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A. D. Rawlins

*Proc. R. Soc. Lond. A* 1975 **346**, 469-484
doi: 10.1098/rspa.1975.0186

*Proc. R. Soc. Lond.* A. **346**, 469–484 (1975)
*Printed in Great Britain*

# The solution of a mixed boundary value problem in the theory of diffraction by a semi-infinite plane

BY A. D. RAWLINS
*Department of Mathematics, The University, Dundee DD1 4HN, Scotland*

(*Communicated by D. S. Jones, F.R.S. – Received 23 December 1974*)

A solution is obtained for the problem of the diffraction of a plane wave sound source by a semi-infinite half plane. One surface of the half plane has a soft (pressure release) boundary condition, and the other surface a rigid boundary condition. Two unusual features arise in this boundary value problem. The first is the edge field singularity. It is found to be more singular than that associated with either a completely rigid or a completely soft semi-infinite half plane. The second is that the normal Wiener–Hopf method (which is the standard technique to solve half plane problems) has to be modified to give the solution to the present mixed boundary value problem.

The mathematical problem which is solved is an approximate model for a rigid noise barrier, one face of which is treated with an absorbing lining. It is shown that the optimum attenuation in the shadow region is obtained when the absorbing lining is on the side of the screen which makes the smallest angle to the source or the receiver from the edge.

## 1. INTRODUCTION

The present work arose in connection with noise abatement by means of noise barriers. In recent years, noise reduction by barriers has become a common measure of environmental protection in heavily built-up areas, Kurze (1974). Traffic noise from motorways, railways, and airports, and other outdoor noises from heavy construction machinery or stationary installations, such as large transfomers or plants, can be shielded by a barrier which intercepts the line-of-sight from the source to a receiver. Noise in an open plan office can also be reduced by means of barriers. The acoustic field in the shadow region of a barrier, when transmission through a barrier is negligible, is due to diffraction at the edge alone.

The design of such noise barriers should meet two important requirements, namely, that they are effective noise attenuators, and that their construction and maintenance should be economical. The latter requirement is not difficult to appreciate when one considers the miles of motorway which run through built-up areas. One possible economic barrier construction is to have a rigid barrier (hence reducing transmitted noise) of cheap material which will not, however, be a good attenuator of edge diffracted noise, and to cover one or both surfaces with an absorbing lining which is a good attenuator of sound. The most economic situation would be where only one side of the barrier is covered with the absorbing lining.

The presence of an acoustically absorbing lining on a surface is usually described by an impedance relation between the pressure ($p$) and the normal velocity fluctuation on the lining surface (Morse & Ingard 1961). This gives rise to a boundary condition on the absorbing lining of the form

$$\frac{\partial p}{\partial n} = \mathrm{i}k\beta p, \quad \mathrm{Re}\,(\beta) > 0,$$

where the sound wave has time harmonic variation $\mathrm{e}^{-\mathrm{i}\omega t}$, and $k = \omega/c$, $c$ is the velocity of sound, $n$ the normal pointing into the lining, and $\beta$ the complex specific admittance of the acoustic lining (see Rawlins 1975). An acoustically hard (or perfectly reflecting) surface has a vanishing admittance, i.e. $|\beta| \to 0$ and an acoustically soft surface (pressure fluctuation vanishing on surface) is given by $|\beta| \to \infty$. The limiting case when the barrier surface is ideally soft is considered for simplicity, and it is suggested, can provide an upper bound to the effects one can expect to obtain with an absorbing surface.

If the wavelength of the sound is much smaller than the length scale associated with the barrier, the diffraction process is governed only by the local conditions at the edge and the solution to be canonical problem of diffraction by a semi-infinite half plane can be applied.

Under the above approximations a mathematical model for an absorbing barrier, whose two faces have acoustic lining, is given by the canonical problem of diffraction by a semi-infinite half plane with a soft boundary condition on both faces. This canonical problem was solved many years ago by Sommerfeld and an elegant solution by the Wiener–Hopf technique is given in Noble's (1958) book. One serious drawback of this model is that since it is completely soft it would have no mechanical support.

Under the same approximations, a mathematical model for a rigid barrier, one face of which is treated with an absorbing lining, is given by the canonical problem of diffraction by a semi-infinite half plane, one side of which is rigid and the other side soft. This model unlike the previous one does have mechanical support. A crude approximation to this model would be a barrier of hard board with foam rubber sheet on one face. We propose to solve this canonical mixed boundary value problem.

In § 2 the canonical boundary value problem is formulated. To ensure a unique solution an 'edge condition' Jones (1964), is imposed. This edge condition is the normal one associated with diffraction theory (i.e. that the sound energy is bounded in a finite region around the edge of the semi-infinite plate). It is found by an application of Meixner's method (see Hönl 1961) that if $\phi(x,y)$ represents the velocity potential then

$$\phi(x,0) \sim O(1), \quad \frac{\partial \phi}{\partial y}(x,0) \sim O(x^{-\frac{3}{4}}) \quad \text{as} \quad x \to 0^+.$$

This is in contrast to the case of a completely rigid or completely soft semi-infinite half plane when

$$\phi(x,0) \sim O(1), \quad \frac{\partial \phi}{\partial y}(x,0) \sim O(x^{-\frac{1}{2}}) \quad \text{as} \quad x \to 0^+.$$

In § 3 a solution is obtained for the boundary value problem. The mathematical method used to solve the problem consists of expressing the field in terms of integral representations. The integrands of the integral representations are obtained explicitly by means of the boundary conditions and a modification of the Wiener–Hopf technique. The approach is heuristic in so far that certain assumptions are made about singularities of the integrands. These assumptions are plausible from the physics of the problem. It is shown that the solution obtained is the (unique) solution to the problem.

Section three gives the asymptotic solutions for the far field. Also the known asymptotic solutions for the completely rigid and completely soft half planes are given. From these some graphs are plotted and comparisons made.

