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Eigenvalues of Ruijsenaars–Schneider models associated with A_{n-1} root system in Bethe ansatz formalism

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Ruijsenaars–Schneider models associated with A_{n-1} root system with a discrete *coupling constant* are studied. The eigenvalues of the Hamiltonian are given in terms of the Bethe ansatz formulas. Taking the “nonrelativistic” limit, we obtain the spectrum of the corresponding Calogero–Moser systems in the *third formulas* of Felder *et al.* © 2004 American Institute of Physics.

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I. INTRODUCTION

Ruijsenaars–Schneider (RS) models^{1,2} are integrable generalization of Calogero–Moser (CM) models^{3,4} at both classical and quantum levels. The integrability of classical RS models associated with various root systems were studied by Lax pair representation for A_{n-1} ,¹ for D_n ,⁵ for C_n and BC_n with one coupling constant.⁶ The commuting conserved quantities for quantum RS models were discussed in Refs. 2, 7, 8, and 9.

It is well known that the Hamiltonian of RS model with degenerate potentials (trigonometric and rational ones) is one of the commuting families of Macdonald operators,¹⁰ which are also called *Ruijsenaars–Macdonald operators*. The eigenfunctions of the *degenerate* Hamiltonian are given by so-called *Macdonald polynomials*.¹⁰ However, an analogous construction for elliptic generalization of *Macdonald polynomials* is still an open problem.

Bethe ansatz method has proved to be the most powerful and (probably) unified method to construct the common eigenvectors of commuting families of operators (usually called *transfer matrices*) in two-dimensional trigonometric and rational integrable models.^{11–15} Recently, after a definition of elliptic quantum groups $E_{\tau,\eta}(\mathfrak{g})$ associated with any simple classical Lie algebra \mathfrak{g} was given,¹⁶ the algebraic Bethe ansatz method has been successfully extended. The method for construction of the eigenvectors of the transfer matrices associated with the module over $E_{\tau,\eta}(sl_2)$ (Ref. 17) is now generalized to apply for those associated with the module over $E_{\tau,\eta}(sl_n)$ with generic n .^{18,19}

In particular, the elliptic *Ruijsenaars–Macdonald operator* (3.7) with a discrete *coupling constant* $\gamma = \sqrt{-1}gl$ (l being a non-negative integer) associated with A_{n-1} root system can be rewritten as the *transfer matrices* associated with the symmetric $n \times l$ tensor product evaluation $E_{\tau,\eta}(sl_n)$ module.²⁰ This enables us to obtain the eigenvalues of the Hamiltonian of elliptic RS models with the discrete *coupling constant* $\gamma = \sqrt{-1}gl$ associated with A_{n-1} root system by the algebraic Bethe ansatz for elliptic quantum group $E_{\tau,\eta}(sl_n)$.

The paper is organized as follows. In Sec. II, we give a brief review of the algebraic Bethe ansatz for elliptic quantum group $E_{\tau,\eta}(sl_n)$ developed in Ref. 19. In Sec. III, choosing the special $E_{\tau,\eta}(sl_n)$ -module $W = V_{\Lambda(n)}(0)$, we give the eigenvalues of elliptic RS model with the discrete *coupling constant* $\gamma = \sqrt{-1}gl$ and the associated Bethe ansatz equations. Taking the degenerate

(trigonometric and rational) limit, we obtain the eigenvalues of Hamiltonian of the degenerate RS models associated with A_{n-1} root system. In Sec. IV, taking the “nonrelativistic” limit,²¹ we obtain the eigenvalues of elliptic, trigonometric and rational types of CM models associated with A_{n-1} root system with an integer coupling constant $\gamma=l+1$ in the Bethe ansatz formulas or the third formulas in the sense of Felder *et al.*²²

II. ALGEBRAIC BETHE ANSATZ FOR ELLIPTIC QUANTUM GROUP $E_{\tau,\eta}(sl_n)$

A. Elliptic quantum group associated with A_{n-1}

We first review the definition of the elliptic quantum group $E_{\tau,\eta}(sl_n)$ associated with A_{n-1} .¹⁶ Let $\{\epsilon_i|i=1,2,\dots,n\}$ be the orthonormal basis of the vector space \mathbb{C}^n such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. The A_{n-1} simple roots are $\{\alpha_i = \epsilon_i - \epsilon_{i+1}|i=1,\dots,n-1\}$ and the fundamental weights $\{\Lambda_i|i=1,\dots,n-1\}$ satisfying $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ are given by

$$\Lambda_i = \sum_{k=1}^i \epsilon_k - \frac{i}{n} \sum_{k=1}^n \epsilon_k.$$

Set

$$\hat{i} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{n} \sum_{k=1}^n \epsilon_k, \quad i = 1, \dots, n, \quad \text{then } \sum_{i=1}^n \hat{i} = 0. \tag{2.1}$$

For each dominant weight $\Lambda = \sum_{i=1}^{n-1} a_i \Lambda_i$, $a_i \in \mathbb{Z}^+$, there exists an irreducible highest weight finite-dimensional representation V_Λ of A_{n-1} with the highest vector $|\Lambda\rangle$. For example, the fundamental vector representation is V_{Λ_1} . In this paper, we consider only the symmetric tensor-product representation of $\overbrace{V_{\Lambda_1} \otimes V_{\Lambda_1} \cdots \otimes V_{\Lambda_1}}^l$ (or, the higher spin- l representation of A_{n-1}), namely, the one parameter series of highest weight representations $V_{\Lambda^{(l)}}$, with

$$\Lambda^{(l)} = l\Lambda_1, \quad l \in \mathbb{Z} \quad \text{and } l > 0. \tag{2.2}$$

This corresponds to the Young diagram $\overbrace{\square \square \cdots \square}^l$.

Let \mathfrak{h} be the Cartan subalgebra of A_{n-1} and \mathfrak{h}^* be its dual. A finite dimensional diagonalizable \mathfrak{h} -module is a complex finite dimensional vector space W with a weight decomposition $W = \bigoplus_{\mu \in \mathfrak{h}^*} W[\mu]$, so that \mathfrak{h} acts on $W[\mu]$ by $xv = \mu(x)v$, ($x \in \mathfrak{h}, v \in W[\mu]$). For example, the fundamental vector representation $V_{\Lambda_1} = \mathbb{C}^n$, the nonzero weight spaces $W[i] = \mathbb{C}\epsilon_i$, $i = 1, \dots, n$.

Let us fix τ such that $\text{Im}(\tau) > 0$ and a generic complex number η . For convenience, we introduce another parameter $w = n\eta$ related to η . Let us introduce the following elliptic functions:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (u, \tau) = \sum_{m=-\infty}^{\infty} \exp\{\sqrt{-1} \pi[(m+a)^2 \tau + 2(m+a)(u+b)]\}, \tag{2.3}$$

$$\sigma(u) = \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u, \tau), \quad \zeta(u) = \partial_u \{\ln \sigma(u)\}, \quad \wp(u) = -\partial_u \{\zeta(u)\}. \tag{2.4}$$

These functions have the following properties:

$$\sigma(u) = 0 + u\sigma'(0) + \frac{u^3}{6} \sigma'''(0) + \dots, \quad \text{when } u \rightarrow 0, \tag{2.5}$$

$$\sigma(-u) = -\sigma(u), \quad \zeta(-u) = -\zeta(u), \quad \wp(-u) = \wp(u), \tag{2.6}$$

where $\sigma'(0) = \partial_u \{\sigma(u)\}|_{u=0}$ and $\sigma'''(0) = \partial_u^3 \{\sigma(u)\}|_{u=0}$.

For a generic $\lambda \in \mathbb{C}^n$, define

$$\lambda_i = \langle \lambda, \epsilon_i \rangle, \quad \lambda_{ij} = \lambda_i - \lambda_j = \langle \lambda, \epsilon_i - \epsilon_j \rangle, \quad i, j = 1, \dots, n. \tag{2.7}$$

Let $R(z, \lambda) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the R -matrix given by

$$R(z, \lambda) = \sum_{i=1}^n R_{ii}^{ii}(z, \lambda) E_{ii} \otimes E_{ii} + \sum_{i \neq j} \{R_{ij}^{ij}(z, \lambda) E_{ii} \otimes E_{jj} + R_{ji}^{ji}(z, \lambda) E_{ji} \otimes E_{ij}\}, \tag{2.8}$$

in which E_{ij} is the matrix with elements $(E_{ij})_k^l = \delta_{jk} \delta_{il}$. The coefficient functions are

$$R_{ii}^{ii}(z, \lambda) = 1, \quad R_{ij}^{ij}(z, \lambda) = \frac{\sigma(z) \sigma(\lambda_{ij} + w)}{\sigma(z + w) \sigma(\lambda_{ij})}, \tag{2.9}$$

$$R_{ij}^{ji}(z, \lambda) = \frac{\sigma(w) \sigma(z + \lambda_{ij})}{\sigma(z + w) \sigma(\lambda_{ij})}, \tag{2.10}$$

and λ_{ij} is defined in (2.7). The R -matrix satisfies the dynamical (modified) quantum Yang–Baxter equation.

$$\begin{aligned} R_{12}(z_1 - z_2, \lambda - wh^{(3)}) R_{13}(z_1 - z_3, \lambda) R_{23}(z_2 - z_3, \lambda - wh^{(1)}) \\ = R_{23}(z_2 - z_3, \lambda) R_{13}(z_1 - z_3, \lambda - wh^{(2)}) R_{12}(z_1 - z_2, \lambda), \end{aligned} \tag{2.11}$$

with the initial condition

$$R_{ij}^{kl}(0, \lambda) = \delta_i^l \delta_j^k. \tag{2.12}$$

We adopt the notation $R_{12}(z, \lambda - wh^{(3)})$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R(z, \lambda - w\mu) \otimes Id$ if $v_3 \in W[\mu]$.

