and conditions

$$\partial\sigma_y/\partial y\big|_{y=1}=-\mathrm{d}\,q_1/\mathrm{d}\,x,\quad \partial\sigma_y/\partial y\big|_{y=-1}=-\mathrm{d}\,q_2/\mathrm{d}\,x \tag{13}$$

on the basis of (1) and (5) which are irrespective to the material properties. Apply the Fourier integral transform [6] over x to (5), (6), (12), and (13), we get, thereby, the set of equations

$$\mathrm{D}_y\tilde{\sigma}_y=-s^2\tilde{\sigma},\quad \mathrm{D}_y\left(\frac{\tilde{\sigma}}{\overline{E}}+\overline{\alpha}\tilde{T}\right)=\frac{\tilde{\sigma}_y}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{1}{G}\right),\quad \mathrm{D}_y=\frac{\mathrm{d}^2}{\mathrm{d}y^2}-s^2 \tag{14}$$

and conditions

$$\tilde{\sigma}_y\big|_{y=1}=-\tilde{p}_1,\quad \tilde{\sigma}_y\big|_{y=-1}=-\tilde{p}_2,\quad \mathrm{d}\tilde{\sigma}_y/\mathrm{d}\,y\big|_{y=1}=-is\tilde{q}_1,\quad \mathrm{d}\tilde{\sigma}_y/\mathrm{d}\,y\big|_{y=-1}=-is\tilde{q}_2. \tag{15}$$

Here s denotes the parameter of integral transform, $i=\sqrt{-1}$, and the images of Fourier transform are tilde-indicated. From the first equation of (14) under conditions (15), we arrive the expression

$$\tilde{\sigma}_y=-\tilde{p}_2\cosh s(1+y)-i\tilde{q}_2\sinh s(1+y)-s\int_{-1}^{y}\tilde{\sigma}(\xi)\sinh s(y-\xi)\mathrm{d}\xi \tag{16}$$

and integral conditions

$$\int_{-1}^{1}\tilde{\sigma}\sinh s\xi\mathrm{d}\xi=(\tilde{q}_1+\tilde{q}_2)\frac{i\sinh s}{s}-(\tilde{p}_1-\tilde{p}_2)\frac{\cosh s}{s},\quad \int_{-1}^{1}\tilde{\sigma}\cosh s\xi\mathrm{d}\xi=(\tilde{q}_1-\tilde{q}_2)\frac{i\cosh s}{s}-(\tilde{p}_1+\tilde{p}_2)\frac{\sinh s}{s}. \tag{17}$$

The second equation of (14), accompanied by (16), yields

$$\tilde{\sigma}=\overline{E}\left[A\cosh sy+B\sinh sy+P_2\tilde{p}_2+Q_2\tilde{q}_2-\overline{\alpha}\tilde{T}-(1/2)\int_{-1}^{y}\tilde{\sigma}(\eta)K(\eta,y)\mathrm{d}\eta\right]. \tag{18}$$

Here $K(\eta,y)=\int_{\eta}^{y}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{1}{G}\right)\sinh s(y-\xi)\sinh s(\xi-\eta)\mathrm{d}\xi$, $P_2=-\frac{1}{2s}\int_{-1}^{y}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{1}{G}\right)\cosh s(1+\xi)\sinh s(y-\xi)\mathrm{d}\xi$,

$Q_2=-\frac{i}{2s}\int_{-1}^{y}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{1}{G}\right)\sinh s(1+\xi)\sinh s(y-\xi)\mathrm{d}\xi$.

Following [7,8], we solve the obtained Volterra type integral equation (18) by the simple iteration method. According to this, the expression

$$\tilde{\sigma}^{(m)}=\overline{E}\left[A^{(m)}\cosh sy+B^{(m)}\sinh sy+P_2\tilde{p}_2+Q_2\tilde{q}_2-\overline{\alpha}\tilde{T}-(1/2)\int_{1}^{y}\tilde{\sigma}^{(m-1)}(\eta)K(\eta,y)\mathrm{d}\eta\right] \tag{19}$$

follow from (18) for each the iteration $m\ge 1$. The constants $A^{(m)},B^{(m)}$ can be determined by substitution (19) into (17). For determination of $A^{(1)},B^{(1)}$, it is assumed $\tilde{\sigma}^{(0)}=0$, naturally.

