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Accuracy of Circulation Estimation Schemes Applied to Discretised Velocity Field Data

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Abstract

Numerical experiments are conducted on the velocity field of the Oseen vortex to determine the effect of random errors in the velocity field on the circulation estimate. The circulation is estimated by either a velocity integral or a vorticity integral over a particular region of integration. A novel method for the determination of this region is used. The accuracy of circulation estimation schemes is characterised in terms of the velocity sample spacing, the amount of random noise in the velocity field and the vorticity estimation scheme used. It is found that, in general, the velocity integral outperforms the vorticity integral in terms of reducing total error.

Introduction

Over the last few decades Particle Image Velocimetry (PIV) has become a commonly used experimental technique in fluid mechanics. The availability of the velocity field allows for the estimation of derived quantities, such as vorticity and strain, and integrated features such as streamline patterns and circulation. PIV-derived velocity fields contain errors that are subsequently transferred to any quantity estimated from the velocity field. Much attention has been given to the transmission of errors from the velocity field to the vorticity field. In particular, Fouras and Soria [3] developed a framework by which this transmission is quantified in terms of the velocity sample spacing and the vorticity estimation scheme used.

Fouras and Soria [3] and Etebari and Vlachos [1] used the analytic velocity and vorticity fields of the Oseen vortex to test the accuracy of their respective vorticity estimation schemes. They digitised the analytic velocity field over a grid of known velocity-sample spacing (resolution) and then applied various vorticity estimation schemes to the discretised velocity data. They were then able to compare the resulting vorticity field to the analytic vorticity field and thus quantify the effect of resolution on the vorticity estimate. They then added a known amount of random noise to the discretised velocity field, estimated the vorticity from the noisy velocity data and then compared the vorticity estimate to the analytic value. This allowed them to quantify the transmission of random error from the velocity to vorticity field for various vorticity estimation schemes.

The present study aims to quantify the transmission of random error from the velocity field to various estimates of the circulation. We implement a similar methodology to that of Fouras and Soria [3]. We use the analytic velocity field of the Oseen vortex to generate a discretised velocity field to which we add a known amount of random noise. The circulation is then estimated from the noisy velocity field and compared to the analytic value of the Oseen vortex circulation. The accuracy of the circulation estimation schemes is then characterised in terms of the velocity sample spacing, the amount of random noise in the velocity field and the vorticity estimation scheme used.

Circulation

In two dimensions, circulation, Γ, can be expressed as either a velocity integral (equation 1), or, by way of Stokes' theorem, a vorticity integral (equation 2).

$$\Gamma_{vel} = \oint_c (udx + vdy) \tag{1}$$

$$\Gamma_{vor} = \int\int_S \omega_z dS \tag{2}$$

Here u and v are the velocity components in the x and y directions respectively, $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the out-of-plane vorticity component and S is an area enclosed by a closed curve C. Γ_{vel} and Γ_{vor} are the circulation estimators that are to be tested.

The Oseen vortex

The Oseen vortex is an ideal vortex with analytically defined velocity, vorticity and circulation. The tangential velocity and out of plane vorticity of the Oseen vortex are specified by Saffman [5] , in polar co-ordinates, as equations (3) and (4) respectively.

$$V_\theta(r) = \frac{\Gamma_0}{2\pi r}(1 - e^{-r^2/2L^2}) \tag{3}$$

$$\omega_z(r) = \frac{\Gamma_0}{2\pi L^2}(e^{-r^2/2L^2}) \tag{4}$$

Here, Γ_0 is the circulation and L is a length scale corresponding to one standard deviation of the Gaussian vorticity distribution of the Oseen vortex (equation 4). The radial velocity component is zero.

Vorticity Estimation Schemes

Many different vorticity estimation schemes exist and different schemes have different error propagation. Fouras and Soria [3] demonstrated that errors in vorticity estimation can be decomposed into two, usually counteracting, components. The first is a bias error component which represents the over or underestimation by the scheme. The second is a random error component which characterises the scheme's susceptibility to random noise in the velocity field. Fouras and Soria [3] showed that, for the vorticity estimation schemes they considered, bias error and random error can not be simultaneously reduced.