A representation of the elliptic quantum group $E_{\tau, \eta}(sl_n)$ [an $E_{\tau, \eta}(sl_n)$ -module] is by definition a pair (W, L) where W is a diagonalizable \mathfrak{h} -module and $L(z, \lambda)$ is a meromorphic function of λ and the spectral parameter $z \in \mathbb{C}$, with values in $\text{End}_{\mathfrak{h}}(\mathbb{C}^n \otimes W)$ (the endomorphisms commuting with the action of \mathfrak{h}). It obeys the so-called RLL relation

$$R_{12}(z_1 - z_2, \lambda - wh^{(3)}) L_{13}(z_1, \lambda) L_{23}(z_2, \lambda - wh^{(1)}) = L_{23}(z_2, \lambda) L_{13}(z_1, \lambda - wh^{(2)}) R_{12}(z_1 - z_2, \lambda), \tag{2.13}$$

where the first and second space are auxiliary spaces (\mathbb{C}^n) and the third space plays the role of the quantum space (W). The total weight conservation condition for the L -operator reads

$$[h^{(1)} + h^{(3)}, L_{13}(z, \lambda)] = 0.$$

In terms of the elements of the L -operator defined by

$$L(z, \lambda)(\epsilon_i \otimes v) = \sum_{j=1}^n \epsilon_j \otimes L_i^j(z, \lambda)v, \quad v \in W, \tag{2.14}$$

the above condition can be expressed equivalently as

$$f(h)L_i^j(z, \lambda) = L_i^j(z, \lambda)f(h + \hat{i} - \hat{j}), \tag{2.15}$$

in which $f(h)$ is any meromorphic function of h and h measures the weight of the quantum space (W).

B. Modules over $E_{\tau,\eta}(sl_n)$ and the associated operator algebra

The basic example of an $E_{\tau,\eta}(sl_n)$ -module is (\mathbb{C}^n, L) with $L(z, \lambda) = R(z - z_1, \lambda)$, which is called the fundamental vector representation $V_{\Lambda_1}(z_1)$ with the evaluation point z_1 . It is obvious that the *RLL* relation is satisfied as a consequence of the dynamical Yang–Baxter equation (2.11). Other modules can be obtained by taking tensor products: if $(W_1, L^{(1)})$ and $(W_2, L^{(2)})$ are $E_{\tau,\eta}(sl_n)$ -modules, where $L^{(j)}$ acts on $(\mathbb{C}^n \otimes W_j)$, then also $(W_1 \otimes W_2, L)$ with

$$L(z, \lambda) = L^{(1)}(z, \lambda - wh^{(2)})L^{(2)}(z, \lambda) \quad \text{acting on } \mathbb{C}^n \otimes W_1 \otimes W_2. \tag{2.16}$$

An $E_{\tau,\eta}(sl_n)$ -submodule of an $E_{\tau,\eta}(sl_n)$ -module (W, L) is a pair (W_1, L_1) where W_1 is an \mathfrak{h} -submodule of W such that $\mathbb{C}^n \otimes W_1$ is invariant under the action of all the $L(z, \lambda)$, and $L_1(z, \lambda)$ is the restriction to this invariant subspace. Namely, the $E_{\tau,\eta}(sl_n)$ -submodules are $E_{\tau,\eta}(sl_n)$ -modules.

Using the fusion rule of $E_{\tau,\eta}(sl_n)$ (2.16) one can construct the symmetric $E_{\tau,\eta}(sl_n)$ -submodule of l -tensors of fundamental vector representations:

$$V_{\Lambda^{(l)}}(z_1) = \text{symmetric subspace of } V_{\Lambda_1}(z_1) \otimes V_{\Lambda_1}(z_1 - w) \otimes \cdots \otimes V_{\Lambda_1}(z_1 - (l-1)w),$$

where $\Lambda^{(l)}$ is defined by (2.2). We call such an $E_{\tau,\eta}(sl_n)$ -module the *higher spin- l representation* with the evaluation point z_1 . These series of representations in the case of \mathbb{Z}_n Sklyanin algebra have been studied in Ref. 23 for $n=2$ case and in Refs. 24 and 25 for generic n case.

For any $E_{\tau,\eta}(sl_n)$ -module, as in Ref. 17 one can define an associated operator algebra of difference operators on the space $\text{Fun}(W)$ of meromorphic functions of λ with values in W . The algebra is generated by h and the operators $\tilde{L}(z) \in \text{End}(\mathbb{C}^n \otimes \text{Fun}(W))$ acting as

$$\tilde{L}(z)(\epsilon_i \otimes f)(\lambda) = \sum_{j=1}^n \epsilon_j \otimes L_i^j(z, \lambda) f(\lambda - w\hat{i}). \tag{2.17}$$

One can derive the following exchange relation of the difference operator $\tilde{L}(z)$ from the *RLL* relation (2.13), the weight conservation condition $L_i^j(z, \lambda)$ (2.15) and the fact $[h^{(1)} + h^{(2)}, R_{12}(z, \lambda)] = 0$,

$$R_{12}(z_1 - z_2, \lambda - wh) \tilde{L}_{13}(z_1) \tilde{L}_{23}(z_2) = \tilde{L}_{23}(z_2) \tilde{L}_{13}(z_1) R_{12}(z_1 - z_2, \lambda), \tag{2.18}$$

$$f(h) \tilde{L}_i^j(z) = \tilde{L}_i^j(z) f(h + \hat{i} - \hat{j}), \tag{2.19}$$

where $f(h)$ is any meromorphic function of h . Let $W = V_{\Lambda^{(l_1)}}(z_1) \otimes V_{\Lambda^{(l_2)}}(z_2) \otimes \cdots \otimes V_{\Lambda^{(l_m)}}(z_m)$ and $\Lambda = \Lambda^{(l_1)} + \cdots + \Lambda^{(l_m)}$, then $W[\Lambda] = \mathbb{C}|\Lambda\rangle$ with $|\Lambda\rangle = |\Lambda^{(l_1)}\rangle \otimes \cdots \otimes |\Lambda^{(l_m)}\rangle$.

Theorem 1 (Ref. 19): *With generic evaluation points $\{z_i\}$, W is an irreducible highest weight $E_{\tau,\eta}(sl_n)$ -module and the vector $|\Lambda\rangle$, viewed as a constant function in $\text{Fun}(W)$, obeys the following highest weight conditions:*

$$\tilde{L}_1^1(z)|\Lambda\rangle = A(z, \lambda)|\Lambda\rangle, \quad \tilde{L}_1^i(z)|\Lambda\rangle = 0, \quad i = 2, \dots, n,$$

$$\tilde{L}_j^i(z)|\Lambda\rangle = \delta_j^i D_i(z, \lambda)|\Lambda\rangle, \quad i, j = 2, \dots, n, \quad f(h)|\Lambda\rangle = f(N\hat{1})|\Lambda\rangle.$$

The highest weight functions read

$$A(z, \lambda) = 1, \quad D_i(z, \lambda) = \left\{ \prod_{k=1}^m \frac{\sigma(z - p_k)}{\sigma(z - q_k)} \right\} \frac{\sigma(\lambda_{i1} + Nw)}{\sigma(\lambda_{i1})}, \quad i = 2, \dots, n, \tag{2.20}$$

where

$$p_k = z_k, \quad q_k = z_k - l_k w, \quad N = \sum_{k=1}^m l_k, \quad k = 1, \dots, m. \tag{2.21}$$

The *transfer matrices* associated with an $E_{\tau, \eta}(sl_n)$ -module (W, L) (Ref. 17) are difference operators acting on the space $\text{Fun}(W)[0]$ of meromorphic functions of λ with values in the zero-weight space of W . They are defined by

$$T(u)f(\lambda) = \sum_{i=1}^n \tilde{L}_i^i(u)f(\lambda) = \sum_{i=1}^n L_i^i(u, \lambda)f(\lambda - w\hat{i}). \tag{2.22}$$

The exchange relations of \tilde{L} -operators (2.18) and (2.19) imply that, for any $E_{\tau, \eta}(sl_n)$ -module, the above transfer matrices preserve the space $\mathcal{H} = \text{Fun}(W)[0]$. Moreover, they commute pairwise on $\mathcal{H}: [T(u), T(v)]|_{\mathcal{H}} = 0$.

C. Algebraic Bethe ansatz for $E_{\tau, \eta}(sl_n)$

We fix a highest weight $E_{\tau, \eta}(sl_n)$ -module W of weight Λ , the functions $A(z, \lambda)$, $D_i(z, \lambda)$ (2.20), with the highest weight vector $|\Lambda\rangle$. We assume that $N = \sum_{k=1}^m l_k = n \times l$ with l being an integer, so that the zero-weight space $W[0]$ can be nontrivial and that the algebraic Bethe ansatz method can be applied as in Refs. 26, 27, 28, 17, and 19.

