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Cylindrical-wave diffraction by a rational wedge

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In this paper, new expressions for the field produced by the diffraction of a cylindrical wave by a wedge, whose angle can be expressed as a rational multiple of π, are given. The solutions are expressed in terms of source terms and real integrals that represent the diffracted field. The general result obtained includes as special cases, Macdonald's solution for diffraction by a half plane, a solution for Carslaw's problem of diffraction by a wedge of open angle $\frac{2}{3}\pi$, and a new representation for the solution of the problem of diffraction by a mixed soft–hard half plane.

1. Introduction

This paper is a sequel to the paper Rawlins (1987), in which the solution to the problem of the diffraction of a plane wave by a rational wedge is given in terms of geometrical acoustic terms, and real integrals (involving Bessel functions of fractional order) representing the diffracted field. Here we shall give an analogous solution to the problem of diffraction of a cylindrical acoustic wave by a wedge whose angle can be expressed as a rational multiple of π.

The exact solution of the problem of diffraction by a soft or hard wedge of any angle, in the two-dimensional case of cylindrical acoustic-wave incidence, is due to Macdonald (1902). The solution was given in the form of a complex contour integral, which was obtained by summing the Fourier series representation of the Green function. For the special case of a wedge that reduces to a half-plane, Macdonald showed how the contour integral could be reduced to an elegant form involving real integrals. Although the form of Macdonald's solution is extremely simple, the method used in obtaining it required a considerable amount of analysis. The problem of cylindrical-wave diffraction by a half-plane had been solved earlier by Carslaw (1898) with a method based on that used by Sommerfeld (1896) in considering diffraction by a plane wave. Carslaw's solution, although equivalent to Macdonald's solution, was of a different form. Sommerfeld's method was heuristic, using the physical method of images in various mathematical Riemann sheets associated with a multivalued function. Although the hybridism of the mathematical and physical concepts was considered abstruse, it did produce exact solutions to hitherto insuperable problems in diffraction theory. Carslaw, who was an early convert to Sommerfeld's method, later gave up the idea of Riemann surfaces and instead used the more-modern approach of periodic Green's functions. Before giving up the Sommerfeld approach to solve diffraction problems, he espoused the cause of Sommerfeld by writing some fairly long expository papers on his method with applications. In particular, he gives (Carslaw 1898) a rather lucid description of Sommerfeld's technique by considering the problem of diffraction by a wedge of open angle $\frac{2}{3}\pi$, when the normal method of images breaks down. This particular example is often used elsewhere to describe Sommerfeld's method (Carslaw 1906; Carslaw & Jaeger 1959; Baker & Copson 1950): however, nowhere is the explicit solution given in terms of sources and images and real integrals representing the diffracted field. We shall give such a solution here as a special case of a more general result. Our approach avoids Sommerfeld's use of Riemann surfaces and simply uses the periodic Green function for an arbitrary angle wedge. We then consider the special case of a wedge whose angle can be expressed as a rational multiple of π. It is then shown, by means of an appropriate integral representation for a Bessel function, that the Green function for a cylindrical line source can be derived from the plane-wave Green function for a rational wedge. This enables us to obtain a representation for the Green function for a cylindrical source, in the form of source and image terms and real integrals that are convenient for calculations of the diffracted field. We remark that recently there has been much work done on asymptotics for the wedge (see, for example, Deschamps 1985). The results presented here offer a new approach, in that a wedge of any angle can be approximated to any order of accuracy by a rational wedge of angle $p\pi/q$ (where p and q are integers), and the real integrals obtained in this paper can be asymptotically evaluated without difficulty.

In §2 we shall give the periodic Green function for a cylindrical-wave source and a wedge of arbitrary angle. The Green function is in the form of a complex-contour integral. Some of the important properties of the Green function are stated, and appropriate expressions, in terms of this Green function, are given for various diffraction problems. In §3 we shall consider in detail the special case of evaluating the complex-contour integral representation of the Green function for a wedge whose angle can be expressed as a rational multiple of π. In §4 we shall give expressions for the Green function for special cases of wedge angles. Finally, in §5 we shall give solutions to some specific problems in diffraction theory that are special cases of the more-general result obtained in §4. The first problem is the classical problem of diffraction by soft or hard half-plane of a cylindrical source, whose solution was given in different forms by Carslaw (1898), and Macdonald (1902, 1915). The second is Carslaw's (1898) didactic problem, used to describe Sommerfeld's technique of diffraction by an open wedge of angle $\frac{2}{3}\pi$; no explicit solution has appeared in the literature for this problem. The last is a new result for the problem of diffraction by a hard–soft plane of a cylindrical source (see Rawlins 1975).

So as not to disrupt the flow of the arguments in the main text of the paper, various proofs of results needed have been placed in appendices at the end of the paper. We remark in particular that in Appendix A we derive a useful integral representation for the Hankel function $H_\nu^{(2)}(z)$, $|\arg z| < \frac{1}{2}\pi$, $\operatorname{Re}\nu > -1$. This integral is closely related to a result given by Macdonald (1897), that does not seem to be well known. Macdonald's derivation does not given precise ranges of validity, and Watson's (1944) treatise on Bessel functions seems to have overlooked this integral representation.

2. Periodic Green function for a wedge

The periodic Green function $G_\alpha(r, \theta, r_0, \theta_0; k)$ for a two-dimensional wedge situated in the space $0 \leqslant r < \infty$, $\alpha \leqslant \theta \leqslant 2\pi$ (see figure 1), where (r, θ) are cylindrical polar coordinates has been shown by Carslaw (1919) to be given by

$$G_\alpha(r, \theta, r_0, \theta_0; k) = \frac{1}{2\alpha \mathrm{i}} \int_c H_0^{(2)}[kR(\zeta)] \frac{\sin \pi\zeta/\alpha}{\cos \pi\zeta/\alpha - \cos \pi(\theta - \theta_0)/\alpha} \mathrm{d}\zeta, \tag{1}$$

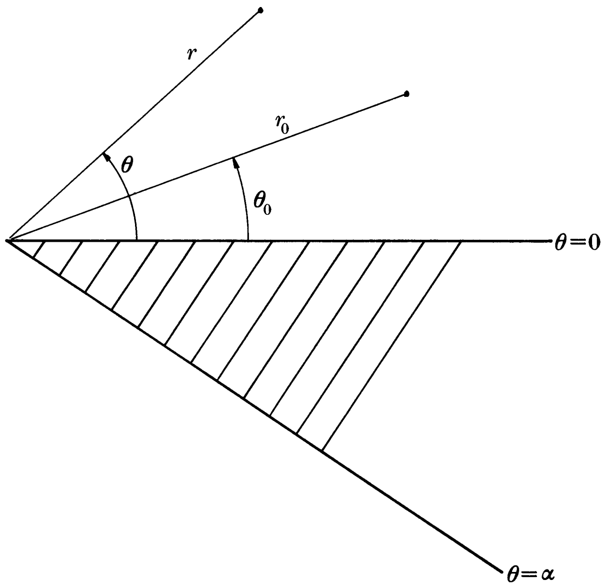

Figure 1. Geometry of the wedge diffraction problem.

where $R(\zeta) = \sqrt{(r^2 + r_0^2 - 2rr_0 \cos \zeta)}$, and the square root is defined by $-\frac{1}{2}\pi \leqslant \arg R(\zeta) \leqslant \frac{1}{2}\pi$. The contour of integration c is such that the starting point is given by $i\infty + c_1$ and the termination point is given by $i\infty + c_2$, where $-\pi < c_1 < 0$, $\pi < c_2 < 2\pi$. The contour of integration c lies below the branch point $\zeta = \beta = \operatorname{arcosh}((r^2 + r_0^2)/2rr_0)$, and does not intersect the branch cut: $\mathrm{Re}\, \zeta = 0$ $\beta < \mathrm{Im}\, \zeta < \infty$ (see figure 2).

