

A perturbation solution for forced convection in a porous saturated duct

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Abstract

Fully developed forced convection through a porous medium bounded by two isoflux parallel plates is investigated analytically on the basis of a Brinkman-Forchheimer model. The matched asymptotic expansion method is applied for small values of the Darcy number. For the case of large Darcy number the solution for the Brinkman-Forchheimer momentum equation is found in terms of an asymptotic expansion. With the velocity distribution determined, the energy equation is solved using the same asymptotic technique. The results for limiting cases are found to be in good agreement with those available in the literature and the numerical results obtained here.

Keywords: Forced convection, porous media, pipe flow, Brinkman-Forchheimer model

Nomenclature

C	constant defined in Eq. 25
C_F	inertial coefficient
c_P	specific heat at constant pressure
Da	Darcy number, K/H^2
F	Forchheimer number
G	negative of the applied pressure gradient
H	half channel distance
k	effective thermal conductivity
K	permeability
M	μ_{eff} / μ
Nu	Nusselt number defined by Eq. (11)
O	symbol for order of magnitude
Pe	Péclet number defined by Eq. (3)
q''	wall heat flux
s	porous media shape parameter, $(M Da)^{-1/2}$
T^*	temperature
T_m	bulk mean temperature
T_w	downstream wall temperature
u	$\mu u^* / GH^2$
u^*	filtration velocity
\hat{u}	u^* / U
U	mean velocity

x^* longitudinal coordinate

x x^*/PeH

y^* transverse coordinate

y y^*/H

Greek symbols

η stretched variable

θ $(T^* - T_w)/(T_m - T_w)$

μ fluid viscosity

μ_{eff} effective viscosity in the Brinkman term

ρ fluid density

Subscripts

0, 1 term sequence in asymptotic expansion

i node number

Superscripts

in, out inner and outer expansion

n iteration number

1. INTRODUCTION

The problem of forced convection in a porous duct is a classical one (at least for the case of slug flow (Darcy model)). However, the incorporation of the boundary and inertial effects on the fully developed momentum transfer equation in porous media requires a major change in approach since the velocity field is no more prescribed. This change results in an algebraic equation problem being replaced by a second order non-linear ordinary differential one, since the boundary effect can be included by adding the viscous shear stress term (due to Brinkman, cf. Nield [1]) and a nonlinear term should be added to account for the form drag effects, as noted by Nield and Bejan [2]. For flow through parallel plate porous channels Nield et al. [3] and Vafai and Kim [4] have reported theoretical analysis of the problem considering both boundary and inertial effects. Applying a boundary layer approach, some authors have reported analytical solution for the Brinkman-Forchheimer momentum equation [4,5]. Nield et al. [3] have revisited the problem and noted that the solution reported in [4] is not accurate for the case that the non-linear term is dropped and the momentum equation is a Brinkman one. Nield et al. [3] have added that for small values of the Darcy number the closed form solution by Vafai and Kim [4] is not predicting the answer to the physical problem. On the other hand, Nield et al. [3] have found the velocity gradient by direct integration and continued by numerical integration to determine the velocity profile in the duct cross section. Absent in their study was a closed form solution of the velocity profile, and consequently the temperature distribution, in terms of the Darcy number and the Forchheimer number at a given transverse location.

Finding analytical solution for the velocity distribution with negligible form drag effects is an easy task in the light of [6]. However, when it comes to consider the effects of form drag by adding a non-linear term, the problem becomes more complicated and one no more expects a uniformly valid exact solution without involving numerical integration.

In order to by-pass the full analysis of this problem, an asymptotic solution is presented here that considers two limiting values of the porous media shape parameter (see Eq.7), namely very large and very small values compared to unity. Based on previous investigations [3-5], one expects that for the former case the velocity profile be a slug-like one with a boundary layer near the wall (where the velocity changes can be felt in a thin near-wall region) while for the latter, one expects that the velocity be increased monotonically from the wall to the duct center. For this reason, one designs two different solutions for these two ranges of the porous media shape parameter which have significant physical importance. Basic asymptotic techniques used here to simplify solving the governing equations are mentioned and discussed in Nayfeh [7].

Previous work on the forced convection in ducts, in the case of fluids clear of solid material, has been surveyed by Shah and London [8].

