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The Lax pairs for elliptic C_n and BC_n Ruijsenaars–Schneider models and their spectral curves

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We study the elliptic C_n and BC_n Ruijsenaars–Schneider models which are elliptic generalization of systems given in previous paper by the present authors [Chen *et al.*, J. Math. Phys. **41**, 8132 (2000)]. The Lax pairs for these models are constructed by Hamiltonian reduction technology. We show that the spectral curves can be parametrized by the involutive integrals of motion for these models. Taking nonrelativistic limit and scaling limit, we verify that they lead to the systems corresponding to Calogero–Moser and Toda types. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1389091]

I. INTRODUCTION

The Ruijsenaars–Schneider (RS) and Calogero–Moser (CM) models, as integrable many-body models, recently have attracted remarkable attention and have been extensively studied. They describe one-dimensional N-particle systems with pairwise interaction. Their importance lies in various fields ranging from lattice models in statistics physics,[1,2] to the field theory and gauge theory,[3,4] to the Seiberg–Witten theory,[5] etc. In particular, the study of the RS model is of great importance since it is the integrable relativistic generalization of the corresponding CM model.[6,7]

The Lax pairs for the CM models in various root systems have been constructed by Olshanetsky, and Perelomov[8] using reduction on symmetric space, and are further given by Inozemtsev in Ref. 9 without spectral parameter. It was almost 20 years until D'Hoker and Phong[10] constructed the Lax pairs with a spectral parameter for each of the finite dimensional Lie algebras, and the untwisted and twisted Calogero–Moser systems were introduced. Subsequently, Bordner *et al.*[11–13] succeeded in giving two types of Lax pairs associated to all of the Lie algebra: the root type and the minimal type, with and without spectral parameters. Even for all of the Coxeter group, the construction has been obtained in Ref. 14. In Ref. 15, Hurtubise and Markman utilized a so-called "structure group," which combines a semisimple group and Weyl group, to construct CM systems associated with the Hitchin system, which in some degree generalizes the result of Refs. 10–14. Furthermore, the quantum version of the generalization has been developed in Refs. 16 and 17 at least for degenerate potentials of trigonometry after the works of Olshanetsky and Perelomov.[18]

So far as for the RS model, only the Lax pair of the A_{N-1} type RS model was obtained[6,2,19–22] and succeeded in recovering it by applying the Hamiltonian reduction procedure on a two-dimensional current group.[23] Although the commutative operators for the RS model based on various type Lie algebras have been given by Komori and co-workers,[24,25] Diejen,[26,27] and Hasegawa *et al.*,[1,28] the Lax integrability (or Lax pair representation) of the other type of RS model is still an open problem[5] except for a few degenerate cases.[27,30]

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In Refs. 29 and 30, we succeeded in constructing the Lax pair for C_n and BC_n RS systems only with the degenerate case (without spectral parameters). The r-matrix structure for them have been derived by Avan *et al.*[31] In this paper, we study the Lax pair for the most general C_n and BC_n RS models—the elliptic C_n and BC_n RS models. We shall give the explicit forms of Lax pairs for these systems by Hamiltonian reduction. We calculate the spectral curves for these systems, which are shown to be parametrized by a set of involutive integrals of motion. In particular, taking their nonrelativistic limit and scaling limit, we shall recover the systems of corresponding CM and Toda types, respectively. The other various degenerate cases are also be discussed and the connection between the Lax pair with a spectral parameter and the one without the spectral parameter is commented on.

The paper is organized as follows. The basic materials of the A_{N-1} RS model are reviewed in Sec. II, where we propose a Lax pair associating with the Hamiltonian which has a reflection symmetry with respect to the particles in the origin. This includes construction of a Lax pair for the A_{N-1} RS system together with its symmetry analysis. The main results are shown in Secs. III and IV. In Sec. III, we present the Lax pairs for the elliptic C_n and BC_n RS models by reducing the A_{N-1} RS model. The explicit forms for the Lax pairs are given in Sec. IV. Section V is devoted to deriving the spectral curves for these systems and their nonrelativistic counterpart, the Calogero–Moser model and scaling limit of the Toda model. Section VI shows the various degenerate limits: the trigonometric, hyperbolic, and rational cases. The last section is a brief summary and discussion.

II. THE A_{N-1}-TYPE RUIJSENAARS–SCHNEIDER MODEL

As a relativistic-invariant generalization of the A_{N-1}-type nonrelativistic Calogero–Moser model, the A_{N-1}-type Ruijsenaars–Schneider systems are completely integrable. The system's integrability was first shownd by Ruijsenaars.[6,7] The Lax pair for this model has been constructed in Refs. 6, 2, 19–22. Recent progress has shown that the compactification of higher dimension SUSY Yang–Mills theory and Seiberg–Witten theory can be described by this model.[5] Instanton correction of the prepotential associated with the sl_2 RS system has been calculated in Ref. 32.

A. Model and equations of motion

Let us briefly give the basics of this model. In terms of the canonical variables p_i, $x_i(i,j=1,\ldots,N)$ enjoying the canonical Poisson bracket

$$\{p_i,p_j\}=\{x_i,x_j\}=0,\quad \{x_i,p_j\}=\delta_{ij}, \tag{II.1}$$

the Hamiltonian of the A_{N-1} RS system reads

$$\mathcal{H}_{A_{N-1}}=\sum_{i=1}^{N}\left(e^{p_i}\prod_{k\neq i}f(x_i-x_k)+e^{-p_i}\prod_{k\neq i}g(x_i-x_k)\right), \tag{II.2}$$

where

$$\begin{gathered} f(x):=\frac{\sigma(x-\gamma)}{\sigma(x)}, \\ g(x):=f(x)\big|_{\gamma\to-\gamma},\quad x_{ik}:=x_i-x_k, \end{gathered} \tag{II.3}$$

and γ denotes the coupling constant. Here, $\sigma(x)$ is the Weierstrass σ function which is an entire, odd and quasiperiodic function with a fixed pair of the primitive quasiperiods $2\omega_1$ and $2\omega_3$. It can be defined as the infinite product

$$\sigma(x)=x\prod_{w\in\Gamma\setminus\{0\}}\left(1-\frac{x}{w}\right)\exp\left[\frac{x}{w}+\frac{1}{2}\left(\frac{x}{w}\right)^2\right],$$

where $\Gamma = 2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}$ is the corresponding period lattice. Defining a third dependent quasiperiod $2\omega_2 = -2\omega_1 - 2\omega_3$, one has

$$\sigma(x+2\omega_k) = -\sigma(x)e^{2\eta_k(x+\omega_k)}, \quad \zeta(x+2\omega_k) = \zeta(x) + 2\eta_k, \quad k=1,2,3,$$

where

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}, \quad \wp(x) = -\zeta'(x),$$

and $\eta_k = \zeta(\omega_k)$ satisfy $\eta_1\omega_3 - \eta_3\omega_1 = \pi i/2$.