It has been shown by Carslaw that $G_\alpha(r, \theta, r_0, \theta_0; k)$ has the following properties:

(i) $$(\nabla^2 + k^2) G_\alpha = 0, \quad \text{where} \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

for all points $(r, \theta) \neq (r_0, \theta_0)$;

(ii) $$G_\alpha(r, \theta, r_0, \theta_0; k) = G_\alpha(r, \theta + 2\alpha, r_0, \theta_0; k);$$

(iii) $G_\alpha(r, \theta, r_0, \theta_0; k)$ is finite and continuous for all $(r, \theta) \neq (r_0, \theta_0)$;

(iv) $$G_\alpha(r, \theta, r_0, \theta_0; k) \sim H_0^{(2)}[kR(\theta - \theta_0)], \quad \text{as} \quad (r, \theta) \to (r_0, \theta_0),$$
$$\sim 0, \quad \text{as} \quad r \to \infty.$$

(2)

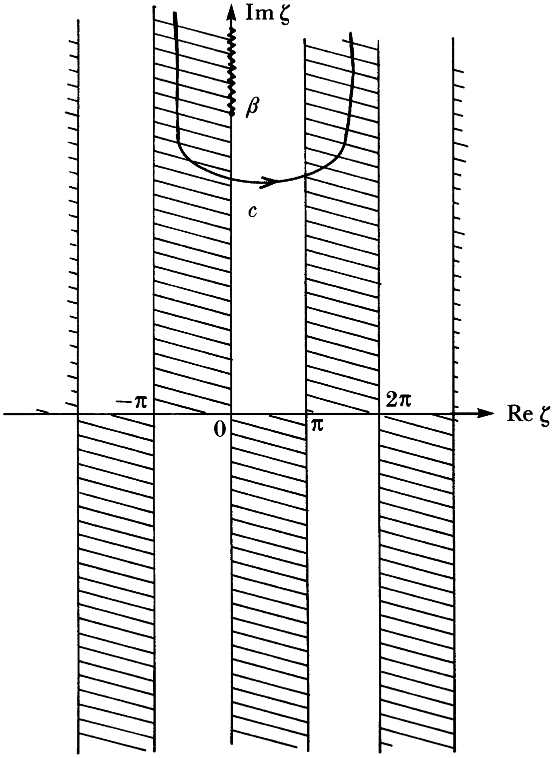

FIGURE 2. The contour of integration c in the complex ζ-plane. The wavy line represents a branch cut of the multivalued function $R(\zeta)$.

The Green function given above enables one to derive solutions to various diffraction problems in wedge-shaped regions. To be specific, we shall discuss acoustic waves. The solution U_{h} or U_{s} of the problem of a cylindrical wave†

$$U_0 = H_0^{(2)}[kR(\theta-\theta_0)], \tag{3}$$

diffracted by a rigid wedge ($\partial U_{\mathrm{h}}/\partial\theta = 0$ for $\theta = 0$ and $\theta = \alpha$) or a soft wedge ($U_{\mathrm{s}} = 0$ for $\theta = 0$ and $\theta = \alpha$) is given by

$$U_{\mathrm{h}} = G_\alpha(r, \theta, r_0, \theta_0;\, k) + G_\alpha(r, \theta, r_0, -\theta_0;\, k), \tag{4}$$

or

$$U_{\mathrm{s}} = G_\alpha(r, \theta, r_0, \theta_0;\, k) - G_\alpha(r, \theta, r_0, -\theta_0;\, k), \tag{5}$$

respectively.

The solution $U_{\mathrm{h,\,s}}$ of the problem of a cylindrical wave (3) diffracted by a wedge whose face $\theta = 0$ is rigid ($\partial U_{\mathrm{h,\,s}} | \partial\theta = 0$) and whose face $\theta = \alpha$ is soft ($U_{\mathrm{h,\,s}} = 0$) is given by

$$U_{\mathrm{h,\,s}} = G_{2\alpha}(r, \theta, r_0, \theta_0;\, k) + G_{2\alpha}(r, \theta, r_0, -\theta_0;\, k) - G_{2\alpha}(r, \theta, r_0, 2\alpha-\theta_0;\, k) - G_{2\alpha}(r, \theta, r_0, -2\alpha+\theta_0;\, k). \tag{6}$$

† The wave is assumed to have time harmonic variation $\mathrm{e}^{\mathrm{i}\omega t}$, but will not be shown explicitly in the rest of the paper.

3. Line-source Green function for a rational wedge

If the wedge angle α is a rational multiple of π i.e., $\alpha = p\pi/q$, where p and q are integers, the line-source Green function (1) becomes

$$G_{p\pi/q}(r,\theta,r_0,\theta_0;k) = \frac{1}{2\pi \mathrm{i} p}\int_c H_0^{(2)}[kR(\zeta)]\frac{q\sin(q\zeta/p)\,\mathrm{d}\zeta}{\cos(\zeta q/p)-\cos((\theta-\theta_0)q/p)}. \tag{7}$$

From the integral representation for the Hankel function, (A 4) of Appendix A with $\nu = 0$, we have

$$H_0^{(2)}[kR(\zeta)] = \frac{1}{\pi \mathrm{i}}\int_{\infty+\mathrm{i}a}^{0}\exp\left\{-\tfrac{1}{2}\mathrm{i}\left(t+\frac{k^2(r^2+r_0^2)}{t}\right)+\mathrm{i}\,\frac{k^2rr_0}{t}\cos\zeta\right\}\frac{\mathrm{d}t}{t}, \tag{8}$$

where $a>0$ and the contour of integration is as shown in figure 5.

Substituting (8) into (7), and interchanging the order of integration (which is permissible because integrals are uniformly convergent) gives

$$G_{p\pi/q}(r,\theta,r_0,\theta_0;k) = \frac{1}{\pi \mathrm{i}}\int_{\infty+\mathrm{i}a}^{0}\exp\left\{-\tfrac{1}{2}\mathrm{i}\left(t+\frac{k^2(r^2+r_0^2)}{t}\right)\right\}G_{p\pi/q}\left(r,\theta,\theta_0;\frac{k^2r_0}{t}\right)\frac{\mathrm{d}t}{t}, \tag{9}$$

where

$$G_{p\pi/q}(r,\theta,\theta_0;k) = \frac{1}{2\pi \mathrm{i} p}\int_c \mathrm{e}^{\mathrm{i}kr\cos\zeta}\frac{q\sin(q\zeta/p)\,\mathrm{d}\zeta}{\cos(\zeta q/p)-\cos((\theta-\theta_0)q/p)}, \tag{10}$$

is the plane-wave Green function for a rational wedge. It has been shown (Rawlins 1987) that the integral (10) can be written in the alternative form

$$\begin{aligned}
G_{p\pi/q}(r,\theta,\theta_0;k) &= \sum_{m=0}^{q-1}\sum_N H[\pi-|\theta-\theta_0+2\pi mp/q+2\pi pN|]\,\mathrm{e}^{\mathrm{i}kr\cos(\theta-\theta_0+2\pi mp/q)}\\
&+\frac{1}{2p}\sum_{m=0}^{q-1}\exp\{\mathrm{i}kr\cos(\theta-\theta_0+2\pi mp/q)-\mathrm{i}\pi/(2p)\}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\\
&\times\int_\infty^{kr}\mathrm{e}^{-\mathrm{i}x\cos(\theta-\theta_0+2\pi mp/q)}\,H_{1/p}^{(2)}(x)\,\mathrm{d}x\\
&+\frac{1}{2\mathrm{i}p}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\frac{\mathrm{e}^{\mathrm{i}kr\cos(\theta-\theta_0+2\pi mp/q)}}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\\
&\times\Big\{\mathrm{e}^{\mathrm{i}n\pi/(2p)}\sin((n+1)(\theta-\theta_0+2\pi mp/q)/p)\sin(n\pi/p)\\
&\times\int_\infty^{kr}\mathrm{e}^{-\mathrm{i}x\cos(\theta-\theta_0+2\pi mp/q)}\,H_{(p-n)/p}^{(2)}(x)\,\mathrm{d}x\\
&-\mathrm{e}^{\mathrm{i}(n+1)\pi/(2p)}\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p)\\
&\times\int_\infty^{kr}\mathrm{e}^{-\mathrm{i}x\cos(\theta-\theta_0+2\pi mp/q)}\,H_{(p-1-n)/p}^{(2)}(x)\,\mathrm{d}x\Big\},
\end{aligned} \tag{11}$$