## 2. Formulation of the boundary value problem

We consider the scattering of acoustic waves from a plane wave source by a semi-infinite soft–hard half plane. Let the half plane be infinitely thin and occupy $y = 0$, $x \leqslant 0$ so that its edge lies along the $z$-axis, see figure 1. Let the plane wave source potential be described by

$$\phi_0(x, y) = \exp\{-\mathrm{i}[k(x\cos\theta_0 + y\sin\theta_0) + \omega t]\}, \tag{1}$$

where $k = \omega/c$, and $c$ is the velocity of sound. In future expressions the time harmonic variation $e^{-\mathrm{i}\omega t}$ will be suppressed. Hence we require, from the dynamic equations of motion, that the sound potential function $\phi(x, y)$ satisfy

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right\}\phi(x, y) = 0. \tag{2}$$

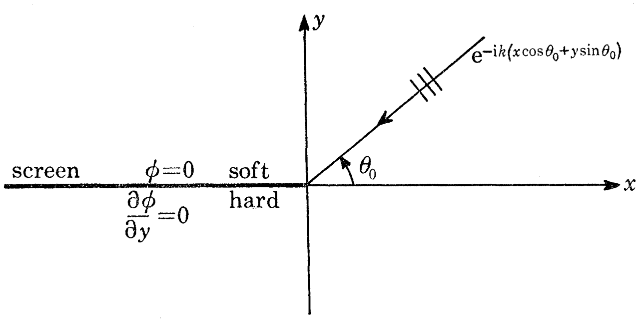



Figure 1.

In order to take account of the semi-infinite soft–hard screen we impose the boundary conditions,

$$\left.\begin{aligned} \phi(x, 0^+) &= 0, \\ \frac{\partial\phi}{\partial y}(x, 0^-) &= 0; \end{aligned}\right\} x < 0, \tag{3, 4}$$

that is the upper surface is soft and the lower surface is rigid. The boundary condition on the extension of the screen is the continuity of pressure and particle displacement, i.e.

$$\phi(x,0^+) = \phi(x,0^-), \qquad (5)$$

$$x > 0,$$

$$\frac{\partial \phi}{\partial y}(x,0^+) = \frac{\partial \phi}{\partial y}(x,0^-). \qquad (6)$$

We also require for a causal solution that the total field minus the incident plane wave should be radiating outgoing waves at infinity like a line source. This radiation condition requires that

$$\lim_{r\to\infty} \{\phi(x,y)-\phi_0(x,y)\} \sim O\left(\frac{e^{ikr}}{\sqrt{(kr)}}\right), \qquad (7)$$

where $r^2 = x^2+y^2$.

Finally for a unique solution to the above outlined problem we require an edge condition. The edge condition requires that the edge itself does not radiate energy. This requires that the derivative of $\phi(x,y)$ must have an integrable singularity at the origin. Applying the method of Meixner (see, for example, Karp & Karal 1962) to the present problem it is not difficult to show that

$$\phi(x,0) \sim O(1), \quad \frac{\partial \phi}{\partial y}(x,0) \sim O(x^{-\frac{3}{4}}) \quad \text{as} \quad x \to 0^+. \qquad (8)$$

We now find a solution to the problem outlined in §2.

## 3. Solution of the boundary value problem

For analytic convenience in the ensuing calculations we shall let $k = k_r + ik_i$, $k_r, k_i > 0$. In the final result we can let $k_i = 0$.

We shall assume that a solution for $\phi(x,y)$ can be written as

$$\phi(x,y) = \phi_0(x,y) + \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \left\{\Phi_+(\sqrt{(k+\nu)})\operatorname{sgn}(y) + \frac{\Psi_+(\sqrt{(k+\nu)})}{\sqrt{(k-\nu)}}\right\} \frac{e^{i(\nu x+\kappa|y|)}}{(\nu+k\cos\theta_0)}\,d\nu, \qquad (9)$$

where $\kappa = \sqrt{(k^2-\nu^2)}$ is defined to be on the cut sheet such that $\operatorname{Im}(\kappa) > 0$ for $|\operatorname{Im}(\nu)| < k_i$ see figure 2; and $\operatorname{sgn}(y) = \pm 1$ for $y \gtrless 0$. For analytic convenience (which will be apparent later in the analysis) the unknown quantities $\Phi_+$, $\Psi_+$ have arguments of the form $\sqrt{(k+\nu)}$. The expression (9) is certainly a solution of the wave equation (2). From the radiation condition (7) the integral representation (9) will exist provided $-k_i\cos\theta_0 < \tau < k_i$. The edge condition (8) will be satisfied provided

$$\Phi_+(\sqrt{(k+\nu)}) \sim |\nu|^{-\frac{1}{4}}, \quad \Psi_+(\sqrt{(k+\nu)}) \sim |\nu|^{\frac{1}{4}}, \qquad (10)$$

as $|\nu| \to \infty$, $\operatorname{Im}(\nu) > -k_i\cos\theta_0$.

The plus sign subscripts denote that the (as yet unknown) functions $\Phi_+$, $\Psi_+$ are regular and analytic functions of $\nu$ in the domain $\operatorname{Im}(\nu) > -k_i\cos\theta_0$ (n.b., an analytic function of an analytic function is an analytic function!).

We shall now show that under these conditions (9) automatically satisfies the boundary conditions (5) and (6). Thus substituting (9) into (5) and (6) gives

$$\left.\begin{aligned} \int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau} \frac{\Phi_+(\sqrt{(k+\nu)})}{(\nu+k\cos\theta_0)}\,\mathrm{e}^{\mathrm{i}\nu x}\,\mathrm{d}\nu = 0, \quad (11)\\ \int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau} \frac{\sqrt{(k+\nu)}\,\Psi_+(\sqrt{(k+\nu)})}{(\nu+k\cos\theta_0)}\,\mathrm{e}^{\mathrm{i}\nu x}\,\mathrm{d}\nu = 0. \quad (12) \end{aligned}\right\} x>0,$$

An application of Cauchy's theorem, and Jordan's Lemma to an infinite closed semi-circle in the upper $\nu$-plane (using the growth estimates (10)) shows that the expressions (11) and (12) are satisfied identically because the integrands are regular and analytic in the upper $\nu$-plane, $\mathrm{Im}\,(\nu) > -k_1\cos\theta_0$.

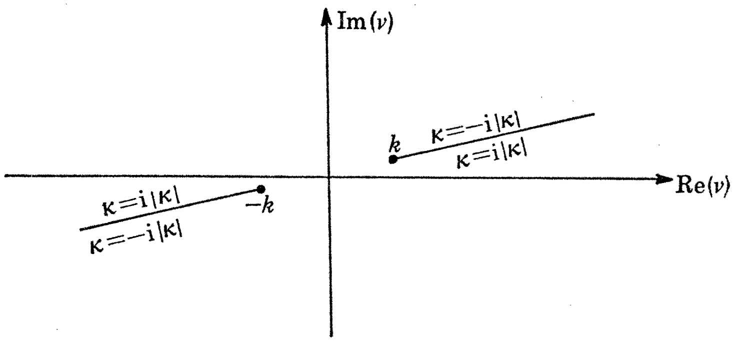



FIGURE 2.