Let us adopt the standard notation for convenience:

$$\mathcal{A}(u) = \tilde{L}_1^1(u), \quad \mathcal{B}_i(u) = \tilde{L}_i^1(u), \quad i = 2, \dots, n, \tag{2.23}$$

$$\mathcal{C}_i(u) = \tilde{L}_1^i(u), \quad \mathcal{D}_i^j(u) = \tilde{L}_i^j(u), \quad i, j = 2, \dots, n. \tag{2.24}$$

The transfer matrices $T(u)$ become

$$T(u) = \mathcal{A}(u) + \sum_{i=2}^n \mathcal{D}_i^i(u). \tag{2.25}$$

Any nonzero vector $|\Omega\rangle \in \text{Fun}(W)[\Lambda]$ is of form $|\Omega\rangle = g(\lambda)|\Lambda\rangle$, for some meromorphic function $g \neq 0$. When $N = n \times l$, the weight Λ can be written in the form

$$\Lambda = nl\Lambda_1 = l \sum_{k=1}^{n-1} (\epsilon_1 - \epsilon_{k+1}). \tag{2.26}$$

Noting (2.19), the zero-weight vector space is spanned by the vectors of the following form:

$$\mathcal{B}_{i_{N_1}}(v_{N_1}) \mathcal{B}_{i_{N_1-1}}(v_{N_1-1}) \cdots \mathcal{B}_{i_1}(v_1) |\Omega\rangle, \tag{2.27}$$

where $N_1 = (n-1) \times l$ and among the indices $\{i_k | k = 1, \dots, N_1\}$, the number of $i_k = j$, denoted by $\#(j)$, should be

$$\#(j) = l, \quad j = 2, \dots, n. \tag{2.28}$$

The above states (2.27) actually belong to the zero-weight space $W[0]$.¹⁹ Let us introduce the following set of integers for convenience:

$$N_i = (n-i) \times l, \quad i = 1, 2, \dots, n-1, \tag{2.29}$$

and $[n(n-1)/2]l$ parameters $\{\{v_k^{(i)} | k = 1, 2, \dots, N_{i+1}\}, i = 0, 1, \dots, n-2\}$. The parameters $\{\{v_k^{(i)}\}\}$ will be specified later by the Bethe ansatz equations (2.40). We will seek the common eigenvectors of the *transfer matrices* $T(u)$ in the form

$$|\lambda; \{v_k\}\rangle = \sum_{i_1, \dots, i_{N_1}=2}^n F^{i_1, i_2, \dots, i_{N_1}}(\lambda; \{v_k\}) \mathcal{B}_{i_{N_1}}(v_{N_1}) \mathcal{B}_{i_{N_1-1}}(v_{N_1-1}) \cdots \mathcal{B}_{i_1}(v_1) |\Omega\rangle, \quad (2.30)$$

with the restriction condition (2.28). We adopt hereafter the convention

$$v_k = v_k^{(0)}, \quad k = 1, 2, \dots, (n-1) \times l. \quad (2.31)$$

Let us introduce n parameters $\{\alpha^{(i)} | i = 1, \dots, n\}$ to specify quasivacua of each step of the nested Bethe ansatz,¹⁹ and another set of parameters related to them,

$$\bar{\alpha}^{(i)} = \frac{1}{(n-i-1)} \left\{ \alpha^{(i+1)} - \frac{\sum_{k=i+1}^n \alpha^{(k)}}{n-i} \right\}, \quad i = 0, \dots, n-2. \quad (2.32)$$

Choosing the function of $g(\lambda)$,

$$g(\lambda) = e^{\sqrt{-1}\pi(\alpha^{(1)} \epsilon_1, \lambda)} \prod_{j=2}^n \left\{ \prod_{k=1}^l \frac{\sigma(\lambda_{j1} + kw)}{\sigma(w)} \right\}, \quad (2.33)$$

then we have the following.

Theorem 2 (Ref. 19): *With properly chosen coefficients $F^{i_1, i_2, \dots, i_{N_1}}(\lambda; \{v_k\})$, we obtain eigenvectors of the transfer matrices*

$$T(u) |\lambda; \{v_k\}\rangle = t(u; \{v_k\}) |\lambda; \{v_k\}\rangle, \quad (2.34)$$

with the eigenvalue

$$t(u; \{v_k\}) = e^{\sqrt{-1}\pi(1-n)\bar{\alpha}w} \left\{ \prod_{k=1}^{N_1} \frac{\sigma(v_k - u + w)}{\sigma(v_k - u)} \right\} + e^{\sqrt{-1}\pi\bar{\alpha}w} \left\{ \prod_{k=1}^{N_1} \frac{\sigma(u - v_k + w)}{\sigma(u - v_k)} \right\} \\ \times \left\{ \prod_{k=1}^m \frac{\sigma(u - p_k)}{\sigma(u - q_k)} \right\} t^{(1)}(u; \{v_k^{(1)}\}). \quad (2.35)$$

The functions $t^{(i)}(u; \{v_k^{(i)}\})$ are given recursively,

$$t^{(i)}(u; \{v_k^{(i)}\}) = e^{\sqrt{-1}\pi(i+1-n)\bar{\alpha}^{(i)}w} \left\{ \prod_{k=1}^{N_{i+1}} \frac{\sigma(v_k^{(i)} - u + w)}{\sigma(v_k^{(i)} - u)} \right\} + e^{\sqrt{-1}\pi\bar{\alpha}^{(i)}w} \left\{ \prod_{k=1}^{N_{i+1}} \frac{\sigma(u - v_k^{(i)} + w)}{\sigma(u - v_k^{(i)})} \right\} \\ \times \left\{ \prod_{k=1}^{N_i} \frac{\sigma(u - p_k^{(i)})}{\sigma(u - q_k^{(i)})} \right\} t^{(i+1)}(u; \{v_k^{(i+1)}\}), \quad i = 0, 1, \dots, n-2, \quad (2.36)$$

$$t^{(n-1)}(u) = 1, \quad t^{(0)}(u; \{v_k^{(0)}\}) = t(u; \{v_k\}), \quad (2.37)$$

where $\bar{\alpha}^{(i)}$, $i = 0, 1, \dots, n-2$ are given by (2.32), $\bar{\alpha}^{(0)} = \bar{\alpha}$, $N_0 = m$ and

$$p_k^{(0)} = p_k = z_k, \quad q_k^{(0)} = q_k = z_k - l_k w, \quad k = 1, 2, \dots, m, \quad (2.38)$$

$$p_k^{(i)} = v_k^{(i-1)}, \quad q_k^{(i)} = v_k^{(i-1)} - w, \quad i = 1, 2, \dots, n-2, \quad k = 1, 2, \dots, N_i. \quad (2.39)$$

The $\{\{v_k^{(i)}\}\}$ satisfy the following Bethe ansatz equations:

$$e^{\sqrt{-1}\pi(i-n)\bar{\alpha}^{(i)}w} \left\{ \prod_{k=1, k \neq s}^{N_{i+1}} \frac{\sigma(v_k^{(i)} - v_s^{(i)} + w)}{\sigma(v_k^{(i)} - v_s^{(i)} - w)} \right\} = \left\{ \prod_{k=1}^{N_i} \frac{\sigma(v_s^{(i)} - p_k^{(i)})}{\sigma(v_s^{(i)} - q_k^{(i)})} \right\} t^{(i+1)}(v_s^{(i)}; \{v_k^{(i+1)}\}). \tag{2.40}$$

In principle one can construct explicit expression of the coefficients of $F^{i_1, i_2, \dots, i_{N_i}}(\lambda; \{v_k\})$ (for details we refer the reader to Ref. 19).

We conclude this section with some remarks on functional dependence of the states $|\lambda; \{v_k\}\rangle$. By construction (2.23), the operators $\{\mathcal{B}_i\}$ are the functions of $\{\lambda_i - \lambda_j\}$ because of the definition of the R -matrix (2.9)–(2.10), and the states can be written in the following form:

$$|\lambda; \{v_k\}\rangle = \exp\left\{ \sum_{i=1}^n \sqrt{-1} \pi \alpha^{(i)} \lambda_i \right\} \overline{|\lambda; \{v_k\}\rangle}. \tag{2.41}$$

Here $\overline{|\lambda; \{v_k\}\rangle}$ is a meromorphic function of $\{\lambda_i\}$ and has the following properties:

$$\overline{|\lambda_1 + c, \dots, \lambda_n + c; \{v_k\}\rangle} = \overline{|\lambda_1, \dots, \lambda_n; \{v_k\}\rangle}, \quad \text{for } \forall c \in \mathbb{C}, \tag{2.42}$$

$$\overline{|\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_n; \{v_k\}\rangle} = (-1)^{l(n-1)} \overline{|\lambda_1, \dots, \lambda_n; \{v_k\}\rangle}, \quad i = 1, \dots, n. \tag{2.43}$$