where the summation over N is for all integer values of N that can make the argument of the Heaviside step function

$$H[x] = \begin{cases} 1 & x > 0 \\ \tfrac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$

non-negative. Thus on substituting (11) into (9) and interchanging the order of integrations results in evaluating integrals of the form

$$\frac{1}{\pi \mathrm{i}} \int_{\infty + \mathrm{i}a}^{0} \exp\left\{-\tfrac{1}{2}\mathrm{i}\left(t + \frac{k^2(r^2+r_0^2)}{t}\right)\right\} \exp\left\{\frac{ik^2 r r_0}{t}\cos\psi\right\} \int_{\infty}^{k^2 r r_0/t} \mathrm{e}^{-\mathrm{i}x\cos\psi} H_\nu^{(2)}(x)\,\mathrm{d}x \,\frac{\mathrm{d}t}{t},$$

which is shown in Appendix B to be equal to

$$-\frac{2}{\pi}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\nu\pi} \int_0^\infty \frac{\cosh \nu t}{\cosh t + \cos\psi}\, H_0^{(2)}[kR(\pi - \mathrm{i}t)]\,\mathrm{d}t.$$

Thus

$$\begin{aligned} &G_{p\pi/q}(r, \theta, r_0, \theta_0; k) \\ &= \sum_{m=0}^{q-1} \sum_N H[\pi - |\theta - \theta_0 + 2\pi m p/q + 2\pi p N|]\, H_0^{(2)}[kR(\theta - \theta_0 + 2\pi m p/q)] \\ &\quad - \frac{1}{\pi p} \sum_{m=0}^{q-1} \frac{\sin(\theta - \theta_0 + 2\pi m p/q)\sin(\pi/p)}{\sin((\theta - \theta_0 + 2\pi m p/q)/p)} \int_0^\infty \frac{\cosh(t/p)\, H_0^{(2)}[kR(\pi - \mathrm{i}t)]}{\cosh t + \cos(\theta - \theta_0 + 2\pi m p/q)}\,\mathrm{d}t \\ &\quad - \frac{1}{\pi p} \sum_{m=0}^{q-1} \sum_{n=1}^{p-2} \left\{ \frac{\sin((n+1)(\theta - \theta_0 + 2\pi m p/q)/p)\sin(n\pi/p)}{\sin((\theta - \theta_0 + 2\pi m p/q)/p)} \right. \\ &\quad \times \int_0^\infty \frac{\cosh((p-n)t/p)\, H_0^{(2)}[kR(\pi - \mathrm{i}t)]}{\cosh t + \cos(\theta - \theta_0 + 2\pi m p/q)}\,\mathrm{d}t \\ &\quad - \frac{\sin(n(\theta - \theta_0 + 2\pi m p/q)/p)\sin((n+1)\pi/p)}{\sin((\theta - \theta_0 + 2\pi m p/q)/p)} \\ &\quad \left. \times \int_0^\infty \frac{\cosh((p-1-n)t/p)\, H_0^{(2)}[kR(\pi - \mathrm{i}t)]}{\cosh t + \cos(\theta - \theta_0 + 2\pi m p/q)}\,\mathrm{d}t \right\}, \end{aligned} \tag{12}$$

where the summation $\sum_N$ is performed for all values of N that satisfy the inequality $-\pi \leqslant \theta - \theta_0 + 2\pi m p/q + 2\pi p N \leqslant \pi$. Thus the solution $U(r, \theta)$ of the problem of diffraction of the cylindrical wave $U_0 = H_0^{(2)}[kR(\theta - \theta_0)]$ by a soft or hard wedge of open angle $\alpha = p\pi/q$ is given by

$$U_\mathrm{s}(r, \theta) = G_{p\pi/q}(r, \theta, r_0, \theta_0; k) - G_{p\pi/q}(r, \theta, r_0, -\theta_0; k), \tag{13}$$

and

$$U_\mathrm{h}(r, \theta) = G_{p\pi/q}(r, \theta, r_0, \theta_0; k) + G_{p\pi/q}(r, \theta, r_0, -\theta_0; k), \tag{14}$$

respectively, where $G_{p\pi/q}$ is given by (12). Similarly, the solution of the problem of diffraction of the wave $U_0 = H_0^{(2)}[kR(\theta-\theta_0)]$ by a wedge whose face $\theta = 0$ is hard, and whose other face $\theta = p\pi/q$ is soft is given by

$$U_{h,s}(r,\theta) = G_{2p\pi/q}(r,\theta,r_0,\theta_0;k) + G_{2p\pi/q}(r,\theta,r_0,-\theta_0;k) - G_{2p\pi/q}(r,\theta,r_0,-2p\pi/q+\theta_0;k) - G_{2p\pi/q}(r,\theta,r_0,2p\pi/q-\theta_0;k), \quad (15)$$

where $G_{2p\pi/q}(r,\theta,r_0,\theta_0;k)$ is given by (12) with p replaced by $2p$.

An asymptotic expression for $G_{p\pi/q}(r,\theta,r_0,\theta_0;k)$ can be obtained, from expression (20) of Rawlins (1987), by applying the techniques outlined in the Appendix C. Thus for $kr \to \infty$ we have

$$\begin{aligned} &G_{p\pi/q}(r,\theta,r_0,\theta_0;k) \\ &= \sum_{m=0}^{q-1}\sum_{N} H[\pi - |\theta-\theta_0+2\pi mp/q+2\pi pN|]\, H_0^{(2)}[kR(\theta-\theta_0+2\pi mp/q)] \\ &+ \frac{i}{\pi p}\sum_{m=0}^{q-1} \frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)\,|\cos\frac{1}{2}(\theta-\theta_0+2\pi mp/q)|} \\ &\times \int_{\infty}^{\xi(\theta-\theta_0+2\pi mp/q)} e^{-ikR(\theta-\theta_0+2\pi mp/q)\cosh\xi}\, d\xi \\ &+ \frac{i}{\pi p}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2} \frac{1}{\sin((\theta-\theta_0+2\pi mp/q)/p)\,|\cos\frac{1}{2}(\theta-\theta_0+2\pi mp/q)|} \\ &\times \Big\{ \sin((n+1)(\theta-\theta_0+2\pi mp/q)/p)\sin(n\pi/p) \\ &\times \int_{\infty}^{\xi(\theta-\theta_0+2\pi mp/q)} e^{-ikR(\theta-\theta_0+2\pi mp/q)\cosh\xi}\, d\xi \\ &- \sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p) \\ &\times \int_{\infty}^{\xi(\theta-\theta_0+2\pi mp/q)} e^{-ikR(\theta-\theta_0+2\pi mp/q)\cosh\xi}\, d\xi \Big\} + O[(kR)^{-\frac{3}{2}}], \end{aligned} \quad (16)$$

where $$\xi(\psi) = \operatorname{arsinh}\{2\surd(rr_0)\,|\cos\tfrac{1}{2}\psi| R(\psi)\}. \quad (17)$$

The integrals appearing in (16) can be further expressed asymtotically in terms of Fresnel integrals, whose properties are well known; for details, see Jones (1986).