2. ANALYSIS

2.1 Basic equations

For the steady-state fully developed situation there exists a unidirectional flow in the x^* -direction inside a channel with impermeable walls at $y^* = \pm H$, as illustrated in Fig. 1. For $x^* > 0$, the heat flux at the tube wall is held constant at the value q'' .

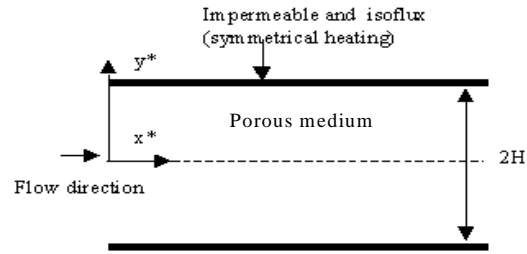


Fig. 1. Definition sketch

The Brinkman-Forchheimer momentum equation is

$$\mu_{eff} \frac{d^2 u^*}{dy^{*2}} - \frac{\mu}{K} u^* - \frac{C_F \rho u^{*2}}{\sqrt{K}} + G = 0, \quad (1)$$

where μ_{eff} is an effective viscosity, μ is the fluid viscosity, K is the permeability, ρ is the fluid density, C_F is the inertial coefficient, and G is the negative of the applied pressure gradient.

The dimensionless variables are defined as

$$x = \frac{x^*}{PeH}, \quad y = \frac{y^*}{H}, \quad u = \frac{\mu u^*}{GH^2}. \quad (2)$$

Here the Péclet number Pe is defined by

$$Pe = \frac{\rho c_p H U}{k}. \quad (3)$$

The dimensionless form of Eq. (1) is then

$$M \frac{d^2 u}{dy^2} - \frac{u}{Da} - \frac{M F u^2}{\sqrt{Da}} + 1 = 0. \quad (4)$$

The viscosity ratio M , the Darcy number Da , and the Forchheimer number F are defined by

$$M = \frac{\mu_{eff}}{\mu}, \quad Da = \frac{K}{H^2}, \quad F = \frac{C_F \rho G H^3}{\mu_{eff} \mu}. \quad (5)$$

Eq. (4) can be rewritten as

$$\frac{d^2u}{dy^2} - s^2u - Fsu^2 + \frac{1}{M} = 0, \quad (6)$$

where the porous media shape parameter is defined as

$$s = \left(\frac{1}{MDa} \right)^{1/2}. \quad (7)$$

Eq. (6) is to be solved subject to no slip boundary condition, i.e. $u = 0$ at $y = 1$, and

the symmetry condition or $\frac{du}{dy} = 0$ at $y = 0$.

The mean velocity U and the bulk mean temperature T_m are defined by

$$U = \frac{1}{H} \int_0^H u^* dy^*, \quad T_m = \frac{1}{HU} \int_0^H u^* T^* dy^*. \quad (8)$$

Further dimensionless variables are introduced as

$$\hat{u} = \frac{u^*}{U}, \quad (9)$$

and

$$\theta = \frac{T^* - T_w}{T_m - T_w}. \quad (10)$$

The Nusselt number Nu is

$$Nu = \frac{2Hq''}{k(T_w - T_m)}. \quad (11)$$

Local thermal equilibrium and homogeneity is assumed. The steady state thermal energy equation in the absence of heat source terms, axial conduction and thermal dispersion is then

$$\rho c_p u^* \frac{\partial T^*}{\partial x^*} = k \frac{\partial}{\partial y^*} \left(\frac{\partial T^*}{\partial y^*} \right). \quad (12)$$

It follows from the First Law of Thermodynamics that

$$\frac{\partial T^*}{\partial x^*} = \frac{2q''}{\rho c_p HU}. \quad (13)$$

As noted in [9], though the local temperature T^* is a function of both axial and radial coordinates the dimensionless temperature profile in the fully developed region, θ , is a function of the radial coordinate (y^*) only, while the bulk mean temperature is a function of the axial coordinate (x^*) only.