III. RUIJSENAARS–SCHNEIDER MODELS ASSOCIATED WITH A_{n-1} ROOT SYSTEM

A. The elliptic case

Let us choose an $E_{\tau, \eta}(sl_n)$ -module W , a special one which corresponds to the Young diagram $\overbrace{\square \square \dots \square}^{nl}$,

$$W = V_{\Lambda^{(nl)}}(0), \tag{3.1}$$

in which the evaluation point z_1 is set to 0. Then the zero-weight space of this module is one-dimensional: $|\lambda; \{v_k\}\rangle = \tilde{\Phi}_{RS}(\lambda; \{\alpha^{(i)}\}) e_0$, $e_0 \in W[0]$ and it does not depend on λ_i . The associated *transfer matrices* can be written as²⁰

$$T(u) = \frac{\sigma(u + lw)}{\sigma(u + nlw)} M. \tag{3.2}$$

The operator M is independent of u and is given by

$$M = \sum_{i=1}^n \left\{ \prod_{j \neq i} \frac{\sigma(\lambda_{ij} + lw)}{\sigma(\lambda_{ij})} \Gamma_i \right\}. \tag{3.3}$$

Here $\{\Gamma_i\}$ are difference operators: $\Gamma_i f(\lambda) = f(\lambda - w\hat{i})$. Noting (2.41)–(2.43) and (3.14), we find

$$M \tilde{\Phi}_{RS} = \tilde{H}_{RS} \tilde{\Phi}_{RS} = \epsilon_{RS} \tilde{\Phi}_{RS}. \tag{3.4}$$

The difference operator \tilde{H}_{RS} is given by

$$\tilde{H}_{RS} = \sum_{i=1}^n \left\{ \prod_{j \neq i} \frac{\sigma(\lambda_{ij} + lw)}{\sigma(\lambda_{ij})} e^{-w(\partial/\partial \lambda_i)} \right\}. \tag{3.5}$$

In order to apply Theorem 2 to RS model, hereafter we restrict the parameters τ and w as follows:

$$\tau = \sqrt{-1} \kappa, \quad \kappa \in \mathbb{R}, \quad \kappa > 0, \quad w = \sqrt{-1} g, \tag{3.6}$$

where g is a real number. This is necessary for the reality of the Hamiltonian. Because the parameters $\{\lambda_i\}$ will play the role of the canonical coordinates, we further restrict $\lambda_i \in \mathbb{R}$. In terms of the specified parameters, \tilde{H}_{RS} becomes

$$\tilde{H}_{RS} = \sum_{i=1}^n \left\{ \prod_{j \neq i} \frac{\sigma(\lambda_{ij} + \sqrt{-1}gl)}{\sigma(\lambda_{ij})} e^{-\sqrt{-1}g(\partial/\partial\lambda_i)} \right\}. \tag{3.7}$$

The resulting difference operator \tilde{H}_{RS} will be the Hamiltonian of elliptic A_{n-1} type RS model² with the special coupling constant $\gamma = \sqrt{-1}gl$, up to conjugation by a function.^{25,29} Suppose $\tilde{\Phi}_{RS}$ and ϵ_{RS} are an eigenfunction and the corresponding eigenvalue of \tilde{H}_{RS} ,

$$\tilde{H}_{RS}\tilde{\Phi}_{RS} = \epsilon_{RS}\tilde{\Phi}_{RS}. \tag{3.8}$$

Let us introduce another function Φ_{RS} ,

$$\Phi_{RS} = e^{-\Psi_{RS}}\tilde{\Phi}_{RS}, \quad \Psi_{RS} = \frac{1}{2} \ln \prod_{i \neq j} \left\{ \prod_{k=1}^l \frac{\sigma(\lambda_{ij} - \sqrt{-1}gk)}{\sigma(\sqrt{-1}g)} \right\}, \tag{3.9}$$

associated to $\tilde{\Phi}_{RS}$. Then Φ_{RS} is an eigenfunction of the similarity transformed Hamiltonian H_{RS} with the same eigenvalue ϵ_{RS} ,

$$H_{RS}\Phi_{RS} = \epsilon_{RS}\Phi_{RS}, \quad H_{RS} = e^{-\Psi_{RS}}\tilde{H}_{RS}e^{\Psi_{RS}}, \tag{3.10}$$

$$H_{RS} = \sum_{i=1}^n \left\{ \prod_{j \neq i} \frac{\sigma(\lambda_{ji} - \sqrt{-1}gl)}{\sigma(\lambda_{ji})} \right\}^{1/2} e^{-\sqrt{-1}g(\partial/\partial\lambda_i)} \left\{ \prod_{j \neq i} \frac{\sigma(\lambda_{ij} - \sqrt{-1}gl)}{\sigma(\lambda_{ij})} \right\}^{1/2}. \tag{3.11}$$

One finds that H_{RS} is the Hamiltonian of elliptic A_{n-1} type RS model² with the special coupling constant $\gamma = \sqrt{-1}gl$.

Now, we consider the spectrum of \tilde{H}_{RS} . Theorem 2 enables us to obtain the spectrum of the Hamiltonian of the elliptic A_{n-1} Ruijsenaars–Schneider model as well as the eigenfunctions, in terms of the associated transfer matrices (3.2). Since we have already taken the special $E_{\tau,\gamma}(sl_n)$ -module $W = V_{\Lambda(n \times l)}(0)$, thanks to Theorem 2, the eigenvalues are given by (2.34) and (2.35) but with special values of the parameters

$$m = N_0 = 1, \quad p_1^{(0)} = z_1 = 0, \quad q_1^{(0)} = -\sqrt{-1}gnl. \tag{3.12}$$

Since M is independent of u , we can evaluate the eigenvalues of $T(u)$ at $u = z_1 = 0$. Then the expression of the eigenvalue $t(u; \{v_k\})$ simplifies drastically, for the second term on the right-hand side of (2.35) [the one depending on the eigenvalue of the reduced transfer matrices $t^{(1)} \times (u; \{v_k^{(1)}\})$] vanishes because $u - p_1^{(0)} = 0$.

Note that $\tilde{H}_{RS}(H_{RS})$ has periodic coefficients with $\lambda_i \rightarrow \lambda_i + 1$, and therefore preserves the space of Bloch functions such that

$$\psi(\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_n) = \pm (-1)^{l(n-1)} \psi(\lambda_1, \dots, \lambda_n). \tag{3.13}$$

The (quasi)periodicity requires integer $\alpha^{(i)}$, $\alpha^{(i)} \in \mathbb{Z}$. Noting the relation (2.32), in order to get one-to-one correspondence between $\{\alpha^{(i)}\}$ and $\{\bar{\alpha}^{(i)}\}$, we need further choose

$$\alpha^{(i)} \in \mathbb{Z}^+, \quad i = 1, \dots, n-1, \quad \text{and} \quad \alpha^{(n)} = -\sum_{k=1}^{n-1} \alpha^{(k)}. \tag{3.14}$$

Finally, we obtain the eigenvalues $\epsilon_{RS}(\{v_k\})$ of \tilde{H}_{RS} (3.4):

$$e^{\pi(n-1)\bar{\alpha}g} \frac{\sigma(\sqrt{-1}gnl)}{\sigma(\sqrt{-1}gl)} \left\{ \prod_{k=1}^{(n-1) \times l} \frac{\sigma(v_k + \sqrt{-1}g)}{\sigma(v_k)} \right\}, \tag{3.15}$$

where $\{v_k^{(i)}\}$ satisfy the Bethe ansatz equations

$$e^{\pi(n-i)\bar{\alpha}^{(i)}g} \left\{ \prod_{k=1, k \neq s}^{N_{i+1}} \frac{\sigma(v_k^{(i)} - v_s^{(i)} + \sqrt{-1}g)}{\sigma(v_k^{(i)} - v_s^{(i)} - \sqrt{-1}g)} \right\} = \left\{ \prod_{k=1}^{N_i} \frac{\sigma(v_s^{(i)} - p_k^{(i)})}{\sigma(v_s^{(i)} - q_k^{(i)})} \right\} t^{(i+1)}(v_s^{(i)}; \{v_k^{(i+1)}\}),$$

$$i = 0, 1, \dots, n-2. \tag{3.16}$$

The functions $t^{(i)}(u)$ appearing in (3.16) are given by the same recurrence relations as (2.36)–(2.37), but with the special $N_0=1$, $p_1^{(0)}=0$, and $q_1^{(0)}=-\sqrt{-1}gnl$ and replacing w by $\sqrt{-1}g$. Substituting the expression of the function $t^{(i+1)}(u)$ (2.36) into the Bethe ansatz equations (3.16), noting the conditions (3.12) and (2.39), we have the following.