Substituting (9) into the remaining unsatisfied boundary conditions (3) and (4) gives

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\left\{\Phi_+(\sqrt{(k+\nu)})+\frac{\Psi_+(\sqrt{(k+\nu)})}{\sqrt{(k-\nu)}}\right\}\frac{\mathrm{e}^{\mathrm{i}\nu x}}{(\nu+k\cos\theta_0)}\,\mathrm{d}\nu = -\mathrm{e}^{-\mathrm{i}kx\cos\theta_0}, \tag{13}$$

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\left\{\Phi_+(\sqrt{(k+\nu)})-\frac{\Psi_+(\sqrt{(k+\nu)})}{\sqrt{(k-\nu)}}\right\}\frac{\mathrm{i}\kappa\,\mathrm{e}^{\mathrm{i}\nu x}}{(\nu+k\cos\theta_0)}\,\mathrm{d}\nu = \mathrm{i}k\sin\theta_0\,\mathrm{e}^{-\mathrm{i}kx\cos\theta_0}. \tag{14}$$

The solution of the equations (13) and (14) is given by the solution of the following auxiliary equations

(*a*)
$$\Phi_+\{\sqrt{(k(1-\cos\theta_0))}\}+\frac{\Psi_+\{\sqrt{(k(1-\cos\theta_0))}\}}{\sqrt{(k(1+\cos\theta_0))}} = 1, \tag{15}$$

$$\Phi_+\{\sqrt{(k(1-\cos\theta_0))}\}-\frac{\Psi_+\{\sqrt{(k(1-\cos\theta_0))}\}}{\sqrt{(k(1+\cos\theta_0))}} = -\operatorname{sgn}(\theta_0); \tag{16}$$

(*b*)
$$\Phi_+(\sqrt{(k+\nu)})+\frac{\Psi_+(\sqrt{(k+\nu)})}{\sqrt{(k-\nu)}} = -Y_-(\nu), \tag{17}$$

$$\left(\Phi_+(\sqrt{(k+\nu)})-\frac{\Psi_+(\sqrt{(k+\nu)})}{\sqrt{(k-\nu)}}\right)\sqrt{(k+\nu)} = \Lambda_-(\nu). \tag{18}$$

The functions $Y_-(\nu)$, $\Lambda_-(\nu)$ are regular and analytic in $\mathrm{Im}(\nu) < k_i$. The equations (*a*) account for the source singularity at $\nu = -k\cos\theta_0$ which is in the lower $\nu$-plane, $\tau > -k_i\cos\theta_0$. The satisfaction of (15) and (16) will cancel the right-hand side of (13) and (14). Adding and subtracting (15) and (16) gives the more compact requirement

$$\Phi_+\{\surd(k(1-\cos\theta_0))\} = \tfrac{1}{2}(1-\mathrm{sgn}(\theta_0)), \tag{19}$$

$$\Psi_+\{\surd(k(1-\cos\theta_0))\} = \surd(k(1+\cos\theta_0)) \times \tfrac{1}{2}(1+\mathrm{sgn}(\theta_0)). \tag{20}$$

The equations (*b*) mean that the left-hand side of (17) and (18) are regular analytic functions in $\mathrm{Im}(\nu) < k_i$. These equations consititute a system of Wiener–Hopf equations of the matrix form

$$\boldsymbol{K}(\nu)\,\boldsymbol{\Phi}_+(\nu) = \boldsymbol{\Phi}_-(\nu).$$

Although Krein & Gohberg (1960) established the theory of simultaneous Wiener–Hopf equations and proved the existence of the factorization of the kernel matrix $\boldsymbol{K}(\nu)$, they were unable to find a general procedure to carry out the factorization. To my knowledge the factorization of square matrices into the form

$$\boldsymbol{K}(\nu) = \boldsymbol{K}_+(\nu)\,\boldsymbol{K}_-(\nu)$$

still remains an unsolved problem. Occasionally one can uncouple the equations and then the problem is reduced to solving two separate standard Wiener–Hopf equations. The system (*b*) considered here cannot be uncoupled. We shall obtain a solution to (*a*) and (*b*) by an *ad-hoc* method which can be considered as a new way of looking at the Wiener–Hopf equations.

The expressions (17) and (18) can be interpreted as meaning that on any closed contour in $\mathrm{Im}(\nu) < k_i$ in one complete circuit the left-hand side of (17) and (18) must return to its original value. Hence

$$\Phi_+(\surd(k+\nu)) + \frac{\Psi_+(\surd(k+\nu))}{\surd(k-\nu)} = \Phi_+(-\surd(k+\nu)) + \frac{\Psi_+(-\surd(k+\nu))}{\surd(k-\nu)}, \tag{21}$$

and
$$\Phi_+(\surd(k+\nu)) - \frac{\Psi_+(\surd(k+\nu))}{\surd(k-\nu)} = -\left(\Phi_+(-\surd(k+\nu)) - \frac{\Psi_+(-\surd(k+\nu))}{\surd(k-\nu)}\right), \tag{22}$$

where we have used the fact that the argument of $\surd(k+\nu)$ on one side of the branch cut $\nu = -k$ differs by $\pi$ from its value on the other side of the branch cut, see appendix A.

In effect the equations (21) and (22) are a statement of the fact that across the branch cut $\nu = -k$ (17) and (18) are regular and analytic. We have, of course, assumed that $\Phi_+(\surd(k+\nu))$, $\Psi_+(\surd(k+\nu))$ have no poles or branch points (other than $\nu = -k$) in $\mathrm{Im}(\nu) < k_i$, and on physical grounds this is a plausible assumption. By adding and subtracting (21) and (22) we see that these equations are satisfied if

$$\Phi_+(\surd(k+\nu)) = \frac{\Psi_+(-\surd(k+\nu))}{\surd(k-\nu)} \tag{23}$$

and
$$\Phi_+(-\surd(k+\nu)) = \frac{\Psi_+(\surd(k+\nu))}{\surd(k-\nu)}.$$

It can be shown that a solution of (23) which satisfies (19) and (20) and also has the correct growth conditions (see (10)) as $|\nu| \to \infty$ is given by (see appendix B)

$$\Phi_+(\sqrt{(k+\nu)}) = \frac{\{\sqrt{(k+\nu)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}\sqrt{\{\sqrt{(2k)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}}}{2\sqrt{(k+\nu)}\,\chi_1(\nu)}, \tag{24}$$

$$\Psi_+(\sqrt{(k+\nu)}) = \frac{\{\sqrt{(k+\nu)} + \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}\,\chi_1(\nu)\sqrt{\{\sqrt{(2k)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}}}{2\sqrt{(k+\nu)}}, \tag{25}$$

where $\chi_1(\nu) = \sqrt{(\sqrt{(2k)} + \sqrt{(k+\nu)})}$. The function $\chi_1(\nu)$ is defined to be such that $\chi_1(0) = \sqrt{(\sqrt{(2k)} + \sqrt{(k)})}$. If $\chi_2(\nu) = \sqrt{(\sqrt{(2k)} - \sqrt{(k+\nu)})}$, and $\chi_2(0) = \sqrt{(\sqrt{(2k)} - \sqrt{k})}$ then on the cut sheet we are using, for all $\nu$, we have the factorization

$$\sqrt{(k-\nu)} = \chi_1(\nu)\,\chi_2(\nu), \tag{26}$$

see appendix A.