In non-dimensional form Eq. (12) becomes (when Eqns. (8-11) are used)

$$2 \frac{d^2 \theta}{dy^2} + \hat{u} \text{Nu} = 0, \quad (14)$$

where the boundary conditions are as follows

$$\left. \frac{d\theta}{dy} \right|_{y=0} = 0 \quad \text{and} \quad \theta|_{y=1} = 0. \quad (15)$$

3. ASYMPTOTIC SOLUTIONS

For some practical application of porous media one has $F=O(1)$, see [5] and [10] for example. The same range for the value of F is assumed in the present work. Two limiting cases of very small and very large s values are considered, the former being applicable to hyperporous cases and the latter being relevant to a low permeability porous medium. For more details on the topic one can consult [11].

3.1 Large Darcy number case

Considering the case of large Darcy number, one writes the following asymptotic expansion for the velocity distribution

$$u = u_0 + su_1 + \dots, \quad (16)$$

with the assumption that

$$s \ll 1 \quad (17)$$

Regular asymptotic expansions are written to solve Eq. (6) subject to the no-slip and symmetry boundary conditions, on the assumption that s is a small parameter as described in [7] and [12-14]. For the sake of brevity, the mathematical details of the problem are neglected and the results for the two first orders are presented as

$$u = \frac{1-y^2}{2M} + \frac{Fs}{120M^2} (15y^2 - 5y^4 + y^6 - 11) + O(s^2), \quad (18)$$

The zeroth-order solution is the familiar one that corresponds to the plane Poiseuille flow or a fluid clear of solid materials. Using Eq. (8), the mean velocity is found to be

$$U = \frac{1}{3M} \left(1 - \frac{6Fs}{35M} \right). \quad (19)$$

This implies that

$$\hat{u} = \frac{3}{2} (1 - y^2) + \frac{Fs}{280M} (33y^2 - 35y^4 + 7y^6 - 5) \quad (20)$$

Referring back to Eq. (14), one proceeds to find the temperature distribution as

$$\theta = \theta_0 + s\theta_1 + \dots \quad (21)$$

Using this expansion and solving the two first order solutions leads to

$$\theta = \frac{Nu}{16} \left(y^4 - 6y^2 + 5 - \frac{Fs}{840M} (3y^8 - 28y^6 + 66y^4 - 60y^2 + 19) \right). \quad (22)$$

Finally the Nusselt number can be found by substituting for \hat{u} and θ (and using Eqs. (20) and (22)) in the compatibility condition

$$\int_0^1 \hat{u}\theta \, dy = 1. \quad (23)$$

The solution is readily completed, and one finds that

$$Nu = \frac{70}{17} \left(1 + 0.009854 \frac{Fs}{M} \right) \quad (24)$$

Recovering the known analytical solution in the literature, one can check the new solution. For relatively large values of Da , i.e. $Da \rightarrow \infty$ or $s \rightarrow 0$, Eq. (18) gives $\hat{u} \rightarrow 3(1-y^2)/2$, as expected for plane Poiseuille flow in a duct clear of solid material.

Further, by Eq. (22), $\theta = \frac{Nu}{16}(y^4 - 6y^2 + 5)$ and by (24), $Nu=70/17$ or approximately

4.117. This agrees with the well-known value of Nu for the clear-fluid problem [8].

3.2 Small Darcy number case

When the Darcy number is sufficiently small ($s \gg 1$) the highest order derivative is multiplied by an small parameter, which is s^{-2} , as

$$s^{-2} \frac{d^2 u}{dy^2} - u - Fu^2 s^{-1} + \frac{1}{s^2 M} = 0, \quad (25-a)$$

One notes that the boundary layer is located near the wall, i.e. at $y=1$. According to Bush [12], one can find the outer expansion in regions far away from the wall by regular expansion. In this problem the outer expansion is found to be

$$u^{out} = \frac{1}{Ms^2}, \quad (25-b)$$

and the inner solution can be found by applying the stretched variable as

$$\eta = s(1 - y) \quad (25-c)$$

which, after neglecting smaller terms to obtain a one term solution [7], leads to the following form for the momentum equation

$$\frac{d^2 u^{in}}{d\eta^2} - u^{in} = 0, \quad (25-d)$$

The solution to the above equation is

$$u^{in} = C \exp(s(y-1)), \quad (25-e)$$

where the constant C should be found by matching. Prandtl's matching condition, as described in [12], is applied to find the composite expansion as