Proposition 1: The eigenvalues of the Hamiltonian (3.11) of the elliptic RS model associated with A_{n-1} root system with the discrete coupling constant $\gamma = \sqrt{-1}gl$ (l being an integer) are

$$\epsilon_{RS} = e^{\pi(n-1)\bar{\alpha}g} \frac{\sigma(\sqrt{-1}gnl)}{\sigma(\sqrt{-1}gl)} \left\{ \prod_{k=1}^{(n-1) \times l} \frac{\sigma(v_k + \sqrt{-1}g)}{\sigma(v_k)} \right\}. \tag{3.17}$$

The $[n(n-1)/2]l$ parameters $\{v_k^{(i)}\}$ satisfy the Bethe ansatz equations

$$e^{\pi n \bar{\alpha} g} \prod_{k=1, k \neq s}^{N_1} \frac{\sigma(v_k - v_s + \sqrt{-1}g)}{\sigma(v_k - v_s - \sqrt{-1}g)} = e^{\pi(n-2)\bar{\alpha}^{(1)}g} \frac{\sigma(v_s)}{\sigma(v_s + \sqrt{-1}gnl)} \prod_{k=1}^{N_2} \frac{\sigma(v_k^{(1)} - v_s + \sqrt{-1}g)}{\sigma(v_k^{(1)} - v_s)}, \tag{3.18}$$

$$e^{\pi(n-i)\bar{\alpha}^{(i)}g} \prod_{k=1, k \neq s}^{N_{i+1}} \frac{\sigma(v_k^{(i)} - v_s^{(i)} + \sqrt{-1}g)}{\sigma(v_k^{(i)} - v_s^{(i)} - \sqrt{-1}g)} = e^{\pi(n-i-2)\bar{\alpha}^{(i+1)}g} \prod_{k=1}^{N_i} \frac{\sigma(v_s^{(i)} - v_k^{(i-1)})}{\sigma(v_s^{(i)} - v_k^{(i-1)} + \sqrt{-1}g)}$$

$$\times \prod_{k=1}^{N_{i+2}} \frac{\sigma(v_k^{(i+1)} - v_s^{(i)} + \sqrt{-1}g)}{\sigma(v_k^{(i+1)} - v_s^{(i)})}, \quad i = 1, \dots, n-3, \tag{3.19}$$

$$e^{2\pi\bar{\alpha}^{(n-2)}g} \prod_{k=1, k \neq s}^{N_{n-1}} \frac{\sigma(v_k^{(n-2)} - v_s^{(n-2)} + \sqrt{-1}g)}{\sigma(v_k^{(n-2)} - v_s^{(n-2)} - \sqrt{-1}g)} = \prod_{k=1}^{N_{n-2}} \frac{\sigma(v_s^{(n-2)} - v_k^{(n-3)})}{\sigma(v_s^{(n-2)} - v_k^{(n-3)} + \sqrt{-1}g)}. \tag{3.20}$$

The parameters $\bar{\alpha}^{(i)}$, $i = 0, 1, \dots, n-2$ and $\bar{\alpha}^{(0)} = \bar{\alpha}$ are given by the relation (2.32) from $n-1$ arbitrary non-negative integers $\{\alpha^{(i)} \in \mathbb{Z}^+ | i = 1, \dots, n-1\}$.

Our formula of eigenvalues is the elliptic generalization of the *third formulas* in the sense of Felder *et al.*²² Taking complex conjugation of the Bethe ansatz equations (3.18)–(3.20), noting the property (A5), we find that the solutions $\{v_k^{(i)}\}$ are all pure imaginary numbers. This ensures that the eigenvalues ϵ_{RS} are real. By construction from the nested Bethe ansatz method and the relation (3.9), we know that the corresponding eigenfunction is a meromorphic function of $\{\lambda_i\}$ and satisfies the following properties:

$$\Phi_{RS}(\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_n; \{\alpha^{(i)}\}) = (-1)^{\alpha^{(i)}} \Phi_{RS}(\lambda; \{\alpha^{(i)}\}). \tag{3.21}$$

B. The trigonometric case

Here we consider the trigonometric RS model associated with A_{n-1} root system. The corresponding Hamiltonian with the discrete coupling constant $\gamma = \sqrt{-1}gl$ is given

$$H_{RS} = \sum_{i=1}^n \left\{ \prod_{j \neq i} \frac{\sin \pi(\lambda_{ji} - \sqrt{-1}gl)}{\sin \pi(\lambda_{ji})} \right\}^{1/2} e^{-\sqrt{-1}g(\partial/\partial \lambda_i)} \left\{ \prod_{j \neq i} \frac{\sin \pi(\lambda_{ij} - \sqrt{-1}gl)}{\sin \pi(\lambda_{ij})} \right\}^{1/2}. \tag{3.22}$$

Taking the trigonometric limit $\kappa \rightarrow +\infty$ ($\tau \rightarrow +\sqrt{-1}\infty$), one finds that

$$\frac{\sigma(x)}{\sigma(y)} \rightarrow \frac{\sin \pi x}{\sin \pi y}, \quad \text{when } \kappa \rightarrow +\infty, \tag{3.23}$$

from the product expression (A1) of the σ -function. The trigonometric Hamiltonian (3.22) can be obtained from the elliptic one (3.11) by taking limit $\kappa \rightarrow +\infty$. Consequently, we can find the spectrum of the Hamiltonian of the trigonometric RS model associated with A_{n-1} type root system by taking the trigonometric limit of the elliptic one. Noting that the solutions to the Bethe ansatz equations (3.18)–(3.20) $\{\{v_k^{(i)}\}\}$ are all pure imaginary numbers, let us introduce $[n(n-1)/2]l$ real parameters $\{\{\bar{v}_k^{(i)}\}\}$ associated to $[n(n-1)/2]l$ pure imaginary parameters $\{\{v_k^{(i)}\}\}$ as follows:

$$v_k^{(i)} = \sqrt{-1}\bar{v}_k^{(i)}, \quad \bar{v}_k^{(i)} \in \mathbb{R}. \tag{3.24}$$

Finally, we have

Proposition 2: The eigenvalues of the Hamiltonian (3.22) of the trigonometric RS model associated with A_{n-1} root system with the discrete coupling constant $\gamma = \sqrt{-1}gl$ are

$$\epsilon_{RS} = e^{\pi(n-1)\bar{\alpha}g} \frac{\sinh \pi(nlg)}{\sinh \pi(lg)} \left\{ \prod_{k=1}^{(n-1) \times l} \frac{\sinh \pi(\bar{v}_k^{(0)} + g)}{\sinh \pi(\bar{v}_k^{(0)})} \right\}. \tag{3.25}$$

The $[n(n-1)/2]l$ real parameters $\{\{\bar{v}_k^{(i)}\}\}$ satisfy the Bethe ansatz equations

$$e^{\pi n \bar{\alpha} g} \left\{ \prod_{k=1, k \neq s}^{N_1} \frac{\sinh \pi(\bar{v}_k^{(0)} - \bar{v}_s^{(0)} + g)}{\sinh \pi(\bar{v}_k^{(0)} - \bar{v}_s^{(0)} - g)} \right\} = e^{\pi(n-2)\bar{\alpha}^{(1)}g} \frac{\sinh \pi(\bar{v}_s^{(0)})}{\sinh \pi(\bar{v}_s^{(0)} + nlg)} \times \left\{ \prod_{k=1}^{N_2} \frac{\sinh \pi(\bar{v}_k^{(1)} - \bar{v}_s^{(0)} + g)}{\sinh \pi(\bar{v}_k^{(1)} - \bar{v}_s^{(0)})} \right\}, \tag{3.26}$$

$$e^{\pi(n-i)\bar{\alpha}^{(i)}g} \left\{ \prod_{k=1, k \neq s}^{N_{i+1}} \frac{\sinh \pi(\bar{v}_k^{(i)} - \bar{v}_s^{(i)} + g)}{\sinh \pi(\bar{v}_k^{(i)} - \bar{v}_s^{(i)} - g)} \right\} = e^{\pi(n-i-2)\bar{\alpha}^{(i+1)}g} \left\{ \prod_{k=1}^{N_i} \frac{\sinh \pi(\bar{v}_s^{(i)} - \bar{v}_k^{(i-1)})}{\sinh \pi(\bar{v}_s^{(i)} - \bar{v}_k^{(i-1)} + g)} \right\} \times \left\{ \prod_{k=1}^{N_{i+2}} \frac{\sinh \pi(\bar{v}_k^{(i+1)} - \bar{v}_s^{(i)} + g)}{\sinh \pi(\bar{v}_k^{(i+1)} - \bar{v}_s^{(i)})} \right\}, \tag{3.27}$$

$i = 1, \dots, n-3,$

$$e^{2\pi\bar{\alpha}^{(n-2)}g} \left\{ \prod_{k=1, k \neq s}^{N_{n-1}} \frac{\sinh \pi(\bar{v}_k^{(n-2)} - \bar{v}_s^{(n-2)} + g)}{\sinh \pi(\bar{v}_k^{(n-2)} - \bar{v}_s^{(n-2)} - g)} \right\} = \left\{ \prod_{k=1}^{N_{n-2}} \frac{\sinh \pi(\bar{v}_s^{(n-2)} - \bar{v}_k^{(n-3)})}{\sinh \pi(\bar{v}_s^{(n-2)} - \bar{v}_k^{(n-3)} + g)} \right\}. \tag{3.28}$$

Here the parameters $\bar{\alpha}^{(i)}$, $i = 0, 1, \dots, n-2$ and $\bar{\alpha} = \bar{\alpha}^0$ are given by the relation (2.32) from $n-1$ arbitrary non-negative integers $\{\alpha^{(i)} \in \mathbb{Z}^+ | i = 1, \dots, n-1\}$.

Our formula of eigenvalues is the trigonometric generalization of the third formulas in the sense of Felder *et al.*²² Similarly to the elliptic case, we know that the corresponding eigenfunction has the same quasiperiodic properties (3.21).