Notice that in Ref. 6 Ruijsenaars used another "gauge" of the momenta such that two are connected by the following canonical transformation:

$$x_i \to x_i, \quad p_i \to p_i + \frac{1}{2}\ln\prod_{j\neq i}^{N} \frac{f(x_{ij})}{g(x_{ij})}. \tag{II.4}$$

The Lax matrix for this model has the form(for the general elliptic case)

$$L(\lambda) = \sum_{i,j=1}^{N} \frac{\Phi(x_i - x_j + \gamma, \lambda)}{\Phi(\gamma,\lambda)} \exp(p_j) b_j E_{ij}, \tag{II.5}$$

where

$$\Phi(x,\lambda) := \frac{\sigma(x+\lambda)}{\sigma(x)\sigma(\lambda)}, \quad b_j := \prod_{k\neq j} f(x_j - x_k), \quad (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \tag{II.6}$$

and λ is the spectral parameter. It is shown in Refs. 21, 33, 34 that the Lax operator satisfies the quadratic fundamental Poisson bracket

$$\{L_1, L_2\} = L_1 L_2 a_1 - a_2 L_1 L_2 + L_2 s_1 L_1 - L_1 s_2 L_2, \tag{II.7}$$

where $L_1 = L_{A_{N-1}} \otimes Id, L_2 = Id \otimes L_{A_{N-1}}$ and the four matrices read

$$\begin{gathered} a_1 = a + w, \quad s_1 = s - w, \\ a_2 = a + s - s^* - w, \quad s_2 = s^* + w. \end{gathered} \tag{II.8}$$

The forms of a,s,w are

$$\begin{gathered} a(\lambda,\mu) = -\zeta(\lambda-\mu)\sum_{k=1}^{N} E_{kk}\otimes E_{kk} - \sum_{k\neq j} \Phi(x_j - x_k, \lambda-\mu) E_{jk}\otimes E_{kj}, \\ s(\lambda) = \zeta(\lambda)\sum_{k=1}^{N} E_{kk}\otimes E_{kk} + \sum_{k\neq j} \Phi(x_j - x_k, \lambda) E_{jk}\otimes E_{kk}, \\ w = \sum_{k\neq j} \zeta(x_k - x_j) E_{kk}\otimes E_{jj}. \end{gathered} \tag{II.9}$$

The asterisk means

$$r^* = \Pi r \Pi \ \text{ with } \ \Pi = \sum_{k,j=1}^{N} E_{kj}\otimes E_{jk}. \tag{II.10}$$

Noticing that

$$L(\lambda)^{-1}{}_{ij}=\frac{\sigma(\gamma+\lambda)\sigma(\lambda+(N-1)\gamma)}{\sigma(\lambda)\sigma(\lambda+N\gamma)}\times\sum_{i,j=1}^{N}\frac{\Phi(x_i-x_j-\gamma,\lambda+N\gamma)}{\Phi(-\gamma,\lambda+N\gamma)}\exp(-p_i)b_j'E_{ij}, \tag{II.11}$$

$$b_j':=\prod_{k\neq j}g(x_j-x_k) \tag{II.12}$$

(the proof of the above-given identity is sketched in the Appendix) one can get the characteristic polynomials of $L_{A_{N-1}}$ (Refs. 35 and 34),

$$\det(L(\lambda)-v\cdot Id)=\sum_{j=0}^{N}\Phi(\gamma,\lambda)^{-j}(-v)^{N-j}\frac{\mathcal{H}_j^+}{\sigma^j(\gamma)}\times\frac{\sigma(\lambda+j\gamma)}{\sigma(\lambda)}, \tag{II.13}$$

and that of $L_{A_{N-1}}^{-1}$ by using formula given in Eq. (A8),

$$\det\left(\frac{\sigma(\lambda)\sigma(\lambda-N\gamma)}{\sigma(\lambda-\gamma)\sigma(\lambda-(N-1)\gamma)}\times L(\lambda-N\gamma)^{-1}-v\cdot Id\right)$$
$$=\sum_{j=0}^{N}\Phi(-\gamma,\lambda)^{-j}(-v)^{N-j}\times\frac{(\mathcal{H}_j^-)}{\sigma^j(-\gamma)}\frac{\sigma(\lambda-j\gamma)}{\sigma(\lambda)}, \tag{II.14}$$

where $(\mathcal{H}_0^{\pm})_{A_{N-1}}=(\mathcal{H}_N^{\pm})_{A_{N-1}}=1$, and

$$(\mathcal{H}_i^+)_{A_{N-1}}=\sum_{\substack{J\subset\{1,\ldots,N\}\\|J|=i}}\exp\left(\sum_{j\in J}p_j\right)\prod_{\substack{j\in J\\k\in\{1,\ldots,N\}\setminus J}}f(x_j-x_k), \tag{II.15}$$

$$(\mathcal{H}_i^-)_{A_{N-1}}=\sum_{\substack{J\subset\{1,\ldots,N\}\\|J|=i}}\exp\left(\sum_{j\in J}-p_j\right)\prod_{\substack{j\in J\\k\in\{1,\ldots,N\}\setminus J}}g(x_j-x_k). \tag{II.16}$$

Defining

$$(\mathcal{H}_i)_{A_{N-1}}=(\mathcal{H}_i^+)_{A_{N-1}}+(\mathcal{H}_i^-)_{A_{N-1}}, \tag{II.17}$$

from the fundamental Poisson bracket Eq. (II.7), we can verify that

$$\{(\mathcal{H}_i)_{A_{N-1}},(\mathcal{H}_j)_{A_{N-1}}\}=\{(\mathcal{H}_i^{\varepsilon})_{A_{N-1}},(\mathcal{H}_j^{\varepsilon'})_{A_{N-1}}\}=0,\quad \varepsilon,\varepsilon'=\pm,\quad i,j=1,\ldots,N. \tag{II.18}$$

In particular, the Hamiltonian Eq. (II.2) can be rewritten as

$$\mathcal{H}_{A_{N-1}}\equiv\mathcal{H}_1=(\mathcal{H}_1^+)_{A_{N-1}}+(\mathcal{H}_1^-)_{A_{N-1}}$$
$$=\sum_{j=1}^{N}(e^{p_j}b_j+e^{-p_j}b_j')$$
$$=\mathrm{Tr}\left(L(\lambda)+\frac{\sigma(\lambda)\sigma(\lambda+N\gamma)}{\sigma(\gamma+\lambda)\,\sigma(\lambda+(N-1)\gamma)}L(\lambda)^{-1}\right). \tag{II.19}$$

It should be remarked that the set of integrals of motion Eq. (II.17) has a reflection symmetry which is the key property for the later reduction to C_n and BC_n cases, i.e., if we set

$$p_i \leftrightarrow -p_i, \quad x_i \leftrightarrow -x_i, \tag{II.20}$$

then the Hamiltonians flows $(\mathcal{H}_i)_{A_{N-1}}$ are invariant with respect to this symmetry.

The canonical equations of motion associated with the Hamiltonian flows $\mathcal{H}_1^+$ in its generic (elliptic) form read

$$\ddot{x}_i = \sum_{j \neq i} \dot{x}_i \dot{x}_j (V(x_{ij}) - V(x_{ji})), \quad i = 1, \ldots, N, \tag{II.21}$$

where the potential $V(x)$ is given by

$$V(x) = \zeta(x) - \zeta(x + \lambda), \tag{II.22}$$

in which $\zeta(x) = \sigma'(x)/\sigma(x)$. Here, $x_i = x_i(t)$, $p_i = p_i(t)$, and the superimposed dot denotes t differentiation.

B. The construction of Lax pair for the A_{N-1} RS model

As for the A_{N-1} RS model, a generalized Lax pair has been given in Refs. 6, 2, and 19–22. But there is a common character that the time evolution of the Lax matrix $L_{A_{N-1}}$ is associated with the Hamiltonian $(\mathcal{H}_1^+)_{A_{N-1}}$. We will see in Sec. III that the Lax pair cannot reduce from that kind of forms directly. Instead, we give a new Lax pair in which the evolution of $L_{A_{N-1}}$ is associated with the Hamiltonian $\mathcal{H}_{A_{N-1}}$,

$$\dot{L}_{A_{N-1}} = \{L_{A_{N-1}}, \mathcal{H}_{A_{N-1}}\} = [M_{A_{N-1}}, L_{A_{N-1}}], \tag{II.23}$$

where $M_{A_{N-1}}$ can be constructed with the help of (r,s) matrices as follows:

$$M_{A_{N-1}} = \mathrm{Tr}_2\left((s_1 - a_2)\left(1 \otimes \left(L(\lambda) - \frac{\sigma(\lambda)\sigma(\lambda + N\gamma)}{\sigma(\gamma+\lambda)\,\sigma(\lambda + (N-1)\gamma)} L(\lambda)^{-1} \right)\right)\right). \tag{II.24}$$