Commonly used schemes include explicit and implicit approximations to the the first derivative, direct approximations via Stokes theorem and analytic differentiation of fitted surfaces. Five vorticity estimation schemes are tested with the two circulation estimators, Γ_{vel} and Γ_{vor}. With the schemes presented here the velocity data is assumed to be on a uniform grid of velocity sample spacing, Δ. The indices i and j denote the x and y directions respectively.

The central difference (CD) scheme directly estimates the first derivative by $f'_i \simeq \frac{f_{i+1}-f_{i-1}}{2\Delta}$. Raffel *et al.* [4] present a "least squares" (LS) scheme that estimates the first derivative by $f'_i \simeq \frac{2f_{i+2}+f_{i+1}-f_{i-1}-2f_{i-2}}{10\Delta}$. Ferziger and Peric [2] describe an estimate to the first derivative based on fitting a fourth order polynomial to to five adjacent data points resulting in a finite difference scheme given by $f'_i \simeq \frac{-f_{i+2}+8f_{i+1}-8f_{i-1}+f_{i-2}}{12\Delta}$. We refer to this as the POLY4 scheme.

A direct approximation to the vorticity at a point can by obtained by equating the circulation within a region to the product of the average vorticity and the region area as presented in Raffel *et al.* [4]. This yields $(\overline{\omega}_z)_{i,j} = \Gamma_{i,j}/A_{i,j}$, where $A_{i,j} = 4\Delta^2$ and

$$\begin{aligned}\Gamma_{ij} &= \frac{1}{2}\Delta(u_{i-1,j-1} + 2u_{i,j-1} + u_{i+1,j-1}) \\ &+ \frac{1}{2}\Delta(v_{i+1,j-1} + 2v_{i+1,j} + v_{i+1,j+1}) \\ &- \frac{1}{2}\Delta(u_{i+1,j+1} + 2u_{i,j+1} + u_{i-1,j+1}) \\ &- \frac{1}{2}\Delta(v_{i-1,j+1} + 2v_{i-1,j} + v_{i-1,j-1}). \qquad (5)\end{aligned}$$

We refer to this as the CIRC scheme.

Etebari and Vlachos [1] developed a fourth-order, hybrid, compact-Richardson extrapolation (CR4) finite-difference scheme for estimating first derivatives. The CR4 method has two stages. First-stage approximations are obtained by solving the following implicit finite-difference equation for f_i' with i in the range $k+1 \leq i \leq N-k$:

$$\frac{1}{4}f_{i-k}' + f_i' + \frac{1}{4}f_{i+k}' = \frac{3}{2}\left(\frac{f_{i+k} - f_{i-k}}{2k\Delta}\right). \qquad (6)$$

This equation is solved three times with $k = 1$, $k = 2$ and $k = 4$ giving three different approximations ${}_1f_i'$, ${}_2f_i'$ and ${}_4f_i'$. A weighted average of these gives the second-stage estimate of the derivative:

$${}_{\mathrm{CR4}}f_i' = \frac{272}{1239}.({}_1f_i') + \frac{1036}{1239}.({}_2f_i') - \frac{69}{1239}.({}_4f_i'). \qquad (7)$$

With the CD, LS, POLY4 and CR4 schemes, the first derivative estimates are used to estimate $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ and thus estimate the vorticity, ω.

Numerical Experiments

Numerical simulations are conducted to determine errors in the circulation estimates, Γ_{vel} and Γ_{vor}. These simulations are conducted for differing values of added noise magnitude, ε_u, and resolution, Δ/L, and for different vorticity estimation schemes. It should be noted that the nature of the the CIRC vorticity estimation scheme should result in both Γ_{vel} and Γ_{vor} being equal when evaluated with the CIRC scheme.

Region of Integration, ROI, Determination

In order to evaluate Γ_{vel} and Γ_{vor} it is required that an appropriate path or region of integration be chosen. Other authors such as Fouras and Soria [3] and Weigand and Gharib [6] used a rectangular box that encloses the vortex core as the closed path for evaluating the circulation. This box, however, may contain features not pertaining to the vortex core under examination. This is especially true in the case complex flows where structures of opposite vorticity are in close proximity to each other. A more appropriate choice of path is an iso-vortical contour encompassing the required vortex core. The area of this region can be used as the area S in the evaluation of Γ_{vor} and its perimeter can be used as the closed curve C in Γ_{vel}. This region is dubbed the region of integration, ROI.