C. The rational case

The Hamiltonian of the rational RS model associated with A_{n-1} root system reads as

$$H_{RS} = \sum_{i=1}^n \left\{ \prod_{j \neq i} \left(1 - \frac{\sqrt{-1}gl}{\lambda_{ji}} \right) \right\}^{1/2} e^{-\sqrt{-1}g(\partial/\partial\lambda_i)} \left\{ \prod_{j \neq i} \left(1 - \frac{\sqrt{-1}gl}{\lambda_{ij}} \right) \right\}^{1/2}. \tag{3.29}$$

If one rescales

$$\lambda_i \rightarrow \delta\lambda_i, \quad \bar{\alpha}^{(i)} \rightarrow \frac{1}{\pi\delta} \bar{\alpha}^{(i)}, \quad g \rightarrow \delta g, \tag{3.30}$$

$$\bar{v}_k^{(i)} \rightarrow \delta \bar{v}_k^{(i)}, \tag{3.31}$$

and takes the limit: $\delta \rightarrow 0$ (we call it the rational limit), the Hamiltonian (3.29) of the rational RS model can be obtained from the trigonometric one (3.22). Therefore we can find the spectrum of the Hamiltonian of the rational RS model associated with A_{n-1} root system by taking the rational limit of Proposition 2. Finally we have the following.

Proposition 3: The eigenvalues of the Hamiltonian (3.29) of the rational RS model associated with A_{n-1} root system with the discrete coupling constant $\gamma = \sqrt{-1}gl$ are

$$\epsilon_{RS} = n e^{(n-1)\bar{\alpha}g} \left\{ \prod_{k=1}^{(n-1)l} \frac{\bar{v}_k^{(0)} + g}{\bar{v}_k^{(0)}} \right\}. \tag{3.32}$$

The $[n(n-1)/2]l$ real parameters $\{\{\bar{v}_k^{(i)}\}\}$ satisfy the Bethe ansatz equations

$$e^{n\bar{\alpha}g} \left\{ \prod_{k=1, k \neq s}^{N_1} \frac{(\bar{v}_k^{(0)} - \bar{v}_s^{(0)} + g)}{(\bar{v}_k^{(0)} - \bar{v}_s^{(0)} - g)} \right\} = e^{(n-2)\bar{\alpha}^{(1)}g} \frac{\bar{v}_s^{(0)}}{\bar{v}_s^{(0)} + nl} \left\{ \prod_{k=1}^{N_2} \frac{(\bar{v}_k^{(1)} - \bar{v}_s + g)}{(\bar{v}_k^{(1)} - \bar{v}_s)} \right\}, \tag{3.33}$$

$$e^{(n-i)\bar{\alpha}^{(i)}g} \left\{ \prod_{k=1, k \neq s}^{N_{i+1}} \frac{(\bar{v}_k^{(i)} - \bar{v}_s^{(i)} + g)}{(\bar{v}_k^{(i)} - \bar{v}_s^{(i)} - g)} \right\} = e^{(n-i-2)\bar{\alpha}^{(i+1)}g} \left\{ \prod_{k=1}^{N_i} \frac{(\bar{v}_s^{(i)} - \bar{v}_k^{(i-1)})}{(\bar{v}_s^{(i)} - \bar{v}_k^{(i-1)} + g)} \right\} \\ \times \left\{ \prod_{k=1}^{N_{i+2}} \frac{(\bar{v}_k^{(i+1)} - \bar{v}_s^{(i)} + g)}{(\bar{v}_k^{(i+1)} - \bar{v}_s^{(i)})} \right\}, \quad i = 1, \dots, n-3, \tag{3.34}$$

$$e^{2\bar{\alpha}^{(n-2)}g} \left\{ \prod_{k=1, k \neq s}^{N_{n-1}} \frac{(\bar{v}_k^{(n-2)} - \bar{v}_s^{(n-2)} + g)}{(\bar{v}_k^{(n-2)} - \bar{v}_s^{(n-2)} - g)} \right\} = \left\{ \prod_{k=1}^{N_{n-2}} \frac{(\bar{v}_s^{(n-2)} - \bar{v}_k^{(n-3)})}{(\bar{v}_s^{(n-2)} - \bar{v}_k^{(n-3)} + g)} \right\}. \tag{3.35}$$

Here the parameters $\bar{\alpha}^{(i)}$, $i = 0, 1, \dots, n-2$ and $\bar{\alpha}^{(0)} = \bar{\alpha}$ are given by the relation (2.32) from $n-1$ arbitrary non-negative real numbers $\{\alpha^{(i)} \in \mathbb{R}^+ | i = 1, \dots, n-1\}$.

Our formula of eigenvalues is the rational generalization of the *third formulas* in the sense of Felder *et al.*²² The rescalings (3.30) and (3.31) and the rational limit lead to that the coefficients of the Hamiltonian (3.29) are no longer periodic (cf. the elliptic and trigonometric case). The (quasi)periodic properties (3.21) of the eigenfunction now are replaced by the following asymptotic properties:

$$\Phi_{RS} \propto e^{\sqrt{-1}\alpha^i \lambda_i}, \quad \lambda_i \rightarrow \infty. \tag{3.36}$$

Namely, the corresponding eigenfunctions are bounded when $\lambda_i \rightarrow \infty$ on the real axis.

IV. CALOGERO–MOSER SYSTEMS ASSOCIATED WITH A_{n-1} ROOT SYSTEM

In this section, we will study all types of CM models associated with A_{n-1} root system by taking “nonrelativistic” limit:²¹ $g \rightarrow 0$ of the corresponding RS models which have already been studied in Sec. III.

A. Elliptic potential

Taking the *nonrelativistic limit* of the Hamiltonian (3.7) and noting the asymptotic properties of σ -function (2.5), we obtain

$$\tilde{H}_{RS} = n + \frac{g^2}{2} \tilde{H}_{CM} + O(g^3), \quad g \rightarrow 0. \tag{4.1}$$

The resulting differential operator \tilde{H}_{CM} is given by

$$\tilde{H}_{CM} = - \sum_{i=1}^n \frac{\partial^2}{(\partial \lambda_i)^2} + 2l \sum_{i \neq j}^n \zeta(\lambda_{ij}) \frac{\partial}{\partial \lambda_i} - l^2 \sum_{i \neq j}^n \frac{\sigma''(\lambda_{ij})}{\sigma(\lambda_{ij})} - l^2 \sum_{i \neq j \neq k}^n \zeta(\lambda_{ij}) \zeta(\lambda_{ik}), \tag{4.2}$$

where $\sigma''(t) = [\partial^2 / (\partial u)^2] \sigma(u)|_{u=t}$ and the function ζ is defined in (2.4). We can further transform \tilde{H}_{CM} to a more familiar form. Let us suppose $\tilde{\Phi}$ and ϵ_{CM} are an eigenfunction and the corresponding eigenvalue of \tilde{H}_{CM} , namely,

$$\tilde{H}_{CM} \tilde{\Phi} = \epsilon_{CM} \tilde{\Phi}. \tag{4.3}$$

At the same time, we introduce another function Φ ,

$$\Phi = e^{-\Psi} \tilde{\Phi}, \quad \Psi = \ln \prod_{i < j} (\sigma(\lambda_{ij}))^l, \tag{4.4}$$

associated to $\tilde{\Phi}$. Then Φ is an eigenfunction of the Hamiltonian H_{CM} with the same eigenvalue ϵ_{CM} ,

$$H_{CM} \Phi = \epsilon_{CM} \Phi, \quad H_{CM} = e^{-\Psi} \tilde{H}_{CM} e^{\Psi} = - \sum_{i=1}^n \frac{\partial^2}{(\partial \lambda_i)^2} + l(l+1) \sum_{i \neq j}^n \wp(\lambda_{ij}), \tag{4.5}$$

where the function \wp is defined in (2.4). One finds that H_{CM} is exactly the Hamiltonian of elliptic CM model associated with A_{n-1} root system^{3,4} with the *coupling constant* $l+1$. (Traditionally, the *coupling constant* of the Hamiltonian: $H_{CM} = - \sum_{i=1}^n [\partial^2 / (\partial \lambda_i)^2] + \gamma(\gamma-1) \sum_{i \neq j} \wp(\lambda_{ij})$ of CM model is set to γ .)