The explicit expression of entries for $M_{A_{N-1}}$ is

$$M_{ij} = \Phi(x_{ij}, \lambda) e^{p_j} b_j - \Phi(x_{ij}, \lambda + N\gamma) e^{-p_i} b'_j, \quad i \neq j, \tag{II.25}$$

$$M_{ii} = (\zeta(\lambda) + \zeta(\gamma)) e^{p_i} b_i - (\zeta(\lambda+\gamma) - \zeta(\gamma)) e^{-p_i} b'_i \tag{II.26}$$

$$+ \sum_{j \neq i} ((\zeta(x_{ij} + \gamma) - \zeta(x_{ij})) e^{p_j} b_j$$

$$+ \frac{\Phi(x_{ji} + \gamma, \lambda)}{\Phi(\gamma, \lambda)} \Phi(x_{ij}, \lambda + N\gamma) e^{-p_i} b'_j). \tag{II.27}$$

III. HAMILTONIAN REDUCTION OF C_n AND BC_n RS MODELS FROM A_{N-1}-TYPE MODELS

Let us first mention some results about the integrability of Hamiltonian (II.2). In Ref. 7 Ruijsenaars demonstrated that the symplectic structure of the C_n- and BC_n-types of RS systems can be proved integrable by embedding their phase space to a submanifold of the A_{2n-1} and A_{2n} type RS ones, respectively, while in Refs. 26, 27, and 25, Diejen and Komori, respectively, gave a series of commuting difference operators which led to their quantum integrability. However, there are not any results about their Lax representations so far except for the special degenerate case.[29,30] In this section, we concentrate our treatment on the exhibition of the explicit forms for general C_n and BC_n RS systems.

For the convenience of analysis of symmetry, let us first give vector representation of A_{N-1} Lie algebra. Introducing an N-dimensional orthonormal basis of $\mathbb{R}^N$,

$$e_j \cdot e_k = \delta_{j,k}, \quad j,k=1,\ldots,N. \tag{III.1}$$

Then the sets of roots and vector weights of A_{N-1} type are,

$$\Delta = \{e_j - e_k : j,k=1,\ldots,N\}, \tag{III.2}$$

$$\Lambda = \{e_j : j=1,\ldots,N\}. \tag{III.3}$$

The dynamical variables are canonical coordinates $\{x_j\}$ and their canonical conjugate momenta $\{p_j\}$ with the Poisson brackets of Eq. (II.1). In a general sense, we denote them by N-dimensional vectors x and p,

$$x=(x_1,\ldots,x_N)\in \mathbb{R}^N, \quad p=(p_1,\ldots,p_N)\in \mathbb{R}^N,$$

so that the scalar products of x and p with the roots $\alpha \cdot x$, $p \cdot \beta$, etc., can be defined. The Hamiltonian of Eq. (II.2) can be rewritten as

$$\mathcal{H}_{A_{N-1}} = \sum_{\mu\in\Lambda} \left(\exp(\mu\cdot p) \prod_{\Delta \ni \beta=\mu-\nu} f(\beta\cdot x) + \exp(-\mu\cdot p) \prod_{\Delta\ni\beta=-\mu+\nu} g(\beta\cdot x) \right). \tag{III.4}$$

Here, the condition $\Delta \ni \beta = \mu - \nu$ means that the summation is over roots β such that for $\exists \nu \in \Lambda$,

$$\mu - \nu = \beta \in \Delta. \tag{III.5}$$

So does for $\Delta \ni \beta = -\mu + \nu$.

A. The C_n model

The set of C_n roots consists of two parts, long roots and short roots:

$$\Delta_{C_n} = \Delta_L \cup \Delta_S, \tag{III.6}$$

in which the roots are conveniently expressed in terms of an orthonormal basis of $\mathbb{R}^n$:

$$\begin{aligned} \Delta_L &= \{\pm 2 e_j : \quad j=1,\ldots,n\}, \\ \Delta_S &= \{\pm e_j \pm e_k, : \quad j,k=1,\ldots,n\}. \end{aligned} \tag{III.7}$$

In the vector representation, vector weights Λ are

$$\Lambda_{C_n} = \{e_j, -e_j : \quad j=1,\ldots,n\}. \tag{III.8}$$

The Hamiltonian of the C_n model is given by

$$\mathcal{H}_{C_n} = \frac{1}{2} \sum_{\mu\in\Lambda_{C_n}} \left(\exp(\mu\cdot p) \prod_{\Delta_{C_n} \ni \beta=\mu-\nu} f(\beta\cdot x) + \exp(-\mu\cdot p) \prod_{\Delta_{C_n}\ni\beta=-\mu+\nu} g(\beta\cdot x) \right). \tag{III.9}$$

From the above-mentioned data, we notice that either for A_{N-1} or C_n Lie algebra, any root $\alpha \in \Delta$ can be constructed in terms with vector weights as $\alpha = \mu - \nu$ where $\mu, \nu \in \Lambda$. By simple comparison of representation between A_{N-1} and C_n, one can find that if replacing e_{j+n} with $-e_j$

in the vector weights of A_{2n-1} algebra, we can obtain the vector weights of C_n algebra. This also holds for the corresponding roots. This gives us a hint that it is possible to get the C_n model by this kind of reduction.

For the A_{2n-1} model let us set restrictions on the vector weights with

$$e_{j+n}+e_j=0, \quad \text{for } j=1,\ldots,n, \tag{III.10}$$

which correspond to the following constraints on the phase space of the A_{2n-1}-type RS model with

$$\begin{aligned} G_i &\equiv (e_{i+n}+e_i)\cdot x = x_i + x_{i+n}=0, \\ G_{i+n} &\equiv (e_{i+n}+e_i)\cdot p = p_i + p_{i+n} = 0, \quad i=1,\ldots,n. \end{aligned} \tag{III.11}$$

Following Dirac's method,[36] we can show

$$\{G_i,\mathcal{H}_{A_{2n-1}}\}\simeq 0, \quad \text{for } \forall i \in \{1,\ldots,2n\}, \tag{III.12}$$

i.e., $\mathcal{H}_{A_{2n-1}}$ is the first class Hamiltonian corresponding to the constraints in Eq. (III.11). Here the ''weak equal'' symbol$\simeq$represents that only after calculating the result of the left-hand side of the identity could we use the conditions of constraints. It should be pointed out that the most necessary condition ensuring Eq. (III.12) is the symmetry property of formula (II.20) for the Hamiltonian Eq. (II.2). So for an arbitrary dynamical variable A, we have

$$\begin{aligned} \dot{A} &= \{A,\mathcal{H}_{A_{2n-1}}\}_D = \{A,\mathcal{H}_{A_{2n-1}}\} - \{A,G_i\}\Delta_{ij}^{-1}\{G_j,\mathcal{H}_{A_{2n-1}}\} \\ &\simeq \{A,\mathcal{H}_{A_{2n-1}}\}, \quad i,j=1,\ldots,2n, \end{aligned} \tag{III.13}$$

where

$$\Delta_{ij} = \{G_i,G_j\} = 2\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}, \tag{III.14}$$

and $\{,\}_D$ denotes the Dirac bracket. By straightforward calculation, we have the nonzero Dirac brackets of

$$\begin{aligned} \{x_i,p_j\}_D &= \{x_{i+n},p_{j+n}\}_D = \tfrac{1}{2}\delta_{i,j}, \\ \{x_i,p_{j+n}\}_D &= \{x_{i+n},p_j\}_D = -\tfrac{1}{2}\delta_{i,j}. \end{aligned} \tag{III.15}$$

Using the above-mentioned data together with the fact that $\mathcal{H}_{A_{N-1}}$ is the first class Hamiltonian-[see Eq. (III.12)], we can directly obtain a Lax representation of the C_n RS model by imposing constraints G_k on Eq. (II.23),

$$\begin{aligned} \{L_{A_{2n-1}},\mathcal{H}_{A_{2n-1}}\}_D &= \{L_{A_{2n-1}},\mathcal{H}_{A_{2n-1}}\}|_{G_k,k=1,\ldots,2n}, \\ &= [M_{A_{2n-1}},L_{A_{2n-1}}]|_{G_k,k=1,\ldots,2n} = [M_{C_n},L_{C_n}], \end{aligned} \tag{III.16}$$

$$\{L_{A_{2n-1}},\mathcal{H}_{A_{2n-1}}\}_D = \{L_{C_n},\mathcal{H}_{C_n}\}, \tag{III.17}$$

where

$$\mathcal{H}_{C_n} = \tfrac{1}{2}\mathcal{H}_{A_{2n-1}}|_{G_k,k=1,\ldots,2n},$$

$$L_{C_n}=L_{A_{2n-1}}|_{G_k,k=1,\ldots,2n},$$

$$M_{C_n}=M_{A_{2n-1}}|_{G_k,k=1,\ldots,2n}, \tag{III.18}$$

so that

$$\dot{L}_{C_n}=\{L_{C_n},\mathcal{H}_{C_n}\}=[M_{C_n},L_{C_n}]. \tag{III.19}$$

Nevertheless, the $(\mathcal{H}_1^+)_{A_{N-1}}$ is not the first class Hamiltonian, so the Lax pair given by many authors previously cannot reduce to the C_n case directly in this way.

