

5<sup>th</sup> Australasian Congress on Applied Mechanics, ACAM 2007  
10-12 December 2007, Brisbane, Australia

## Investigation of plasticity-induced fatigue crack closure

John Codrington and Andrei Kotousov

School of Mechanical Engineering, The University of Adelaide, SA 5005, Australia

**Abstract:** Plasticity induced crack closure and constraint effects due to finite plate thickness are both fundamental aspects in the mechanics of fatigue cracks. Moreover, plasticity induced crack closure provides an effective first-order correction to the crack driving force, as used in the correlation and prediction of fatigue crack growth. The approach developed in this study utilises the distributed dislocation technique to model fatigue cracks growing under constant amplitude loading in finite thickness plates. Numerical results are obtained through the application of Gauss-Chebyshev quadrature and are presented for the crack opening stress ratio. An excellent agreement is observed with previous three-dimensional finite element studies.

**Keywords:** crack closure, crack opening stress, crack tip plasticity, distributed dislocation technique, edge dislocation, plate thickness effect, through-the-thickness crack.

### 1 Introduction

Under the classic linear elastic approach to fatigue it is assumed that crack growth rates can be described as a function of the stress intensity factor range,  $\Delta K = K_{max} - K_{min}$ . However, this method is unable to account for many phenomena, which are present in almost all practical situations. Such examples include the effects of the loading history, plate thickness, crack size, stress concentrators, etc. The discovery of crack closure [1] provided a new foundation for fatigue analysis. Elber [1] suggested that closure occurs as the result of crack tip plasticity and that significant crack growth will only occur when the crack is fully open. This led to the introduction of an effective stress intensity factor range  $\Delta K_{eff} = K_{max} - K_{open}$ , where  $K_{open}$  refers to the stress intensity factor when the crack just re-opens. Plasticity-induced crack closure has been found able to account for a range of factors including the load ratio, overload effects, and plate thickness [2].

Crack closure has been experimentally investigated by various techniques such as compliance methods [1,3] and by acoustic emission [3] among others. In general these studies have proven the importance of crack closure in fatigue and that plasticity-induced closure is the dominant mechanism under many conditions. Although, there is often some scatter between the crack opening stress values obtained by the different measurement techniques.

Many simplified analytical [4,5] and numerical [6] models have been developed to try and describe the effects of crack closure for various specimen and crack geometries. A large number of these studies are based on the Dugdale strip-yield model [7] modified to include a wake of plasticity left behind as the crack grows. In addition, plane stress conditions are usually assumed, which greatly limits the applicability of the results to actual structures where the crack tip stress field is always three-dimensional. Newman [6] introduced the use of a so-called constraint factor,  $\alpha$ , into his plane stress analysis in an attempt to account for the effects of plate thickness. However, choosing a value for the constraint factor is not straightforward as it requires numerical or experimental results for similar materials, specimen geometry and load conditions. Trial and error, or curve fitting of fatigue crack growth data are often employed, which creates further ambiguity. Finite element (FE) methods have been utilised [8-10] as an alternative to model complex two- and three-dimensional geometry, however, there are many issues including mesh refinement, crack face contact, required computational effort, etc. These issues make FE methods very difficult to apply for most practically important applications.

This paper presents an analytical approach for investigating the effects of plasticity-induced fatigue crack closure in plates of finite thickness. The developed procedure is based on the distributed dislocation technique and the solution for an edge dislocation in a finite thickness plate. Numerical results are obtained via the application of Gauss-Chebyshev quadrature and are presented for the crack opening stress ratio, which is an important parameter used in the correlation and prediction of fatigue crack growth. A comparison with previously published two and three-dimensional FE studies shows a very good agreement.

## 2 Fatigue crack under constant amplitude loading

Consider a through-the-thickness crack of length  $2a$ , which has been growing under constant amplitude loading from an initial crack size  $2a_i$ . The crack lies centrally along the  $x$ -axis ( $-a < x < a$ ) in an infinite plate of thickness  $2h$  and is subjected to the remotely applied stress  $\sigma_{yy}^\infty(x)$ . A rigid perfect-plastic strip yield model is employed along with the assumption that the stresses and plastic zone size are uniform across the plate thickness. To eliminate the need to manually grow the crack from initiation to its final length, which can be a laborious task, we approximate the plastic wake as being a linear function of the final crack half-length [4,5]. It is understood that all of these simplifications are most applicable to the case of plane stress, particularly where the final crack length is several times greater than its original length. The developed methods, however, aim to provide a powerful first-order estimate of the effects of plasticity-induced crack closure in plates of finite thickness. This will allow for more accurate prediction of crack growth rates and structural lifetimes in many practical applications.