By way of checking we shall substitute (24) and (25) directly into the left-hand side of the equation (13) and show that it equals the right-hand side. Thus substituting (24) and (25) into the left-hand side of the equality sign in the expression (13) and letting the result be denoted by $T(x)$ we have

$$\begin{aligned} T(x) = \alpha \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{\mathrm{e}^{\mathrm{i}\nu x}}{(\nu + k\cos\theta_0)} &\left\{ \frac{\sqrt{(k+\nu)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}}{\sqrt{(k+\nu)}\,\sqrt{(\sqrt{(2k)} + \sqrt{(k+\nu)})}} \right. \\ &\left. + \frac{\{\sqrt{(k+\nu)} + \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}\sqrt{(\sqrt{(2k)} + \sqrt{(k+\nu)})}}{\kappa} \right\} \mathrm{d}\nu \\ & -k_{\mathrm{i}}\cos\theta_0 < \tau < k_{\mathrm{i}}, \end{aligned} \tag{27}$$

where
$$\alpha = \frac{1}{4\pi \mathrm{i}}\sqrt{\{\sqrt{(2k)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}}.$$

The evaluation of the integral (27) will be achieved by deforming the path of integration into the lower $\nu$-plane. However, before we can do this we must examine whether or not the function in the curly bracket is regular and analytic in this region. An inspection of the expression in the curly brackets shows that the only possibility of non-regularity is the branch cut at $\nu = -k$ of $\sqrt{(k+\nu)}$. However, if we consider a circuit around the point $\nu = -k$ it will be seen that the term in the curly bracket is regular there. Thus if we start at a point $\nu = -k + \delta\,\mathrm{e}^{\mathrm{i}\vartheta}$, $\delta$ small and fixed, the expression in the curly bracket is

$$\left\{ \frac{\sqrt{(k+\nu)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}}{\sqrt{(k+\nu)}\sqrt{(\sqrt{(2k)} + \sqrt{(k+\nu)})}} + \frac{\{\sqrt{(k+\nu)} + \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}\,\chi_1(\nu)}{\kappa} \right\}_{\vartheta}.$$

In traversing the circle $|\nu + k| = \delta$ from $\nu = -k + \delta\,\mathrm{e}^{\mathrm{i}\vartheta}$ to $\nu = -k + \delta\,\mathrm{e}^{\mathrm{i}(\vartheta + 2\pi)}$, $\sqrt{(k+\nu)}$ changes to $-\sqrt{(k+\nu)}$ and $\sqrt{(k-\nu)}$ is unchanged. Thus the expression in the curly bracket becomes

$$\begin{aligned} &\left\{ \frac{-\sqrt{(k+\nu)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}}{-\sqrt{(k+\nu)}\sqrt{(\sqrt{(2k)} - \sqrt{(k+\nu)})}} \right. \\ &\qquad \left. + \frac{\{-\sqrt{(k+\nu)} - \mathrm{sgn}\,(\theta_0)\sqrt{(k(1-\cos\theta_0))}\}\sqrt{(\sqrt{(2k)} - \sqrt{(k+\nu)})}}{-\kappa} \right\}_{\vartheta + 2\pi}, \end{aligned}$$

which on using the identity (26) gives

$$\left\{\frac{\surd(k+\nu)-\operatorname{sgn}(\theta_0)\surd(k(1-\cos\theta_0))}{\surd(k+\nu)\,\chi_1(\nu)}+\frac{\{\surd(k+\nu)+\operatorname{sgn}(\theta_0)\surd(k(1-\cos\theta_0))\}\,\chi_1(\nu)}{\kappa}\right\}_{\vartheta+2\pi}.$$

Thus we have $\left\{\quad\right\}_{\vartheta}=\left\{\quad\right\}_{\vartheta+2\pi}$.

Hence as far as the function in the curly brackets is concerned no branch cut exists in $\operatorname{Im}(\nu)<\tau$. The contour of integration can now be distorted into the lower half plane and an application of Jordan's Lemma and Cauchy's residue theorem (the pole $\nu=-k\cos\theta_0$ is captured), gives (for each case of $\theta_0>0$ and $\theta_0<0$ considered separately)

$$T(x)=-\mathrm{e}^{-\mathrm{i}kx\cos\theta_0},$$

which agrees with the right-hand side of the equation (13). The satisfaction of the equation (14) follows on exactly the same lines.

Thus the solution to the boundary value problem is given by

$$\begin{aligned}\phi(x,y)&=\phi_0(x,y)\\&\quad+\alpha\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\frac{\mathrm{e}^{\mathrm{i}(\nu x+\kappa|y|)}}{(\nu+k\cos\theta_0)}\left\{\frac{\surd(k-\nu)\{\surd(k+\nu)-\operatorname{sgn}(\theta_0)\surd(k(1-\cos\theta_0))\}}{\chi_1(\nu)}\operatorname{sgn}(y)\right.\\&\quad\left.+\{\surd(k+\nu)+\operatorname{sgn}(\theta_0)\surd(k(1-\cos\theta_0))\}\,\chi_1(\nu)\right\}\frac{\mathrm{d}\nu}{\kappa},\\&\quad-k_1\cos\theta_0<\tau<k_1.\end{aligned}\tag{28}$$