B. The BC_n model

The BC_n root system consists of three parts: long, middle, and short roots:

$$\Delta_{BC_n}=\Delta_L\cup\Delta\cup\Delta_S, \tag{III.20}$$

in which the roots are conveniently expressed in terms of an orthonormal basis of $\mathbb{R}^n$:

$$\Delta_L=\{\pm 2e_j: \quad j=1,\ldots,n\},$$

$$\Delta=\{\pm e_j\pm e_k: \quad j,k=1,\ldots,n\}, \tag{III.21}$$

$$\Delta_S=\{\pm e_j: \quad j=1,\ldots,n\}.$$

In the vector representation, vector weights Λ can be

$$\Lambda_{BC_n}=\{e_j,-e_j,0: \quad j=1,\ldots,n\}. \tag{III.22}$$

The Hamiltonian of the BC_n model is given by

$$\mathcal{H}_{BC_n}=\frac{1}{2}\sum_{\mu\in\Lambda_{BC_n}}\left(\exp(\mu\cdot p)\prod_{\Delta_{BC_n}\ni\beta=\mu-\nu}f(\beta\cdot x)+\exp(-\mu\cdot p)\prod_{\Delta_{BC_n}\ni\beta=-\mu+\nu}g(\beta\cdot x)\right). \tag{III.23}$$

By similar comparison of representations between A_{N-1} and BC_n, one can find that if replacing e_{j+n} with $-e_j$ and e_{2n+1} with 0 in the vector weights of the A_{2n} Lie algebra, we can obtain the vector weights of the BC_n one. The same holds for the corresponding roots. So by the same procedure as the C_n model, we could get the Lax representation of the BC_n model.

For the A_{2n} model, we set restrictions on the vector weights with

$$e_{j+n}+e_j=0 \quad \text{for} \quad j=1,\ldots,n,$$

$$e_{2n+1}=0, \tag{III.24}$$

which correspond to the following constraints on the phase space of the A_{2n}-type RS model with

$$G_i'\equiv(e_{i+n}+e_i)\cdot x=x_i+x_{i+n}=0,$$

$$G_{i+n}'\equiv(e_{i+n}+e_i)\cdot p=p_i+p_{i+n}=0, \quad i=1,\ldots,n,$$

$$G_{2n+1}'\equiv e_{2n+1}\cdot x=x_{2n+1}=0, \tag{III.25}$$

$$G_{2n+2}'\equiv e_{2n+1}\cdot p=p_{2n+1}=0.$$

Similarly, we can show

$$\{G_i,\mathcal{H}_{A_{2n}}\}\simeq 0,\quad \text{for}\quad \forall i\in\{1,\ldots,2n+1,2n+2\}, \tag{III.26}$$

i.e., $\mathcal{H}_{A_{2n}}$ is the first class Hamiltonian corresponding to the above-mentioned constraints Eq. (III.25). So L_{BC_n} and M_{BC_n} can be constructed as follows:

$$\begin{aligned} L_{BC_n}&=L_{A_{2n}}|_{G'_k,k=1,\ldots,2n+2},\\ M_{BC_n}&=M_{A_{2n}}|_{G'_k,k=1,\ldots,2n+2}, \end{aligned} \tag{III.27}$$

while $\mathcal{H}_{BC_n}$ is

$$\mathcal{H}_{BC_n}=\tfrac{1}{2}\mathcal{H}_{A_{2n}}|_{G_k,k=1,\ldots,2n+2}, \tag{III.28}$$

due to the similar derivation of Eqs. (III.13)–(III.19).

IV. LAX REPRESENTATIONS OF THE C_n AND BC_n RS MODELS

A. The C_n model

The Hamiltonian of the C_n RS system is Eq. (III.9), so the canonical equations of motion are

$$\dot{x}_i=\{x_i,\mathcal{H}\}=e^{p_i}b_i-e^{-p_i}b'_i, \tag{IV.1}$$

$$\begin{aligned} \dot{p}_i=\{p_i,\mathcal{H}\}=&\sum_{j\neq i}^{n}(e^{p_j}b_j(h(x_{ji})-h(x_j+x_i))+e^{-p_j}b'_j(\hat{h}(x_{ji})-\hat{h}(x_j+x_i)))\\ &-e^{p_i}b_i\left(2h(2x_i)+\sum_{j\neq i}^{n}(h(x_{ij})+h(x_i+x_j))\right)\\ &-e^{-p_i}b'_i\left(2\hat{h}(2x_i)+\sum_{j\neq i}^{n}(\hat{h}(x_{ij})+\hat{h}(x_i+x_j))\right), \end{aligned} \tag{IV.2}$$

where

$$\begin{aligned} h(x)&:=\frac{d\ln f(x)}{dx},\quad \hat{h}(x):=\frac{d\ln g(x)}{dx},\\ b_i&=f(2x_i)\prod_{k\neq i}^{n}(f(x_i-x_k)f(x_i+x_k)),\\ b'_i&=g(2x_i)\prod_{k\neq i}^{n}(g(x_i-x_k)g(x_i+x_k)). \end{aligned} \tag{IV.3}$$

The Lax matrix for the C_n RS model can be written in the following form:

$$(L_{C_n})_{\mu\nu}=e^{\nu\cdot p}b_\nu\frac{\Phi((\mu-\nu)\cdot x+\gamma,\lambda)}{\Phi(\gamma,\lambda)}, \tag{IV.4}$$

which is a $2n\times 2n$ matrix whose indices are labeled by the vector weights, denoted by $\mu,\nu\in\Lambda_{C_n}$, M_{C_n} can be written as

$$M_{C_n}=D+Y, \tag{IV.5}$$

where D denotes the diagonal part and Y denotes the off-diagonal part

$$Y_{\mu\nu}=e^{\nu\cdot p}b_\nu\Phi(x_{\mu\nu},\lambda)+e^{-\mu\cdot p}b'_\nu\Phi(x_{\mu\nu},\lambda+N\gamma), \tag{IV.6}$$

$$\begin{aligned}D_{\mu\mu}=&(\zeta(\lambda)+\zeta(\gamma))e^{\mu\cdot p}b_\mu-(\zeta(\lambda+\gamma)-\zeta(\gamma))e^{-\mu\cdot p}b'_\mu\\&+\sum_{\nu\neq\mu}\left((\zeta(x_{\mu\nu}+\gamma)-\zeta(x_{\mu\nu}))e^{\nu\cdot p}b_\nu+\frac{\Phi(x_{\nu\mu}+\gamma,\lambda)}{\Phi(\gamma,\lambda)}\Phi(x_{\mu\nu},\lambda+N\gamma)e^{-\mu\cdot p}b'_\nu\right)\end{aligned} \tag{IV.7}$$

and

$$\begin{aligned}b_\mu&=\prod_{\Delta_{C_n}\ni\beta=\mu-\nu}f(\beta\cdot x),\\ b'_\mu&=\prod_{\Delta_{C_n}\ni\beta=\mu-\nu}g(\beta\cdot x),\\ x_{\mu\nu}&:=(\mu-\nu)\cdot x.\end{aligned} \tag{IV.8}$$