An algorithm has been developed to determine the ROI within velocity vector fields of grid spacing Δ. The algorithm first identifies the grid square corresponding to the maximum evaluated vorticity within the measured flow field. This is the stating point from which the ROI will develop. Each grid square adjacent to the starting square is incorporated into the ROI if both of the following conditions are met; (1.) the vorticity in that particular grid square exceeds or is equal to a user-defined vorticity threshold, ω_T and (2.) the horizontal or vertical distance from the starting grid square does not exceed a particular user defined distance, d. This process is then reapplied to each of the newly incorporated grid squares. This process is continued until the ROI stops growing. Each grid square in the ROI is joined to another grid square at a face or a vertex. There are no "stray" agglomerations of grid squares that are not joined to the ROI.

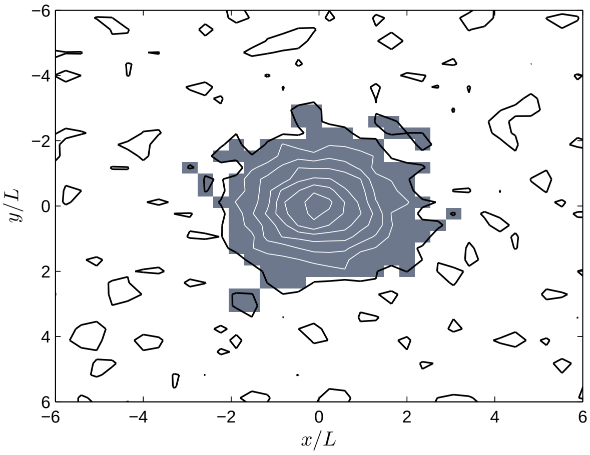

Figure 1: An evaluated ROI overlayed on iso-vortical (CD) contours of an Oseen vortex with added noise. The resolution is $\Delta/L = 0.353$. The lowest vorticity level shown is $\omega = 0.05s^{-1}$ (black lines) and the increment is $\omega = 0.15s^{-1}$ (white lines). The ROI is determined with $\omega_T = 0.05s^{-1}$ and $d = 4L$.

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The circulation estimations are dependent on the vorticity estimation scheme used because the ROI is determined from the vorticity field. The evaluation of Γ_{vel} is only indirectly dependent on the vorticity as the vorticity is used in determining the ROI and hence the path of integration. Γ_{vor} , however, is directly dependent on the vorticity as it is the vorticity that is integrated within the ROI.

Numerical Formulation

For the numerical tests conducted, the Oseen vortex circulation is set to $\Gamma_0 = 1$ mm^2/s and the characteristic length scale was set to $L = 1$ mm. The computational domain is $-8L \leq \{x,y\} \leq 8L$. The estimates of circulation, Γ_{vel} and Γ_{vel}, are obtained by implimenting the following steps.

1. Using equation 3, generate the velocity field, $u_{i,j}$, of the Oseen vortex on the computational domain with grid spacing Δ. This data is of resolution Δ/L.

2. Generate the noise-added velocity field, $\widetilde{u}_{i,j}$, by adding zero-mean white Gaussian noise of a particular magnitude, ε_u, to the velocity field $u_{i,j}$:

$$\widetilde{u}_{i,j} = u_{i,j} + max(u_{i,j})\mathcal{N}(0, \varepsilon_u^2) \qquad (8)$$

Here $max(u_{i,j})$ is the maximum value of $u_{i,j}$ in the entire computational domain and $\mathcal{N}(0, \varepsilon_u^2)$ is a normal random variable with a mean of zero and a variance of ε_u^2.