It is worth comparing this result with the incident field diffracted by a soft screen, ($\phi_s(x,y)$, where $\phi_s(x,y)$ vanishes on both sides of the screen) and a hard screen ($\phi_h(x,y)$, where $\partial\phi_h/\partial y(x,y)$ vanishes on both sides of the screen). These are given in Noble (1958) as

$$\phi_s(x,y)=\phi_0(x,y)+\frac{\surd(k(1+\cos\theta_0))}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\frac{\mathrm{e}^{\mathrm{i}(\nu x+\kappa|y|)}}{\surd(k-\nu)\,(\nu+k\cos\theta_0)}\,\mathrm{d}\nu,\tag{29}$$

$$\phi_h(x,y)=\phi_0(x,y)-\frac{\surd(k(1-\cos\theta_0))}{2\pi\mathrm{i}}\operatorname{sgn}(y)\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\frac{\mathrm{e}^{\mathrm{i}(\nu x+\kappa|y|)}}{\surd(k+\nu)\,(\nu+k\cos\theta_0)}\,\mathrm{d}\nu,$$

$$-k_1\cos\theta_0<\tau<k_1.$$

A 'perfectly absorbing' surface is given by, adding the hard and soft surface solutions together, for the same incident wave, and dividing by two, see Butler (1974). Thus the solution to the problem of plane wave diffraction by a semi-infinite plane which is perfectly absorbing on one surface and rigid on the other is given by linear superposition as

$$\phi_A(x,y)=\tfrac{1}{2}\{\phi(x,y)+\phi_h(x,y)\}.\tag{30}$$

## 4. Asymptotic expressions for the far field

Without going through the detailed analysis the asymptotic far field expressions for $\phi(x,y)$, $\phi_s(x,y)$, and $\phi_h(x,y)$ as $kr \to \infty$, are given below. The technique for obtaining these results can be found in Jones's (1964) book. Thus if

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad -\pi < \theta < \pi,$$

$$I = \mathrm{e}^{-\mathrm{i}kr\cos(\theta-\theta_0)}, \qquad R = \mathrm{e}^{-\mathrm{i}kr\cos(\theta+\theta_0)},$$

$$F(Z) = \mathrm{e}^{-\mathrm{i}Z^2}\int_Z^\infty \mathrm{e}^{\mathrm{i}t^2}\,\mathrm{d}t, \quad \text{and} \quad Q = \sqrt{\left(\frac{kr}{2}\right)}\left(\frac{\cos\theta+\cos\theta_0}{\sin\theta}\right),$$

then the *soft–hard far field* for $-\pi < \theta_0 < \pi$ is given by

$$\begin{aligned} \phi(r,\theta) &= I + D_+, && 0 < \theta < \pi-\theta_0, \\ &= I + \operatorname{sgn}(-\theta_0)R + D_+, && \pi-\theta_0 < \theta < \pi, \\ &= I + D_-, && \theta_0-\pi < \theta < 0, \\ &= D_-, && -\pi < \theta < \theta_0-\pi; \end{aligned} \tag{31}$$

where
$$D_\pm = -\frac{\mathrm{e}^{\mathrm{i}(kr-\frac{1}{4}\pi)}}{\sqrt{(2\pi kr)}}\frac{2|Q|F(|Q|)}{(\cos\theta+\cos\theta_0)}\frac{\sqrt{(\sqrt{2}-\operatorname{sgn}(\theta_0)\sqrt{(1-\cos\theta_0)})}}{2}$$
$$\times\left\{(\sqrt{(1+\cos\theta)}+\operatorname{sgn}(\theta_0)\sqrt{(1-\cos\theta_0)})\sqrt{(\sqrt{2}+\sqrt{(1+\cos\theta)})}\right.$$
$$\left.\pm\frac{\sqrt{(1-\cos\theta)}\,(\sqrt{(1+\cos\theta)}-\operatorname{sgn}(\theta_0)\sqrt{(1-\cos\theta_0)})}{\sqrt{(\sqrt{2}+\sqrt{(1+\cos\theta)})}}\right\};$$

*soft far field* for $0 < \theta_0 < \pi$

$$\begin{aligned} \phi_s(r,\theta) &= I + D_s, && \theta_0-\pi < \theta < \pi-\theta_0, \\ &= I - R + D_s, && \pi-\theta_0 < \theta < \pi, \\ &= D_s, && -\pi < \theta < \theta_0-\pi, \end{aligned} \tag{32}$$

where
$$D_s = -\frac{\mathrm{e}^{\mathrm{i}(kr-\frac{1}{4}\pi)}}{\sqrt{(2\pi kr)}}\frac{\sqrt{(1+\cos\theta_0)}\sqrt{(1+\cos\theta)}}{(\cos\theta+\cos\theta_0)}2|Q|F(|Q|);$$

*hard far field* for $0 < \theta_0 < \pi$

$$\begin{aligned} \phi_h(r,\theta) &= I + D_h, && 0 < \theta < \pi-\theta_0, \\ &= I + R + D_h, && \pi-\theta_0 < \theta < \pi, \\ &= I - D_h, && \theta_0-\pi < \theta < 0, \\ &= -D_h, && -\pi < \theta < \theta_0-\pi, \end{aligned} \tag{33}$$

where
$$D_h = \frac{\mathrm{e}^{\mathrm{i}(kr-\frac{1}{4}\pi)}}{\sqrt{(2\pi kr)}}\frac{\sqrt{(1-\cos\theta)}\sqrt{(1-\cos\theta_0)}}{(\cos\theta+\cos\theta_0)}2|Q|F(|Q|).$$

Expressions (31)–(33) are used to give polar diagrams of the scattered far fields for $|\phi(r,\theta)|$, $|\phi_s(r,\theta)|$, and $|\phi_h(r,\theta)|$ when $kr = 10\pi$. For $|\phi(r,\theta)|$ the angle of incidence is taken as $\pm 90°$ which corresponds to the illumination of the soft and hard face respectively; for $|\phi_s(r,\theta)|$ and $|\phi_h(r,\theta)|$ we need only consider an angle of incidence of 90° because of the symmetry of the problems. These plots are given in graphs (1)–(4). A graph showing the attenuation (for each of the above situations) in the shadow region of the screen is also given in graph (5). Graphs (6) and (7) give the attenuation in the shadow region of a perfectly absorbing-hard screen for $\theta_0 = \pm 90°$, and $\theta_0 = \pm 135°$, i.e. $20 \log_{10} |\phi_A|$.