The L_{C_n},M_{C_n} satisfies the Lax equation

$$\dot{L}_{C_n}=\{L_{C_n},\mathcal{H}_{C_n}\}=[M_{C_n},L_{C_n}], \tag{IV.9}$$

which is equivalent to the equations of motion (IV.1) and (IV.2). The Hamiltonian $\mathcal{H}_{C_n}$ can be rewritten as the trace of L_{C_n},

$$\mathcal{H}_{C_n}=\operatorname{tr}L_{C_n}=\frac{1}{2}\sum_{\mu\in\Lambda_{C_n}}(e^{\mu\cdot p}b_\mu+e^{-\mu\cdot p}b'_\mu). \tag{IV.10}$$

B. The BC_n model

The Hamiltonian of the BC_n model is expressed in Eq. (III.23), so the canonical equations of motion are

$$\dot{x}_i=\{x_i,\mathcal{H}\}=e^{p_i}b_i-e^{-p_i}b'_i, \tag{IV.11}$$

$$\begin{aligned}\dot{p}_i=\{p_i,\mathcal{H}\}=&\sum_{j\neq i}^{n}(e^{p_j}b_j(h(x_{ji})-h(x_j+x_i))+e^{-p_j}b'_j(\hat{h}(x_{ji})-\hat{h}(x_j+x_i)))\\&-e^{p_i}b_i\left(h(x_i)+2h(2x_i)+\sum_{j\neq i}^{n}(h(x_{ij})+h(x_i+x_j))\right)\\&-e^{-p_i}b'_i\left(\hat{h}(x_i)+2\hat{h}(2x_i)+\sum_{j\neq i}^{n}(\hat{h}(x_{ij})+\hat{h}(x_i+x_j))\right)-b_0(h(x_i)+\hat{h}(x_i)),\end{aligned} \tag{IV.12}$$

where

$$b_i = f(x_i) f(2x_i) \prod_{k \neq i}^{n} (f(x_i - x_k) f(x_i + x_k)),$$

$$b_i' g(x_i) g(2x_i) \prod_{k \neq i}^{n} (g(x_i - x_k) g(x_i + x_k)), \tag{IV.13}$$

$$b_0 = \prod_{i=1}^{n} f(x_i) g(x_i).$$

The Lax pair for the BC_n RS model can be constructed as the form of Eqs. (IV.4)–(IV.8) where one should replace the matrices labels with $\mu, \nu \in \Lambda_{BC_n}$, and roots with $\beta \in \Delta_{BC_n}$.

The Hamiltonian $\mathcal{H}_{BC_n}$ can also be rewritten as the trace of L_{BC_n},

$$\mathcal{H}_{BC_n} = \mathrm{tr}\, L_{BC_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{BC_n}} (e^{\mu \cdot p} b_\mu + e^{-\mu \cdot p} b_\mu'). \tag{IV.14}$$

V. SPECTRAL CURVES OF THE C_n AND BC_n RS SYSTEMS

It has recently been pointed out in Refs. 4, 5, 37 and 38, that the SU(N) RS model is related to five-dimensional gauge theories. In the context of Seiberg–Witten theory, the elliptic RS integrable system can be linked with the relevant low energy effective action when a compactification from five dimension to four dimension is imposed with all of the fields belonging to the adjoint representation of the SU(N) gauge group.[5] More evidence for this correspondence between the SYM and RS models is depicted by calculating instanton correction of prepotential for SU(2) Seiberg-Witten theory in Ref. 32.

As for the spectral curve and its relation to the Seiberg–Witten spectral curve, much progress has been made in the correspondence of ''Calogero-Moser integrable theories and gauge theories.'' See the recent reviews in Refs. 39 and 40, and references therein. In the following we will give the spectral curves for C_n and BC_n systems, which are shown to be parametrized by the integrals of motion of the corresponding system. We will also see that the elliptic Calogero–Moser, Toda (affine and nonaffine) ones are particular limits of these systems.

A. Spectral curve of the C_n RS system

Given the Lax operator with spectral parameter for the Calogero–Moser system and of the RS system associated with Lie algebras $\mathcal{G}$, the spectral curve for the given system is defined as

$$\Gamma : R(v, \lambda) = \det(L(\lambda) - v \cdot Id) \equiv 0. \tag{V.1}$$

It is natural that the function $R(v,z)$ is invariant under time evolution,

$$\frac{\mathrm{d}}{\mathrm{d}t} R(v, \lambda) = \{\mathcal{H}, R(v, \lambda)\} = 0. \tag{V.2}$$

Thus, $R(v,\lambda)$ must be a function of only the n independent integrals of motion, which in super-Yang–Mills theory play the role of *moduli*, parametrizing the supersymmetric vacua of the gauge theory. This has been confirmed in the case of the elliptic Calogero–Moser system for general Lie algebra in Refs. 41 and 42 and in the case of the elliptic SU(N) RS system for the perturbative limit and some nonperturbative special point.[5]

As for the C_n RS system, the spectral curve can be generated by the Lax matrix $L(\lambda)_{C_n}$ as follows:

$$\det(L(\lambda)_{C_n}-v\cdot Id)=\sum_{j=0}^{2n}\frac{(\sigma(\lambda))^{(j-1)}\sigma(\lambda+j\gamma)}{(\sigma(\gamma+\lambda))^{j}}(-v)^{2n-j}(H_j)_{C_n}=0, \tag{V.3}$$

where $(\mathcal{H}_0)_{C_n}=(\mathcal{H}_{2n})_{C_n}=1$ and $(\mathcal{H}_i)_{C_n}=(\mathcal{H}_{2n-i})_{C_n}$ who Poisson commute

$$\{(\mathcal{H}_i)_{C_n},(\mathcal{H}_j)_{C_n}\}=0, \quad i,j=1,\ldots,n. \tag{V.4}$$

This can be deduced by verbose but straightforward calculation to verify that the $(\mathcal{H}_i)_{A_{2n-1}}$, $i=1,\ldots,2n$ is the first class Hamiltonian with respect to the constraints (III.11), using Eqs. (II.18), (III.13) and the first formula of Eq. (III.18).

The explicit form of $(\mathcal{H}_l)_{C_n}$ is

$$(\mathcal{H}_l)_{C_n}=\sum_{\substack{J\subset\{1,\ldots,n\},\ |J|\leqslant l\\ \varepsilon_j=\pm 1, j\in J}}\exp(p_{\varepsilon J})\,F_{\varepsilon J;J^c}\,U_{J^c,l-|J|}, \quad l=1,\ldots,n \tag{V.5}$$

with

$$p_{\varepsilon J}=\sum_{j\in J}\varepsilon_j p_j,$$

$$F_{\varepsilon J;K}=\prod_{\substack{j,j'\in J\\ j<j'}}f^2(\varepsilon_j x_j+\varepsilon_j x_j)\prod_{\underline{j\in J k\in K}}f(\varepsilon_j x_j+x_k)f(\varepsilon_j x_j-x_k)\prod_{j\in J}f(2\varepsilon_j x_j), \tag{V.6}$$

$$U_{I,p}=\sum_{\substack{I'\subset I\\ |I'|=[p/2]}}\prod_{\substack{j\in I'\\ k\in I\backslash I'}}f(x_{jk})f(x_j+x_k)g(x_{jk})g(x_j+x_k)\begin{cases}0, & (p\ \text{odd})\\ 1, & (p\ \text{even})\end{cases}.$$