After having found $\tilde{\sigma}=\lim_{m\to\infty}\tilde{\sigma}^{(m)}$, we determine $\tilde{\sigma}_y$ from (16). Applying the inverse Fourier transform, we find the normal stresses σ_y, $\sigma_x=\sigma-\sigma_y$. The shear stress σ_{xy} can be found from the expression

$$4\sigma_{xy} = q_1 + q_2 - \int_{-1}^{1} \frac{\partial \sigma_x}{\partial x} \operatorname{sgn}(y-\xi)\mathrm{d}\xi - \int_{-\infty}^{\infty} \frac{\partial \sigma_y}{\partial y} \operatorname{sgn}(x-\eta)\mathrm{d}\eta$$

which is derived by direct integration of the equilibrium equations (1).

It is worthy to note that if G^{-1} is linear function of y, then equation (18) has an exact analytical solution at $m=1$.

3.2 Analysis of the radially inhomogeneous ring

In order to solve the problems formulated in the section 2.2, we represent the stresses, tractions and temperature as decompositions into Fourier series

$$\{\sigma_r, \sigma_\varphi, \sigma, T, p_j, \sigma_{r\varphi}, q_j\} = \{R_0(\rho), \Phi_0(\rho), \sigma_0(\rho), T_0(\rho), p_{j0}, S_0(\rho), q_{j0}\}$$
$$+ \sum_{n=1}^{\infty} \begin{bmatrix} \{R_n^1(\rho), \Phi_n^1(\rho), \sigma_n^1(\rho), T_n^1(\rho), p_{jn}^1, S_n^2(\rho), q_{jn}^2\}\cos n\varphi \\ + \{R_n^2(\rho), \Phi_n^2(\rho), \sigma_n^2(\rho), T_n^2(\rho), p_{jn}^2, S_n^1(\rho), q_{jn}^1\}\sin n\varphi \end{bmatrix}, \quad j = 3,4 \tag{20}$$

according to well-known decomposition-algorithm [9]. Note that the terms of (20), which are “0”-subscribed, correspond to the particular case, when the tractions, temperature field, and, consequently, the stresses are distributed uniformly with respect to angular variable φ. Solutions for this one-dimensional case have been analyzed in [7,8]. Therefore, we will consider the constructions of the solutions dependent on the angular variable. Using the technique proposed in [3] for direct integration of the equilibrium equations (7), we can express the coefficients of the stress-tensor components in terms of the total stress:

$$S_n^i = \frac{1}{2\rho^2}\left[k^2\chi_n^+(\rho,k)q_{3n}^i + (-1)^i k^2 \chi_n^-(\rho,k)p_{3n}^i - (-1)^i \int_k^\rho \eta\sigma_n^i(\eta)\left(n\chi_n^+(\rho,\eta) + \chi_n^-(\rho,\eta)\right)\mathrm{d}\eta\right],$$

$$R_n^i = \frac{-1}{2\rho^2}\left[(-1)^i k^2\chi_n^-(\rho,k)q_{3n}^i + k^2 \chi_n^+(\rho,k)p_{3n}^i - \int_k^\rho \eta\sigma_n^i(\eta)\left(n\chi_n^-(\rho,\eta) + \chi_n^+(\rho,\eta)\right)\mathrm{d}\eta\right], \ \Phi_n^i = \sigma_n^i - R_n^i \tag{21}$$

where $\chi_n^+(x,y) = x^{-n}y^n + x^n y^{-n}, \chi_n^-(x,y) = x^{-n}y^n - x^n y^{-n}$. Note that the expressions (21) are invariant with respect to the material properties. Moreover, the integral equilibrium conditions

$$\int_k^1 \rho^{1+n}\sigma_n^i \mathrm{d}\rho = \frac{k^{n+2}p_{3n}^i - p_{4n}^i + (-1)^i(k^{n+2}q_{3n}^i - q_{4n}^i)}{n+1}, \ \int_k^1 \rho^{1-n}\sigma_n^i \mathrm{d}\rho = \frac{p_{4n}^i - k^{2-n}p_{3n}^i + (-1)^i(k^{2-n}q_{3n}^i - q_{4n}^i)}{n-1} \tag{22}$$