Special cases of wedge angles

Case 1: $p = 1$

$$G_{\pi/p}(r,\theta,r_0,\theta_0;k) = \sum_{m=0}^{q-1}\sum_{N} H[\pi - |\theta-\theta_0+2\pi m/q+2\pi N|]\, H_0^{(2)}[kR(\theta-\theta_0+2\pi m/q)]. \quad (18)$$

Case 2: $q = 1$

$$G_{p\pi}(r,\theta,r_0,\theta_0;k) = \sum_N H[\pi - |\theta-\theta_0+2\pi pN|]\, H_0^{(2)}[kR(\theta-\theta_0)]$$

$$-\frac{1}{\pi p}\frac{\sin(\theta-\theta_0)\sin(\pi/p)}{\sin((\theta-\theta_0)/p)}\int_0^\infty \frac{\cosh(t/p)\, H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$

$$-\frac{1}{\pi p}\sum_{n=1}^{p-2}\left\{\frac{\sin((n+1)(\theta-\theta_0)/p)\sin(n\pi/p)}{\sin((\theta-\theta_0)/p)}\int_0^\infty \frac{\cosh((p-n)t/p)\, H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t\right.$$

$$\left.-\frac{\sin(n(\theta-\theta_0)/p)\sin((n+1)\pi/p)}{\sin((\theta-\theta_0)/p)}\int_0^\infty \frac{\cosh((p-1-n)t/p)\, H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t\right\}. \tag{19}$$

Case 3: $p = 2$

$$G_{2\pi/q}(r,\theta,r_0,\theta_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi-|\theta-\theta_0+4\pi m/q+4\pi N|]\, H_0^{(2)}[kR(\theta-\theta_0+4\pi m/q)]$$

$$-\frac{1}{\pi}\sum_{m=0}^{q-1}\cos\tfrac{1}{2}(\theta-\theta_0+4\pi m/q)\int_0^\infty \frac{\cosh\frac{1}{2}t\, H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0+4\pi m/q)}\,\mathrm{d}t. \tag{20}$$

The last expression, (20), can be put in an alternative form by using the results of Appendix D. Thus

$$G_{2\pi/q}(r,\theta,r_0,\theta_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi-|\theta-\theta_0+4\pi m/q+4\pi N|]\, H_0^{(2)}[kR(\theta-\theta_0+4\pi m/q)]$$

$$+\frac{\mathrm{i}}{\pi}\sum_{m=0}^{q-1}\operatorname{sgn}[\cos\{\tfrac{1}{2}(\theta-\theta_0+4\pi m/q)\}]\int_\infty^{\xi(\theta-\theta_0+4\pi m/q)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0+4\pi m/q)\cosh\xi}\,\mathrm{d}\xi. \tag{21}$$

4. Some specific problems in diffraction theory

Macdonald's solution for a half-plane

In terms of Green's function, the solution for the problem of diffraction of a cylindrical wave $U_0(r,\theta) = H_0^{(2)}[kR(\theta-\theta_0)]$ by a soft, or hard, half-plane is given by

$$\left.\begin{aligned} U_{\mathrm{h}}(r,\theta,r_0,\theta_0) &= G_{2\pi}(r,\theta,r_0,\theta_0;k)+G_{2\pi}(r,\theta,r_0,-\theta_0;k),\\ U_{\mathrm{s}}(r,\theta,r_0,\theta_0) &= G_{2\pi}(r,\theta,r_0,\theta_0;k)-G_{2\pi}(r,\theta,r_0,-\theta_0;k),\end{aligned}\right\} \tag{22}$$

respectively.

Putting $q = 1$ in (21) gives

$$G_{2\pi}(r,\theta,r_0,\theta_0;k) = \sum_N H[\pi-|\theta-\theta_0+4\pi N|]\, H_0^{(2)}[kR(\theta-\theta_0)]$$

$$+\frac{\mathrm{i}}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta-\theta_0)]\int_\infty^{|\xi(\theta-\theta_0)|} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi. \tag{23}$$

Now for $0 < \theta_0 < 2\pi$, and $0 < \theta < 2\pi$, then $|\theta-\theta_0| < 2\pi$, so that the argument of the Heaviside step function in (23) can only be positive if $N = 0$. Hence

$$
\begin{aligned}
&G_{2\pi}(r,\theta,r_0,\theta_0;k) \\
&= H[\pi-|\theta-\theta_0|]\,H_0^{(2)}[kR(\theta-\theta_0)]+\frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta-\theta_0)]\int_{\infty}^{|\xi(\theta-\theta_0)|} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi, \\
&= H[\cos\tfrac{1}{2}(\theta-\theta_0)]\,H_0^{(2)}[kR(\theta-\theta_0)]+\frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta-\theta_0)]\int_{\infty}^{|\xi(\theta-\theta_0)|} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi.
\end{aligned}
\tag{24}
$$

If $\cos\frac{1}{2}(\theta-\theta_0) > 0$, then

$$G_{2\pi}(r,\theta,r_0,\theta_0;k) = H_0^{(2)}[kR(\theta-\theta_0)]+\frac{\mathrm{i}}{\pi}\int_{\infty}^{\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi.$$

Now using the fact that

$$H_0^{(2)}[kR(\theta-\theta_0)] = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi,$$

we can write

$$
\begin{aligned}
G_{2\pi}(r,\theta,r,\theta;k) &= \frac{\mathrm{i}}{\pi}\left\{\int_{-\infty}^{\infty}+\int_{\infty}^{\xi(\theta-\theta_0)}\right\} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi, \\
&= \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi.
\end{aligned}
\tag{25}
$$

If $\cos\frac{1}{2}(\theta-\theta_0) < 0$, then

$$
\begin{aligned}
G_{2\pi}(r,\theta,r_0,\theta_0;k) &= -\frac{\mathrm{i}}{\pi}\int_{\infty}^{-\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi, \\
&= \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi.
\end{aligned}
\tag{26}
$$

Hence for any sign of $\cos\frac{1}{2}(\theta-\theta_0)$ we have

$$G_{2\pi}(r,\theta,r_0,\theta_0;k) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi. \tag{27}$$

The expression for $G_{2\pi}(r,\theta,r_0,-\theta_0;k)$ can be found in exactly the same manner for $0 < 0+\theta_0 < 4\pi$, i.e.

$$
\begin{aligned}
G_{2\pi}(r,\theta,r_0,-\theta_0;k) = {}& H[\pi-|\theta+\theta_0|]\,H_0^{(2)}[kR(\theta+\theta_0)] \\
&+H[\pi-|\theta+\theta_0-4\pi|]\,H_0^{(2)}[kR(\theta+\theta_0)] \\
&+\frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta+\theta_0)]\int_{\infty}^{|\xi(\theta+\theta_0)|} \mathrm{e}^{-\mathrm{i}kR(\theta+\theta_0)\cosh\xi}\,\mathrm{d}\xi.
\end{aligned}
\tag{28}
$$

Hence

$$G_{2\pi}(r,\theta,r_0,-\theta_0;k) = H[\cos\tfrac{1}{2}(\theta+\theta_0)]\,H_0^{(2)}[kR(\theta+\theta_0)] + \frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta+\theta_0)]\int_{\infty}^{\xi(\theta+\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta+\theta_0)\cosh\xi}\,\mathrm{d}\xi \tag{29}$$

$$= \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta+\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta+\theta_0)\cosh\xi}\,\mathrm{d}\xi. \tag{30}$$

Thus the solution of the problem of diffraction of a cylindrical wave by a hard or soft half-plane is given by substituting (27) and (30) into (22) giving

$$\left.\begin{aligned} U_{\mathrm{h}}(r,\theta,r_0,\theta_0) &= \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi + \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta+\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta+\theta_0)\cosh\xi}\,\mathrm{d}\xi,\\ U_{\mathrm{s}}(r,\theta,r_0,\theta_0;k) &= \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta-\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi}\,\mathrm{d}\xi - \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\xi(\theta+\theta_0)} \mathrm{e}^{-\mathrm{i}kR(\theta+\theta_0)\cosh\xi}\,\mathrm{d}\xi, \end{aligned}\right\} \tag{31}$$

where $\xi(\theta\pm\theta_0)$ are given by (24). This result agrees with that of Macdonald (1915).