Here, $[p/2]$ denotes the integer part of $p/2$. As an example, for the C_2 RS model, the independent Hamiltonian flows $(\mathcal{H}_1)_{C_2}$ and $(\mathcal{H}_2)_{C_2}$ generated by the Lax matrix L_{C_2} are[29]

$$\begin{aligned}(\mathcal{H}_1)_{C_2}=\mathcal{H}_{C_2}&=e^{p_1}f(2x_1)\,f(x_{12})f(x_1+x_2)+e^{-p_1}g(2x_1)\,g(x_{12})g(x_1+x_2)\\ &\quad+e^{p_2}f(2x_2)\,f(x_{21})f(x_2+x_1)+e^{-p_2}g(2x_2)\,g(x_{21})g(x_2+x_1),\end{aligned} \tag{V.7}$$

$$\begin{aligned}(\mathcal{H}_2)_{C_2}&=e^{p_1+p_2}f(2x_1)\,(f(x_1+x_2))^2f(2x_2)+e^{-p_1-p_2}g(2x_1)\,(g(x_1+x_2))^2g(2x_2)\\ &\quad+e^{p_1-p_2}f(2x_1)\,(f(x_{12}))^2f(-2x_2)+e^{p_2-p_1}g(2x_1)\,(g(x_{12}))^2g(-2x_2)\\ &\quad+2f(x_{12})\,g(x_{12})\,f(x_1+x_2)g(x_1+x_2).\end{aligned} \tag{V.8}$$

Similar to the form of "gauge" transformation of Eq. (II.4), we can check that the involutive Hamiltonians of Eq. (V.5) are identical to the one given by Diejen in Ref. 26 with the following transformation

$$x_i\to x_i, \quad p_i\to p_i+\frac{1}{2}\ln\left(\frac{f(2x_i)}{g(2x_i)}\prod_{j\neq i}^{n}\frac{f(x_{ij})f(x_i+x_j)}{g(x_{ij})g(x_i+x_j)}\right). \tag{V.9}$$

B. Spectral curve of the BC_n model

Similar to the calculation of the C_n case, the spectral curve of the BC_n RS system can be generated by Lax matrix $L(\lambda)_{BC_n}$ as follows

$$\det(L(\lambda)_{BC_n} - v\cdot Id) = 0. \tag{V.10}$$

The explicit form of the spectral curve is

$$\det(L(\lambda)_{BC_n} - v\cdot Id) = \sum_{j=0}^{2n+1} \frac{(\sigma(\lambda))^{(j-1)}\sigma(\lambda+j\gamma)}{(\sigma(\gamma+\lambda))^{j}} (-v)^{2n+1-j}(\mathcal{H}_j)_{BC_n} = 0, \tag{V.11}$$

where $(\mathcal{H}_0)_{BC_n} = (\mathcal{H}_{2n})_{BC_n} = 1$ and $(\mathcal{H}_i)_{BC_n} = (\mathcal{H}_{2n+1-i})_{BC_n}$ Poisson commute

$$\{(\mathcal{H}_i)_{BC_n}, (\mathcal{H}_j)_{BC_n}\} = 0, \quad \forall i,j \in \{1,\ldots,n\}. \tag{V.12}$$

This can be deduced similarly to the C_n case to verify that $(\mathcal{H}_i)_{A_{2n}}$, $i=1,\ldots,2n$ is the first class Hamiltonian with respect to the constraints (III.25).

The explicit forms of $(\mathcal{H}_l)_{BC_n}$ are

$$(\mathcal{H}_l)_{BC_n} = \sum_{\substack{J\subset\{1,\ldots,n\},\ |J|\le l\\ \varepsilon_j=\pm 1,\ j\in J}} \exp(p_{\varepsilon J})\, F_{\varepsilon J;J^c} U_{J^c, l-|J|}, \quad l=1,\ldots,n \tag{V.13}$$

with

$$p_{\varepsilon J} = \sum_{j\in J} \varepsilon_j p_j,$$

$$F_{\varepsilon J;K} = \prod_{\substack{j,j'\in J\\ j<j'}} f^2(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) \prod_{\substack{j\in J\\ k\in K}} f(\varepsilon_j x_j + x_k) f(\varepsilon_j x_j - x_k) \prod_{j\in J} f(2\varepsilon_j x_j) \prod_{j\in J} f(\varepsilon_j x_j), \tag{V.14}$$

$$U_{I,p} = \sum_{\substack{I'\subset I\\ |I'|=[p/2]}} \prod_{\substack{j\in I'\\ k\in I\setminus I'}} f(x_{jk}) f(x_j+x_k) g(x_{jk}) g(x_j+x_k) \begin{cases} \prod_{i\in I\setminus I'} f(x_i)g(x_i), & (p \ \ \text{odd}) \\ \prod_{i\in I} f(x_i)g(x_i), & (p \ \ \text{even}) \end{cases}.$$

It is similar to the C_n case for the relation between $(\mathcal{H}_l)_{BC_n}$ with the one given in Ref. 26:

$$x_i \to x_i, \quad p_i \to p_i + \tfrac{1}{2}\ln\left(\frac{f(x_i)}{g(x_i)} \frac{f(2x_i)}{g(2x_i)} \prod_{j\neq i}^{n} \frac{f(x_{ij}) f(x_i+x_j)}{g(x_{ij}) g(x_i+x_j)} \right). \tag{V.15}$$

Remarks: So far we have Lax matrices with the spectral parameter of Eq. (IV.4) for the C_n and BC_n RS models, and the corresponding spectral curve equation of Eqs. (V.3) and (V.11). It is expected that they will be useful to study the relation between the $5d$ SUSY gauge theory and these integrable models which have been pointed out in Ref. 5. More exactly, it is expected that these spectral curves would be identical to the complex curve in the context of SUSY gauge theory associated with the corresponding gauge group. On the other hand, these nonsimple laced models may be potential candidates which are connected with orientifold in brane theory, corresponding to the fact that the A_{n-1} RS model is connected with orbifold. This exact correspondence in these directions is missed and certainly desires further investigation.

C. Limit to the Calogero–Moser system and Toda system

The Calogero–Moser system can be achieved by taking the so-called "nonrelativistic limit." The procedure is by rescaling $p_\mu \mapsto \beta p_\mu$, $\gamma \mapsto \beta\gamma$ and letting $\beta \mapsto 0$, followed by making a canonical transformation

$$p_\mu \mapsto p_\mu + \gamma \sum_{\Delta \ni \eta = \mu - \nu} \zeta(\eta \cdot x). \tag{V.16}$$

Here $p_\mu = \mu \cdot p$, such that

$$L \mapsto Id + \beta L_{\mathrm{CM}} + O(\beta^2), \tag{V.17}$$

and

$$\mathcal{H} \mapsto N + 2\beta^2 \mathcal{H}_{\mathrm{CM}} + O(\beta^2), \tag{V.18}$$

where $N=2n$ for the C_n model and $N=2n+1$ for BC_n model.

L_{CM} can be expressed as

$$L_{\mathrm{CM}} = p \cdot H + X, \tag{V.19}$$

where

$$H_{\mu\nu} = \mu \delta_{\mu\nu}, \quad X_{\mu\nu} = \gamma \Phi(x_{\mu\nu}, \lambda)(1 - \delta_{\mu\nu}). \tag{V.20}$$

The Hamiltonian $\mathcal{H}_{\mathrm{CM}}$ of the CM model can be given by

$$\mathcal{H}_{\mathrm{CM}} = \frac{1}{2} p^2 - \frac{\gamma^2}{2} \sum_{\alpha \in \Delta} \wp(\alpha \cdot x) = \frac{1}{4} \mathrm{tr}\, L^2 + \mathrm{const}, \tag{V.21}$$

where

$$\mathrm{const} = -\frac{N(N-1)\gamma^2}{4} \wp(\lambda).$$

All of the above-mentioned results are identified with the results of Refs. 8, 10, 12–15 up to a suitable choice of coupling parameters. Now the degenerate RS spectral curve reduces to

$$\Gamma: \;\; R(v,\lambda) = \det(L(\lambda)_{\mathrm{CM}} - v \cdot Id) \equiv 0, \tag{V.22}$$

which is exactly identified with the spectral curve analyzed in Refs. 39 and 41.

Starting from the CM system to the Toda system is more directly due to the progress that the limit to Toda for the general Lie algebra has been studied extensively in Refs. 43–45. The main idea is making a suitable scaling limit with the following parametrization:

$$\omega_1 = -i\pi, \quad \omega_3 \in \mathbf{R}_+, \quad \tau \equiv \frac{\omega_3}{\omega_1} = i\omega_3/\pi, \tag{V.23}$$

and shifting the dynamical variable x,

$$\begin{aligned} x &\to Q - 2\omega_3\, \delta \rho^\vee, \quad p \to P, \\ \lambda &\to \log Z - \omega_3, \quad Z \in \mathbf{R}_+, \end{aligned} \tag{V.24}$$

TABLE I. Root system of A_{n-1}, C_n and BC_n types.