$$u = \frac{1 - \exp(s(y-1))}{Ms^2} \quad (26-a)$$

Integration by parts (as described by Nayfeh [7]) of Eq. (8) leads to the following value for the mean velocity

$$U = \frac{s-1}{Ms^3}. \quad (26-b)$$

It implies that

$$\hat{u} = \left(1 + \frac{1}{s}\right)(1 - \exp(-s(1-y))). \quad (27)$$

where the terms smaller than $O(s^{-1})$ are neglected in the above equation. This velocity distribution is now used to find the temperature distribution using Eq. (14) subject to the aforementioned boundary conditions. It is found that

$$\theta = \left(\frac{1}{4} + \frac{1}{4s}\right)(1 - y^2)\text{Nu} \quad (28)$$

Using the compatibility condition, the Nusselt number is found to be

$$\text{Nu} = 6\left(1 - \frac{2}{s}\right) \quad (29)$$

As a check on this solution, one examines the limiting case of $s \rightarrow \infty$, to see that the velocity distribution tends to a slug flow one. Explicitly, $u^* \rightarrow KG/\mu$ or $\hat{u} \rightarrow 1$, and so by Eq. (28), $\theta \rightarrow 3(1 - y^2)/2$ and $\text{Nu} \rightarrow 6$. Clearly the results are in good agreement with those of the Darcy flow model, see for example [2]. To present the results in the forgoing discussion the value of M is fixed ($M=1$) to work in terms of s instead of Da , however, in table 1 other values of M are applied to compare the results with those of [3].

4. NUMERICAL SOLUTION

A CDS finite difference scheme has been employed to integrate the governing equations (Eqs. (6), (14), and (23)) similar to what reported by Hooman and Ranjbar-Kani [13]. However, the present numerical scheme accounts for the form drag (nonlinear) term inclusion. To solve this nonlinear equation using the same SY subroutine (which solves a tri-diagonal system of equations), a linearization procedure is required. For this reason the nonlinear term is discretized as

$$\left(u_i^n\right)^2 = u_i^n u_i^{n-1} \quad (30)$$

in which the superscripts show iteration number. A uniform velocity profile is assumed as the initial guess. Next the system of algebraic equations, which emerges as a result of discretization, is solved applying the subroutine SY [13]. This newly obtained solution is then compared with that of the previous iteration (initial guess for $n=1$) and this procedure is continued until the maximum relative error in the values of the local velocity between two successive iterations become less than 10^{-5} . The other steps to achieve the numerical results are the same as those of [13] and, for this reason, are not repeated here. All runs were performed with 190 grid points while it was observed that a finer mesh (380 grid point) will not alter the results to three significant figures. Accuracy of the numerical results was verified as shown in table 1.

5. RESULTS AND DISCUSSION

5.1 Hydrodynamic aspects

The velocity field is presented in Fig. 2. This figure shows the effect of the parameter s on the fully developed velocity profile, which contains a relatively flat portion

located around the centerline. This is similar to results of Hooman and Merrikh [15] for flow through a duct of rectangular cross section. When $s \rightarrow \infty$, the velocity profile tends to that of the Darcy model, and when this parameter decreases to zero the velocity tends to the plane Poiseuille flow, as expected. For general values of the porosity, permeability, viscosity and the length-scale, the velocity profile is bounded by these two limiting curves. One observes that our theoretical predictions are in good agreement with numerical counterparts. Moreover, better agreement is observed for higher values of s .