Schematic diagrams of the strip-yield models for a fatigue crack at maximum and minimum applied loads in the cycle are shown in Figures 1a and 1b, respectively. Here the size of the tensile plastic zone at maximum load is given by  $r_p$  and the maximum crack tip stretch by  $\delta_M$ . At minimum applied load the size of the reverse plastic zone is given by  $r_{p,c}$ ,  $\beta$  is the half-length of the region of the crack that remains open and  $\delta_R$  is the residual crack tip stretch.

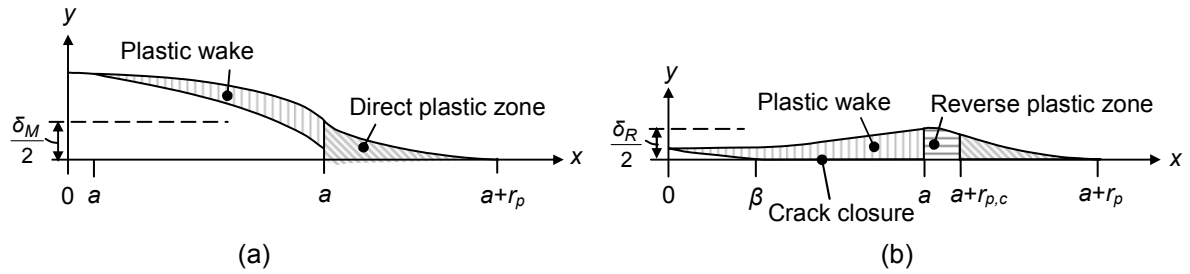


Figure 1. Schematic of a fatigue crack at (a) maximum applied load and (b) minimum applied load.

## 3 Solution procedure

### 3.1 Maximum applied load

The distributed dislocation technique involves replacing the crack and plastic zones with a continuous distribution of edge dislocations. By taking advantage of the symmetry of the problem about the  $y$ -axis, the following singular integral equation can be derived for the  $y$ -stresses along the  $x$ -axis:

$$\sigma(x) = \frac{1}{\pi} \int_0^{a+r_p} B_y(\xi) [G(x, \xi) - G(x, -\xi)] d\xi + \sigma_{yy}^\infty(x), \quad (1)$$

where  $B_y(\xi)$  is the unknown dislocation density function and  $G(x, \xi)$  is the dislocation influence function. Here the influence function represents the non-dimensional stress at a point  $x$  due to a dislocation with a unit Burgers vector located at  $\xi$ . The second term in the kernel, that is  $-G(x, -\xi)$ , is simply due to the symmetry condition. The separation of the crack faces,  $g(x)$ , can be determined from the dislocation density through the relationship:

$$\frac{dg(\xi)}{d\xi} = -B_y(\xi). \quad (2)$$

The influence functions for a through-the-thickness edge dislocation in a finite thickness plate are given as [11]:

$$G_{yy}(x, \xi) = -\frac{E}{4(1-\nu^2)\rho} \frac{1}{\left[ \frac{4\nu^2}{(\lambda\rho)^2} - (1-\nu^2) - 2\nu^2 K_0(\lambda|\rho|) - \frac{2(2+\lambda^2\rho^2)\nu^2 K_1(\lambda|\rho|)}{\lambda|\rho|} \right]}, \quad (3)$$

and:

$$G_{zz}(x, \xi) = \frac{E}{2(1-\nu^2)} \lambda\nu K_1(\lambda|\rho|) \frac{\rho}{|\rho|}, \quad (4)$$

for the  $y$  and  $z$ -directions, respectively. In these equations  $\rho = x - \xi$ ,  $E$  is Young's modulus,  $\nu$  is Poisson's ratio,  $K_0$  and  $K_1$  are modified Bessel functions of the second kind, and the thickness parameter is:

$$\lambda = \frac{1}{h} \sqrt{\frac{6}{1-\nu}}. \quad (5)$$

In the limiting cases of plane stress and plane strain the  $x$  and  $y$  influence functions are given as [12]:

$$G_{xx}(x, \xi) = G_{yy}(x, \xi) = \frac{2\mu}{\kappa + 1} \frac{1}{\rho}. \quad (6)$$

Furthermore, in plane stress use is made of the fact that  $\sigma_{zz} = 0$  and in plane strain  $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ .

By employing a Tresca yield criterion and assuming that  $\sigma_{yy} \geq \sigma_{xx} \geq \sigma_{zz}$ , the stresses in the plastic zone can be written as  $|\sigma_{yy} - \sigma_{zz}| = \sigma_o$ , where  $\sigma_o$  is the flow stress of the material. The boundary conditions for the governing integral (1) at maximum applied load,  $\sigma_{yy}^\infty(x) = \sigma_{\max}^\infty$ , therefore become:

$$G(x, \xi) = G_{yy}(x, \xi) \text{ and } \sigma(x) = \sigma_{yy}(x) = 0, \quad \text{for } 0 \leq x < a, \quad (7)$$

$$G(x, \xi) = G_{yy}(x, \xi) - G_{zz}(x, \xi) \text{ and } \sigma(x) = \sigma_{yy}(x) - \sigma_{zz}(x) = \sigma_o, \quad \text{for } a \leq x \leq a + r_p, \quad (8)$$

For  $x \geq a + r_p$  the condition  $g(x) = 0$  applies and this has already been incorporated into the integral (1).

In order to improve the efficiency of the numerical solution, which will be achieved via the application of Gauss-Chebyshev quadrature, the equation (1) is separated into two integrals. The first is taken over the half crack length and the second over the length of the plastic zone, such that:

$$\sigma(x) = \sigma_1(x) + \sigma_2(x) + \sigma_{yy}^\infty(x), \quad (9)$$

$$\sigma_1(x) = \frac{1}{\pi} \int_0^a B_y(\xi) [G(x, \xi) - G(x, -\xi)] d\xi, \quad (10)$$

and:

$$\sigma_2(x) = \frac{1}{\pi} \int_a^{a+r_p} B_y(\xi) [G(x, \xi) - G(x, -\xi)] d\xi. \quad (11)$$

Separating the integrals as above provides control over the number of integration points placed within each interval. In addition, this method allows for exact placement of the edge of the plastic zone rather than at the integration points, for example if only the one interval was used over the whole length [5].

The equations (10) and (11) are then transformed to integrals over the range -1 to 1, which provides:

$$\sigma_1(x) = \frac{1}{\pi} \int_{-1}^1 \bar{B}_y(s_1) [\bar{G}(x, s_1) - \bar{G}(x, -s_1)] \frac{a}{2} ds_1 \text{ where } \xi = \frac{a}{2}(s_1 + 1), \quad (12)$$

and:

$$\sigma_2(x) = \frac{1}{\pi} \int_{-1}^1 \bar{B}_y(s_2) [\bar{G}(x, s_2) - \bar{G}(x, -s_2)] \frac{r_p}{2} ds_2 \text{ where } \xi = \frac{r_p}{2}(s_2 + 1) + a. \quad (13)$$

We now introduce the function  $\bar{\varphi}(s)$ , which is related to the dislocation density function by:

$$\bar{B}_y(s) = \bar{\varphi}(s)(1+s)^{+1/2}(1-s)^{-1/2}. \quad (14)$$

Through the application of Gauss-Chebyshev quadrature the integrals (12) and (13) can each be written as a linear series summation. This gives:

$$\sigma_1(x) = \frac{a}{2n_1 + 1} \sum_{i=1}^{n_1} \bar{\varphi}_1(s_{1,i})(s_{1,i} + 1) [\bar{G}(x, s_{1,i}) - \bar{G}(x, -s_{1,i})], \quad (15)$$

and:

$$\sigma_2(x) = \frac{r_p}{2n_2 + 1} \sum_{i=1}^{n_2} \bar{\varphi}_2(s_{2,i})(s_{2,i} + 1) [\bar{G}(x, s_{2,i}) - \bar{G}(x, -s_{2,i})], \quad (16)$$

where the integration points within each interval are determined by:

$$s_{j,i} = \cos\left(\pi \frac{2i-1}{2n_j+1}\right), \quad i = 1 \dots n_j, \quad j = 1, 2. \quad (17)$$

Within the crack and plastic zones the discrete stress functions are only valid at the collocation points, which are defined as:

$$t_{j,k} = \cos\left(\pi \frac{2k}{2n_j+1}\right), \quad k = 1 \dots n_j, \quad j = 1, 2, \quad (18)$$

where the  $x$ - $t$  transformations within each interval are equivalent to those for the  $\xi$ - $s$  transformations. When  $x > a + r_p$  equations (15) and (16) may be evaluated at any point.