$\mathcal{G}$	All roots	Simple roots Π	$h_{\mathcal{G}}$	Dual Weyl vector $\rho^\vee$	Vector weights
A_{n-1}	$\pm e_i \pm e_j$, $1 \leqslant i,\ j \leqslant n$, $i \neq j$	$e_i - e_{i+1}$, $i=1,\dots,n-1$	n	$\Sigma_{j=1}^n (n-j) e_j$	e_i, $i=1,\dots,n$
C_n	$\pm e_i \pm e_j, \pm 2e_i$, $1 \leqslant i,\ j \leqslant n$, $i \neq j$	$e_i - e_{i+1}, 2e_n$, $i=1,\dots,n-1$	$2n$	$\Sigma_{j=1}^n (n+\frac{1}{2}-j) e_j$	$e_i, -e_i$, $i=1,\dots,n$
BC_n	$\pm e_i \pm e_j, \pm 2e_i, \pm e_i$ $1 \leqslant i,\ j \leqslant n$, $i \neq j$	$e_i - e_{i+1}, e_n$, $i=1,\dots,n-1$	$2n+1$	$\Sigma_{j=1}^n (n+1-j) e_j$ (define)	$e_i, -e_i, 0$, $i=1,\dots,n$

in which $h_{\mathcal{G}}$ is the Coxeter number for the corresponding root system $\mathcal{G}$, $\rho^\vee$ the dual of the Weyl vector defined as $\rho^\vee = \frac{1}{2} \Sigma_{\alpha \in \Delta_+} 2\alpha/\alpha^2$, and δ satisfies $\delta \leqslant 1/h_{\mathcal{G}}$.

For convenience, we give the basics of these root systems as shown in Table I.

As for the C_n model, selecting $\rho^\vee = \rho^\vee_{C_n}$, $\gamma = im\, e^{\omega_3 \delta}$, one has the nonaffine C_n Toda model from the Hamiltonian of the CM model Eq. (V.21),

$$\mathcal{H}^{\text{Toda}}_{C_n} = \frac{1}{2} P^2 + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{2Q_n}, \tag{V.25}$$

for $\delta < 1/h_{C_n}$ and $C_n^{(1)}$ Toda model

$$\mathcal{H}^{\text{Toda}}_{C_n^{(1)}} = \frac{1}{2} P^2 + m^2 e^{-2Q_1} + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{2Q_n} \tag{V.26}$$

for $\delta = 1/h_{C_n}$.

The same holds for the BC_n model. Selecting $\rho^\vee = \rho^\vee_{BC_n}$, $\gamma = im\, e^{\omega_3 \delta}$, one has the nonaffine B_n Toda model from the Hamiltonian of the CM model Eq. (V.21),

$$\mathcal{H}^{\text{Toda}}_{B_n} = \frac{1}{2} P^2 + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{Q_n} \tag{V.27}$$

for $\delta < 1/h_{BC_n}$ and BC_n Toda model

$$\mathcal{H}^{\text{Toda}}_{BC_n} = \frac{1}{2} P^2 + m^2 e^{-2Q_1} + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{Q_n} \tag{V.28}$$

for $\delta = 1/h_{BC_n}$.

If we use the following gauge for $\Phi(x,\lambda)$:[46]

$$\Phi(x,\lambda) \to \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)} \exp(\zeta(\lambda)x), \tag{V.29}$$

it does not destroy the validity[45] for the Lax pair, We have the following limit for $\gamma\Phi(\alpha \cdot x, \lambda)$:

$$
\begin{aligned}
\gamma\Phi(\alpha\cdot x,\lambda) &\to -m\exp\left(\frac{\alpha\cdot Q}{2}\right) \quad \text{for} \quad \alpha\in\Pi \quad (\delta\leqslant 1/h_{\mathcal{G}})\\
&\to mZ\exp\left(-\frac{\alpha\cdot Q}{2}\right) \quad \text{for} \quad \alpha=\alpha_h \quad (\delta=1/h_{\mathcal{G}})\\
&\to 0 \quad \text{otherwise,}\\
\gamma\Phi(-\alpha\cdot x,\lambda) &\to m\exp\left(\frac{\alpha\cdot Q}{2}\right) \quad \text{for} \quad \alpha\in\Pi(\delta\leqslant 1/h_{\mathcal{G}})\\
&\to -\frac{m}{Z}\exp\left(-\frac{\alpha\cdot Q}{2}\right), \quad \text{for} \quad \alpha=\alpha_h(\delta=1/h_{\mathcal{G}})\\
&\to 0 \quad \text{otherwise.}
\end{aligned}
\tag{V.30}
$$

So the Lax operator now reads

$$
L_{\text{Toda}}=P\cdot H-im\sum_{\alpha\in\Pi}\exp\left(\frac{\alpha\cdot Q}{2}\right)[E(\alpha)-E(-\alpha)]+im\exp\left(\frac{\alpha_0\cdot Q}{2}\right)[ZE(-\alpha_0)-Z^{-1}E(\alpha_0)],
\tag{V.31}
$$

where $E(\alpha)_{\mu\nu}=\delta_{\mu-\nu,\alpha}$. This Lax operator holds for all the root systems of $A_{n-1}(A_{n-1}^{(1)})$, $C_n(C_n^{(1)})$, $B_n(BC_n)$ and coincides with the standard form given in Ref. 8. It is not difficult to find that the parameter Z now plays the role of a spectral parameter for the affine Toda model based on $\mathcal{G}^{(1)}$. When we refer to the Toda models based on a finite Lie algebra $\mathcal{G}$, we should only drop the terms containing the affine root α_0.

So the degenerate spectral curve for the Toda $A_{n-1}^{(1)}$, $C_n^{(1)}$, and $BC_n(A_{2n}^{(2)})$ systems can be defined

$$
\Gamma: R(v,\lambda)=\det(L(\lambda)_{\text{Toda}}-v\cdot Id)\equiv 0,
\tag{V.32}
$$

which is identical to the one given in Refs. 47 and 48.

VI. DEGENERATE CASES

Let us now consider the other various special degenerate cases. As is well known, if one or both of the periods of the Weierstrass sigma function $\sigma(x)$ become infinite, there will occur three degenerate cases associated with trigonometric, hyperbolic, and rational systems. The degenerate limits of the functions $\Phi(x,\lambda)$, $\sigma(x)$, and $\zeta(x)$ will give corresponding Lax pairs which include spectral parameter. Moreover, when the spectral parameter value is on a certain limit, the Lax pairs without spectral parameter will be derived.

A. Trigonometric limit

The limit can be obtained by sending ω_3 to $i\infty$ with $\omega_1=\pi/2$, so that

$$
\begin{aligned}
\sigma(x) &\to e^{(1/6)x^2}\sin x,\\
\zeta(x) &\to \cot x+\tfrac{1}{3}x,
\end{aligned}
\tag{VI.1}
$$

and the function

$$
\Phi(x,\lambda)\equiv\frac{\sigma(x+\lambda)}{\sigma(x)\sigma(\lambda)}
$$

reduces to

$$\Phi(x,\lambda)\rightarrow(\cot\lambda-\cot x)e^{(1/3)\,xu}. \tag{VI.2}$$

By replacing the corresponding functions $\Phi(x,\lambda)$, $\sigma(x)$, and $\zeta(x)$ with the above-given form for the Lax pairs, we will get the corresponding spectral parameter dependent Lax pairs. For simplicity, we notice that the exponential part of the above-mentioned functions can be removed by applying suitable "gauge" transformation of the Lax matrix on which condition the functions can be valued as follows:

$$\begin{aligned}\sigma(x)&\rightarrow\sin x,\\ \zeta(x)&\rightarrow\cot x,\\ \Phi(x,\lambda)&\rightarrow(\cot\lambda-\cot x).\end{aligned} \tag{VI.3}$$

As for the spectral parameter independent Lax pair, furthermore, we can take the limit $\lambda\rightarrow i\infty$, so the function

$$\Phi(x,\lambda)\rightarrow\frac{1}{\sin x}, \tag{VI.4}$$

while the corresponding Lax matrix becomes

$$L_{\mu\nu}=e^{\nu\cdot p}b_\nu\frac{\sin\gamma}{\sin((\mu-\nu)\cdot x+\gamma)}, \tag{VI.5}$$

which is exactly the same as the spectral parameter independent Lax matrix given in Ref. 30.