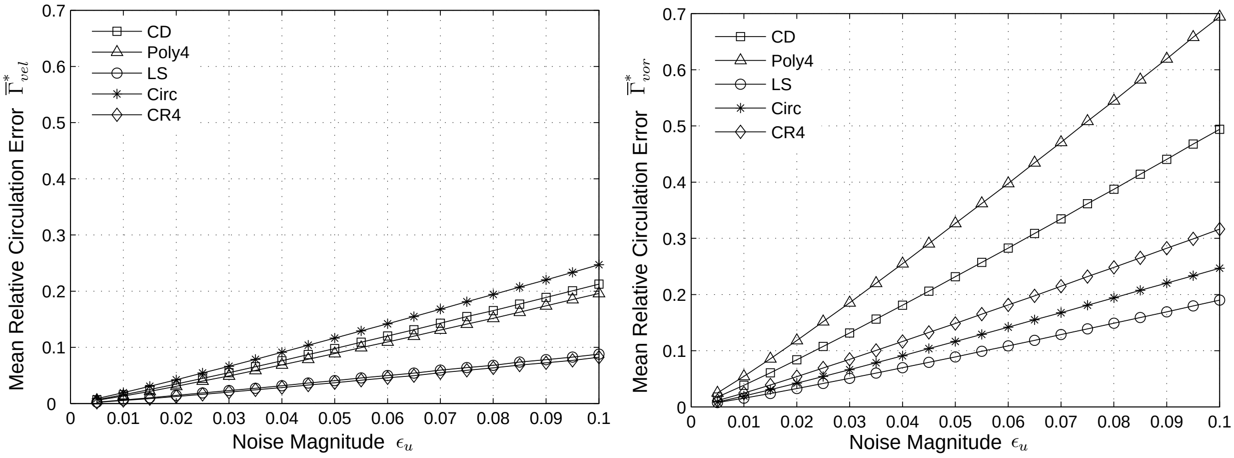

Figure 2: Mean relative circulation error, $\overline{\Gamma}^*$, verses input noise magnitude, ε_u, at a resolution of $\Delta/L = 0.25$ for Γ_{vel} and Γ_{vor}.

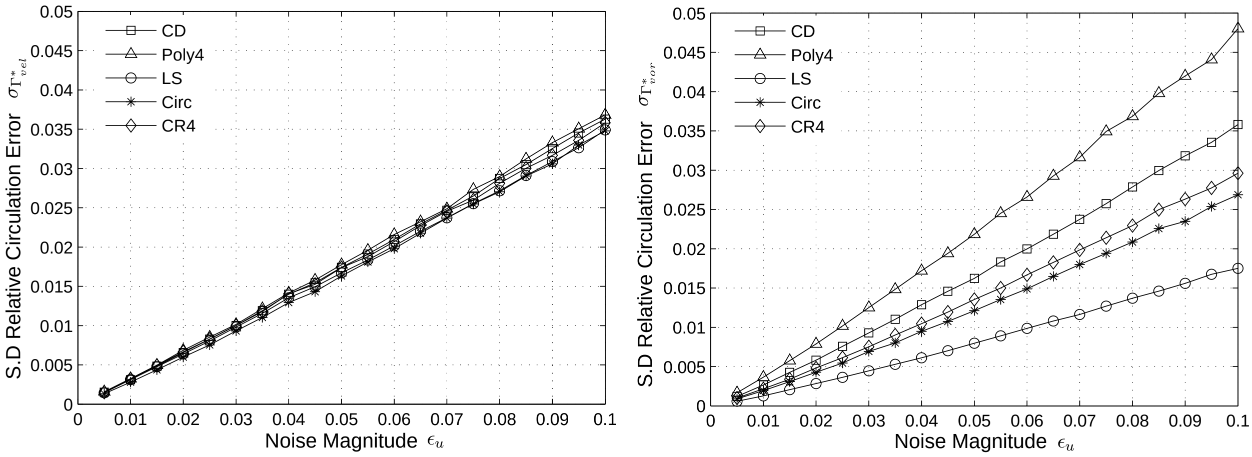

Figure 3: Standard deviations of relative circulation error, $\sigma_{\overline{\Gamma}^*}$, verses input noise magnitude, ε_u, at a resolution of $\Delta/L = 0.25$ for Γ_{vel} and Γ_{vor}.

3. Generate five noise-laden vorticity fields by applying the five vorticity estimation schemes (CD, LS, POLY4, CIRC & CR4) to the noise-added velocity field, $\widetilde{u}_{i,j}$.

4. For each vorticity field, find the corresponding region of integration (ROI) using a vorticity threshold of $\omega_t = 0\ s^{-1}$ and $d = 4L$.

5. Estimate Γ_{vel} by numerically integrating the noise-added velocity field $\widetilde{u}_{i,j}$ around the perimeter of each of the five regions of integration. This gives five estimates of Γ_{vel}.

6. Estimate Γ_{vor} by numerically integrating each noise-laden vorticity field within the area of the corresponding regions of integration. This gives five estimates of Γ_{vor}.