Solution of Carslaw's problem for a wedge angle $\alpha = \frac{2}{3}\pi$

The solution for the problem of diffraction of a cylindrical wave $U_0(r,\theta,r_0,\theta_0) = H_0^{(2)}[kR(\theta-\theta_0)]$ by a soft or hard wedge of open angle $\alpha = \frac{2}{3}\pi$ is given by

$$\left.\begin{aligned} U_{\mathrm{s}}(r,\theta,r_0,\theta_0) &= G_{\frac{2}{3}\pi}(r,\theta,r_0,\theta_0;k) - G_{\frac{2}{3}\pi}(r,\theta,r_0,-\theta_0;k),\\ U_{\mathrm{h}}(r,\theta,r_0,\theta_0) &= G_{\frac{2}{3}\pi}(r,\theta,r_0,\theta_0;k) + G_{\frac{2}{3}\pi}(r,\theta,r_0,-\theta_0;k), \end{aligned}\right\} \tag{32}$$

where, from (21) with $q = 3$,

$$\begin{aligned} G_{\frac{2}{3}\pi}(r,\theta,r_0,\theta_0;k) &= \sum_{N_1} H[\pi-|\theta-\theta_0+4\pi N_1|]\,H_0^{(2)}[kR(\theta-\theta_0)]\\ &+\sum_{N_2} H[\pi-|\theta-\theta_0+\tfrac{4}{3}\pi+4\pi N_2|]\,H_0^{(2)}[kR(\theta-\theta_0+\tfrac{4}{3}\pi)]\\ &+\sum_{N_3} H[\pi-|\theta-\theta_0+\tfrac{8}{3}\pi+4\pi N_3|]\,H_0^{(2)}[kR(\theta-\theta_0+\tfrac{8}{3}\pi)]\\ &+\frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta-\theta_0)]\int_{\infty}^{\xi(\theta-\theta_0)} \exp\{-\mathrm{i}kR(\theta-\theta_0)\cosh\xi\}\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta-\theta_0+\tfrac{4}{3}\pi)]\int_{\infty}^{\xi(\theta-\theta_0+\frac{3}{4}\pi)} \exp\{-\mathrm{i}kR(\theta-\theta_0+\tfrac{4}{3}\pi)\cosh\xi\}\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}}{\pi}\,\mathrm{sgn}\,[\cos\tfrac{1}{2}(\theta-\theta_0+\tfrac{8}{3}\pi)]\int_{\infty}^{\xi(\theta-\theta_0+\frac{8}{3}\pi)} \exp\{-\mathrm{i}kR(\theta-\theta_0+\tfrac{8}{3}\pi)\cosh\xi\}\,\mathrm{d}\xi. \end{aligned} \tag{33}$$

It is not difficult to show that for $-\frac{2}{3}\pi < \theta - \theta_0 < \frac{2}{3}\pi$, then $N_1 = 0$, $N_2 = 0$, $N_3 = -1$. Hence

$$\begin{aligned}
G_{\frac{2}{3}\pi}(r, \theta, r_0, \theta_0; k) = {} & H_0^{(2)}[kR(\theta-\theta_0)] + H[\pi - |\theta-\theta_0+\tfrac{4}{3}\pi|]\, H_0^{(2)}[kR(\theta-\theta_0+\tfrac{4}{3}\pi)] \\
& + H[\pi - |\theta-\theta_0-\tfrac{4}{3}\pi|]\, H_0^{(2)}[kR(\theta-\theta_0-\tfrac{4}{3}\pi)] \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta-\theta_0)] \int_{\infty}^{\xi(\theta-\theta_0)} \exp\{-ikR(\theta-\theta_0)\cosh\xi\}\,d\xi \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta-\theta_0+\tfrac{4}{3}\pi)] \int_{\infty}^{\xi(\theta-\theta_0+\frac{4}{3}\pi)} \exp\{-ikR(\theta-\theta_0+\tfrac{4}{3}\pi)\cosh\xi\}\,d\xi \\
& + \frac{i}{\pi}[\operatorname{sgn}\cos\tfrac{1}{2}(\theta-\theta_0-\tfrac{4}{3}\pi)] \int_{\infty}^{\xi(\theta-\theta_0-\frac{4}{3}\pi)} \exp\{-ikR(\theta-\theta_0-\tfrac{4}{3}\pi)\cosh\xi\}\,d\xi. \qquad (34)
\end{aligned}$$

We also have from (21),

$$\begin{aligned}
G_{\frac{2}{3}\pi}(r, \theta, r_0, -\theta_0; k) = {} & \sum_{N_1} H[\pi - |\theta+\theta_0+4\pi N_1|]\, H_0^{(2)}[kR(\theta+\theta_0)] \\
& + \sum_{N_2} H[\pi - |\theta+\theta_0+\tfrac{4}{3}\pi+4\pi N_2|]\, H_0^{(2)}[kR(\theta+\theta_0+\tfrac{4}{3}\pi)] \\
& + \sum_{N_3} H[\pi - |\theta+\theta_0+\tfrac{8}{3}\pi+4\pi N_3|]\, H_0^{(2)}[kR(\theta+\theta_0+\tfrac{8}{3}\pi)] \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta+\theta_0)] \int_{\infty}^{\xi(\theta-\theta_0)} \exp\{-ikR(\theta+\theta_0)\cosh\xi\}\,d\xi \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta+\theta_0+\tfrac{4}{3}\pi)] \int_{\infty}^{\xi(\theta+\theta_0+\frac{4}{3}\pi)} \exp\{-ikR(\theta+\theta_0+\tfrac{4}{3}\pi)\cosh\xi\}\,d\xi \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta+\theta_0+\tfrac{8}{3}\pi)] \int_{\infty}^{\xi(\theta+\theta_0+\frac{8}{3}\pi)} \exp\{-ikR(\theta+\theta_0+\tfrac{8}{3}\pi)\cosh\xi\}\,d\xi. \qquad (35)
\end{aligned}$$

For the range of values $0 < \theta + \theta_0 < \frac{4}{3}\pi$ is it not difficult to show that $N_1 = 0$, N_2 takes no values, $N_3 = -1$, so that

$$\begin{aligned}
G_{\frac{2}{3}\pi}(r, \theta, r_0, -\theta_0; k) = {} & H[\pi - |\theta+\theta_0|]\, H_0^{(2)}[kR(\theta+\theta_0)] \\
& + H[\pi - |\theta+\theta_0-\tfrac{4}{3}\pi|]\, H_0^{(2)}[kR(\theta+\theta_0-\tfrac{4}{3}\pi)] \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta+\theta_0)] \int_{\infty}^{\xi(\theta+\theta_0)} \exp\{-ikR(\theta+\theta_0)\cosh\xi\}\,d\xi \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta+\theta_0+\tfrac{4}{3}\pi)] \int_{\infty}^{\xi(\theta+\theta_0+\frac{4}{3}\pi)} \exp\{-ikR(\theta+\theta_0+\tfrac{4}{3}\pi)\cosh\xi\}\,d\xi \\
& + \frac{i}{\pi}\operatorname{sgn}[\cos\tfrac{1}{2}(\theta+\theta_0-\tfrac{4}{3}\pi)] \int_{\infty}^{\xi(\theta+\theta_0-\frac{4}{3}\pi)} \exp\{-ikR(\theta+\theta_0-\tfrac{4}{3}\pi)\cosh\xi\}\,d\xi. \qquad (36)
\end{aligned}$$

Substituting (34) and (36) into (32) gives the solution to the problem of diffraction of the cylindrical wave $U_0(r, \theta, r_0, \theta_0)$ by a soft or hard wedge of open angle $\frac{2}{3}\pi$.