follow from (9). For angle-dependent terms, the equation (11) can be solved to have

$$\frac{\sigma_n^i}{\overline{E}} + \overline{\alpha}T_n^i = A_n^i\rho^{-n} + B_n^i\rho^n - \frac{1}{4n}\int_k^\rho \eta R_n^i(\eta)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(\frac{1}{G}\right)\chi_n^-(\rho,\eta)\mathrm{d}\eta - \frac{1}{4n}\int_k^\rho \Phi_n^i(\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G}\right)\chi_n^-(\rho,\eta)\mathrm{d}\eta\,. \tag{23}$$

The Volterra type integral equation

$$\sigma_n^i = \overline{E}\left[A_n^i\rho^{-n} + B_n^i\rho^n + P_n^i + Q_n^i - \overline{\alpha}T_n^i + \frac{1}{4}\int_k^\rho \xi\sigma_n^i(\xi)K_n(\rho,\xi)\mathrm{d}\xi\right] \tag{24}$$

follow from (23) by employing (21). Here

$$K_n(\rho,\xi) = \int_\xi^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G}\right)\left((n+1)\rho^n\xi^n\eta^{-2(n+1)} - (n-1)\rho^{-n}\xi^{-n}\eta^{2(n-1)}\right)\mathrm{d}\eta\,, \ P_n^i = -\frac{kp_{3n}^i}{4}\left(\frac{\chi_n^-(\rho,k)}{n}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\bigg|_{\rho=k}\right.$$

$$\left. + k\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G}\right)\left(k^{-n}\rho^{-n}\frac{\eta^{2n}}{\eta^2} + k^n\rho^n\frac{\eta^2}{\eta^{2n}}\right)\mathrm{d}\eta\right), \ Q_n^i = \frac{(-1)^i k^2 q_{3n}^i}{4}\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G}\right)\left(k^{-n}\rho^{-n}\eta^{2(n-1)} - k^n\rho^n\eta^{-2(n+1)}\right)\mathrm{d}\eta\,.$$

We construct the solution of equation (24) as the limit $\sigma_n^i = \lim_{m\to\infty}\sigma_n^{i(m)}$ of the simple iterations

$$\sigma_n^{i(m)} = \overline{E}\left[A_n^{i(m)}\rho^{-n} + B_n^{i(m)}\rho^n + P_n^i + Q_n^i - \overline{\alpha}T_n^i + \frac{1}{4}\int_k^\rho \xi\sigma_n^{i(m-1)}(\xi)K_n(\rho,\xi)\,\mathrm{d}\xi\right],$$

where $\sigma_n^{i(0)} \equiv 0$, the expressions $A_n^{i(m)} = (I_1 F_{1n}^{i(m)} - I_n^+ F_{2n}^{i(m)})/\gamma_n$, $B_n^{i(m)} = (I_1 F_{2n}^{i(m)} - I_n^- F_{1n}^{i(m)})/\gamma_n$ follow from the conditions (22) for $n \ge 2$, and

$$F_{1n}^{i(m)} = \frac{1}{n+1}\left[k^{n+2} p_{3n}^i - p_{4n}^i + (-1)^i \left(k^{n+2} q_{3n}^i - q_{4n}^i\right)\right] - \int_k^1 \rho^{n+1} \overline{E}(\rho) F_n^{i(m)}(\rho)\,\mathrm{d}\rho\,,$$

$$F_{2n}^{i(m)} = \frac{1}{n-1}\left[-k^{-n+2} p_{3n}^i + p_{4n}^i + (-1)^i \left(k^{-n+2} q_{3n}^i - q_{4n}^i\right)\right] - \int_k^1 \rho^{-n+1} \overline{E}(\rho) F_n^{i(m)}(\rho)\,\mathrm{d}\rho\,,$$

$$F_n^{i(m)}(\rho) = P_n^i + Q_n^i - \overline{\alpha} T_n^i + \frac{1}{4}\int_k^{\rho} \xi \sigma_n^{i(m-1)}(\xi) K_n(\rho,\xi)\,\mathrm{d}\xi,\ \gamma_n = I_1^2 - I_n^- I_n^+,\ \ \left\{I_n^+, I_n^-\right\} = \int_k^1 \rho\left\{\rho^{2n}, \rho^{-2n}\right\}\overline{E}(\rho)d\rho.$$