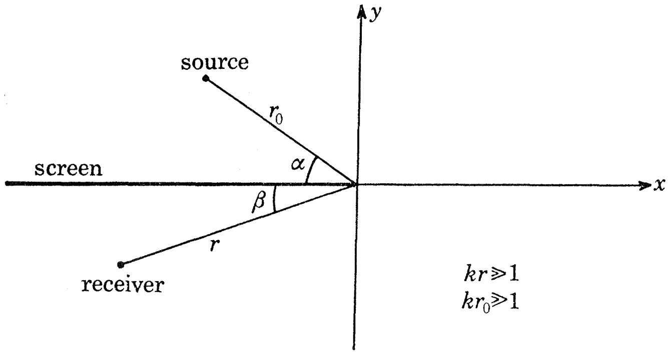
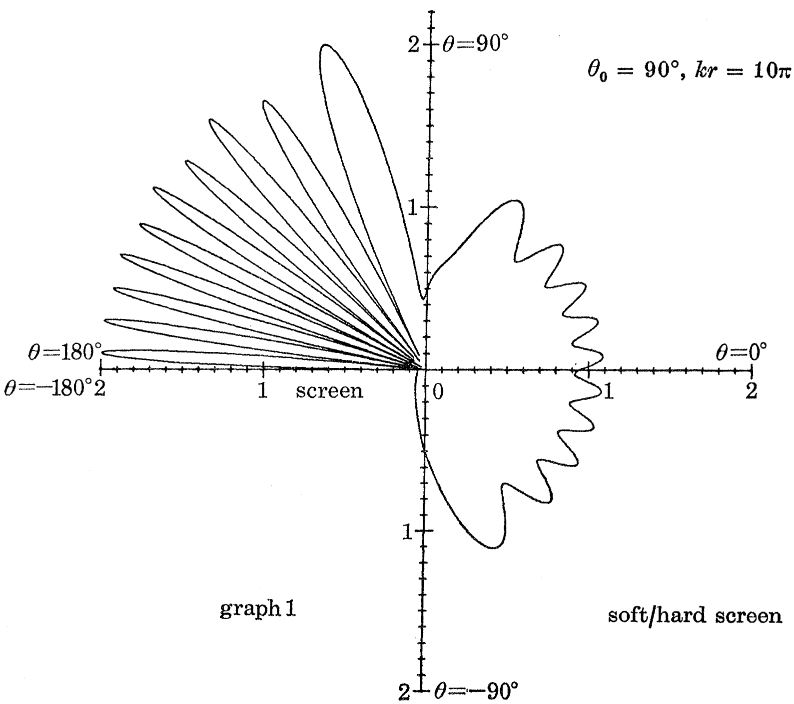
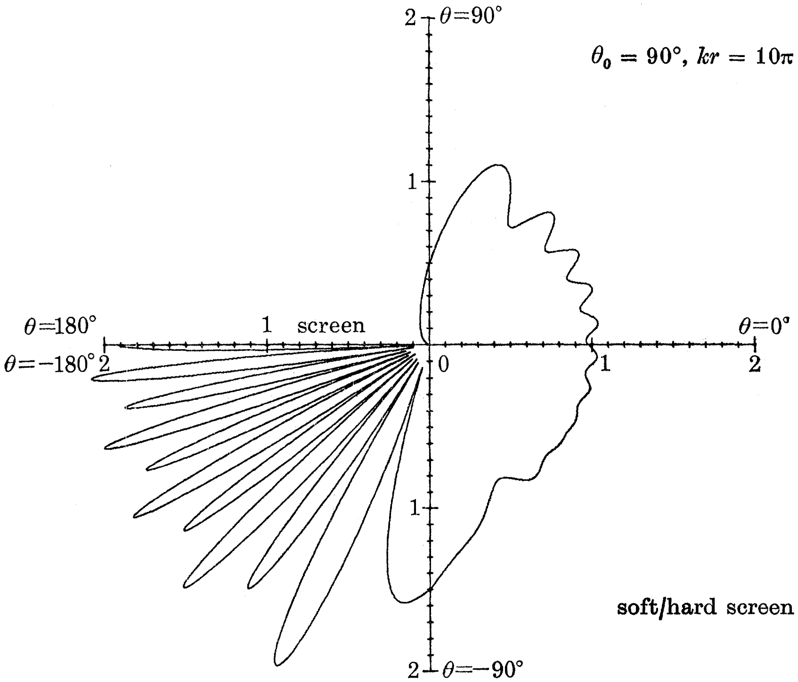
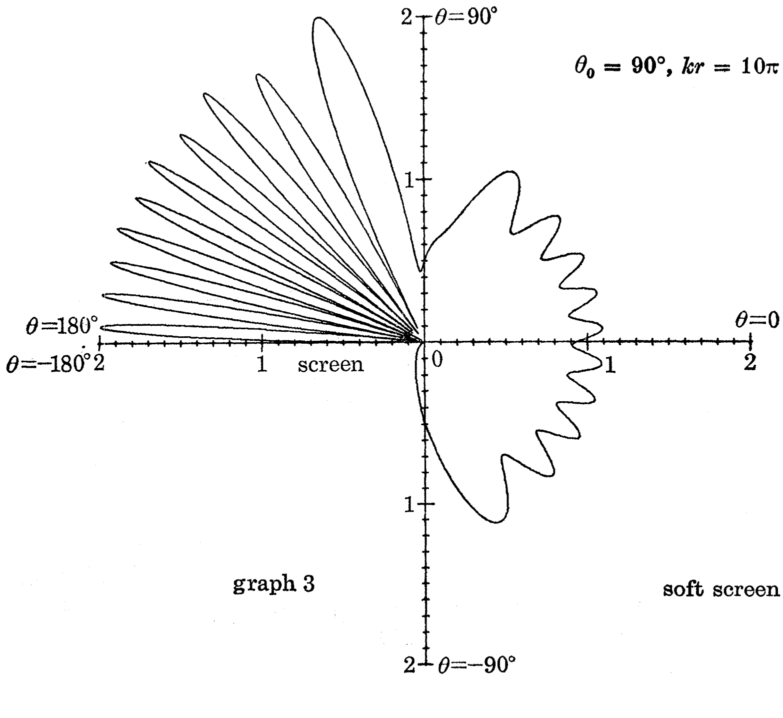



FIGURE 3.