B. Hyperbolic limit

In this case, the periods can be chosen by sending ω_1 to $i\infty$ with $\omega_3=\pi/2$, so following all the procedures in achieving the result of the trigonometric case, we can find the hyperbolic Lax pairs by simple replacement of the functions appearing in the trigonometric Lax pair as follows:

$$\begin{aligned}\sin x&\rightarrow\sinh x,\\ \cos x&\rightarrow\cosh x,\\ \cot x&\rightarrow\coth x.\end{aligned} \tag{VI.6}$$

The same as for the trigonometric case, we can get the Lax pairs with and without spectral parameter.

C. Rational limit

As far as the form of the Lax pair for the rational-type system is concerned, we can achieve it by making the following substitutions:

$$\begin{aligned}\sigma(x)&\rightarrow x,\\ \zeta(x)&\rightarrow\frac{1}{x},\\ \Phi(x,\lambda)&\rightarrow\frac{1}{x}+\frac{1}{\lambda}\end{aligned} \tag{VI.7}$$

for the spectral parameter dependent Lax pair, while furthermore, taking the limit $\lambda \to i\infty$, we can obtain the spectral parameter independent Lax pair. The explicit form of Lax matrix without spectral parameter is

$$L_{\mu\nu} = e^{\nu \cdot p} b_{\nu} \frac{\gamma}{(\mu-\nu)\cdot x + \gamma}. \tag{VI.8}$$

which completely coincides with the spectral parameter independent Lax matrix given in Ref. 30.

Remark: As for the various degenerate cases for the CM and Toda systems, one can follow the same procedure as for the RS model [please refer to Eqs. (VI.1)–(VI.8)].

VII. CONCLUDING REMARKS

In this paper, we have proposed the Lax pairs for elliptic C_n and BC_n RS models. The spectral parameter dependent and independent Lax pairs for the trigonometric, hyperbolic, and rational systems can be derived as the degenerate limits of the elliptic potential case. The spectral curves of these systems are given and shown depicted by the complete sets of involutive constant integrals of motion. They are expected be related to the five-dimensional gauge theory[4,5] and even to brane theory, which desires further study. In the nonrelativistic limit(scaling limit), these systems lead to CM(Toda) systems associated with the root systems of C_n and BC_n. There are still many open problems. For example, it seems to be a challenging subject to carry out a Lax pair with as many independent coupling constants as independent Weyl orbits in the set of roots, as done for the Calogero–Moser systems(see Refs. 8, 11–15). What is also interesting is to generalize the results obtained in this paper to the systems associated with all other Lie Algebras even to those associated with all the finite reflection groups.[14] Moreover, the issue of getting the r-matrix structure for these systems is deserved due to the success of calculating for the trigonometric BC_n RS system by Avan *et al.* in Ref. 31.

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APPENDIX

In this appendix we prove the identity equation (II.11) and then derive the relation between the Lax operator $L(\lambda)$ and its inverse of $L(\lambda)^{-1}$.

Using the result given in Ref. 6 of Eq. (B5), we have the following conclusion:

Let

$$C_{ij} = \frac{\sigma(q_i - r_j + \lambda)}{\sigma(q_i - r_j + \mu)}, \quad i,j = 1,\ldots,N, \tag{A1}$$

then one has

$$\det(C) = \sigma(\lambda-\mu)^{N-1}\sigma(\lambda+(N-1)\mu+\Sigma)\times\prod_{i<j}\sigma(q_i-q_j)\sigma(r_j-r_i)\prod_{i,j}\frac{1}{\sigma(q_i-r_j+\mu)}, \tag{A2}$$

where

$$\Sigma = \sum_{i=1}^{N}(q_i - r_j). \tag{A3}$$

So it is straightforward to compute the inverse of matrix C,

$$(C^{-1})_{ij}=\text{the cofactor of } C \text{ with respect to } C_{ji}$$
$$=\frac{\sigma(\lambda+(N-2)\mu+q_i-q_j)}{\sigma(\lambda-\mu)\sigma(\lambda+(N-1)\mu)\sigma(q_i-q_j-\mu)}\times\frac{\Pi_l\sigma(q_j-q_l+\mu)\Pi_l\sigma(q_i-q_l-\mu)}{\Pi_{k\neq i}\sigma(q_i-q_k)\Pi_{k\neq j}\sigma(q_j-q_k)}. \tag{A4}$$

From Eq. (II.5), we have

$$L(\lambda)=\sum_{i,j=1}^{N}\frac{\Phi(x_{ij}+\gamma,\lambda)}{\Phi(\gamma,\lambda)}\exp(p_j)b_jE_{ij}$$
$$=\frac{1}{\Phi(\gamma,\lambda)}\sum_{i,j=1}^{N}\frac{\sigma(x_{ij}+\gamma+\lambda)}{\sigma(x_{ij}+\gamma)\sigma(\lambda)}\exp(p_j)b_j$$
$$=\frac{1}{\Phi(\gamma,\lambda)}\sum_{i,j=1}^{N}G_{ij}\exp(p_j)b_j, \tag{A5}$$

where

$$G_{ij}:=\Phi(x_{ij}+\gamma,\lambda)=\frac{\sigma(x_{ij}+\gamma+\lambda)}{\sigma(x_{ij}+\gamma)\sigma(\lambda)},$$

with the help of Eq. (A4), one has

$$(G^{-1})_{ij}=\frac{\sigma(\lambda+(N-1)\gamma+x_{ij})}{\sigma(\lambda+N\gamma)\sigma(x_{ij}-\gamma)}\times\frac{\Pi_k\sigma(x_{jk}+\gamma)\Pi_k\sigma(x_{ik}-\gamma)}{\Pi_{k\neq i}\sigma(x_{ik})\Pi_{k\neq j}\sigma(x_{jk})}, \tag{A6}$$

so that

$$L(\lambda)^{-1}{}_{ij}=\Phi(\gamma,\lambda)(G^{-1})_{ij}b_j^{-1}\exp(-p_i))E_{ij}$$
$$=\frac{-\sigma(\gamma)^2\,\sigma(\lambda+\gamma)\sigma(\lambda+(N-1)\gamma+x_{ij})}{\sigma(\lambda)\sigma(\gamma)\sigma(\lambda+N\gamma)\sigma(x_{ij}-\gamma)}\times\exp(-p_i)\prod_{k\neq j}\frac{\sigma(x_{jk}+\gamma)}{\sigma(x_{jk})}$$
$$=\frac{\sigma(\gamma+\lambda)\,\sigma(\lambda+(N-1)\gamma)}{\sigma(\lambda)\sigma(\lambda+N\gamma)}\times\frac{\Phi(x_{ij}-\gamma,\lambda+N\gamma)}{\Phi(-\gamma,\lambda+N\gamma)}\exp(-p_i)b_j'E_{ij}. \tag{A7}$$

By comparing the forms of $L(\lambda)$ and $L(\lambda)^{-1}{}_{ij}$, we find $L(\lambda)^{-1}{}_{ij}$ can be expressed with $L(\lambda)$ as

$$L(\lambda)^{-1}{}_{ij}=L(\lambda)_{ij}\big|_{\gamma\to-\gamma,\lambda\to\lambda+N\gamma}\times\frac{\sigma(\gamma+\lambda)\,\sigma(\lambda+(N-1)\gamma)}{\sigma(\lambda)\sigma(\lambda+N\gamma)}\exp(-p_i-p_j). \tag{A8}$$

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