The unknown functions  $\bar{\varphi}_j(s_{j,i})$ , for  $j = 1, 2$  and  $i = 1 \dots n_j$ , can now be determined by substituting (15) and (16) back into (9) and by enforcing the stress boundary conditions over the length of the crack and direct plastic zone. This gives a total system of  $n_1 + n_2$  linear equations in the  $n_1 + n_2$  unknowns. The size of the direct plastic zone,  $r_p$ , is determined by first making an initial guess and then iterating based on the requirement that  $K_I(a + r_p) = 0$ . Here  $K_I$  is the mode I stress intensity factor and is found through an asymptotic analysis of the stress field ahead of the crack tip as:

$$K_I = \sqrt{2\pi r_p} \frac{E}{4(1-\nu^2)} \bar{\varphi}_2(s_{2,1}). \quad (19)$$

### 3.2 Minimum applied load

In this study it is assumed that the compressive flow stress is of equal magnitude to the tensile flow stress. Furthermore, to enable comparison with previous crack closure investigations, the out-of-plane constraint is removed during reverse plastic yielding. This means that compressive yielding will occur when  $|\sigma_{yy}| = \sigma_o$ . The boundary conditions for  $R \geq 0$ , where  $R = \sigma_{\min}^{\infty} / \sigma_{\max}^{\infty}$ , at minimum applied load,

$\sigma_{yy}^{\infty}(x) = \sigma_{\min}^{\infty}$ , therefore become:

$$G(x, \xi) = G_{yy}(x, \xi) \text{ and } \sigma(x) = \sigma_{yy}(x) = 0, \quad \text{for } x < \beta, \quad (20)$$

$$\frac{dg(x)}{dx} = \frac{\delta_R}{a - a_i}, \quad \beta \leq x < a, \quad (21)$$

$$G(x, \xi) = G_{yy}(x, \xi) \text{ and } \sigma(x) = \sigma_{yy}(x) - \sigma_{zz}(x) = -\sigma_o, \quad a \leq x < a + r_{p,c}, \quad (22)$$

and:

$$\frac{dg(x)}{dx} = \frac{dg_{\max}(x)}{dx}, \quad a + r_{p,c} \leq x < a + r_p, \quad (23)$$

where the subscript max refers to the displacement curve at maximum applied load.

A similar procedure to the maximum load case is then followed. That is, the governing integral (1) is separated into several intervals along the  $x$ -axis, namely 0 to  $\beta$ ,  $\beta$  to  $a$ ,  $a$  to  $a + r_{p,c}$ , and  $a + r_{p,c}$  to  $a + r_p$ . Solution to the problem is again found through the application of Gauss-Chebyshev quadrature and the appropriate coordinate transformations, to provide a new system of linear equations. Full details, however, are not included here. The size of the reverse plastic zone,  $r_{p,c}$ , and the region of non-closure,  $\beta$ , are determined via iteration by minimising and maximising the residual crack tip stretch,  $\delta_R$ , respectively. This condition arises from the requirement that the slope of the deflection/plastic stretch curve must be continuous. An initial guess is necessary for each of the unknowns including  $\delta_R$ , which then converges very rapidly through back substitution into (21). The deflection and plastic stretch curves, and thus  $\delta_R$ , are determined by numerical integration of (2) and (14).