This process is conducted 8,900 times for each of 400 different combinations of $0.05 \leq \varepsilon_u \leq 0.1$ and $0.15 \leq \Delta/L \leq 0.5$.

Results

The circulation estimates are used to generate the relative circulation error, $\Gamma^* = \frac{\Gamma - \Gamma_0}{\Gamma_0}$, where Γ is the circulation estimate and Γ_0 is the circulation of the analytic noise-free Oseen vortex. The mean relative circulation error, $\overline{\Gamma}^*$, and the corresponding standard deviation, σ_{Γ^*}, for both Γ_{vel} and Γ_{vor} are determined for the vorticity estimation schemes considered. The bias error of the circulation estimate is $\overline{\Gamma}^*$ and the random error component is σ_{Γ^*}. Both are functions of the resolution, Δ/L, and and the noise magnitude, ε_u. Figures 2 and 3 show the dependence of $\overline{\Gamma}^*$ and σ_{Γ^*} on noise magnitude, ε_u at a fixed resolution of $\Delta/L = 0.25$. In general the bias error is an order of magnitude larger than the random error.

It is clear from figure 2 that, for every vorticity scheme considered, $\overline{\Gamma}^*_{vel}$ is always lower than $\overline{\Gamma}^*_{vor}$ except in the case of the CIRC method where they are equal. The CR4 and LS schemes give the lowest $\overline{\Gamma}^*_{vel}$ values.

Every vorticity estimation scheme transmits random error in the velocity field to the vorticity field. Γ_{vel} integrates velocity components to estimate circulation and only indirectly incorporates the vorticity estimation scheme in determining the ROI. This is the reason why $\sigma_{\Gamma^*_{vel}}$ (figure 3) seems to have similar values for all vorticity estimation schemes.

Γ_{vor} sums vorticity values to estimate circulation so it is directly dependent on the vorticity estimation scheme used. The transmition of random error in velocity to vorticity was characterised by Fouras and Soria [3] as a nondimensional ratio, λ_0. For LS, CIRC and CR4, $\lambda_0 < 1$ meaning theses schemes reduce the random error from the velocity to vorticity. CD has

$\lambda_0 = 1$ meaning noise is neither amplified nor diminished and Poly4 has $\lambda_0 > 1$ meaning it amplifies noise. This is directly manifested by the dependancy of $\sigma_{\Gamma^*_{vor}}$ (figure 3) on the vorticity estimation scheme. Poly4 shows a larger value of $\sigma_{\Gamma^*_{vor}}$ than $\sigma_{\Gamma^*_{vel}}$ because $\lambda_0 > 1$. As LS, CIRC and CR4 have $\lambda_0 < 1$, $\sigma_{\Gamma^*_{vor}}$ is less than $\sigma_{\Gamma^*_{vel}}$ for those schems.

The different ways in which Γ_{vel} and Γ_{vor} estimate circulation is also the reason why, for every vorticity scheme considered, $\overline{\Gamma}^*_{vel}$ is always lower than or equal to $\overline{\Gamma}^*_{vor}$ (figure 2). Fouras and Soria [3] demonstrated that vorticity estimated from a noisy velocity field will always contain both random and bias errors. Summing vorticity to obtain circulation will always result in higher bias error, $\overline{\Gamma}^*$, when compared to numerically integrating velocity components because of the error-adding step of obtaining the vorticity from the velocity field.

If, however, low random error ,σ_{Γ^*}, is of more importance than bias error, $\overline{\Gamma}^*$, then Γ_{vor} can be used. This is only true if Γ_{vor} is used with a vorticity estimation scheme that reduces random noise ($\lambda_0 < 1$) such as LS (figure 3 for $\sigma_{\Gamma^*_{vor}}$).

Total error

The accuracy of each circulation estimate can be characterised by the total error, $\Lambda = \sqrt{\overline{\Gamma}^{*2} + \sigma^2_{\Gamma^*}}$, that incorporates both the bias and random error components.

The lowest values of Λ, over the computational domain, were observed for Γ_{vel} used with the CR4 scheme and are shown in figure 4. The total error can be seen to be up to about 13% of the true circulation value. It is interesting to note that for a fixed noise magnitude, larger total errors are observed for low values of Δ/L which correspond to a high resolution. This is due to the fact that more noise-laden velocity values are incorporated in the integration about the ROI at a higher resolution than at a lower resolution.