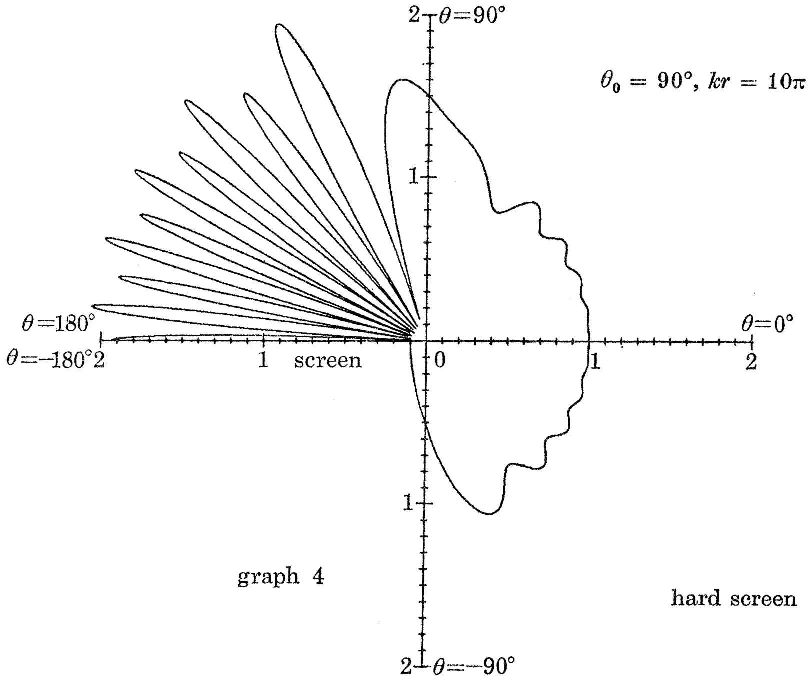




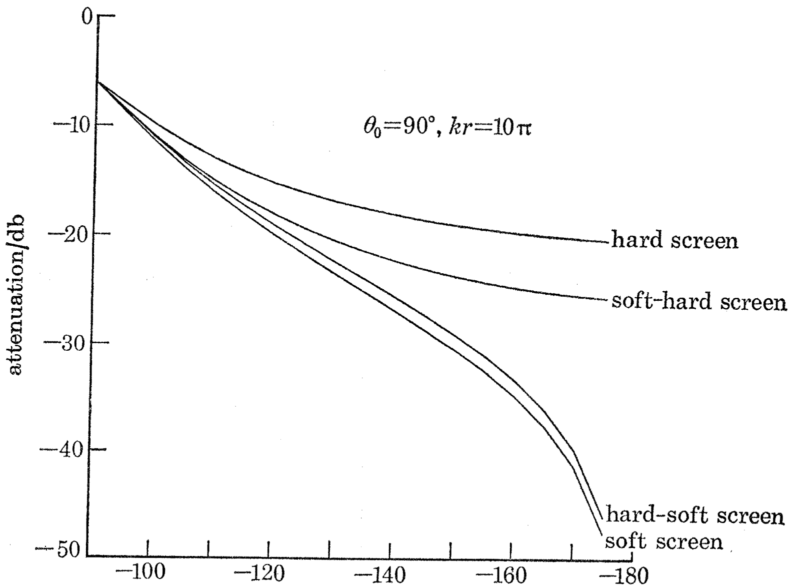

graph 2



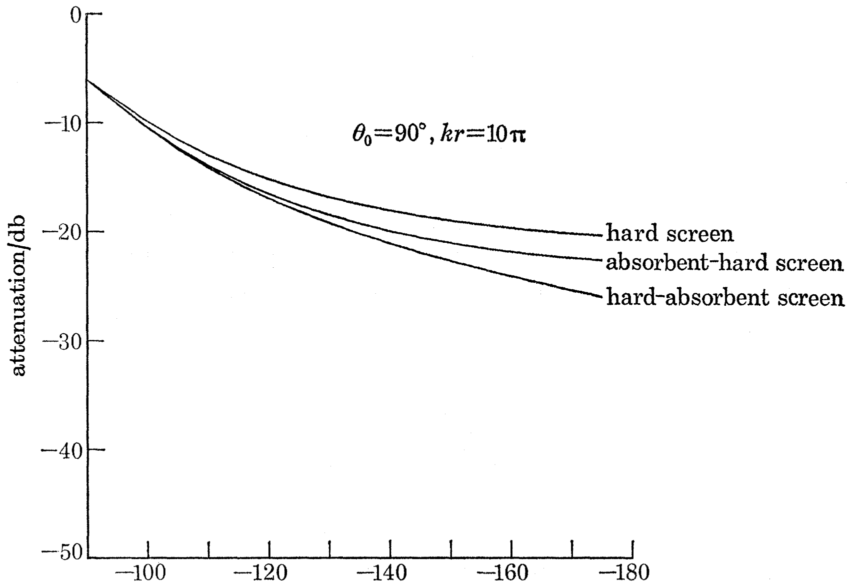

graph 3



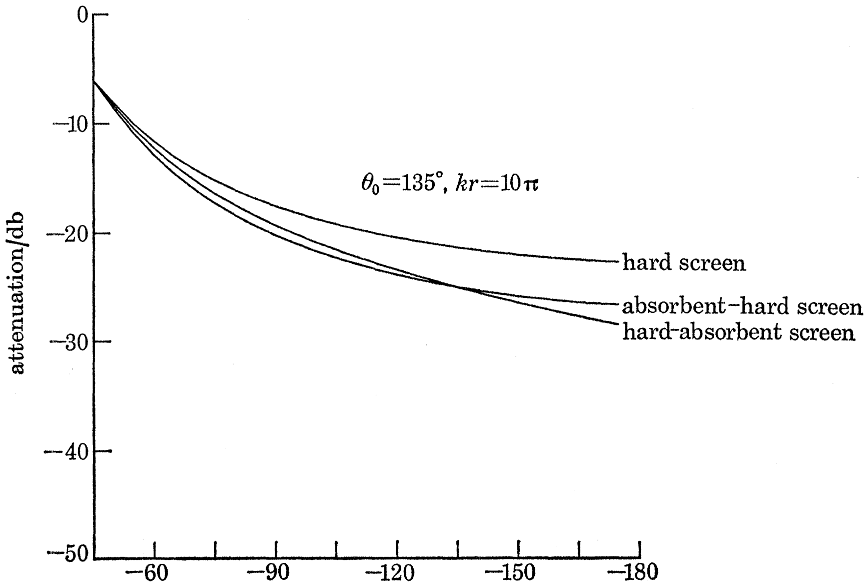

graph 4




graph 5




graph 6




graph 7

## 5. Graphical results

The most important graphical results from the point of noise reduction in the shadow region are graphs (5)–(7). From these graphs it can be seen that a semi-infinite half plane which has a soft or absorbing boundary condition on one side and a rigid boundary condition on the other gives better attenuation in the shadow region than a wholly rigid screen. What is very interesting is the fact that the greatest attenuation of sound in the shadow region is obtained when the soft or absorbing surface of the screen is on the shadow side of the screen for $|\theta| > |\theta_0|$, and on the illuminated side of the screen for $|\theta| < |\theta_0|$. From graphs (1) and (2) it can also be seen that the back scattered sound on the illuminated side of the screen is smallest when the soft surface is illuminated.