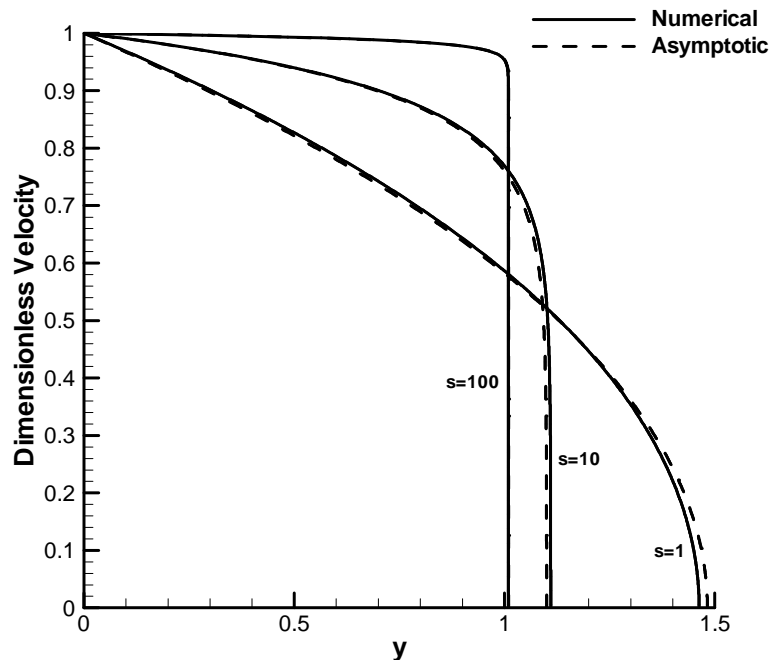


Fig. 2: Dimensionless velocity profiles for some values of s ($F=M=1$)

5.2 Heat transfer aspects

The variation of the Nusselt number as a function of the parameter s is shown in Fig. 3. As mentioned before, the value of Nu lies between its values for the cases of plane Poiseuille flow and slug flow, i.e. between 4.117 and 6. Fig. 3-b shows the Nusselt

number versus F for small values of s . One observes that Nu increases with F and this is in line with the results of previous investigations [3-5]. One notes that, similar to what Vafai and Kim [4] reported, for very large values of s , the problem is not very sensitive to the value of F . Finally the temperature distribution is shown in Fig. 4 for some values of s . The results show that the centerline temperature increases with increase in s .

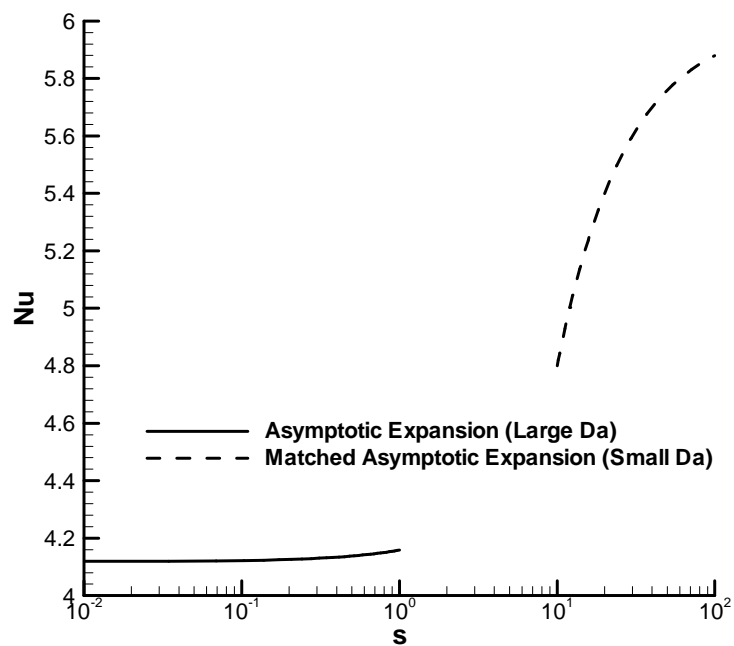


Fig. 3-a. The Nusselt number versus small and large s values ($F=M=1$)