To demonstrate the relative accuracy of the schemes, the total errors for all combinations of Γ_{vel} and Γ_{vor} with all the vorticity estimation schemes are compared at fixed resolution and noise magnitude. Figure 5 shows the total error (Λ) for all combinations at $\varepsilon = 0.05$ and $\Delta/L = 0.25$, relative to the total error for Γ_{vel} used with the CR4 scheme (Λ_0). The combinations are ranked according to increasing relative total error ratio ($\frac{\Lambda}{\Lambda_0}$) and this ranking is consistent across almost all combinations of noise magnitude and resolution. As expected, the CIRC scheme resulted in equal values for Γ_{vel} and Γ_{vor}. When used with the POLY4 scheme, Γ_{vor} can result in total error eight times that of Γ_{vel} with CR4 resulting in a total error over 30%.

Based on the presented data, it is recommended that Γ_{vel} be used over Γ_{vor} in estimating circulation with either the CR4 or LS schemes for estimating vorticity. It should be kept in mind that, in this situation, vorticity is only estimated to determine the ROI at low vorticity levels.

Conclusions

Numerical experiments are conducted to determine the effect of random errors in the velocity field on the circulation determined by a velocity integral (Γ_{vel}) or a vorticity integral (Γ_{vor}). It is found that Γ_{vel} shows a lower bias error ($\overline{\Gamma}^*$) than Γ_{vor}. The random error of Γ_{vel} ($\sigma_{\Gamma^*_{vel}}$) is less susceptible to the choice of vorticity estimation scheme than that of Γ_{vor} ($\sigma_{\Gamma^*_{vor}}$). Of the schemes considered, Γ_{vel} used with the CR4 scheme displayed the minimum total error (Λ) over almost all combinations of noise magnitude and resolution. In terms of total error, Γ_{vel} is superior to Γ_{vor} in estimating circulation for the vorticity estimation schemes considered.

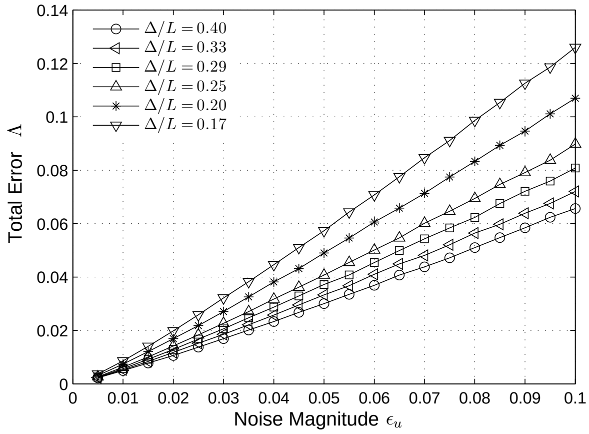

Figure 4: Total error, Λ, in Γ_{vel} used with the CR4 scheme, verses noise magnitude, ε_u, over a range of resolutions, Δ/L.

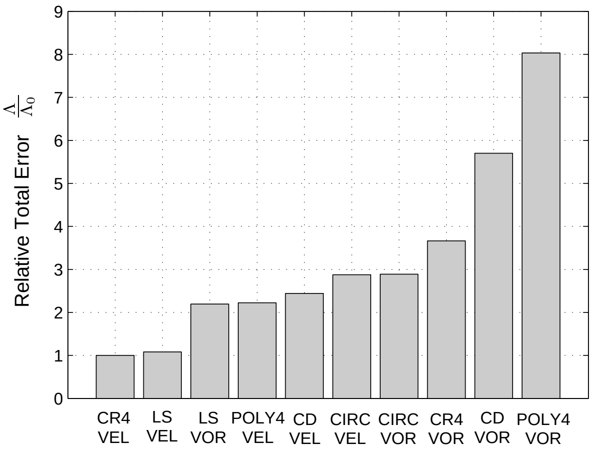

Figure 5: Total error, Λ, relative to the total error for Γ_{vel} used with CR4, Λ_0, for all combinations, evaluated at $\varepsilon = 0.05$ and $\Delta/L = 0.25$

Acknowledgements

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