Diffraction by a hard–soft half-plane

In terms of the Green function, the solution for the problem of the diffraction of the wave $U_0(r,\theta,r_0,\theta_0) = H_0^{(2)}[kR(\theta-\theta_0)]$ by a hard–soft half plane is given by

$$U_{\mathrm{h,s}}(r,\theta,r_0,\theta_0) = G_{4\pi}(r,\theta,r_0,\theta_0;k)+G_{4\pi}(r,\theta,r_0,-\theta_0;k)$$
$$-G_{4\pi}(r,\theta,r_0,4\pi-\theta_0;k)-G_{4\pi}(r,\theta,r_0,-4\pi+\theta_0;k). \quad (37)$$

By putting $p = 4$ in (19) we obtain

$$G_{4\pi}(r,\theta,r_0,\theta_0;k) = \sum_N H[\pi-|\theta-\theta_0+8\pi N|]\,H_0^{(2)}[kR(\theta-\theta_0)]$$
$$-\frac{1}{4\pi\sqrt{2}}\frac{\sin(\theta-\theta_0)}{\sin\frac{1}{4}(\theta-\theta_0)}\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$-\frac{1}{4\pi}\left\{\frac{\sin\frac{1}{2}(\theta-\theta_0)}{\sqrt{2}\sin\frac{1}{4}(\theta-\theta_0)}\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t\right.$$
$$\left.-\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t\right\}$$
$$-\frac{1}{4\pi}\left\{\frac{\sin\frac{3}{4}(\theta-\theta_0)}{\sin\frac{1}{4}(\theta-\theta_0)}\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t\right.$$
$$\left.-\frac{\sin\frac{1}{2}(\theta-\theta_0)}{\sqrt{2}\sin\frac{1}{4}(\theta-\theta_0)}\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t\right\}. \quad (38)$$

For $-2\pi < \theta-\theta_0 < 2\pi$ the only value of N that satisfies $-\pi < \theta-\theta_0+8\pi N < \pi$ is $N = 0$. Hence

$$G_{4\pi}(r,\theta,r_0,\theta_0;k) = H[\pi-|\theta-\theta_0|]\,H_0^{(2)}[kR(\theta-\theta_0)]$$
$$+\frac{1}{4\pi}\left(1-\frac{\sin\frac{3}{4}(\theta-\theta_0)}{\sin\frac{1}{4}(\theta-\theta_0)}\right)\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t$$
$$-\frac{\sqrt{2}}{4\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$+\frac{\sqrt{2}}{4\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta-\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t, \quad (39)$$

$$= H[\pi-|\theta-\theta_0|]\,H_0^{(2)}[kR(\theta-\theta_0)]$$
$$-\frac{1}{2\pi}\cos\tfrac{1}{2}(\theta-\theta_0)\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$-\frac{\sqrt{2}}{4\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$+\frac{\sqrt{2}}{4\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta-\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t. \quad (40)$$

In a similar manner it is not difficult to show that

$$G_{4\pi}(r,\theta,r_0,-\theta_0;k) = H[\pi-|\theta+\theta_0|]\,H_0^{(2)}[kR(\theta+\theta_0)]$$
$$-\frac{1}{2\pi}\cos\tfrac{1}{2}(\theta+\theta_0)\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t$$
$$-\frac{\surd 2}{4\pi}\cos\tfrac{1}{4}(\theta+\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t$$
$$+\frac{\surd 2}{4\pi}\cos\tfrac{1}{4}(\theta+\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta+\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t. \quad (41)$$

$$G_{4\pi}(r,\theta,4\pi-\theta_0;k) = H[\pi-|\theta+\theta_0-4\pi|]\,H_0^{(2)}[kR(\theta+\theta_0)]$$
$$-\frac{1}{2\pi}\cos\tfrac{1}{2}(\theta+\theta_0)\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t$$
$$+\frac{\surd 2}{4\pi}\cos\tfrac{1}{4}(\theta+\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t$$
$$-\frac{\surd 2}{4\pi}\cos\tfrac{1}{4}(\theta+\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta+\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t. \quad (42)$$

$$G_{4\pi}(r,\theta,r_0,-4\pi+\theta_0;k) = -\frac{1}{2\pi}\cos\tfrac{1}{2}(\theta+\theta_0)\int_0^\infty \frac{\cosh\frac{1}{2}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$+\frac{\surd 2}{4\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$-\frac{\surd 2}{4\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta-\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cos t+\cos(\theta-\theta_0)}\,\mathrm{d}t. \quad (43)$$

Substituting (40)–(43) into (37) gives the solution for diffraction by a hard–soft half-plane as

$$U_{\mathrm{h,s}}(r,\theta,r_0,\theta_0) = H[\pi-|\theta-\theta_0|]\,H_0^{(2)}[kR(\theta-\theta_0)]+H[\pi-|\theta+\theta_0|]\,H_0^{(2)}[kR(\theta+\theta_0)]$$
$$-H[\pi-|\theta+\theta_0-4\pi|]\,H_0^{(2)}[kR(\theta+\theta_0)]$$
$$-\frac{1}{\surd 2\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$-\frac{1}{\surd 2\pi}\cos\tfrac{1}{4}(\theta+\theta_0)\int_0^\infty \frac{\cosh\frac{3}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t$$
$$+\frac{1}{\surd 2\pi}\cos\tfrac{1}{4}(\theta-\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta-\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta-\theta_0)}\,\mathrm{d}t$$
$$+\frac{1}{\surd 2\pi}\cos\tfrac{1}{4}(\theta+\theta_0)\,(1-2\cos\tfrac{1}{2}(\theta+\theta_0))\int_0^\infty \frac{\cosh\frac{1}{4}t\,H_0^{(2)}[kR(\pi-\mathrm{i}t)]}{\cosh t+\cos(\theta+\theta_0)}\,\mathrm{d}t. \quad (44)$$

Appendix A

Here we derive a contour integral representation for $H_\nu^{(2)}(z)$ for $\nu > -1$, $-\frac{1}{2}\pi \leqslant \arg z \leqslant \frac{1}{2}\pi$, namely

$$H_\nu^{(2)}(z) = z^\nu \, e^{\frac{1}{2}i\nu\pi} \frac{1}{\pi i} \int_{\infty + ia}^{0 \exp\{i(\arg z - \frac{1}{2}\pi)\}} e^{-i(t + z^2/4t)} \frac{dt}{t^{\nu+1}}, \quad a > 0. \tag{A 1}$$

From Watson (1944) we have the integral representation

$$H_\nu^{(2)}(z) = -\frac{1}{\pi i} \int_{0\, e^{i0}}^{\infty\, e^{-i\pi}} e^{\frac{1}{2}z(u - u^{-1})} \frac{du}{u^{\nu+1}}, \quad -\frac{1}{2}\pi \leqslant \arg z \leqslant \frac{1}{2}\pi, \tag{A 2}$$

where the contour of integration is shown in figure 3. Let $zu = 2t\, e^{-\frac{1}{2}i\pi}$, then

$$H_\nu^{(2)}(z) = -(\tfrac{1}{2}z)^\nu \, e^{\frac{1}{2}i\nu\pi} \frac{1}{\pi i} \int_{0 \exp\{i(\arg z + \frac{1}{2}\pi)\}}^{\infty \exp\{i(\arg z - \frac{1}{2}\pi)\}} e^{-i(t + z^2/4t)} \frac{dt}{t^{\nu+1}}. \tag{A 3}$$

Because $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$, then $-\pi < \arg z - \frac{1}{2}\pi < 0$ and $0 < \arg z + \frac{1}{2}\pi < \pi$, meaning that the upper limit of integration lies in the lower-half t-plane, and the lower limit of integration lies in the upper-half t-plane (see figure 4).