It is easy to see that the second condition (22) degenerates at $n = 1$. Using (10), we determine

$$4A_1^{i(m)} = kp_{31}^i\left[k\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\Bigg|_{\rho=k} - \frac{1}{G(k)} - \frac{1+k^2}{4}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\Bigg|_{\rho=1}\right] + (-1)^i kq_{31}^i\left[\frac{1}{G(k)} + \frac{1-k^2}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\Bigg|_{\rho=1}\right]$$

$$+\frac{p_{41}^i}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\Bigg|_{\rho=1} + \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\Bigg|_{\rho=1}\int_k^1 \rho^2 \sigma_1^{i(m-1)}\,\mathrm{d}\rho,\ i = 1,2\,.$$

Then, the constants $B_1^{i(m)}$ can be determined from the first condition (22). After the total stress is determined, it is easy to find the radial, angular, and shearing stresses with the aid of (21). Note that equation (24) guarantees an exact solution by first iteration in the event when the shear modulus, G, is a constant.

4 Numerical examples and discussions

The presented computations are provided for the plane strain case in absence of the temperature, $T = 0$.

To analyze the exact solution for the strip, we consider the example of inhomogeneity, when G and E are constants, and $\nu = 1 - 2/(3 - ay)$, a is an arbitrary parameter. The strip is stressed by the tractions $p_1 = p_2 = \exp(-x^2)/2$.

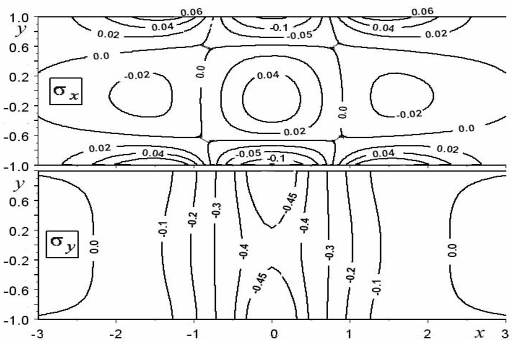

Figure 1

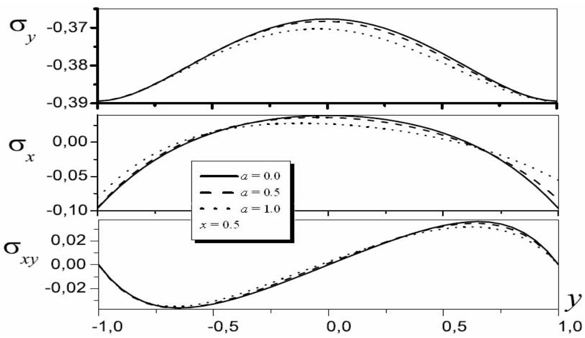

Figure 2

The full-field distributions of the normal stresses are shown in Fig. 1 for $a = 1$. Figure 2 demonstrates the distribution of the stresses in the strip Ω_1 for different values of a. As expected, there is effect of the material inhomogeneity for the stresses. However, this effect is comparatively small due to the Young's and shear moduli are constants.

To analyze a radially inhomogeneous ring, we consider the example when the ring Ω_2 of inner radius $k = 0.5$ is uniformly pressurized with intensity *p* in sectors $[\pi/4, 3\pi/4]$ and $[5\pi/4, 7\pi/4]$ of the rim. The calculations are provided for the Young's modulus vary exponentially, $E(\rho) = E_0 \exp(\beta\rho)$, and as the power function where $E(\rho) = E_0 \rho^{\beta}$, $E_0, \beta = \mathrm{const}$, $\nu = 0.3$. As it is observed in the numerical computation, the third iteration is enough to get the 'engineering' accuracy in the considered examples. The full-field distribution of the dimensionless radial stress in the ring for the exponential Young's modulus for $\beta = 1.5$ is depicted in Fig. 3. Figure 4 shows the effect of Young's modulus in the exponential form (*a*) and in the form of power function (*b*) on the distributions of radial stress in the ring for different values of β.