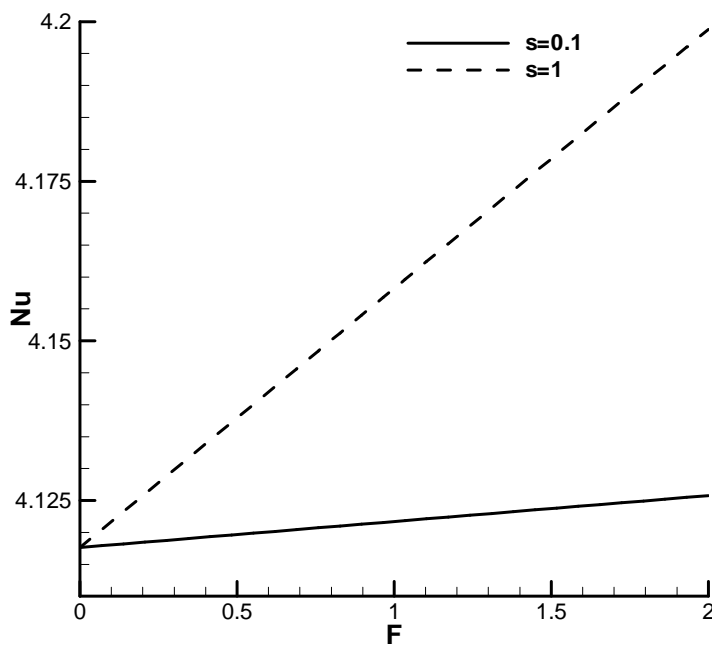


Fig. 3-b. The Nusselt number versus F for some small values of s .

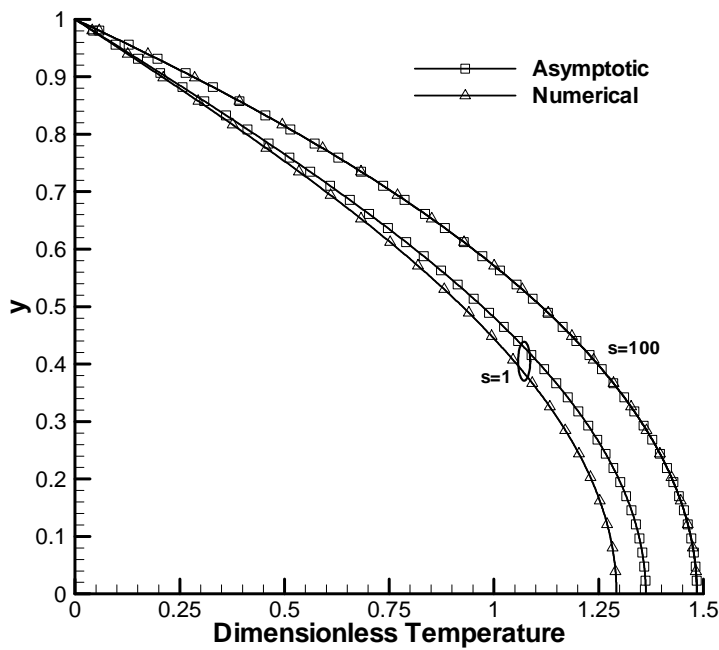


Fig. 4. Dimensionless temperature distribution for some values of s ($F=1$).

One can check the accuracy of the results versus those in the literature with Brinkman flow model, for example [16-20], or the Darcy model [21-24], but the most appropriate results for comparison purposes are those of [3-5] among which [3] has a similar dimensionless parameters that makes the comparison easier. Moreover, there is no limitation on the results of [3] for a wide range of permeability and porosity from natural porous medium to hyperporous cases as stated by the authors.

Table 1 A comparison between present results with those of Nield et al. [3]

Parameters	$M=s=F=1$	$M=10,s=0.316,F=1$	$M=s=10, F=1$	$M=F=1, s=100$
Present (asymptotic)	4.158	4.119	4.8	5.962
Present (numerical)	4.181	4.1319	5.139	5.8935
Ref. [3]	4.159	4.122	5.129	-

6. SUMMARY

Fully developed forced convection in a porous-saturated parallel plate channel, with the inclusion of boundary and inertial effects, were solved asymptotically and numerically. The most important observations are as follows.

- The velocity profile depends strongly on the parameters F and s , when s is very small compared to unity. It is worth noting that, within the range of our approximations, these two parameters affect the velocity profile in a similar way. As s increases, the central region containing a relatively uniform velocity distribution spreads further toward the walls and the effects of form drag

becomes less significant. At large s , the velocity profile is confined to a very thin layer adjacent to the walls and as $s \rightarrow \infty$ the limiting slug flow is observed.

- The value of the Nusselt number increases with an increase in either s or F . For small values of s the Nusselt number tends to be higher for higher F values, however, for large values of s no change is inspected in Nu as the value of F varied.
- The shape of the temperature profile does not change significantly with s or F but the centerline temperature enhances as s increases, as shown in Fig. 4.

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