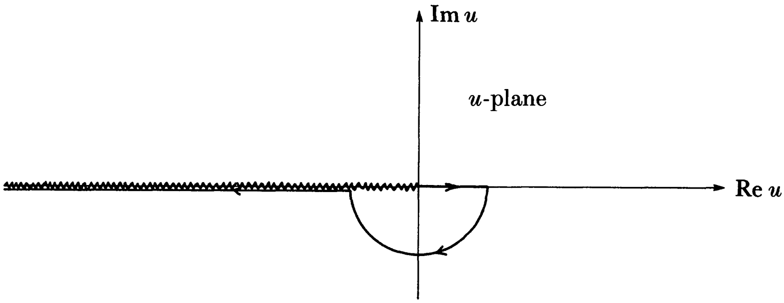

Figure 3. The contour of integration for the integral (A 2). The wavy line represents a branch cut.

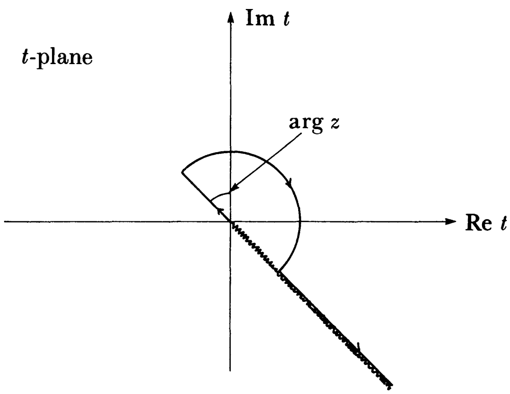

Figure 4. The contour of integration for the integral (A 3).

Provided $\mathrm{Re}\,\nu > -1$ we can apply Jordan's lemma to distort the path of integration to run along a path parallel to the real axis at a distance $a > 0$ as $t \to \infty$ (see Jeffreys & Jeffreys 1966). Thus (see figure 5),

$$H_\nu^{(2)}(z) = (\tfrac{1}{2}z)^\nu \,\mathrm{e}^{\frac{1}{2}\mathrm{i}\nu\pi} \frac{1}{\pi \mathrm{i}} \int_{\infty + \mathrm{i}a}^{0 \exp\{\mathrm{i}(\arg z + \frac{1}{2}\pi)\}} \mathrm{e}^{-\mathrm{i}(t + z^2/4t)} \frac{\mathrm{d}t}{t^{\nu+1}},$$

$$= z^\nu \,\mathrm{e}^{\frac{1}{2}\mathrm{i}\nu\pi} \frac{1}{\pi \mathrm{i}} \int_{\infty + \mathrm{i}a}^{0 \exp\{\mathrm{i}(\arg z + \frac{1}{2}\pi)\}} \mathrm{e}^{-\frac{1}{2}\mathrm{i}(t + z^2/t)} \frac{\mathrm{d}t}{t^{\nu+1}},$$

$$-\tfrac{1}{2}\pi < \arg z < \tfrac{1}{2}\pi,\; a > 0,\; \mathrm{Re}\,\nu > -1. \tag{A 4}$$

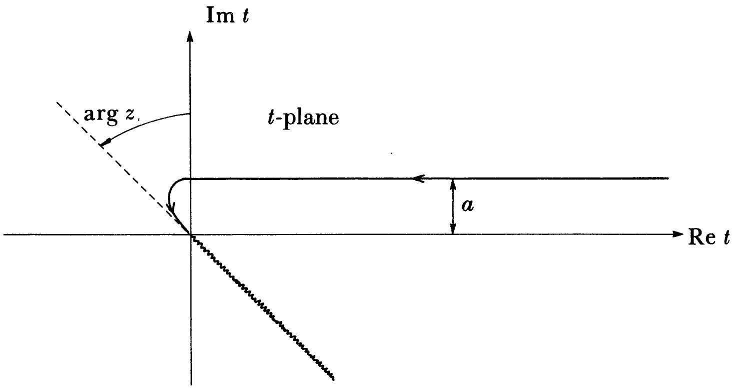

FIGURE 5. The contour of integration for the integral (A 4).

APPENDIX B

Here we derive an alternative representation for the double integral

$$I = \frac{1}{\pi \mathrm{i}} \int_{\infty + \mathrm{i}a}^{0} \exp\{-\tfrac{1}{2}\mathrm{i}(t + k^2(r^2 + r_0^2)/t)\} \Big\{ \mathrm{e}^{\mathrm{i}(k^2 r r_0/t)\cos\psi} \times \int_{\infty}^{k^2 r r_0/t} \mathrm{e}^{-\mathrm{i}x\cos\psi} H_\nu^{(2)}(x)\,\mathrm{d}x \Big\} \frac{\mathrm{d}t}{t}, \quad 0 < \nu < 1. \tag{B 1}$$

Let
$$I_1 = \mathrm{e}^{\mathrm{i}(k^2 r r_0/t)\cos\psi} \int_{\infty}^{k^2 r r_0/t} \mathrm{e}^{-\mathrm{i}x\cos\psi} H_\nu^{(2)}(x)\,\mathrm{d}x; \tag{B 2}$$

then by using the integral representation (Lebedev 1965)

$$H_\nu^{(2)}(z) = -2\frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\nu\pi}}{\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-\mathrm{i}x\cosh u} \cosh \nu u\,\mathrm{d}u, \quad \mathrm{Im}\,z \leqslant 0, \quad |\mathrm{Re}\,\nu| < 1, \tag{B 3}$$

the expression (B 2) can be written (because $\mathrm{Im}\,(k^2rr_0/t) \leqslant 0$) as

$$I_1 = -\frac{2e^{\frac{1}{2}\nu\pi i}}{\pi i}\int_0^\infty \left\{e^{ik^2rr_0/t}\int_\infty^{k^2rr_0/t} e^{-ix(\cos\psi+\cosh u)}\,dx\right\}\cosh\nu u\,du,$$

$$= -\frac{2e^{\frac{1}{2}\nu\pi i}}{\pi i}\int_0^\infty \frac{e^{-(ik^2rr_0/t)\cosh u}}{(\cosh u+\cos\psi)}\cosh\nu u\,du. \qquad (B\ 4)$$

Substituting the last expression (B 4) into (B 1) and interchanging the order of integration gives

$$I = -\frac{2e^{\frac{1}{2}\nu\pi i}}{\pi}\int_0^\infty \frac{\cosh\nu u}{\cosh u+\cos\psi}\left\{\frac{1}{\pi i}\int_{\infty+ia}^0 \left\{\exp\tfrac{1}{2}i\left(t+\frac{k^2(r^2+r_0^2+2rr_0\cosh u)}{t}\right)\right\}\frac{dt}{t}\right\}du,$$

$$= -\frac{2e^{\frac{1}{2}\nu\pi i}}{\pi}\int_0^\infty \frac{\cosh\nu u}{\cosh u+\cos\psi}H_0^{(2)}(k\surd(r^2+r_0^2+2rr_0\cosh u)]\,du,$$

$$= -\frac{2e^{\frac{1}{2}\nu\pi i}}{\pi}\int_0^\infty \frac{\cosh\nu u}{\cosh u+\cos\psi}H_0^{(2)}[kR(\pi-iu)]\,du. \qquad (B\ 5)$$

Appendix C

Here we evaluate the integral

$$I = \frac{1}{\pi i}\int_{\infty+ia}^0 \exp\{-\tfrac{1}{2}i(t+k^2(r^2+r_0^2)/t)\} \times e^{ik^2rr_0\cos\psi/t}\int_\infty^{k\surd(2rr_0/t)\,|\cos\frac{1}{2}\psi|} e^{-iv^2}\,dv\,\frac{dt}{t}. \qquad (C\ 1)$$