## 4 Crack opening stress

An important parameter often utilised in the correlation of fatigue crack growth rates is the effective stress intensity factor range, which can be defined as:

$$\Delta K_{eff} = (\sigma_{max}^{\infty} - \sigma_{open}^{\infty}) Y \sqrt{\pi a}, \quad (24)$$

where  $Y$  is a specific function due to the cracked geometry and  $\sigma_{open}^{\infty}$  is the remotely applied stress at the point when the crack tip re-opens. The effective stress intensity range is therefore related to the linear elastic range by:

$$\Delta K_{eff} = \frac{1 - \sigma_{open}^{\infty} / \sigma_{max}^{\infty}}{1 - R} \Delta K. \quad (25)$$

The crack opening stress can be found using the techniques developed in Section 3, by applying the boundary conditions:

$$G(x, \xi) = G_{yy}(x, \xi) \text{ and } \sigma(x) = \sigma_{yy}(x) = 0, \quad \text{for } x < a, \quad (26)$$

and:

$$\frac{dg(x)}{dx} = \frac{dg_{min}(x)}{dx}, \quad \text{for } a \leq x < a + r_p, \quad (27)$$

with  $\sigma_{yy}^{\infty}(x) = \sigma_{open}^{\infty}$ .

Results for the crack opening stress to maximum stress ratio are presented in Figures 2a and 2b as a function of the normalised direct plastic zone size and load ratio, respectively. Curves for several plate thickness to crack length ratios are given along with the plane stress and plane strain limits. In these calculations the initial crack length was taken as  $a_i = 0$ . It can be seen that as  $\sigma_{max}^{\infty} / \sigma_o \rightarrow 0$  plane strain conditions prevail and as  $\sigma_{max}^{\infty} / \sigma_o \rightarrow 1$  plane stress conditions prevail. Similarly, as  $h/a$  increases the plane strain solution is recovered. Overall the opening stress ratio is quite high, indicating to the significance of accounting for plasticity-induced crack closure in fatigue analyses.

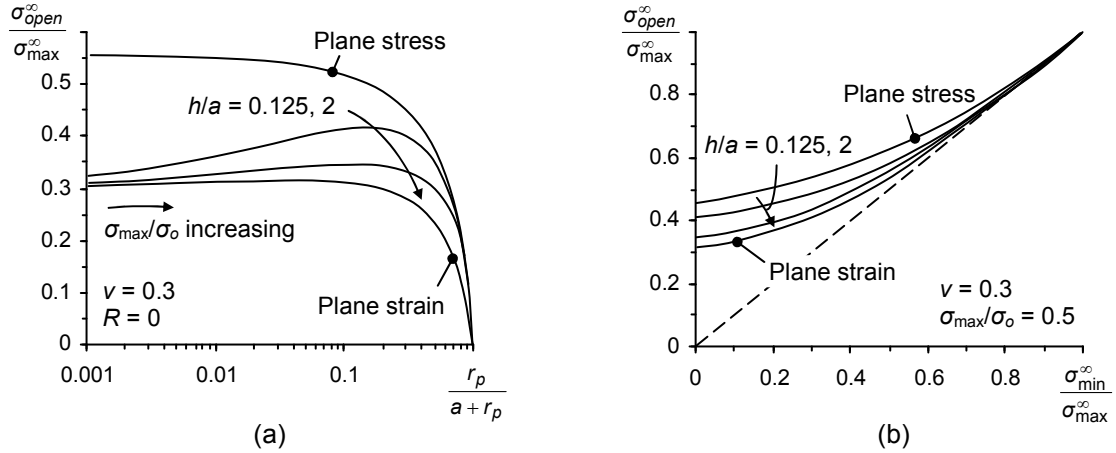


Figure 2. Crack opening stress ratio as a function of (a) the normalised direct plastic zone size and (b) the load ratio.

Figure 3a shows a comparison of the present results for the crack opening stress ratio with those from previous FE investigations in the case of plane stress. Results for the crack opening stress ratio as a function of the plate thickness to crack length ratio are provided in Figure 3b and are compared with the through-the-thickness average values obtained from a three-dimensional FE analysis [9]. To provide for a better comparison the crack length and plate thickness in the current analysis were adjusted in order to keep the crack tip stress intensity factor and the  $h/a$  ratio the same as that for the FE model. An excellent agreement is observed between the present results and the FE values. Any variation between the results can be explained by the different modelling assumptions made in the present analysis compared to the FE studies.