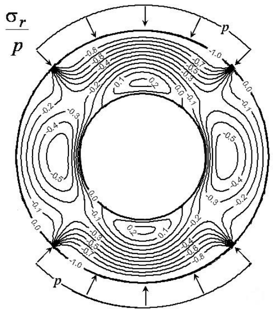

Figure 3

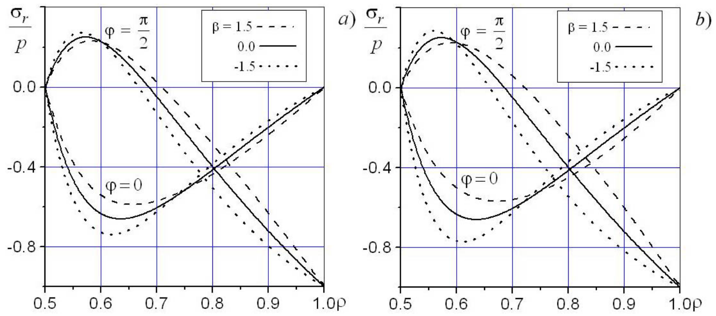

Figure 4

We observe the influence of stress caused by radial dependence of the Young's modulus.

5 Conclusions and recommendations

The method of solutions construction for the plane elasticity and thermoelasticity problems for inhomogeneous solids is presented. The efficiency of the method is demonstrated for two different types of domain. Due to derived relations between stress tensor components, we can simplify calculation of stresses state in inhomogeneous solids considerably, because these relations are irrelative of material properties.

The proposed approach shows the rapid convergence of the iterative scheme used for solution construction. This can be explained by the fact that the solutions of corresponding problems for homogeneous region are considered as the first iteration for solving the problems in inhomogeneous case, taking, however, some various material properties into account. Consequently, the proposed solution scheme promises to be rather useful for practical calculations and analytical investigations. Moreover, for some particular cases, i.e. if the shear modulus is a constant for the polar coordinate system and if its reciprocal is the linear function for the Cartesian coordinate system, the method presents the exact analytical solutions of the corresponding problems.

This approach demonstrates its effectiveness in analyzing the problems for inhomogeneous materials with rather general ranges of dependences of the material properties on coordinates. The computing results show the essential effect of material inhomogeneity on the stress distributions in the solids. Therefore, taking the spatial coordinate dependences of material properties into account is very important for accurate analysis of stresses in practice.

References

[1] Suresh, S., and Mortensen, A., 1998, *Fundamentals of Functionally Graded Materials*, The Institute of Materials, London.

[2] Vihak, V., and Rychahivskyy, A., 2001, Bounded solutions of plane elasticity problems in a semi-plane, *J. Comp. Appl. Mechanics* 2, No. 2, pp. 263–272.

[3] Vihak, V. M., 1996, Solution of the Thermoelastic problem for a cylinder in the case of a two–dimensional nonaxisymmetric temperature field, *Z. Angew. Math. Mech.* **76**, No 1, pp. 35–43.

[4] Timoshenko, S. P., and Goodier J. N., 1970, *Theory of Elasticity*, McGraw-Hill Book Company, New York.

[5] Boley, B. A., and Weiner, J. H., 1960, *Theory of Thermal Stresses*, John Wiley & Sons, Inc., New York.

[6] Brychkov, Yu. A., and Prudnikov, A. P., 1989, *Integral transforms of generalized functions*, Gordon and Breach Science Publishers, New York.

[7] Vihak, V. M., and Kalyniak, B. M., 1999, "Reduction of one-dimensional elasticity and thermoelasticity problems in inhomogeneous and thermal sensitive solids to the solution of integral equation of Volterra type," *Proc. 3rd Int. Congress on Thermal Stresses*. Cracow; Poland; 1999, pp. 457–460.

[8] Kalynyak, B. M., 2000, "Integration of equations of one-dimensional problems of elasticity and thermoelasticity for inhomogeneous cylindrical bodies," *J. Math. Sciences* 99, No 5, pp.1662–1670.

[9] Tolstov, G. P., 1962, *Fourier series*, Prentice-Hall, New Jersey.