Now we study the asymptotic properties of the eigenvalues of \tilde{H}_{RS} (3.17) and the associated Bethe ansatz equations (3.18)–(3.20). Let the solution to the Bethe ansatz equations (3.18)–(3.20) have the following form:

$$v_k^{(i)} = x_k^{(i)} + \sqrt{-1} g y_k^{(i)} - g^2 z_k^{(i)} + O(g^3), \quad g \rightarrow 0. \tag{4.6}$$

Noting the asymptotic properties of σ -function (2.5), the Eq. (3.17) becomes

$$\begin{aligned} \epsilon_{RS} = n + \sqrt{-1} g n \left((1-n) \sqrt{-1} \pi \bar{\alpha} + \sum_{k=1}^{N_1} \zeta(x_k^{(0)}) \right) + \frac{g^2}{2} \left\{ n \sum_{k=1}^{N_1} (2y_k^{(0)} + 1) \wp(x_k^{(0)}) \right. \\ \left. - \frac{(n+1)n(n-1)l^2}{3} \frac{\sigma'''(0)}{\sigma'(0)} - n \left((1-n) \sqrt{-1} \pi \bar{\alpha} + \sum_{k=1}^{N_1} \zeta(x_k^{(0)}) \right)^2 \right\} + O(g^3). \end{aligned} \tag{4.7}$$

The Bethe ansatz equations (3.18)–(3.20) at the first order of g become

$$2 \sum_{k=1, k \neq s}^{N_1} \zeta(x_k^{(0)} - x_s^{(0)}) - n\sqrt{-1} \pi \bar{\alpha} = (2-n)\sqrt{-1} \pi \bar{\alpha}^{(1)} - nl\zeta(x_s^{(0)}) + \sum_{k=1}^{N_2} \zeta(x_k^{(1)} - x_s^{(0)}), \tag{4.8}$$

$$2 \sum_{k=1, k \neq s}^{N_{i+1}} \zeta(x_k^{(i)} - x_s^{(i)}) + (i-n)\sqrt{-1} \pi \bar{\alpha}^{(i)} = (i+2-n)\sqrt{-1} \pi \bar{\alpha}^{(i+1)} - \sum_{k=1}^{N_i} \zeta(x_s^{(i)} - x_k^{(i-1)}) + \sum_{k=1}^{N_{i+2}} \zeta(x_k^{(i+1)} - x_s^{(i)}), \quad i = 1, \dots, n-3, \tag{4.9}$$

$$2 \sum_{k=1, k \neq s}^{N_{n-1}} \zeta(x_k^{(n-2)} - x_s^{(n-2)}) - 2\sqrt{-1} \pi \bar{\alpha}^{(n-2)} = - \sum_{k=1}^{N_{n-2}} \zeta(x_s^{(n-2)} - x_k^{(n-3)}). \tag{4.10}$$

Sum up with s for each equation of (4.8)–(4.10). Then taking the summation of all the equations and noting the parity property of ζ -function (2.4), we find

$$\sum_{k=1}^{N_1} \zeta(x_k^{(0)}) + (1-n)\sqrt{-1} \pi \bar{\alpha} = 0. \tag{4.11}$$

This means that the first order of g term of ϵ_{RS} in (4.7) is vanishing which is in conformity with (4.1).

The Bethe ansatz equations (3.18)–(3.20) at the second order of g are

$$4 \sum_{k=1, k \neq s}^{N_1} (y_s^{(0)} - y_k^{(0)}) \wp(x_k^{(0)} - x_s^{(0)}) = nl(2y_s^{(0)} + nl) \wp(x_s^{(0)}) - \sum_{k=1}^{N_2} (2y_k^{(1)} - 2y_s^{(0)} + 1) \wp(x_k^{(1)} - x_s^{(0)}), \tag{4.12}$$

$$4 \sum_{k=1, k \neq s}^{N_{i+1}} (y_s^{(i)} - y_k^{(i)}) \wp(x_k^{(i)} - x_s^{(i)}) = \sum_{k=1}^{N_i} (2y_s^{(i)} - 2y_k^{(i-1)} + 1) \wp(x_s^{(i)} - x_k^{(i-1)}) - \sum_{k=1}^{N_{i+2}} (2y_k^{(i+1)} - 2y_s^{(i)} + 1) \wp(x_k^{(i+1)} - x_s^{(i)}), \quad i = 1, \dots, n-3, \tag{4.13}$$

$$4 \sum_{k=1, k \neq s}^{N_{n-1}} (y_s^{(n-2)} - y_k^{(n-2)}) \wp(x_k^{(n-2)} - x_s^{(n-2)}) = \sum_{k=1}^{N_{n-2}} (2y_s^{(n-2)} - 2y_k^{(n-3)} + 1) \wp(x_s^{(n-2)} - x_k^{(n-3)}). \tag{4.14}$$

Sum up with s for each equation of (4.12)–(4.14). Then taking the summation of all the equations and noting the parity property of \wp -function (2.4), we find

$$\sum_{s=1}^{N_1} (2y_s^{(0)} + nl) \wp(x_s^{(0)}) = 0. \tag{4.15}$$

Substituting the equations (4.11) and (4.15) into (4.7), we finally have the following.

Proposition 4: The eigenvalues of the Hamiltonian (4.5) of the elliptic CM model associated with A_{n-1} root system with the discrete coupling constants $\gamma = l + 1$ are

$$\epsilon_{CM} = (1-nl)n \sum_{k=1}^{N_1} \wp(x_k^{(0)}) - \frac{(n+1)n(n-1)}{3} l^2 \frac{\sigma'''(0)}{\sigma'(0)}. \tag{4.16}$$

The $[n(n-1)/2]l$ parameters $\{\{x_k^{(i)}\}\}$ satisfy the Bethe ansatz equations

$$2 \sum_{k=1, k \neq s}^{N_1} \zeta(x_k^{(0)} - x_s^{(0)}) - n\sqrt{-1}\pi\bar{\alpha} = (2-n)\sqrt{-1}\pi\bar{\alpha} - nl\zeta(x_s^{(0)}) + \sum_{k=1}^{N_2} \zeta(x_k^{(1)} - x_s^{(0)}), \tag{4.17}$$

$$2 \sum_{k=1, k \neq s}^{N_{i+1}} \zeta(x_k^{(i)} - x_s^{(i)}) + (i-n)\sqrt{-1}\pi\bar{\alpha}^{(i)} = (i+2-n)\sqrt{-1}\pi\bar{\alpha}^{(i+1)} - \sum_{k=1}^{N_i} \zeta(x_s^{(i)} - x_k^{(i-1)}) + \sum_{k=1}^{N_{i+2}} \zeta(x_k^{(i+1)} - x_s^{(i)}), \quad i = 1, \dots, n-3, \tag{4.18}$$

$$2 \sum_{k=1, k \neq s}^{N_{n-1}} \zeta(x_k^{(n-2)} - x_s^{(n-2)}) - 2\sqrt{-1}\pi\bar{\alpha}^{(n-2)} = - \sum_{k=1}^{N_{n-2}} \zeta(x_s^{(n-2)} - x_k^{(n-3)}). \tag{4.19}$$

The parameters $\bar{\alpha}^{(i)}$, $i=0,1,\dots,n-2$ and $\bar{\alpha}^{(0)} = \bar{\alpha}$ are given by the relation (2.32) from $n-1$ arbitrary non-negative integers $\{\alpha^{(i)} \in \mathbb{Z}^+ | i=1,\dots,n-1\}$.

Our result agrees with the *third formulas* (or Bethe ansatz type) of the eigenvalues of the elliptic CM model associated with A_{n-1} root system.²² Taking complex conjugation of the Bethe ansatz equations (4.17)–(4.19), noting the property (2.6) and (A5), we find that the solutions $\{\{x_k^{(i)}\}\}$ to the equations are all pure imaginary numbers. This ensures that the eigenvalues ϵ_{CM} are real and positive up to the ground state energy $\epsilon_0 = -[(n+1)n(n-1)/3]l^2[\sigma'''(0)/\sigma'(0)]$ [the positivity from the expression (A4) of \wp -function when the argument is taken on imaginary axis].

B. Trigonometric potential

Here we consider trigonometric CM models associated with A_{n-1} root system. The corresponding Hamiltonian with the discrete coupling constant $\gamma=l+1$ is given

$$H_{\text{CM}} = - \sum_{i=1}^n \frac{\partial^2}{(\partial\lambda_i)^2} + l(l+1) \sum_{i \neq j} \frac{\pi^2}{\sin^2(\pi\lambda_{ij})}. \tag{4.20}$$

Taking the trigonometric limit $\kappa \rightarrow +\infty$, one finds that

$$\zeta(u) \rightarrow \pi \cot \pi u, \tag{4.21}$$

$$\wp(u) \rightarrow \frac{\pi^2}{\sin^2(\pi u)}, \tag{4.22}$$

from expansions (A2) and (A3). Then the Hamiltonian (4.20) can be obtained from the elliptic type (4.5) by taking the trigonometric limit. Moreover, since the solutions $\{\{x_k^{(i)}\}\}$ to the Bethe ansatz equations (4.17)–(4.19) are all pure imaginary numbers, we can introduce $[n(n-1)/2]l$ real parameters $\{\{\bar{x}_k^{(i)}\}\}$ associated with $\{\{x_k^{(i)}\}\}$,

$$x_k^{(i)} = \sqrt{-1}\bar{x}_k^{(i)}. \tag{4.23}$$

Finally, we can find the spectrum of the Hamiltonian of trigonometric CM model associated with A_{n-1} root system from Proposition 4.