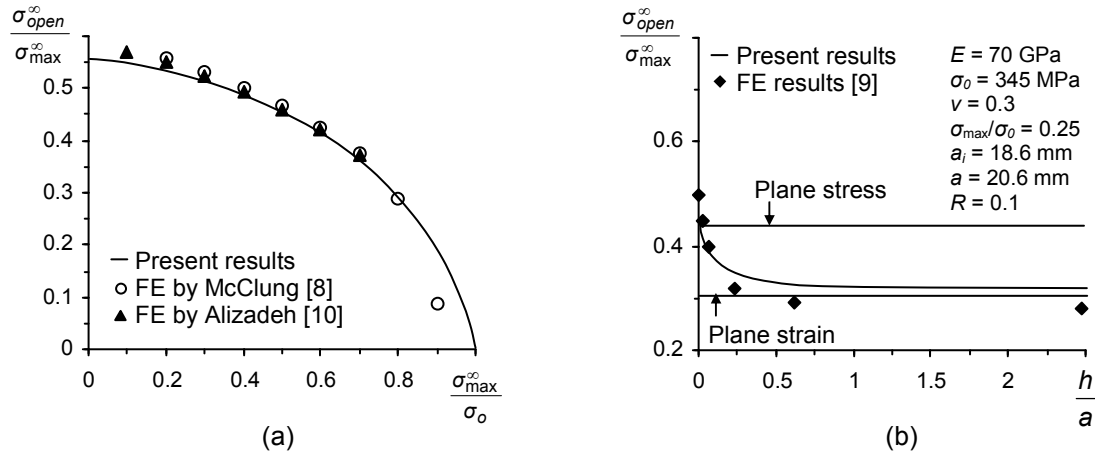


Figure 3. Comparison of the opening stress ratio for (a) plane stress and (b) a finite thickness plate.

## 5 Conclusions

This paper describes a new approach for investigating the effects of plasticity-induced fatigue crack closure in plates of finite thickness. The developed procedure is based on the distributed dislocation technique and the solution for an edge dislocation in a finite thickness plate. Numerical results were obtained via the application of Gauss-Chebyshev quadrature. Results for the crack opening stress ratio were presented and compared with those from previous finite element investigations. A very good agreement was observed. The developed methods therefore offer a powerful alternative to the use of the plastic constraint factor or FE analysis for determining the various parameters, such as the effective stress intensity factor, used in the correlation and prediction of fatigue crack growth.

## Acknowledgements

This work was undertaken with the support of the Australian Research Council (ARC), through research grant no. DP0557124. The support is gratefully acknowledged.

## References

- [1] Elber, W. Fatigue crack closure under cyclic tension. *Eng Fract Mech* 1970;2:37-45.
- [2] Guo W, Wang CH, Rose LRF. The influence of cross-sectional thickness on fatigue crack growth. *Fatigue Fract Eng Mater Struct* 1999;22:437-444.
- [3] Lee CS, Park CG, Chang YW. Precise determination of fatigue crack closure in Al alloys. *Mat Sci Eng* 1996;A216:131-138.
- [4] Rose LRF, Wang CH. Self-similar analysis of plasticity-induced closure of small fatigue cracks. *J Mech Phys Solids* 2001;49:401-429.
- [5] Codrington J, Kotousov A. The distributed dislocation technique for calculating plasticity-induced crack closure in plates of finite thickness. *Int J Fract* 2007;144:285-295.
- [6] Newman JC Jr. A crack closure model for predicting fatigue crack growth under aircraft spectrum loading. ASTM STP 748. American Society for Testing and Materials: Philadelphia, 1981, pp 53-84.
- [7] Dugdale DS. Yielding of steel sheets containing slits. *J Mech Phys Solids* 1960;8:100-104.
- [8] McClung RC, Sehitoglu H. On the finite element analysis of fatigue crack closure – 2 Numerical results. *Eng Fract Mech* 1983;33:253-272.
- [9] Chermahini RG, Shivakumar KN, Newman JC Jr, Blom AF. Three-dimensional aspects of plasticity-induced fatigue crack closure. *Eng Fract Mech* 1989;34:393-401.
- [10] Alizadeh H, Hills DA, de Matos PFP, Nowell D, Pavier MJ, Paynter RJ, Smith DJ, Simandjuntak S. A comparison of two and three-dimensional analyses of fatigue crack closure. *Int J Fatigue* 2007;29:222-231.
- [11] Kotousov A, Wang CH. Fundamental solutions for the generalised plane strain theory. *Int J Eng Sci* 2002;40:1775-1790.
- [12] Hills DA, Kelly PA, Dai DN, Korsunsky AM. Solution of crack problems – the distributed dislocation technique. Dordrecht: Kluwer Academic Publishers, 1996.