## 6. Conclusions

From the graphical results we can see that for the maximum attenuation in the shadow region of the barrier, the absorbing lining should be placed on the surface which makes the smallest angle from the barrier to the source or receiver. Thus in figure **3** if $\alpha < \beta$ then the absorbent material should be place on the source side of the barrier. If $\alpha > \beta$ then it should be place on the receiver side. When $\alpha = \beta$ the lining can be placed on either side. This latter case is a consequence of reciprocity.

Some experimental work has been carried out by Fleischer (1970) when an absorbing lining is on the illuminated side of the barrier. He recorded an increase in attenuation of the order 0–8 dB in the illuminated region and 0–5 dB in the shadow region of the screen. The graph (5) gives much higher levels of attenuation than the experimental results. However this is to be expected because no practical surface is completely soft. The graphs (6) and (7) give much more realistic levels of attenuation. The experimental results are consistent with the theoretical upper bounds predicted in graphs (6) and (7) for a perfectly absorbing surface.

## Appendix A

We shall prove that on the cut sheet for which $\kappa = \sqrt{(k^2-\nu^2)}$ equals $k$ when $\nu = 0$, and $\chi_1(0) = \sqrt{(\sqrt{(2k)}+\sqrt{k})}$, $\chi_2(0) = \sqrt{(\sqrt{(2k)}-\sqrt{k})}$ then the expression (26) is valid for all $\nu$.

*Proof.* If $\kappa_+(\nu) = \sqrt{(k+\nu)}$, and $\kappa_-(\nu) = \sqrt{(k-\nu)}$ then from the way we have defined $\kappa$ on this cut sheet we also have

$$\kappa_+(0) = k^{\frac{1}{2}}, \qquad \kappa_-(0) = k^{\frac{1}{2}};$$

$$\kappa_+(\sigma \pm \mathrm{i}0) = |\kappa_+(\sigma)|, \qquad \kappa_-(\sigma \pm \mathrm{i}0) = \mp \mathrm{i}\,|\kappa_-(\sigma)|;$$

as $\sigma \to +\infty$

$$\kappa_+(0 \pm \mathrm{i}\tau) = |\kappa_+(\pm \mathrm{i}\tau)|\,\mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}, \qquad \kappa_-(0 \pm \mathrm{i}\tau) = |\kappa_-(\pm \mathrm{i}\tau)|\,\mathrm{e}^{\mp\frac{1}{4}\mathrm{i}\pi};$$

as $\tau \to \infty$

$$\kappa_+(-\sigma \pm \mathrm{i}0) = \pm \mathrm{i}\,|\kappa_+(-\sigma)|, \qquad \kappa_-(-\sigma \pm \mathrm{i}0) = |\kappa_-(-\sigma)|; \tag{A 1}$$

$$\text{as } \sigma \to \infty.$$

From the way we have defined $\chi_1(\nu)$ and $\chi_2(\nu)$ at $\nu = 0$ on can obtain by analytic continuation the following results

$$\chi_1(0) = \sqrt{(\sqrt{(2k)} + \sqrt{k})}, \qquad \chi_2(0) = \sqrt{(\sqrt{(2k)} - \sqrt{k})};$$

$$\chi_1(\sigma \pm \mathrm{i}0) = |\chi_1(\sigma)|, \qquad \chi_2(\sigma \pm \mathrm{i}0) = \mp \mathrm{i}\,|\chi_2(\sigma)|;$$

$$\text{as } \sigma \to \infty$$

$$\chi_1(0 \pm \mathrm{i}\tau) = |\chi_1(\pm \mathrm{i}\tau)|\,\mathrm{e}^{\pm\frac{1}{8}\mathrm{i}\pi}, \qquad \chi_2(0 \pm \mathrm{i}\tau) = |\chi_2(\pm \mathrm{i}\tau)|\,\mathrm{e}^{\mp\frac{3}{8}\mathrm{i}\pi};$$

$$\text{as } \tau \to \infty$$

$$\chi_1(-\sigma \pm \mathrm{i}0) = |\chi_1(-\sigma)|\,\mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}, \qquad \chi_2(-\sigma \pm \mathrm{i}0) = |\chi_2(-\sigma)|\,\mathrm{e}^{\mp\frac{1}{4}\mathrm{i}\pi};$$

$$\text{as } \sigma \to \infty. \tag{A 2}$$

From (A 1) and (A 2) it can be seen that $\kappa_-(\nu) = \chi_1(\nu)\,\chi_2(\nu)$.

## Appendix B

The solution of the equations (23) is equivalent to the solution of the solution of the equation

$$\Phi_+(\gamma) = \frac{\Psi_+(-\gamma)}{\sqrt{(2k - \gamma^2)}}, \tag{B 1}$$

where $\gamma = \sqrt{(k + \nu)}$; the square root quantities being defined as in appendix A. We now show how the solutions (24) and (25) are obtained in an *ad-hoc* manner from the equation (B 1).

We may write (B 1) in the form

$$\sqrt{(\sqrt{(2k)} + \gamma)}\,\Phi_+(\gamma) = \frac{\Psi_+(-\gamma)}{\sqrt{(\sqrt{(2k)} - \gamma)}}, \tag{B 2}$$

thence by inspection a particular solution of the equation (B 1) is given by

$$\Phi_+(\gamma) = \frac{1}{\sqrt{(\sqrt{(2k)} + \gamma)}}, \quad \Psi_+(\gamma) = \sqrt{(\sqrt{(2k)} + \gamma)}. \tag{B 3}$$

We now look for a general solution which will take account of the edge condition (see (10)) and the incident wave pole effect (see (19) and (20)). Such a solution is given by the expressions

$$\Phi_+(\gamma) = \frac{f_+(\gamma)}{\sqrt{(\sqrt{(2k)} + \gamma)}}, \quad \Psi_+(\gamma) = f_+(-\gamma)\sqrt{(\sqrt{(2k)} + \gamma)} \tag{B 4}$$

where
$$f_+(\gamma) = \sum_{n=-\infty}^{\infty} A_n \gamma^n. \tag{B 5}$$

We know from the edge conditions that $A_n \equiv 0$ for $n \leqslant -2$, and also since there are only two equations to satisfy for the incident wave pole effect $A_n \equiv 0$, $n \geqslant 1$. Thus the general solution is assumed to take the form

$$\Phi_+(\gamma) = \left(\frac{A_{-1}+A_0\gamma}{\gamma}\right)\frac{1}{\sqrt{(\sqrt{(2k)}+\gamma)}}, \quad \Psi_+(\gamma) = \left(\frac{A_{-1}-A_0\gamma}{-\gamma}\right)\sqrt{(\sqrt{(2k)}+\gamma)}.$$

The two unknown coefficients $A_{-1}$ and $A_0$ are obtained from the equations (19) and (20).

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