Proposition 5: The eigenvalues of the Hamiltonian (4.20) of the trigonometric CM model associated with A_{n-1} root system with the discrete coupling constant $\gamma=l+1$ are

$$\epsilon_{\text{CM}} = (nl - 1)n \sum_{k=1}^{N_1} \frac{\pi^2}{\sinh^2(\pi \bar{x}_k^{(0)})} + \frac{(n+1)n(n-1)}{3} l^2 \pi^2. \tag{4.24}$$

The $[n(n-1)/2]l$ real parameters $\{\{\bar{x}_k^{(i)}\}\}$ satisfy the Bethe ansatz equations

$$2 \sum_{k=1, k \neq s}^{N_1} \coth \pi(\bar{x}_k^{(0)} - \bar{x}_s^{(0)}) + n\bar{\alpha} = (n-2)\bar{\alpha}^{(1)} - nl \coth \pi(\bar{x}_s^{(0)}) + \sum_{k=1}^{N_2} \coth \pi(\bar{x}_k^{(1)} - \bar{x}_s^{(0)}), \tag{4.25}$$

$$2 \sum_{k=1, k \neq s}^{N_{i+1}} \coth \pi(\bar{x}_k^{(i)} - \bar{x}_s^{(i)}) + (n-i)\bar{\alpha}^{(i)} = (n-i-2)\bar{\alpha}^{(i+1)} - \sum_{k=1}^{N_i} \coth \pi(\bar{x}_s^{(i)} - \bar{x}_k^{(i-1)}) + \sum_{k=1}^{N_{i+2}} \coth \pi(\bar{x}_k^{(i+1)} - \bar{x}_s^{(i)}), \quad i = 1, \dots, n-3, \tag{4.26}$$

$$2 \sum_{k=1, k \neq s}^{N_{n-1}} \coth \pi(\bar{x}_k^{(n-2)} - \bar{x}_s^{(n-2)}) + 2\bar{\alpha}^{(n-2)} = - \sum_{k=1}^{N_{n-2}} \coth \pi(\bar{x}_s^{(n-2)} - \bar{x}_k^{(n-3)}). \tag{4.27}$$

Here the parameters $\bar{\alpha}^{(i)}$, $i = 0, 1, \dots, n-2$ and $\bar{\alpha}^{(0)} = \bar{\alpha}$ are given by the relation (2.32) from $n-1$ arbitrary non-negative integers $\{\alpha^{(i)} \in \mathbb{Z}^+ | i = 1, \dots, n-1\}$.

C. Rational potential

Taking further rational limit of the elliptic Hamiltonian (4.5) as in Sec. III C, we can obtain the Hamiltonian of rational CM model associated with A_{n-1} root system

$$H_{\text{CM}} = - \sum_{i=1}^n \frac{\partial^2}{(\partial \lambda_i)^2} + \sum_{i \neq j} \frac{l(l+1)}{(\lambda_i - \lambda_j)^2}. \tag{4.28}$$

Moreover, we have

Proposition 6: The eigenvalues of the Hamiltonian (4.28) of the rational CM model associated with A_{n-1} root system with the discrete coupling constants $\gamma = l+1$ are

$$\epsilon_{\text{CM}} = \sum_{k=1}^{N_1} \frac{(nl-1)n}{(\bar{x}_k^{(0)})^2}, \tag{4.29}$$

where the $[n(n-1)/2]l$ real parameters $\{\{\bar{x}_k^{(i)}\}\}$ satisfy the Bethe ansatz equations

$$2 \sum_{k=1, k \neq s}^{N_1} \frac{1}{\bar{x}_k^{(0)} - \bar{x}_s^{(0)}} + n\bar{\alpha} = (n-2)\bar{\alpha}^{(1)} - \frac{nl}{\bar{x}_s^{(0)}} + \sum_{k=1}^{N_2} \frac{1}{\bar{x}_k^{(1)} - \bar{x}_s^{(0)}}, \tag{4.30}$$

$$2 \sum_{k=1, k \neq s}^{N_{i+1}} \frac{1}{\bar{x}_k^{(i)} - \bar{x}_s^{(i)}} + (n-i)\bar{\alpha}^{(i)} = (n-i-2)\bar{\alpha}^{(i+1)} - \sum_{k=1}^{N_i} \frac{1}{\bar{x}_s^{(i)} - \bar{x}_k^{(i-1)}} + \sum_{k=1}^{N_{i+2}} \frac{1}{\bar{x}_k^{(i+1)} - \bar{x}_s^{(i)}}, \quad i = 1, \dots, n-3, \tag{4.31}$$

$$2 \sum_{k=1, k \neq s}^{N_{n-1}} \frac{1}{\bar{x}_k^{(n-2)} - \bar{x}_s^{(n-2)}} + 2\bar{\alpha}^{(n-2)} = - \sum_{k=1}^{N_{n-2}} \frac{1}{\bar{x}_s^{(n-2)} - \bar{x}_k^{(n-3)}}. \tag{4.32}$$

Here the parameters $\bar{\alpha}^{(i)}$, $i=0,1,\dots,n-2$ and $\bar{\alpha}^{(0)}=\bar{\alpha}$ are given by the relation (2.32) from $n-1$ arbitrary non-negative real numbers $\{\alpha^{(i)} \in \mathbb{R}^+ | i=1,\dots,n-1\}$.

V. SUMMARY AND COMMENTS

Using the nested Bethe ansatz method for $E_{\tau,\eta}(sl_n)$,¹⁹ we obtain the spectrum of the Hamiltonian of all types of (elliptic, trigonometric, rational) RS models associated with A_{n-1} root system with the discrete coupling constant $\gamma=\sqrt{-1}gl$. Eigenvalues are given in the Bethe ansatz formulas (or the *third formulas* in sense of Felder *et al.*²²). The corresponding eigenfunction is a meromorphic function of $\{\lambda_i\}$ and has quasiperiodic properties (3.21) for the elliptic and trigonometric cases, asymptotic properties (3.36) for the rational case. For the special case of $n=2$, our generalized result recovers that of Refs. 17, 30, and 31.

Taking the “*nonrelativistic limit*,” the Hamiltonian of RS model becomes that of the CM model. Then, we give eigenvalues of the Hamiltonian of CM models associated with A_{n-1} root system with the discrete coupling constant $\gamma=l+1$ in the Bethe ansatz formulas. Our formulas coincide with those of Ref. 22 and those of $n=2$ case.^{30,32} The eigenvalues from our formulas are real and positive up to the ground state energy ϵ_0 as physically desired. But, we have not yet got a direct proof of positivity of the eigenvalues of RS models from our formulas. However, we can show that for small coupling constant the eigenvalues of RS model associated with A_{n-1} root system are positive, from their asymptotic expansion (4.1). Moreover, we find that the elliptic and trigonometric RS and CM models have *discrete spectrum* which are parametrized by a set of non-negative integers $\{\alpha^{(i)} | i=1,\dots,n-1\}$. The rational RS and CM models have *continuous spectrum* which are parametrized by a set of non-negative real numbers $\{\alpha^{(i)} | i=1,\dots,n-1\}$.

If one writes the Hamiltonian of CM model with the coupling constant γ as $H_{\text{CM}}(\gamma)$ and the associated eigenvalues as $\epsilon_{\text{CM}}(\gamma)$, from the expression of the Hamiltonian (4.5), (4.20), and (4.28) one can find the following *duality*:

$$H_{\text{CM}}(-\gamma) = H_{\text{CM}}(\gamma-1). \quad (5.1)$$

Then, actually, we have already obtained the spectrum of CM models associated with A_{n-1} root system with the discrete coupling constants $\gamma=l$ ($l \in \mathbb{Z}$). Unfortunately, such a *duality* does not exist for RS models associated with A_{n-1} root systems.

There also exists another way (we call *symmetric polynomials approach*) to get eigenfunctions and the corresponding eigenvalues of the *trigonometric and rational* RS models¹⁰ and CM models.³³⁻³⁶ It would be very interesting to compare our formulas (of trigonometric and rational cases) with those obtained by the *symmetric polynomials approach* (for special A_1 case, it has already been obtained³⁰). However, the *symmetric polynomials approach* fails in the elliptic models.

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APPENDIX: FORMULAS FOR ELLIPTIC FUNCTIONS

In this appendix, we give some useful series expansions of the elliptic functions given by (2.4) when $\tau=\sqrt{-1}\kappa$, $\kappa \in \mathbb{R}$, $\kappa>0$. By (2.3), σ -function can be expressed in terms of product form³⁷ (see Chap. 15)

$$\sigma(u) = q^{1/4} \sin \pi u \prod_{n=1}^{\infty} (1 - q^{2n} e^{2\sqrt{-1}\pi u})(1 - q^{2n} e^{-2\sqrt{-1}\pi u})(1 - q^{2n}), \quad q = e^{-\pi\kappa}. \quad (A1)$$

We can derive the following series expansions from (2.4):

$$\zeta(u) = \frac{\pi \cos \pi u}{\sin \pi u} + \pi \sum_{n=1}^{\infty} \frac{\sin 2 \pi u}{\sin \pi(u + \sqrt{-1} n \kappa) \sin \pi(u - \sqrt{-1} n \kappa)}, \quad (\text{A2})$$

$$\wp(u) = \frac{\pi^2}{\sin^2 \pi u} + \sum_{n=1}^{\infty} \left\{ \frac{\pi^2}{\sin^2 \pi(u + \sqrt{-1} n \kappa)} + \frac{\pi^2}{\sin^2 \pi(u - \sqrt{-1} n \kappa)} \right\}. \quad (\text{A3})$$

Moreover, the functions have the following properties:

$$\wp(\sqrt{-1}u) = - \left\{ \frac{\pi^2}{\sinh^2 \pi u} + \sum_{n=1}^{\infty} \left\{ \frac{\pi^2}{\sinh^2 \pi(u + n \kappa)} + \frac{\pi^2}{\sinh^2 \pi(u - n \kappa)} \right\} \right\}, \quad (\text{A4})$$

$$\sigma^*(u) = \sigma(u^*), \quad \zeta^*(u) = \zeta(u^*), \quad \wp^*(u) = \wp(u^*), \quad (\text{A5})$$

where * stands for the complex conjugation.

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