We can rewrite this as ($v = k\surd(2rr_0/t)\,|\cos\frac{1}{2}\psi|\,u$)

$$I = \frac{1}{\pi i}\int_{\infty+ia}^0 \exp\{-\tfrac{1}{2}i(t+k^2(r^2+r_0^2-2rr_0\cos\psi)/t)\}\,|\cos\tfrac{1}{2}\psi|\,k\surd(2rr_0) \times\int_\infty^1 \exp\{-(i2k^2rr_0\cos^2\tfrac{1}{2}\psi\,u^2/t)\}\,du\,\frac{dt}{t^{\frac{3}{2}}},$$

$$= k\surd(2rr_0)\,|\cos\tfrac{1}{2}\psi|\int_\infty^1 \frac{1}{\pi i}\int_{\infty+ia}^0 \exp\{-\tfrac{1}{2}i(t+k^2(R^2(\psi) + 4rr_0\cos^2\tfrac{1}{2}\psi\,u^2)/t\}\,\frac{dt}{t^{\frac{3}{2}}}\,du,$$

$$= k\surd(2rr_0)\,|\cos\tfrac{1}{2}\psi|\int_\infty^1 e^{-\frac{1}{4}i\pi}\,\frac{H_{\frac{1}{2}}^{(2)}[k(R^2(\psi)+4rr_0\cos^2\frac{1}{2}\psi\,u^2)^{\frac{1}{2}}]}{k^{\frac{1}{2}}(R^2(\psi)+4rr_0\cos^2\frac{1}{2}\psi\,u^2)^{\frac{1}{4}}}\,du.$$

But because $H^{(2)}_{\frac{1}{2}}(z) = \mathrm{i}(2/\pi z)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}z}$, then we get

$$I = \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\, 2\surd(rr_0/\pi)\,|\cos\tfrac{1}{2}\psi| \int_\infty^1 \frac{\exp\{-\mathrm{i}k(R^2(\psi)+4rr_0\cos^2\frac{1}{2}\psi\, u^2)^{\frac{1}{2}}\}}{(R^2(\psi)+4rr_0\cos^2\frac{1}{2}\psi\, u^2)^{\frac{1}{2}}}.$$

Now let $2\surd(rr_0)\,|\cos\frac{1}{2}\psi|\,u = R(\psi)\sinh\xi$, then

$$I = \frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}}{\surd\pi}\int_\infty^{\xi(\psi)} \mathrm{e}^{-\mathrm{i}kR(\psi)\cosh\xi}\,\mathrm{d}\xi, \tag{C 2}$$

where
$$\xi(\psi) = \operatorname{arsinh}\left\{\frac{2\surd(rr_0)\,|\cos\frac{1}{2}\psi|}{R(\psi)}\right\}. \tag{C 3}$$

Appendix D

We shall here give an alternative representation for the integral

$$I = \int_0^\infty \frac{\cosh\frac{1}{2}t}{\cosh t+\cos\psi}\, H_0^{(2)}[k\surd(r^2+r_0^2+2rr_0\cosh t)]\,\mathrm{d}t,$$

$$= 2\int_0^\infty \frac{H_0^{(2)}[k\surd(r^2+r_0^2+2rr_0\cosh t)]}{\cosh t+\cos\psi}\,\frac{\mathrm{d}}{\mathrm{d}t}(\sinh\tfrac{1}{2}t)\,\mathrm{d}t.$$

Let $v = 2\surd(rr_0)\sinh\frac{1}{2}t$, then because $\cosh t = 1+2\sinh^2\frac{1}{2}t$ we get

$$I = 2\surd(rr_0)\int_0^\infty \frac{H_0^{(2)}[k\surd((r+r_0)^2+v^2)]}{v^2+4rr_0\cos^2\frac{1}{2}\psi}\,\mathrm{d}v. \tag{D 1}$$

We now use the representation, see Appendix A,

$$H_0^{(2)}(x) = \frac{1}{\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^0 \mathrm{e}^{\frac{1}{2}(t-x^2/t)}\,\frac{\mathrm{d}t}{t},$$

in the expression (D 1) giving

$$I = \frac{1}{\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^0 \mathrm{e}^{\frac{1}{2}t-k^2(r+r_0)^2/2t}\, 2\surd(rr_0)\int_0^\infty \frac{\mathrm{e}^{-k^2v^2/2t}\,\mathrm{d}v}{v^2+4rr_0\cos^2\frac{1}{2}\psi}\,\frac{\mathrm{d}t}{t}.$$

Now

$$\int_0^\infty \frac{\mathrm{e}^{-\alpha u^2}\,\mathrm{d}u}{u^2+A^2} = \int_0^\infty \mathrm{e}^{-\alpha u^2}\,\mathrm{d}u\int_0^\infty \mathrm{e}^{-(u^2+A^2)\,t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-A^2t}\,\mathrm{d}t\int_0^\infty \mathrm{e}^{-(t+\alpha)\,u^2}\,\mathrm{d}u$$

$$= \tfrac{1}{2}\surd\pi\int_0^\infty \frac{\mathrm{e}^{-A^2t}}{\surd(t+\alpha)}\,\mathrm{d}t = \tfrac{1}{2}\surd\pi\,\mathrm{e}^{A^2\alpha}\int_\alpha^\infty \mathrm{e}^{-A^2x}\,\frac{\mathrm{d}x}{\surd x} = \frac{\surd(\pi\alpha)}{|A|}\,\mathrm{e}^{A^2\alpha}\int_{|A|}^\infty \mathrm{e}^{-\alpha w^2}\,\mathrm{d}w.$$

Hence

$$2\surd(rr_0)\int_0^\infty \frac{e^{-k^2v^2/2t}\,dv}{v^2+4rr_0\cos^2\frac{1}{2}\psi} = \tfrac{1}{2}\surd\pi\,\frac{k\,e^{\frac{1}{2}(4k^2rr_0\cos^2\frac{1}{2}\psi)}}{t^{\frac{1}{2}}|\cos\frac{1}{2}\psi|}\int_{2\surd(rr_0)\,|\cos\frac{1}{2}\psi|}^\infty e^{-k^2w^2/2t}\,dw,$$

so that

$$I = \surd(\tfrac{1}{2}\pi)\,\frac{k}{|\cos\frac{1}{2}\psi|}$$

$$\times\int_{2\surd(rr_0)\,|\cos\frac{1}{2}\psi|}^\infty \frac{1}{\pi i}\int_{a-i\infty}^{0} \exp\{\tfrac{1}{2}(t-k^2\{r^2+r_0^2-2rr_0\cos\psi+w^2\}/t)\}\,\frac{dt}{t^{\frac{3}{2}}}\,dw,$$

$$I = \surd(\tfrac{1}{2}\pi)\,\frac{k}{|\cos\frac{1}{2}\psi|}\int_{2\surd(rr_0)\,|\cos\frac{1}{2}\psi|}^\infty \frac{H^{(2)}_{\frac{1}{2}}\{k\surd(r^2+r_0^2-2rr_0\cos\psi+w^2)\}}{k^{\frac{1}{2}}(r^2+r_0^2-2rr_0\cos\psi+w^2)^{\frac{1}{4}}}\,dw,$$

(see Appendix A). Thus,

$$I = \frac{i}{|\cos\frac{1}{2}\psi|}\int_{2\surd(rr_0)\,|\cos\frac{1}{2}\psi|}^\infty \frac{\exp\{-ik\surd(r^2+r_0^2-2rr_0\cos\psi+w^2)\}}{\surd(r^2+r_0^2-2rr_0\cos\psi+w^2)}\,dw.$$

In the last integral we make the change of variable $w = R(\psi)\sinh\xi$ so that

$$I = \frac{i}{|\cos\frac{1}{2}\psi|}\int_{\xi(\psi)}^\infty e^{-ikR(\psi)\cosh\xi}\,d\xi,$$

where
$$\xi(\psi) = \operatorname{arsinh}\left[\frac{2\surd(rr_0)\,|\cos\frac{1}{2}\psi|}{R(\psi)}\right].$$

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Note added in proof (16 March 1987). Finally we remark that the present method for obtaining solutions to cylindrical-wave diffraction problems can be used to obtain analogous results for spherical-wave diffraction by a rational wedge.